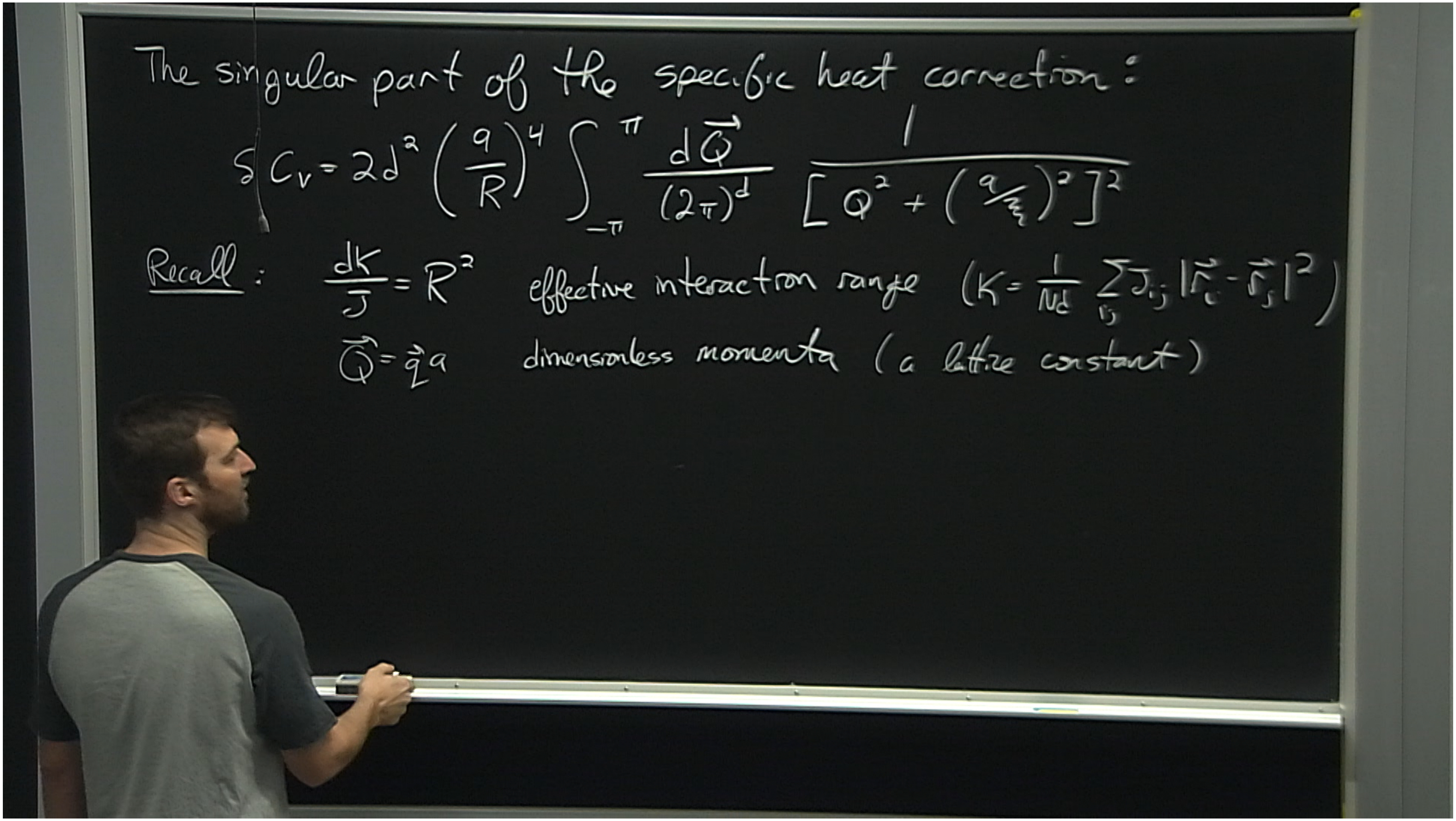
Title: 2016/2017 Statistical Mechanics 2 - Roger Melko - Lecture 7

Date: Jan 25, 2017 10:30 AM

URL: http://pirsa.org/17010053

Abstract:

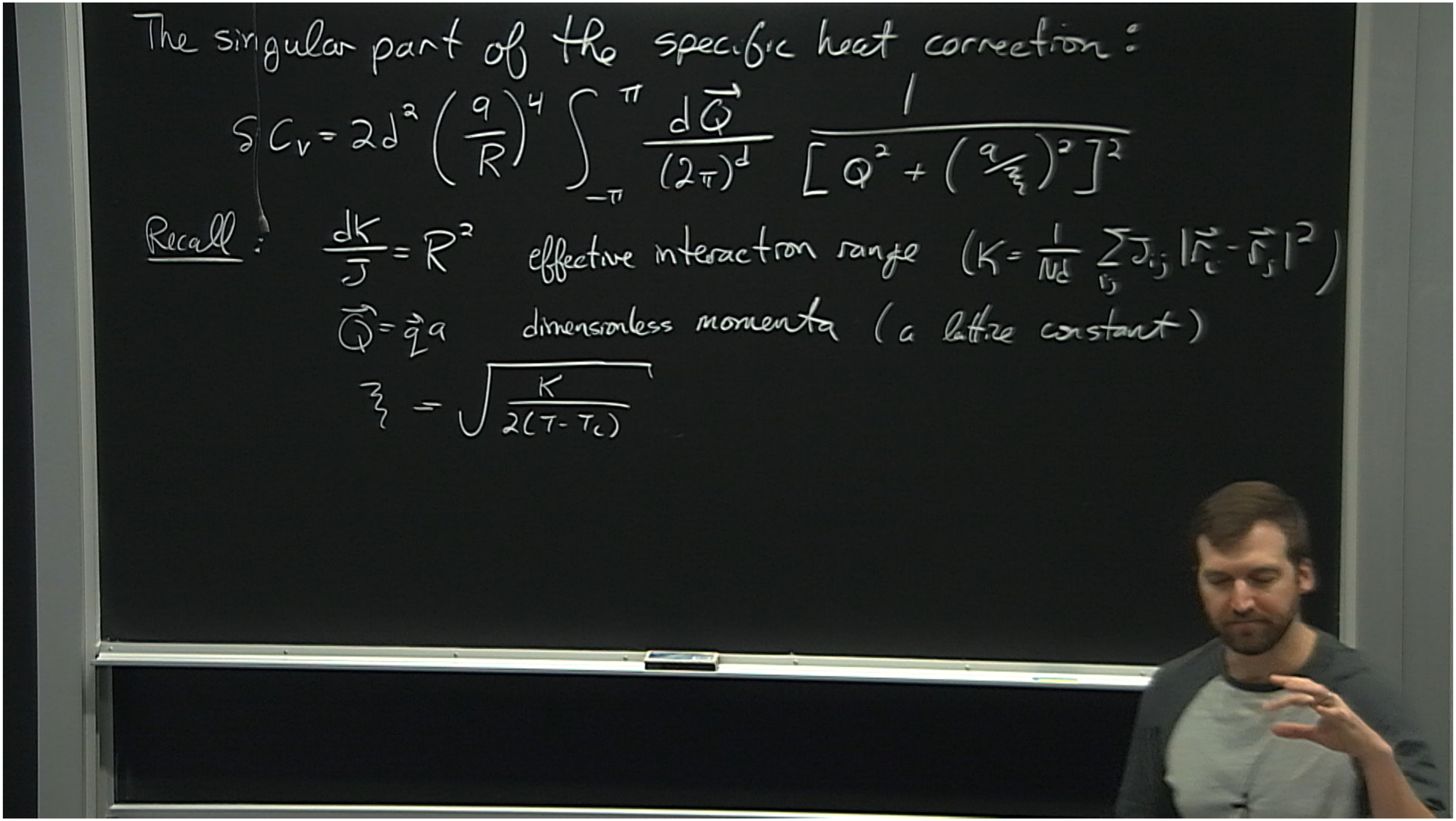


The singular part of the specific heat correction:

$$\delta C_v = 2d^2 \left(\frac{a}{R}\right)^4 \int_{-\pi}^{\pi} \frac{d\vec{Q}}{(2\pi)^d} \frac{1}{\left[Q^2 + \left(a/\xi\right)^2\right]^2}$$

