Title: Spin-Field Correspondence

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Abstract:

Spin-Field Correspondence

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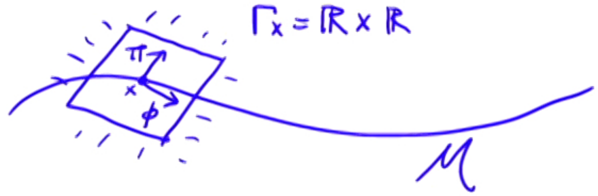
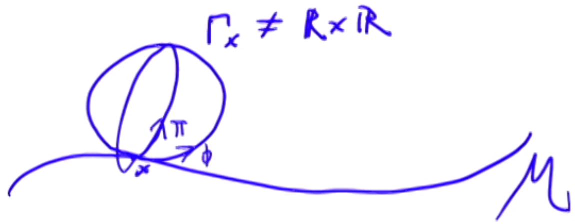
Nonlinear Field Space Theory

Standard Field Theory - Linear Field Space:



Nonlinear Field Space Theory[1]:



[1] J. M. & T. Trześniewski "The Nonlinear Field Space Theory", Physics Letters B **759** (2016) 424.

Relations/inpirations

- A compact field space is a natural way to implement the "Principle of finiteness" of physical theories, which once motivated the Born-Infeld theory (1938). Dynamical constraint on the field values.
- NFST is similar to the case of a relativistic particle, where the maximal speed of propagation is a result of the spacetime geometry.
- Lattice field theories $\rightarrow$ compact field spaces on discrete lattice.
- Non-linear sigma models (GellMann,1960; Witten,1984) - multi-component scalar field (but usually not field velocities or momenta) are constrained to lie on a Riemannian manifold.
- Born reciprocity (1949).
- Relative Locality, curved particle momentum spaces.
- Loop Quantum Gravity, polymer quantization.
- Understanding an origin of the Hamiltonian/Lagrangian functions for fields.

Spherical phase space

The phase space of a classical spin is a two-sphere, S^2.

Kirillov orbit method

"If an orbit is the phase space of a G-invariant classical mechanical system then the corresponding quantum mechanical system ought to be described via an irreducible unitary representation of G." Here, $G = SU(2)$ and the orbit $S^2 = SU(2)/U(1)$.

The phase space is a symplectic manifold and it has to be equipped with the closed symplectic form $\omega = S \sin\theta \, d\phi \wedge d\theta$. Let us consider the following change of coordinates:

$$\begin{aligned} \phi &= \frac{q}{R_1} \in (-\pi, \pi], \\ \theta &= \frac{\pi}{2} - \frac{p}{R_2} \in (0, \pi), \end{aligned}$$

where R_1 and R_2 are constants.

Using the new variables, the symplectic form can be written as

$$\omega = \cos\left(\frac{p}{R_2}\right) dp \wedge dq,$$

where we fixed $R_1 R_2 = S$.

Except the poles $\theta = 0, \pi$, the symplectic form ω is well defined and invertible, allowing for determination of the Poisson tensor $\mathcal{P}^{ij} = (\omega^{-1})^{ij}$, and then we can define the Poisson bracket:

$$\{f, g\} = \mathcal{P}^{ij}(\partial_i f)(\partial_j g) = \frac{1}{\cos(p/R_2)}\left(\frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}\right).$$

Such that the bracket of the canonical (q, p) variables is

$$\{q, p\} = \frac{1}{\cos(p/R_2)}.$$

The Hamilton equation can then be defined as $\frac{d}{dt}f = \{f, H\}$, where f is some phase space function and H is the Hamiltonian.

