Title: RG flows and Boundary States in 2d CFTs

Date: Jan 10, 2017 03:30 PM

URL: http://pirsa.org/17010060

Abstract:

Renormalization Group Flows and Boundary States in Conformal Field Theories

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arXiv:1701.xxxxx

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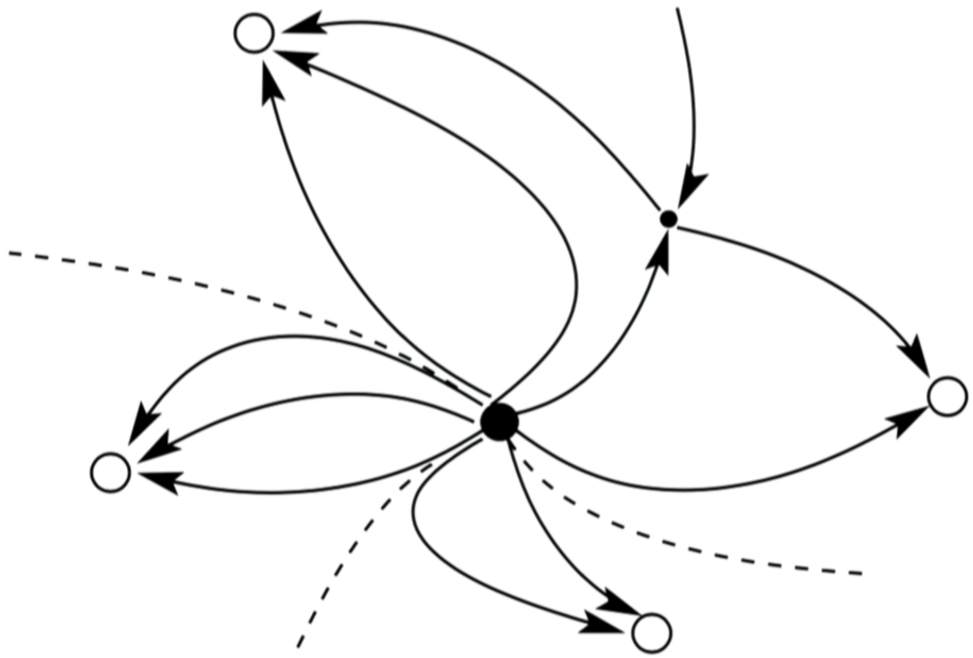
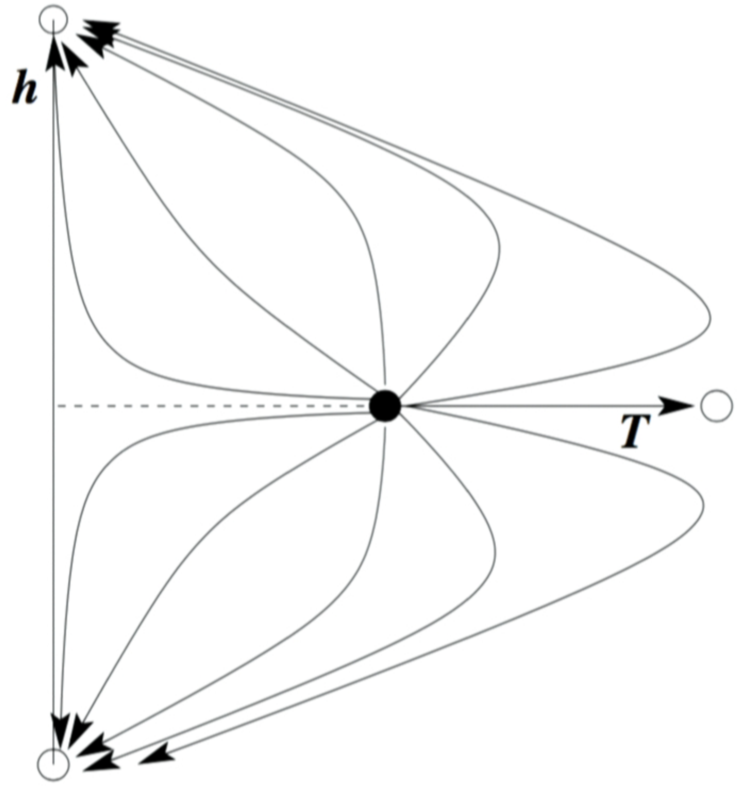
RG fixed points and sinks