Recall: $\frac{dK}{J} = R^2$ effective interaction range $\left(K = \frac{1}{Nd}\sum_{ij} J_{ij} |\vec{r}_i - \vec{r}_j|^2\right)$

$\vec{Q} = \vec{q}a$ dimensionless momenta (a lattice constant)



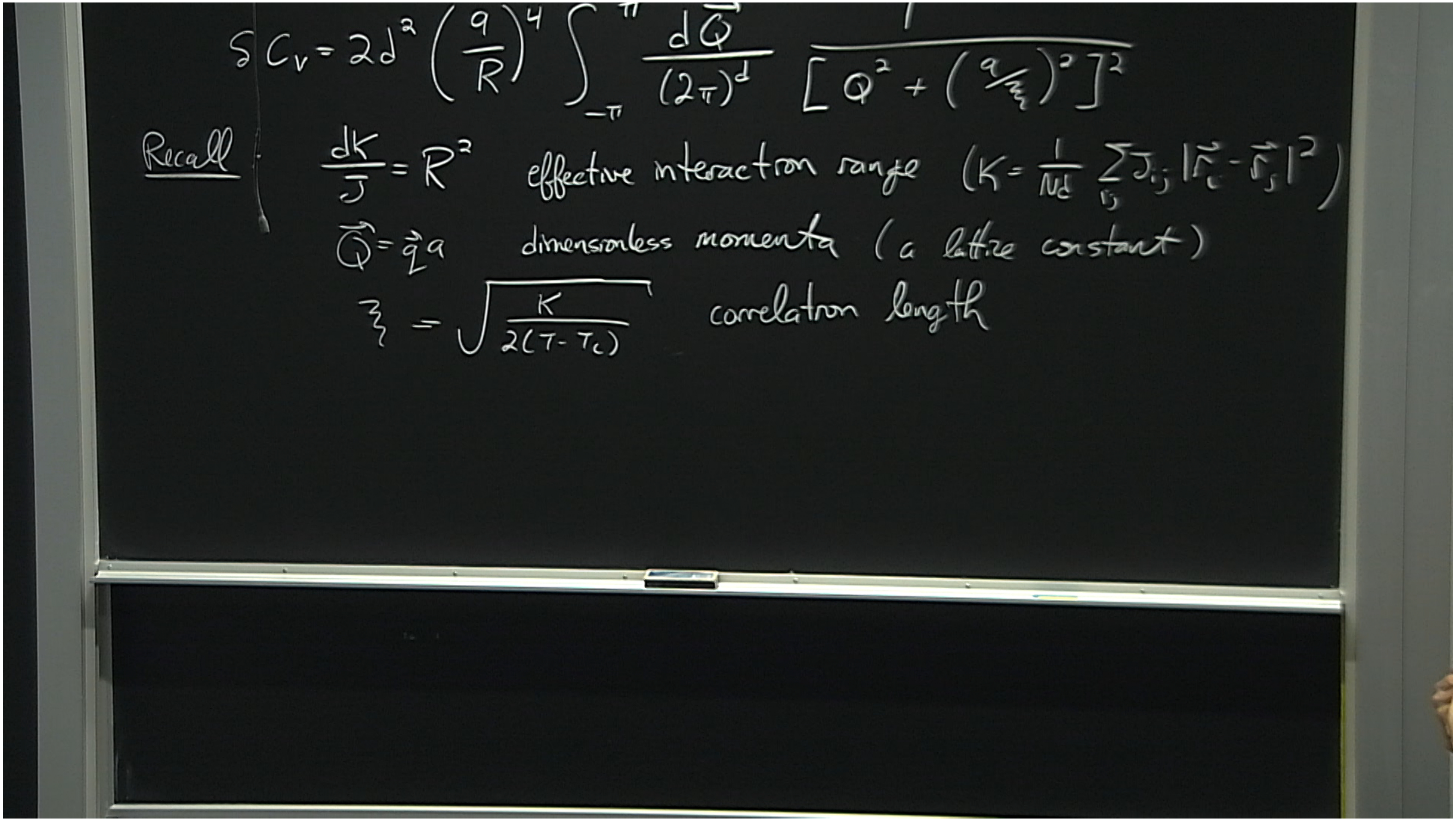
The singular part of the specific heat correction:

$$\delta C_v = 2d^2 \left(\frac{a}{R}\right)^4 \int_{-\pi}^{\pi} \frac{d\vec{Q}}{(2\pi)^d} \frac{1}{\left[Q^2 + \left(a/\xi\right)^2\right]^2}$$

Recall: $\frac{dK}{J} = R^2$ effective interaction range $\left(K = \frac{1}{Nd} \sum_{ij} J_{ij} |\vec{r}_i - \vec{r}_j|^2\right)$

$\vec{Q} = \vec{q}a$ dimensionless momenta (a lattice constant)