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In case of S^2 it is convenient to work with the components of the angular momentum vector $\mathbf{S} = (S_x, S_y, S_z)$, which are globally defined functions:

$$
\begin{aligned}
S_x &:= S\sin\theta\cos\phi = S\cos\left(\frac{p}{R_2}\right)\cos\left(\frac{q}{R_1}\right) \\
&= S\left(1 - \frac{p^2}{2R_2^2} - \frac{q^2}{2R_1^2} + \mathcal{O}(4)\right), \\
S_y &:= S\sin\theta\sin\phi = S\cos\left(\frac{p}{R_2}\right)\sin\left(\frac{q}{R_1}\right) = S\left(\frac{q}{R_1} + \mathcal{O}(3)\right), \\
S_z &:= S\cos\theta = S\sin\left(\frac{p}{R_2}\right) = S\left(\frac{p}{R_2} + \mathcal{O}(3)\right),
\end{aligned}
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together with the condition $S_x^2 + S_y^2 + S_z^2 = S^2 = const$. With use of the Poisson bracket one can easily show that the S_i components satisfy the $su(2)$ algebra bracket $\{S_i, S_j\} = \epsilon_{ijk} S_k$.

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Dynamics - spin precession

In the atomic physics, magnetic moment couples to an external magnetic field $\mathbf{B}$ via the vector $\mathbf{S}$. In such a case, the Hamiltonian of interaction is

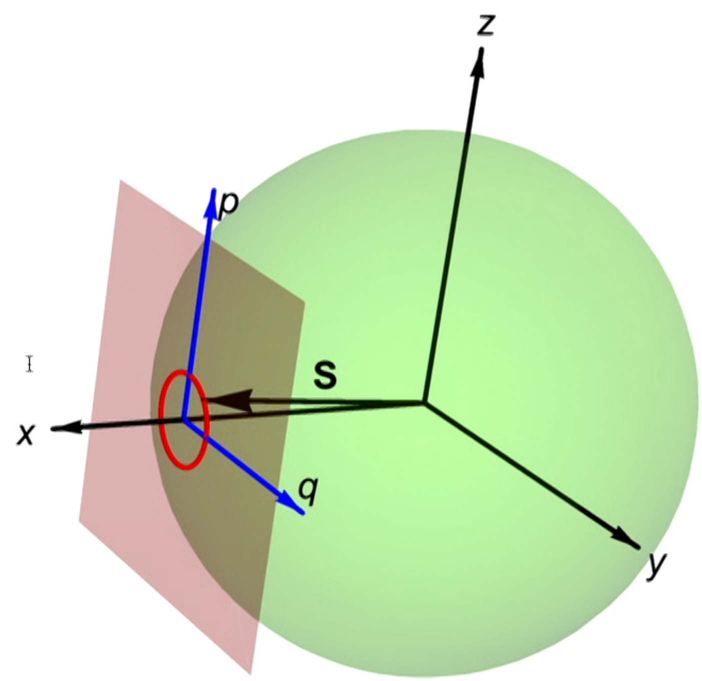
$$H = -\frac{\mu}{S}\mathbf{S}\cdot\mathbf{B},$$

where μ is the value of the magnetic moment, which can be both positive and negative. Based on the above Hamiltonian we obtain the spin precession equation $\dot{\mathbf{S}} = \{\mathbf{S}, H\} = -\frac{\mu}{S}\mathbf{B}\times\mathbf{S}$.

Let us fix $\mathbf{B} = (B_x, 0, 0)$ so that precession takes place around the origin of the (q, p) coordinate system. Then, for small spin displacements from the equilibrium point, the Hamiltonian

$$H \approx -\mu B_x \left(1 - \frac{p^2}{2R_2^2} - \frac{q^2}{2R_1^2}\right) = \frac{p^2}{2m} + \frac{\omega^2 q^2}{2} + const,$$

where the constants $m := \frac{R_2^2}{\mu B_x}$ and $\omega := \frac{\sqrt{\mu B_x}}{R_1}$.



The precession of the vector **S** corresponds to an ellipse in the (q, p) phase space. The precession (for small precession angles) is, therefore, described by harmonic oscillator.