Each stable sink fixed point corresponds to a *phase*

Example: Ising critical point



The general problem

Given a RG fixed point and a set of relevant operators $\{\Phi_j\}$

$$\mathcal{H} = \mathcal{H}^* + \sum_j \sum_x g_j \, \Phi_j(x)$$

where do the RG flows end up for different choices of the $\{g_j\}$?

What is the phase diagram in the vicinity of the critical point?

How do we relate UV and IR physics?

The field theory perspective

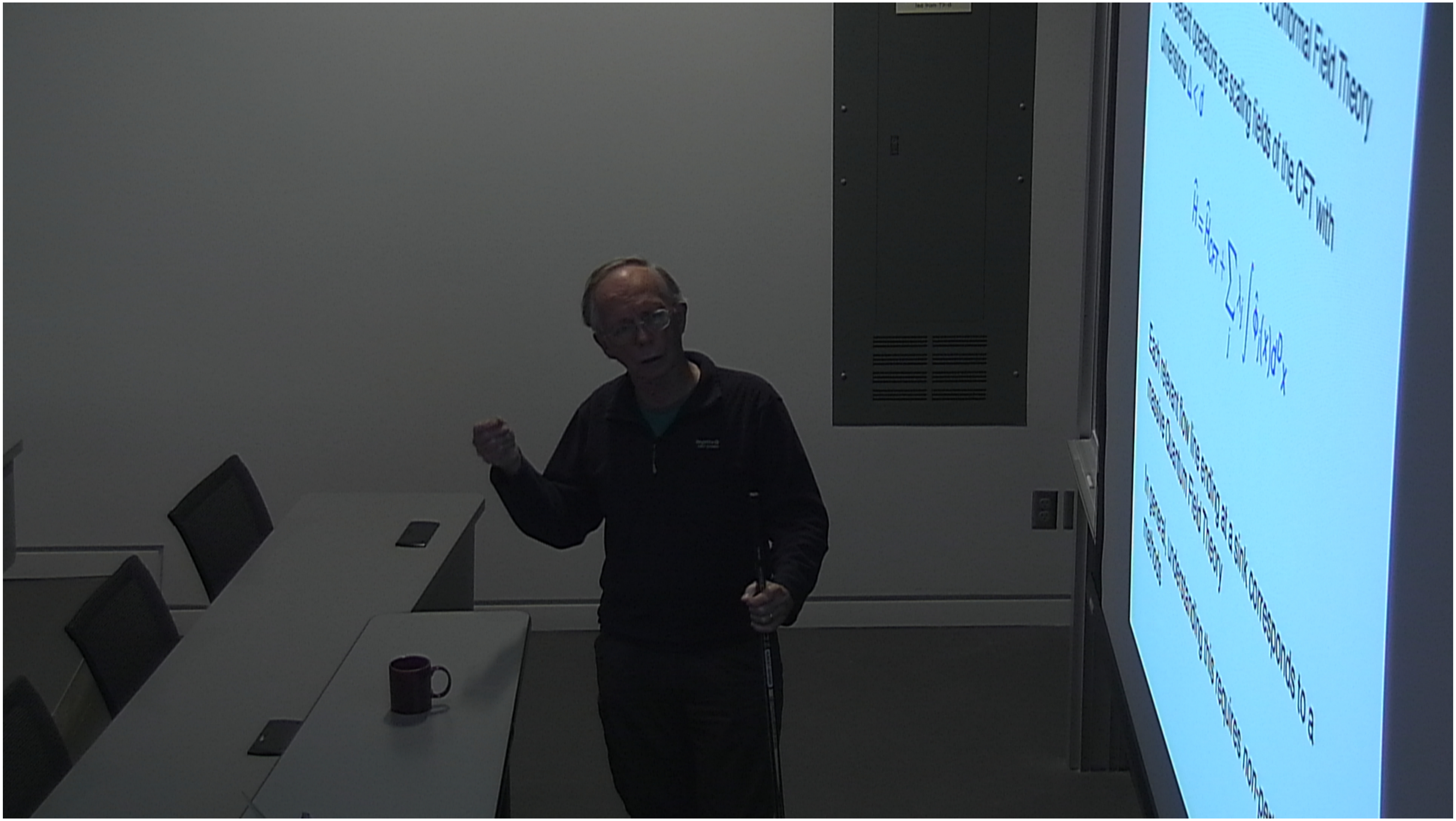
Each RG fixed point corresponds to a Conformal Field Theory