$\xi = \sqrt{\frac{K}{2(T-T_c)}}$

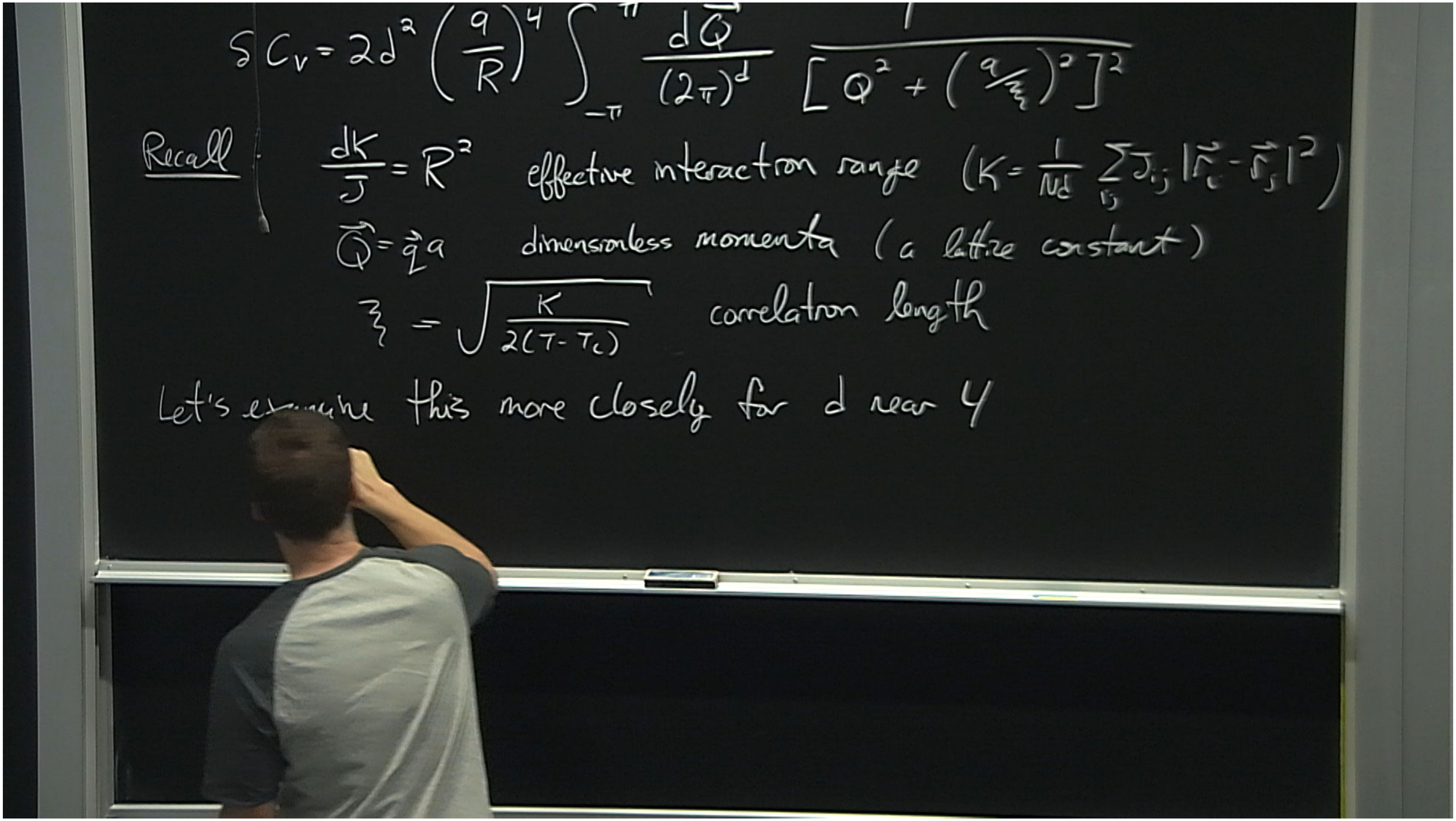


$$\delta C_v = 2d^2 \left(\frac{a}{R}\right)^4 \int_{-\pi}^{\pi} \frac{d\vec{Q}}{(2\pi)^d} \frac{1}{\left[Q^2 + \left(a/\xi\right)^2\right]^2}$$

Recall: $\frac{dK}{J} = R^2$ effective interaction range $\left(K = \frac{1}{Nd} \sum_{ij} J_{ij} |\vec{r}_i - \vec{r}_j|^2\right)$

$\vec{Q} = \vec{q}\,a$ dimensionless momenta (a lattice constant)

$\xi = \sqrt{\frac{K}{2(T - T_c)}}$ correlation length



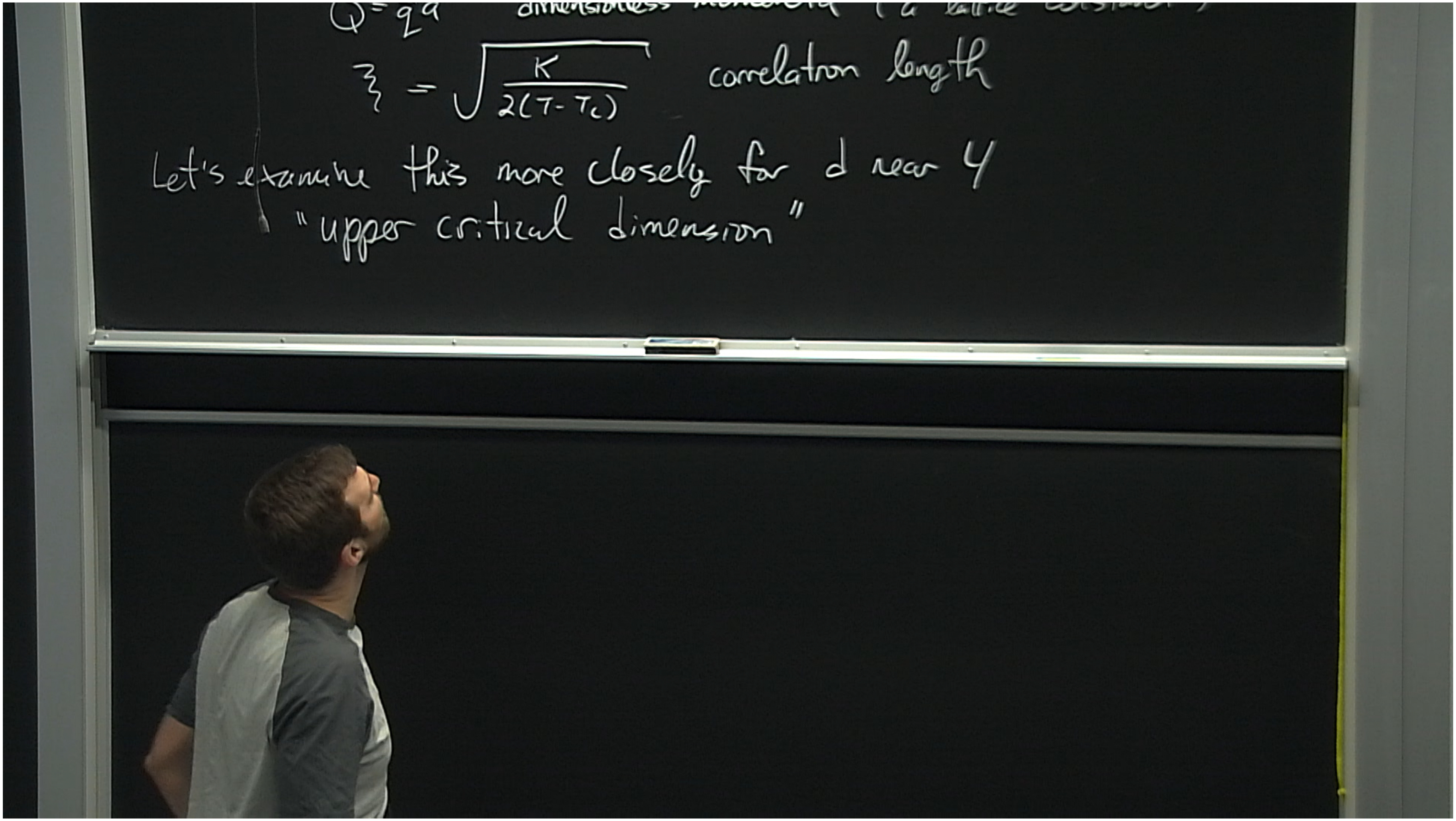
$$\delta C_v = 2d^2\left(\frac{a}{R}\right)^4 \int_{-\pi}^{\pi} \frac{d\vec{Q}}{(2\pi)^d}\,\frac{1}{\left[Q^2+\left(a/\xi\right)^2\right]^2}$$