The picture can be generalized to the area of field theory. For this purpose, let us consider a continuous spin distribution. Then, at any space point the following identification can be performed:

$$q \to \varphi(\mathbf{x}, t),$$
$$p \to \pi_{\varphi}(\mathbf{x}, t).$$

The continuous spin system is, therefore, in correspondence with the scalar field theory with the spherical field phase space. This is basically because dimension of the scalar field phase space $\Gamma^{\varphi}_{\mathbf{x}}$ at any point is equal to the dimension of the spin phase space:

$$\dim(\Gamma^{\varphi}_{\mathbf{x}}) = \dim S^2.$$

Depending on the particular form of the interactions between the spins, different types of the field theories with the bounded field spaces can be reconstructed.

An example: The XXZ Heisenberg model

The XXX Heisenberg model ($\Delta = 1$) ("Spin-Field Correspondence" J. M. 2016 arXiv:1612.04355)

Generalization to the XXZ case (J. M., S. Brahma, J. Bilski, A. Marciano, to appear very soon).

The discrete XXZ Heisenberg model can be introduced by the following Hamiltonian:

$$H_{XXZ} = -J \sum_{ij} \left(S_i^x S_j^x + S_i^y S_j^y + \Delta S_i^z S_j^z \right) - \mu \sum_i \mathbf{S}_i \cdot \mathbf{B},$$

where the first sum is performed over the nearest neighbors and J and μ are the coupling constants. $\mathbf{B}$ denotes an external magnetic field vector.

Continuous case

In the continuum limit the discrete Hamiltonian becomes

$$H^{\rm cont}_{XXZ} = -\tilde{J}\int d^3x\,\left[(\nabla S_x)^2 + (\nabla S_y)^2 + \Delta(\nabla S_z)^2\right] - \tilde{\mu}\int d^3x\,\mathbf{S}\cdot\mathbf{B},$$

For $\mathbf{B} = (B_x, 0, 0)$, the lowest order Hamiltonian in the representation of the field variables is:

$$H_\varphi = \int d^3x\left[\frac{\pi_\varphi^2}{2} + \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2 + \frac{\Delta}{2m^2}(\nabla\pi_\varphi)^2\right],$$

where the constants

$$m := \tilde{\mu}B_x, \quad \tilde{J} = \frac{1}{2Sm},$$

$$R_1 = \sqrt{\frac{S}{m}}, \quad R_2 = \sqrt{Sm},$$

together with the condition $S = R_1R_2$.

The field theoretical Poisson bracket

$$\{f(\mathbf{x}), g(\mathbf{y})\} = \int \frac{d^3\mathbf{z}}{\cos(\pi_\varphi(\mathbf{z})/R_2)} \left(\frac{\delta f(\mathbf{x})}{\delta\varphi(\mathbf{z})} \frac{\delta g(\mathbf{y})}{\delta\pi_\varphi(\mathbf{z})} - \frac{\delta f(\mathbf{x})}{\delta\pi_\varphi(\mathbf{z})} \frac{\delta g(\mathbf{y})}{\delta\varphi(\mathbf{z})} \right),$$

based on which, the leading order equations of motion are:

$$\begin{aligned} \dot{\varphi} &= \{\varphi, H_\varphi\} = \pi_\varphi - \frac{\Delta}{m^2}\nabla^2\pi_\varphi, \\ \dot{\pi}_\varphi &= \{\pi_\varphi, H_\varphi\} = -m^2\varphi + \nabla^2\varphi, \end{aligned}$$

which lead to the following modified version of the Klein-Gordon equation:

$$\ddot{\varphi} - (1+\Delta)\nabla^2\varphi + m^2\varphi + \frac{\Delta}{m^2}\nabla^4\varphi = 0.$$

The relativistic case is recovered in the $\Delta \to 0$ limit (XX or XY Heisenberg model).