The relevant operators are scaling fields of the CFT with dimensions $\Delta < d$

$$\widehat{H} = \widehat{H}_{CFT} + \sum_j \lambda_j \int \hat{\Phi}_j(x) d^D x$$

Each relevant flow line ending at a sink corresponds to a massive Quantum Field Theory

In general, understanding this requires *non-perturbative* methods


Conformal Field Theory
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Boundary states

Another way of understanding the physics is through the different possible *boundary conditions* which may be imposed on the CFT.

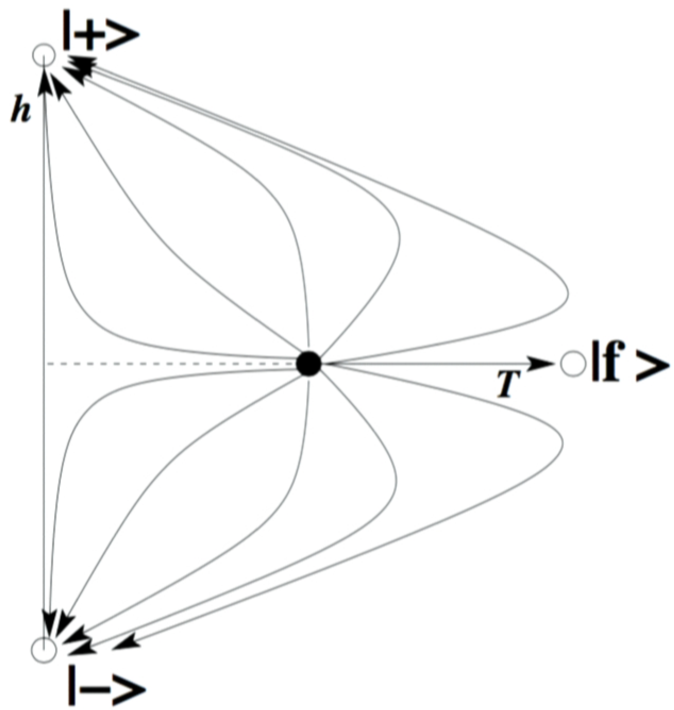
A special set of boundary conditions are *conformal*, corresponding to fixed points of the *boundary* RG flows.

In the language of QFTs in $D+1$ dimensions, these correspond to *boundary states* $|\mathcal{B}\rangle$ satisfying

$$\widehat{T}_{0k}(x)\,|\mathcal{B}\rangle = 0$$

We conjecture that the conformal boundary states label the possible sinks of bulk RG flows,

e.g. for Ising there are 3 such states, $|\text{free}\rangle, |+\rangle, |-\rangle$:



So we can rephrase the question as

"Which boundary state best approximates

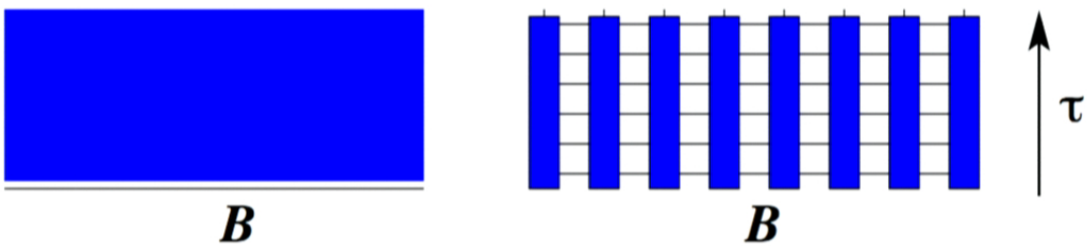
the ground state of $\widehat{H}$ at strong coupling?"