Recall: $\frac{dK}{J} = R^2$ effective interaction range $\left(K = \frac{1}{Nd}\sum_{ij} J_{ij}\,|\vec{r}_i - \vec{r}_j|^2\right)$

$\vec{Q} = \vec{q}a$ dimensionless momenta (a lattice constant)

$\xi = \sqrt{\frac{K}{2(T-T_c)}}$ correlation length

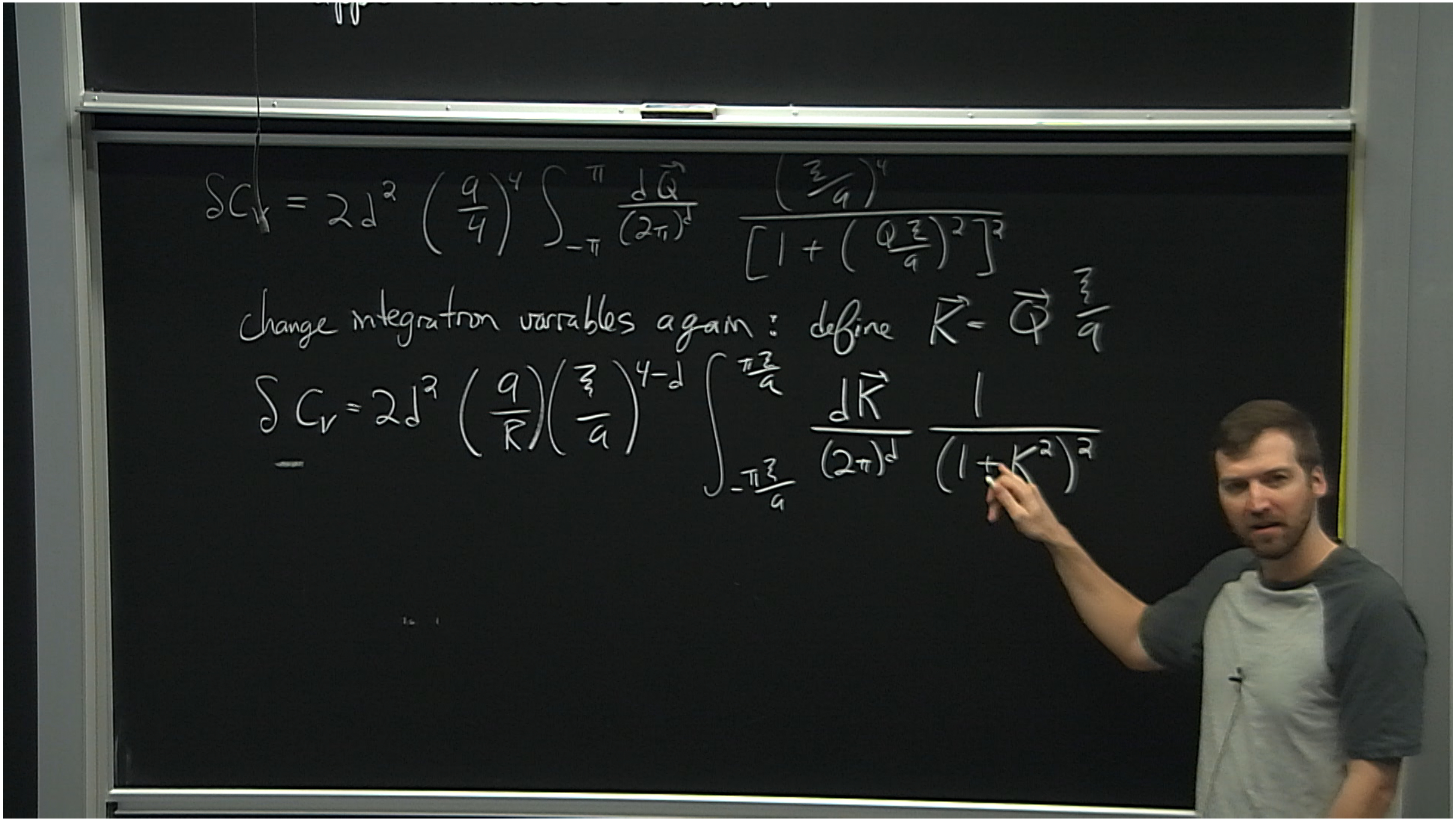
Let's examine this more closely for d near 4



$Q = qa$ dimensionless momentum (a lattice constant)

$$\xi = \sqrt{\frac{K}{2(T-T_c)}}$$ correlation length

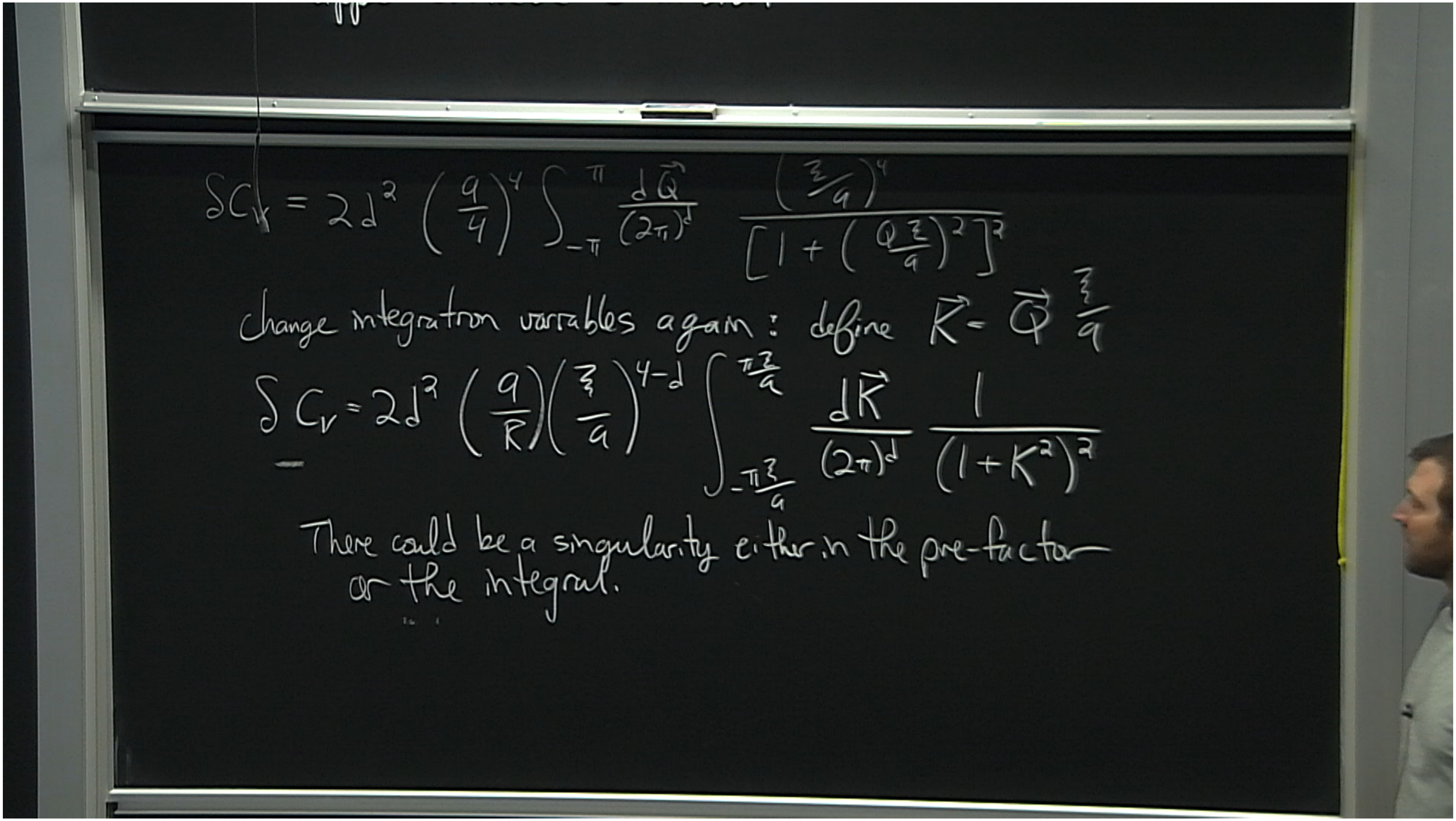
Let's examine this more closely for d near 4
"upper critical dimension"