Performing the Fourier transform

$$\varphi(t,\mathbf{x}) = \int \frac{d^3k d\omega}{(2\pi)^4} \varphi(\omega,\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)},$$

the following dispersion relation

$$\omega^2 = (1+\Delta)\, k^2 + m^2 + \Delta \frac{k^4}{m^2} = (k^2+m^2)\left(1+\Delta\frac{k^2}{m^2}\right).$$

is satisfied. From here, the group velocity

$$v_{gr} := \frac{\partial \omega}{\partial k} = \frac{k}{\omega}\left[1+\Delta\left(1+2\frac{k^2}{m^2}\right)\right],$$

and in consequence, the following relation holds

$$v_{gr}v_{ph} = 1 + \Delta\left(1+2\frac{k^2}{m^2}\right),$$

which might be both greater and smaller that than one depending on the sign of Δ.

The Spin-Field Correspondence conjecture

Observation:

$\dim(\Gamma_x^{\phi}) = \dim S^2$.

Spin precession = a scalar field oscillation at the phase plane.

Spin wave = a scalar field excitation.

Generalization to the different types of fields is possible:

- (s=0) scalar field - $\dim(\Gamma_x) = 2 \Leftrightarrow$ 1-spin (S^2)
- (s=1/2) spinor field - $\dim(\Gamma_x) = 4 \Leftrightarrow$ 2-spins ($S^2 \times S^2$)
- (s=1) vector field - $\dim(\Gamma_x) = 6 \Leftrightarrow$ 3-spins ($S^2 \times S^2 \times S^2$)
- (s=3/2) Rarita-Schwinger field - $\dim(\Gamma_x) = 8 \Leftrightarrow$ 4-spins
- (s=2) tensor field - $\dim(\Gamma_x) = 10 \Leftrightarrow$ 5-spins ($(S^2)^5$)

A method to design condensed matter systems corresponding to given field theories.

Are fields excitations of some more fundamental spin structure?

The Nonlinear Field Space Cosmology

("Nonlinear Field Space Cosmology," J. M. & T. Trześniewski, February 2017)

A cosmology from the condensed matter system Hamiltonian.

The matter Hamiltonian which reduces to the massive scalar field case in the leading order is:

$$
\begin{aligned}
H_{\mathbf{S}} &= mN(S - S_x) = mqN\frac{1}{q}(S - S_x) \\
&= Nq\left(\frac{\pi_\varphi^2}{2q^2} + \frac{1}{2}m^2\varphi^2\right) + \mathcal{O}(4)\,.
\end{aligned}
$$

The Friedmann equation:

$$
H^2 = \left(\frac{1}{3}\frac{\dot{q}}{q}\right)^2 = \frac{\kappa}{3}\rho\,,
$$

with the matter energy density $\rho = \frac{m}{q}(S - S_x)$. Here $q = a^3$.

The equations of motion (for $N = 1$):

$$\dot{S}_x = \{S_x, H_{\text{tot}}\} = -\frac{3}{4}\kappa p \left(-S_y \arctan\frac{S_y}{S_x} + \frac{S_x S_z}{\sqrt{S^2 - S_z^2}} \arcsin\frac{S_z}{S}\right),$$

$$\dot{S}_y = \{S_y, H_{\text{tot}}\} = +m\, S_z$$
$$-\frac{3}{4}\kappa p \left(S_x \arctan\frac{S_y}{S_x} + \frac{S_y S_z}{\sqrt{S^2 - S_z^2}} \arcsin\frac{S_z}{S}\right),$$

$$\dot{S}_z = \{S_z, H_{\text{tot}}\} = -m\, S_y + \frac{3}{4}\kappa p \sqrt{S^2 - S_z^2} \arcsin\frac{S_z}{S},$$

together with

$$\dot{q} = -\frac{3}{2}\kappa p q,$$

$$\dot{p} = \frac{3}{4}\kappa p^2 + m\frac{1}{2q}\left(-S_y \arctan\frac{S_y}{S_x} + \frac{S_x S_z}{\sqrt{S^2 - S_z^2}} \arcsin\frac{S_z}{S}\right).$$