One way around this is to consider *smeared* boundary states

$$e^{-\tau \widehat{H}_{CFT}} \, |\mathcal{B}\rangle$$

These have finite correlation length $\propto \tau$ and finite energy $\propto 1/\tau$

They can be viewed as a continuum version of matrix product states



Motivation: quantum quenches

In a quantum quench, a system is prepared in a state $|\Psi_0\rangle$ and evolves unitarily with a hamiltonian $\widehat{H}$.

One question is whether subsystems reach a stationary state and, if so, what?

In 2006 Calabrese + JC chose $|\Psi_0\rangle$ to be a smeared boundary state, evolved with $\widehat{H}_{CFT}$, and showed that subsystems then *thermalize* after a time $\propto$ their length.

This can be seen as a consequence of the propagation of entangled EPR pairs, a picture which holds much more widely.

A motivation for the current study is which smeared boundary state $|\mathcal{B}\rangle$ should be chosen to best approximate the case when $|\Psi_0\rangle$ is the ground state of a gapped theory?

Back to the problem

The problem then reduces to a variational one: take a general smeared boundary state

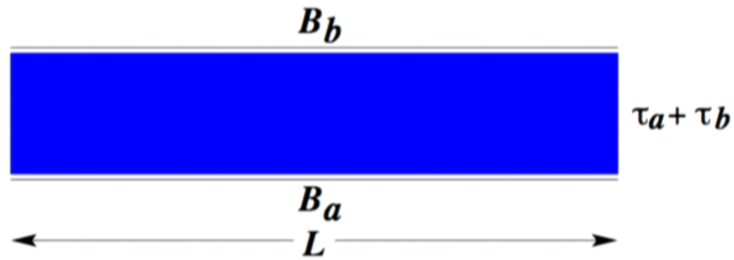
$$|\Psi\rangle = \sum_a \alpha_a \, e^{-\tau_a \widehat{H}_{CFT}} |\mathcal{B}_a\rangle$$

and minimize

$$E_{var} = \frac{\langle\Psi| \, \widehat{H}_{CFT} + \sum_j \lambda_j \int \hat{\Phi}_j(x) d^D x \, |\Psi\rangle}{\langle\Psi|\Psi\rangle}$$

with $\{\alpha_a\}, \{\tau_a\}$ as variational parameters.

Normalization $\langle\Psi|\Psi\rangle$



$$\langle\mathcal{B}_a|e^{-\tau_a H_{CFT}}e^{-\tau_b H_{CFT}}|\mathcal{B}_b\rangle$$

is the partition function Z_{ab} in a long strip.

If $a = b$ this is dominated by the Casimir energy
$Z_{aa} \sim \exp\left(\sigma_a(L/2\tau_a)^D\right)$

For $a \neq b$, Z_{ab} is exponentially smaller than $(Z_{aa}Z_{bb})^{1/2}$ as $L \to \infty$ due to the interfacial energy.

So the off-diagonal terms are suppressed – similarly in the numerator.

Back to the problem

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with $\{\alpha_a\}, \{\tau_a\}$ as variational parameters.

So the problem simplifies for $L \gg \tau_a$:

$$E_{var}/L^D = \sum_a \alpha_a^2 \left(\frac{\sigma_a}{(2\tau_a)^{D+1}} + \sum_j \lambda_j \langle \Phi_j \rangle_a \right)$$

where $\sum_a \alpha_a^2 = 1$ and

$$\langle \Phi_j \rangle_a = \frac{A_j^a}{(2\tau_a)^{\Delta_j}}$$

is the one-point function of Φ_j in the center of a strip of width $2\tau_a$ with boundary condition a on each edge.
A_j^a is a universal amplitude.