$$\delta C_v = 2d^2 \left(\frac{a}{4}\right)^4 \int_{-\pi}^{\pi} \frac{d\vec{Q}}{(2\pi)^d} \frac{\left(\xi/a\right)^4}{\left[1 + \left(\frac{Q\xi}{a}\right)^2\right]^2}$$

change integration variables again: define $\vec{K} = \vec{Q}\,\frac{\xi}{a}$

$$\delta C_v = 2d^2 \left(\frac{a}{R}\right)\left(\frac{\xi}{a}\right)^{4-d} \int_{-\pi\frac{\xi}{a}}^{\pi\frac{\xi}{a}} \frac{d\vec{K}}{(2\pi)^d} \frac{1}{(1+K^2)^2}$$

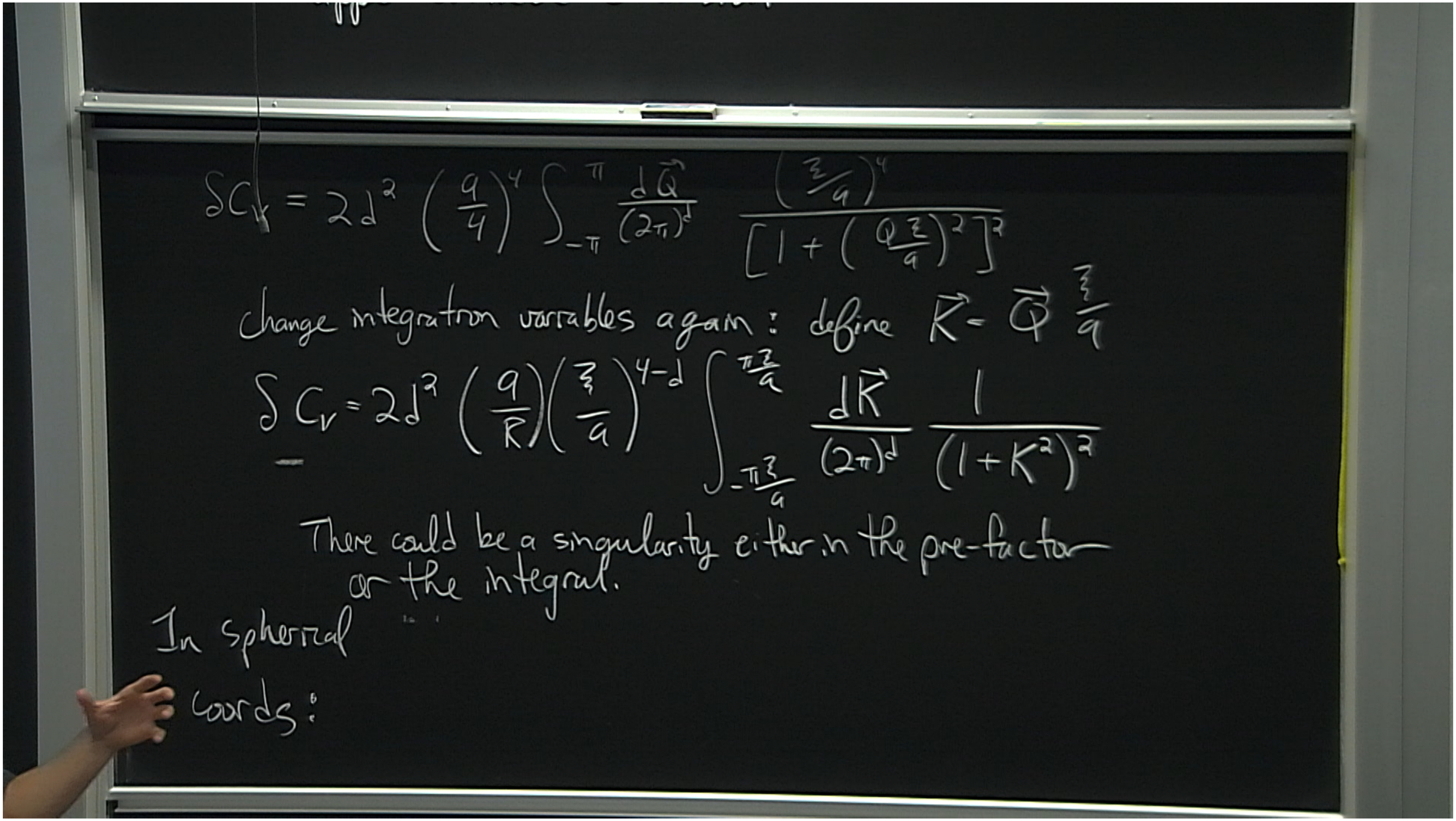


$$\delta C_r = 2d^2 \left(\frac{a}{4}\right)^4 \int_{-\pi}^{\pi} \frac{d\vec{Q}}{(2\pi)^d} \frac{\left(\xi/a\right)^4}{\left[1+\left(\frac{Q\xi}{a}\right)^2\right]^2}$$

change integration variables again: define $\vec{K} = \vec{Q}\,\frac{\xi}{a}$

$$\underline{\delta C_r} = 2d^2 \left(\frac{a}{R}\right)\left(\frac{\xi}{a}\right)^{4-d} \int_{-\pi\frac{\xi}{a}}^{\pi\frac{\xi}{a}} \frac{d\vec{K}}{(2\pi)^d}\,\frac{1}{(1+K^2)^2}$$

There could be a singularity either in the pre-factor or the integral.