A scalar field in the Fourier representation

Assuming that the original spin Hamiltonian is of the form $H \propto \mathbf{S} \cdot \mathbf{B} = S_x B_x$, the scalar field Hamiltonian in the Fourier space can be written as:

$$
\begin{aligned}
H_\phi &= \sum_{\mathbf{k}} H_{\mathbf{k}}\,, \quad \text{where} \\
H_{\mathbf{k}} &:= -Sk \cos\left(\frac{\pi_{\mathbf{k}}}{\sqrt{Sk}}\right) \cos\left(\sqrt{\frac{k}{S}}\phi_{\mathbf{k}}\right) \\
&= -Sk + \frac{1}{2}\left(\pi_{\mathbf{k}}^2 + k^2\phi_{\mathbf{k}}^2\right) - \frac{k}{4S}\phi_{\mathbf{k}}^2\pi_{\mathbf{k}}^2 \\
&- \frac{1}{24Sk}\left(\pi_{\mathbf{k}}^4 + k^4\phi_{\mathbf{k}}^4\right) + \mathcal{O}(S^{-2})\,,
\end{aligned}
$$

together canonical bracket

$$
\{\phi_{\mathbf{k}}, \pi_{\mathbf{k}'}\} = \sec\left(\frac{\pi_{\mathbf{k}}}{\sqrt{Sk}}\right) \delta_{\mathbf{k},\mathbf{k}'}\,.
$$

The obtained Hamiltonian can be perturbatively diagonalized (at least up to the order S^{-1}) with the use of creation and annihilation operators. Due to the deformation of the canonical commutation relation, the expressions for the creation and annihilation operators $\hat{a}^{\dagger}_{\mathbf{k}}$, $\hat{a}_{\mathbf{k}}$ will differ from the usual ones. Furthermore, the $\hat{a}^{\dagger}_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}$ fulfill the q-deformed version of their commutation relation: $\hat{a}_{\mathbf{k}}\hat{a}^{\dagger}_{\mathbf{k}} - q\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}} = \hat{\mathbb{I}}$.

This allows us to express the field operators as follows:

$$\hat{\phi}_{\mathbf{k}} = \sqrt{\frac{\hbar}{2k}} \frac{\left(\hat{a}_{\mathbf{k}} + \hat{a}^{\dagger}_{\mathbf{k}}\right)}{\sqrt{1+\frac{\hbar}{2S}}}, \quad \hat{\pi}_{\mathbf{k}} = -i\sqrt{\frac{\hbar k}{2}} \frac{\left(\hat{a}_{\mathbf{k}} - \hat{a}^{\dagger}_{\mathbf{k}}\right)}{\sqrt{1+\frac{\hbar}{2S}}},$$

where the q-deformation factor:

$$q = \frac{1-\frac{\hbar}{2S}}{1+\frac{\hbar}{2S}} = 1 - \frac{\hbar}{S} + \mathcal{O}(S^{-2}).$$

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$$q = \frac{1-\frac{\hbar}{2S}}{1+\frac{\hbar}{2S}} = 1 - \frac{\hbar}{S} + \mathcal{O}(S^{-2}).$$

The total Hilbert space of the system is $\mathcal{H} = \bigotimes_{\mathbf{k}} \mathcal{H}_{\mathbf{k}}$, where $\mathcal{H}_{\mathbf{k}} = \text{span}\{|0_{\mathbf{k}}\rangle, |1_{\mathbf{k}}\rangle, \dots, |n_{\max,\mathbf{k}}\rangle\}$. The actions of the $\hat{a}^{\dagger}_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}$ operators on the $|n_{\mathbf{k}}\rangle$ basis states are found to have the form:

$$\hat{a}^{\dagger}_{\mathbf{k}}|n\rangle = \sqrt{\frac{1-q^{n+1}}{1-q}}|n+1\rangle\,, \quad \hat{a}_{\mathbf{k}}|n\rangle = \sqrt{\frac{1-q^{n}}{1-q}}|n-1\rangle\,,$$

giving the q-deformed expression for the occupation number operator $\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}}|n_{\mathbf{k}}\rangle = \frac{1-q^n}{1-q}|n_{\mathbf{k}}\rangle$. Based on this, the Hamiltonian can be expanded as follows:

$$\begin{aligned}\hat{H}_{\mathbf{k}} &= -Sk\,\hat{\mathbb{I}} + \left(\frac{1}{2} - \frac{\hbar}{4S}\right) k\hbar\,\hat{\mathbb{I}} + k\hbar\left(1 - \frac{\hbar}{S}\right)\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}} \\ &+ \frac{k\hbar}{24}\frac{\hbar}{S}\left(\hat{a}^4_{\mathbf{k}} + (\hat{a}^{\dagger}_{\mathbf{k}})^4 - 6(\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}})^2 - 6\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}} - 6\hat{\mathbb{I}}\right) \\ &+ \mathcal{O}(S^{-2})\,.\end{aligned}$$

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Propagator

Assuming statistical isotropy of the spatial field configurations, the two-point correlation function is given by

$$\langle 0|\hat{\phi}(\mathbf{x},t)\hat{\phi}(\mathbf{y},t')|0\rangle = \frac{1}{V}\sum_{\mathbf{k},n}|c_n|^2 e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})-i\Delta E_n(t-t')}$$
$$= \frac{1}{V}\sum_{\mathbf{k}}\int\frac{d\omega}{2\pi}D_{(\omega,\mathbf{k})}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})-i\omega(t-t')},$$

where (for a given wave number) $\Delta E_n = E_n^{(1)} - E_0^{(1)}$ and, denoting $p^2 = -\omega^2 + k^2$, we calculate the propagator:

$$D_{(\omega,\mathbf{k})} = \frac{i\left(1-\frac{2}{S}\right)}{-\omega^2 + k^2\left(1-\frac{3}{S}\right) + i\epsilon} + \mathcal{O}(S^{-2})$$
$$= \frac{i}{-\omega^2+k^2} + \frac{i}{S}\frac{k^2+2\omega^2}{(-\omega^2+k^2)^2} + \mathcal{O}(S^{-2}).$$

Renormalized constants

From the propagator given as the single term one can deduce that the "renormalized" speed of light reads

$$c_{\text{ren}} = 1 - \frac{3}{2}\frac{\hbar}{S} + \mathcal{O}(S^{-2}).$$

Furthermore, the propagator can be used to predict the form of interaction potential between two point sources of the scalar field:

$$V(r) = 4\pi i \int \frac{d^3k}{(2\pi\hbar)^3} e^{i\mathbf{k}\cdot\mathbf{r}} D_{(0,\mathbf{k})} Q_0 = -\frac{Q_0}{r}\left(1 + \frac{\hbar}{S} + \mathcal{O}(S^{-2})\right),$$

where Q_0 is the charge of a field source. The difference with the standard case can be absorbed into "renormalized" charge

$$Q_{\text{ren}} = Q_0\left(1 + \frac{\hbar}{S} + \mathcal{O}(S^{-2})\right).$$

Summary

- NFTS - linear field space is only an approximation.
- Compactness of the field space allows to implement "Principle of finiteness".
- Spin-Field correspondence - $\dim(\Gamma_x^{\phi}) = \dim S^2$. Spin precession = scalar field oscillations.
- Generalization to the different types of fields is naturally possible - to be done.
- Numerous interesting predictions, including: generalization of the uncertainty relations, algebra deformations, constrained maximal occupation number, shifting of the vacuum energy and renormalization of constants, deformation of the Lorentz covariance.
- Relation of field theories with the condensed matter physics.
- Nonlinear Field Space Cosmology.