The minimum occurs when all but one of the $\{\alpha_a\}$ vanish (i.e. a pure physical state.)

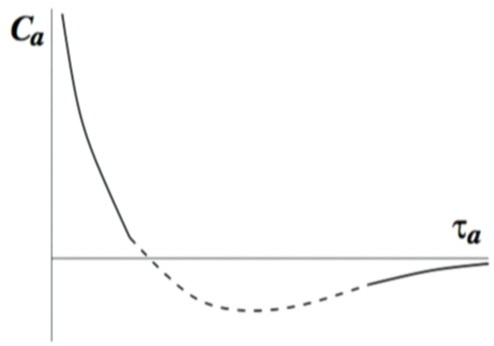
We should minimize each term

$$C_a = \frac{\sigma_a}{(2\tau_a)^{D+1}} + \sum_j \lambda_j \frac{A_j^a}{(2\tau_a)^{\Delta_j}}$$

wrt τ_a and choose the a which gives the smallest value.

Since $\Delta_j < D+1$, $C_a \to +\infty$ as $\tau_a \to 0$

As $\tau_a \to \infty$ $C_a \to 0$ and is dominated by the most relevant operator with $\lambda_j \neq 0$. [At least in 2d] we can show that there always exists an a such that the approach is from below, so that there is always a minimum at finite τ_a



RG flows

$$C_a = \frac{\sigma_a}{(2\tau_a)^{D+1}} + \sum_j \lambda_j \frac{A_j^a}{(2\tau_a)^{\Delta_j}}$$

C_a scales multiplicatively under

$$\lambda_j \to e^{(D+1-\Delta_j)\ell}\lambda_j\,, \qquad \tau_a \to e^{-\ell}\tau_a$$

so once we have found the absolute minimum a for a particular set of couplings $\{\lambda_j\}$, it is the same along the RG trajectory ☺

2d minimal CFTs

Unitary 2d CFTs with $c < 1$ are well understood, and give the scaling limits of simple 2d universality classes.

Bulk operators Φ_j are labelled by entries $j = (r, s)$ in the Kac table with $1 \leq s \leq r \leq m - 1$, with m an integer ≥ 3 and $c = 1 - 6/m(m + 1)$.

In the diagonal A_m models each value of (r, s) occurs just once.

The physical boundary states $\mathcal{B}_a$ are also labelled by entries in the Kac table, one for each value of (r, s).

1-point amplitudes are also known [Lewellen + JC 1991]:

$$A_a^j = \frac{S_a^j}{S_a^0}\left(\frac{S_0^0}{S_j^0}\right)^{1/2}$$

where S_a^j is the modular S-matrix – symmetric, orthogonal, with $S_j^0 > 0$

$$S_{r,s}^{r',s'} \propto (-1)^{(r+s)(r'+s')} \sin\frac{\pi rr'}{m} \sin\frac{\pi ss'}{m+1}$$

Note that for any j we can always choose a so that $\lambda_j A_a^j < 0$, so there is always a minimum for some a.

We can also show that for a particular state b there is a choice of the $\{\lambda_j\}$ so that

$$\sum_j \lambda_j A_a^j < 0 \ (a = b); \quad \sum_j \lambda_j A_a^j > 0 \ (a \neq b)$$

So all boundary states b represent an achievable RG sink.

Example: the Ising model

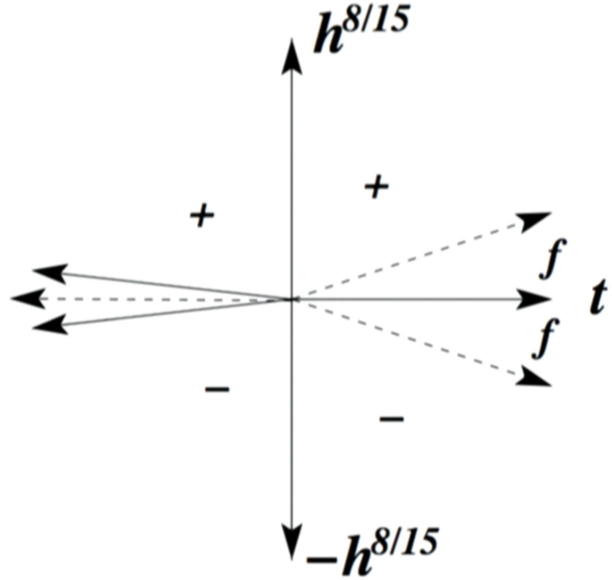
$\widehat{H} = \widehat{H}_{CFT} + t\int \varepsilon dx + h\int \sigma dx$