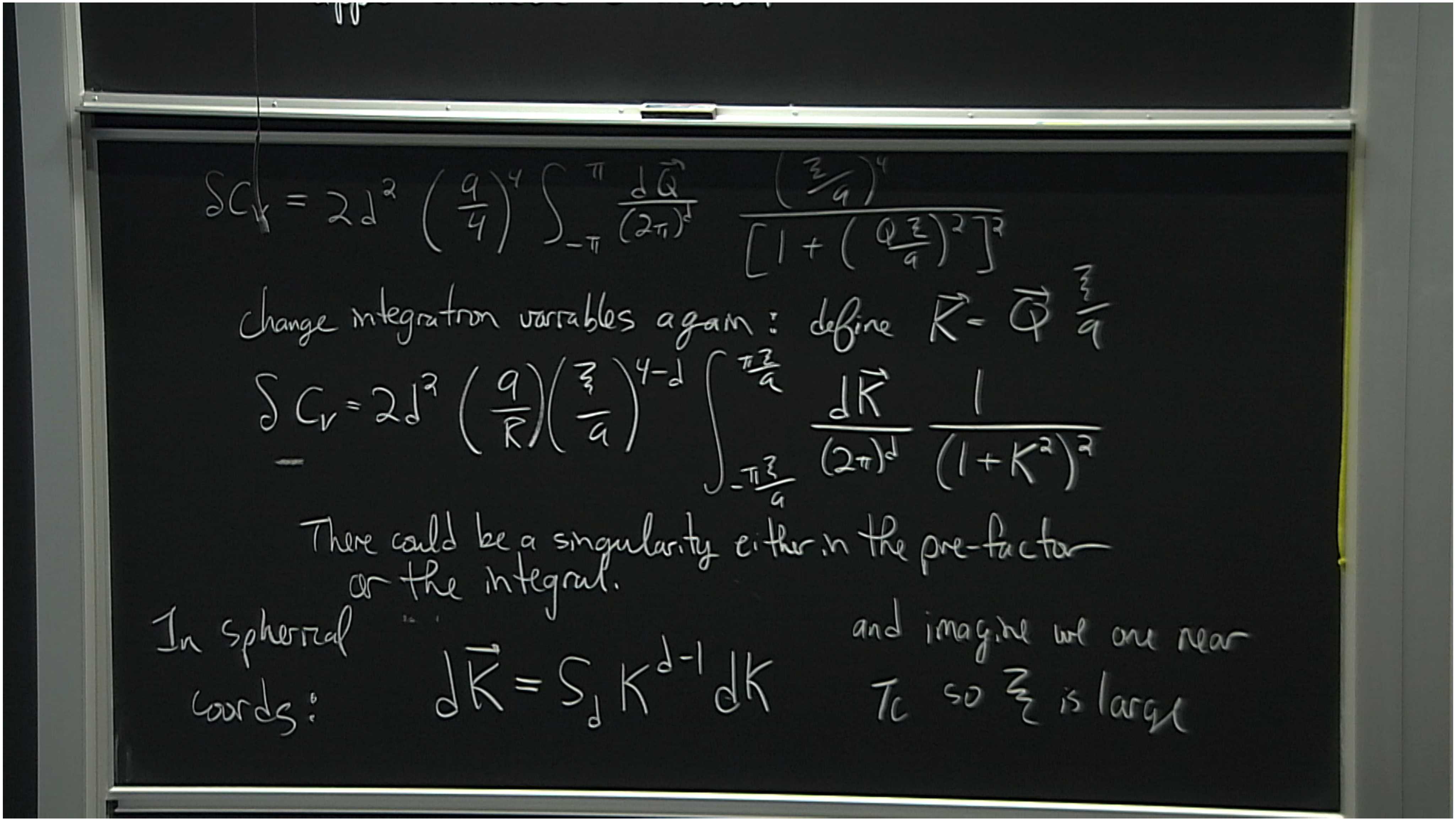
$$\delta C_v = 2d^2 \left(\frac{a}{4}\right)^4 \int_{-\pi}^{\pi} \frac{d\vec{Q}}{(2\pi)^d} \frac{\left(\xi/a\right)^4}{\left[1+\left(\frac{Q\xi}{a}\right)^2\right]^2}$$

change integration variables again: define $\vec{K} = \vec{Q}\,\frac{\xi}{a}$

$$\delta C_v = 2d^2 \left(\frac{a}{R}\right)\left(\frac{\xi}{a}\right)^{4-d} \int_{-\pi\frac{\xi}{a}}^{\pi\frac{\xi}{a}} \frac{d\vec{K}}{(2\pi)^d} \frac{1}{(1+K^2)^2}$$

There could be a singularity either in the pre-factor or the integral.

In spherical coords:



$$\delta C_v = 2d^2 \left(\frac{a}{4}\right)^4 \int_{-\pi}^{\pi} \frac{d\vec{Q}}{(2\pi)^d} \frac{\left(\xi/a\right)^4}{\left[1+\left(\frac{Q\xi}{a}\right)^2\right]^2}$$

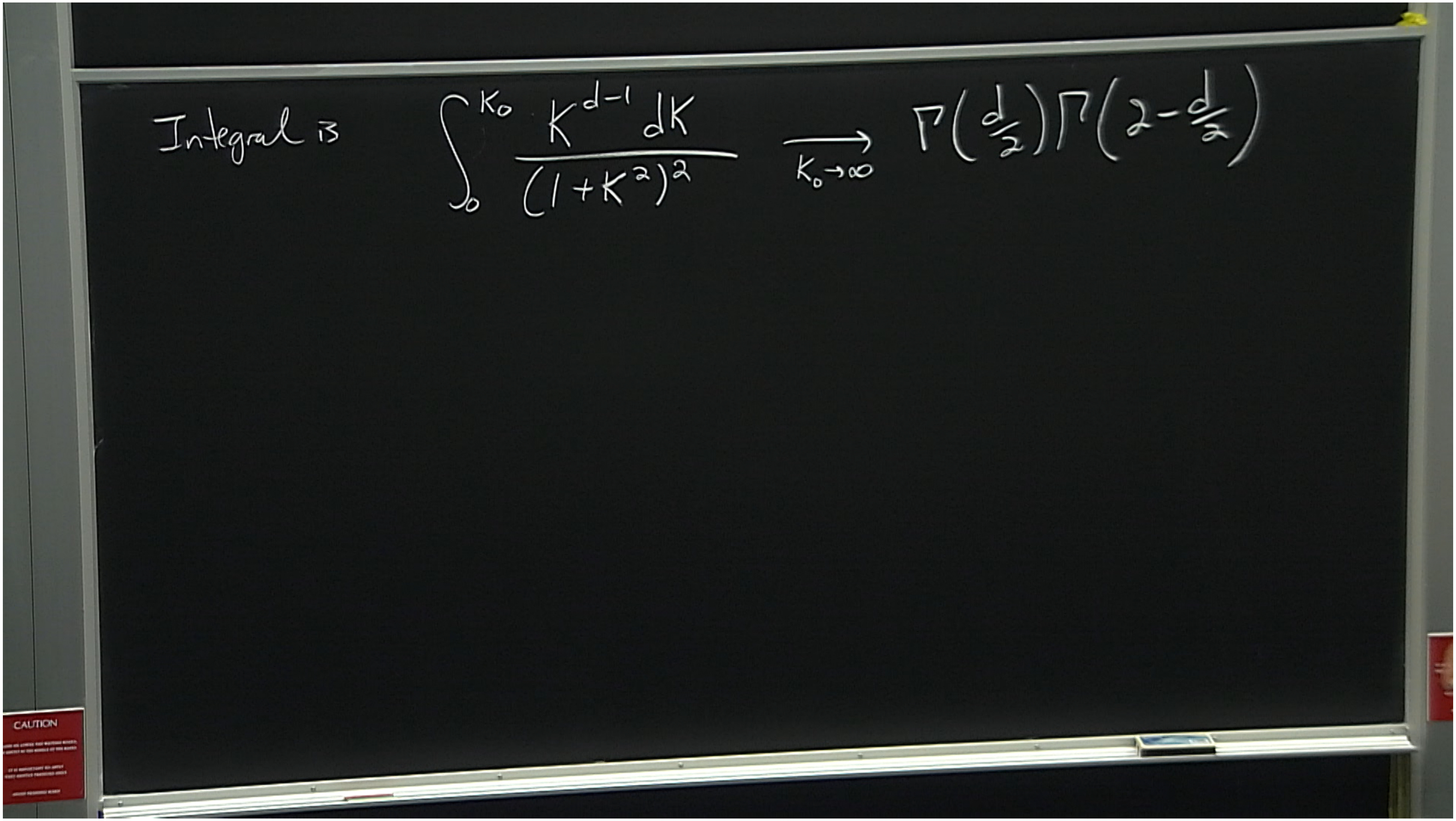
change integration variables again: define $\vec{K} = \vec{Q}\,\frac{\xi}{a}$