$\{\Phi_j\} = (\varepsilon, \sigma)$, boundary states $(+, -, f)$.

$$C_+ = \frac{1}{48\tau^2} + \frac{t}{\tau} - 2^{1/4}\frac{h}{\tau^{1/8}}$$
$$C_- = \frac{1}{48\tau^2} + \frac{t}{\tau} + 2^{1/4}\frac{h}{\tau^{1/8}}$$
$$C_f = \frac{1}{48\tau^2} - \frac{t}{\tau}$$

[In units where $2\pi = 1$.]

For $t > 0$, $h = 0$, f wins
For $t < 0$, $h > 0$, $-$ wins
For $t < 0$, $h < 0$, $+$ wins.

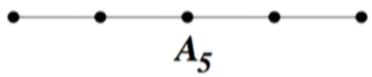


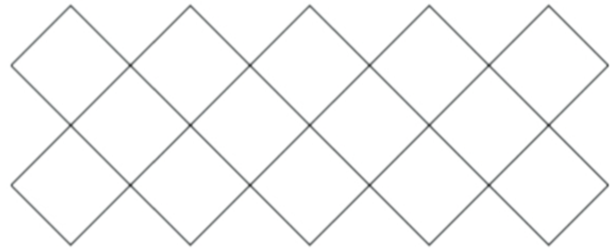
The $\pm$ sinks do not extend all the way to $h = 0$ for $t > 0$
There is an unphysical phase boundary along $h^{8/15}/t \approx 0.1$.

A general feature of this simple variational approximation:
1st-order transitions between different sinks. ☹☹

A_m lattice models

The A_m RSOS models are simple integrable lattice realizations of the diagonal A_m 2d CFTs.





At each site r of a square lattice is a height $h(r) \in A_m$ Dynkin diagram.
Neighboring heights satisfy RSOS condition $|h(r) - h(r')| = 1$.

Boltzmann weights and local operators are defined in terms of the matrix s_a^b of eigenvectors of the adjacency matrix

$$s_a^b \propto \sin \frac{\pi ab}{m+1}$$

UV divergences

If we switch on a single operator $\lambda\Phi$ of dimension Δ, simple scaling implies

$$\langle E\rangle/L \propto \lambda^{2/(2-\Delta)}$$

and this is what comes out of the variational approach (with a definite value for the coefficient).

However, although for $\Delta < 2$ there are no new UV divergences in correlation functions, there are in the ground state energy. E.g. to second order

$$\delta E/L = -\frac{\lambda^2}{2}\int \frac{d^2x}{|x|^{2\Delta}}$$

which is UV divergent for $\Delta \geq 1$.

So the variational calculation is bounding something which is in fact $-\infty$

The solution is to incorporate these as counterterms $\propto \lambda^2(\tau/\epsilon)^{2-2\Delta}$ in the variational energy, where ϵ is the UV cut-off.

When taken into account, they give the expected terms in the energy which are analytic in λ.

For the thermal perturbation of the Ising model ($\Delta = 1$), they give the well-known $t^2 \log|t|$ behavior.

Summary

- smeared boundary states give a simple way of understanding the end points of relevant RG flows for CFTs
- they give a rigorous upper bound on the free energy (ground state energy) of the massive theory
- for 2d minimal models every boundary state corresponds to the end point of an RG flow, but these have finite width with possibly unphysical first-order transitions between them
- the variational states could be improved, and this feature possibly removed, at a considerable cost in computational effort.