$$\delta C_v = 2d^2 \left(\frac{a}{R}\right)\left(\frac{\xi}{a}\right)^{4-d} \int_{-\pi\frac{\xi}{a}}^{\pi\frac{\xi}{a}} \frac{d\vec{K}}{(2\pi)^d} \frac{1}{(1+K^2)^2}$$

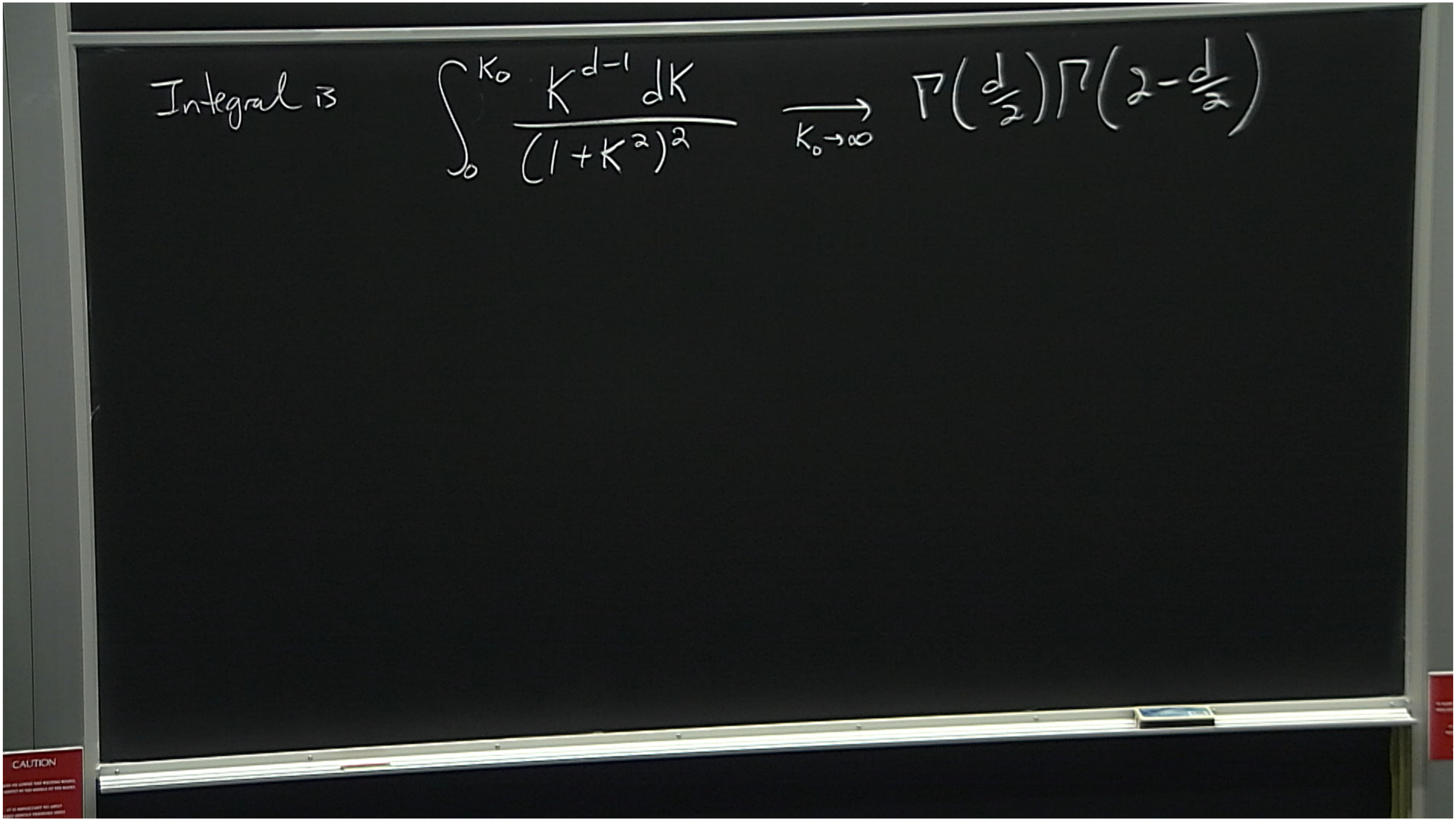
There could be a singularity either in the pre-factor or the integral.

In spherical coords: $d\vec{K} = S_d K^{d-1} dK$

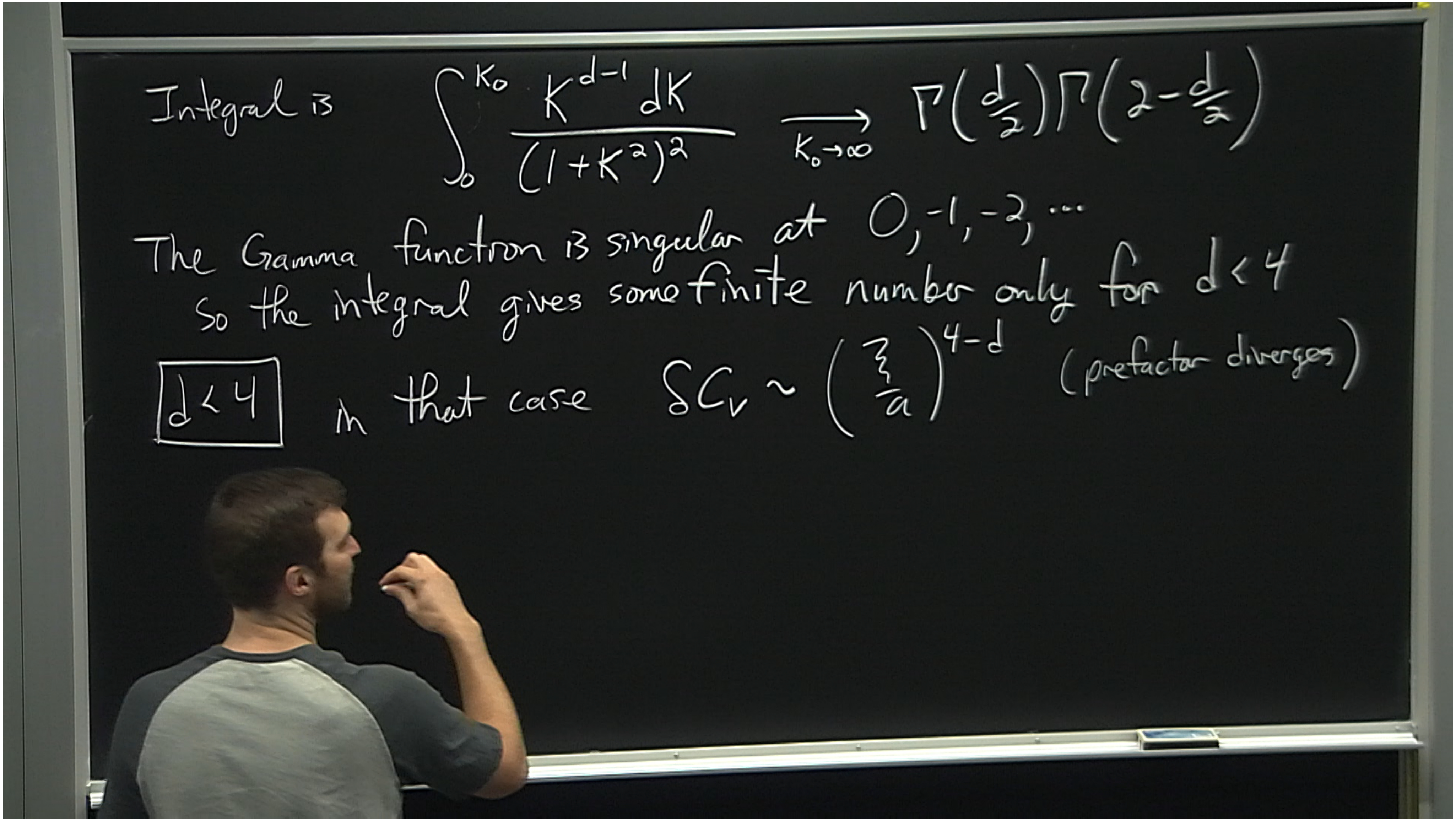
and imagine we are near T_c so ξ is large



Integral is $\int_0^{K_0} \frac{K^{d-1}\,dK}{(1+K^2)^2} \xrightarrow[K_0\to\infty]{} \Gamma\left(\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)$



Integral is $\int_0^{K_0} \frac{K^{d-1}\,dK}{(1+K^2)^2} \xrightarrow[K_0\to\infty]{} \Gamma\left(\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)$

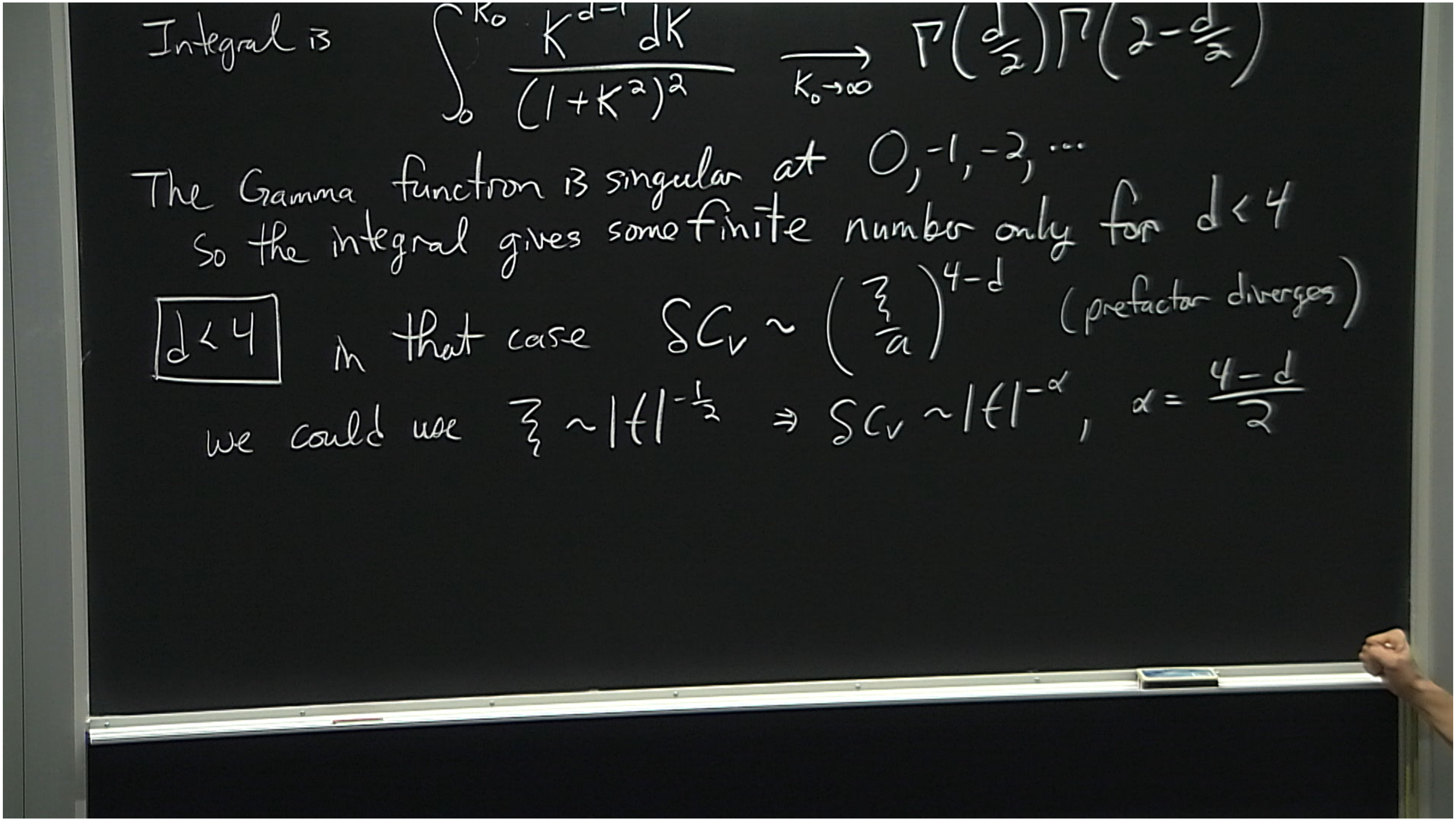


Integral is $\int_0^{K_0} \frac{K^{d-1}\,dK}{(1+K^2)^2} \xrightarrow[K_0\to\infty]{} \Gamma\left(\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)$

The Gamma function is singular at $0, -1, -2, \cdots$

So the integral gives some finite number only for $d<4$

$\boxed{d<4}$ in that case $\delta C_v \sim \left(\frac{\xi}{a}\right)^{4-d}$ (prefactor diverges)



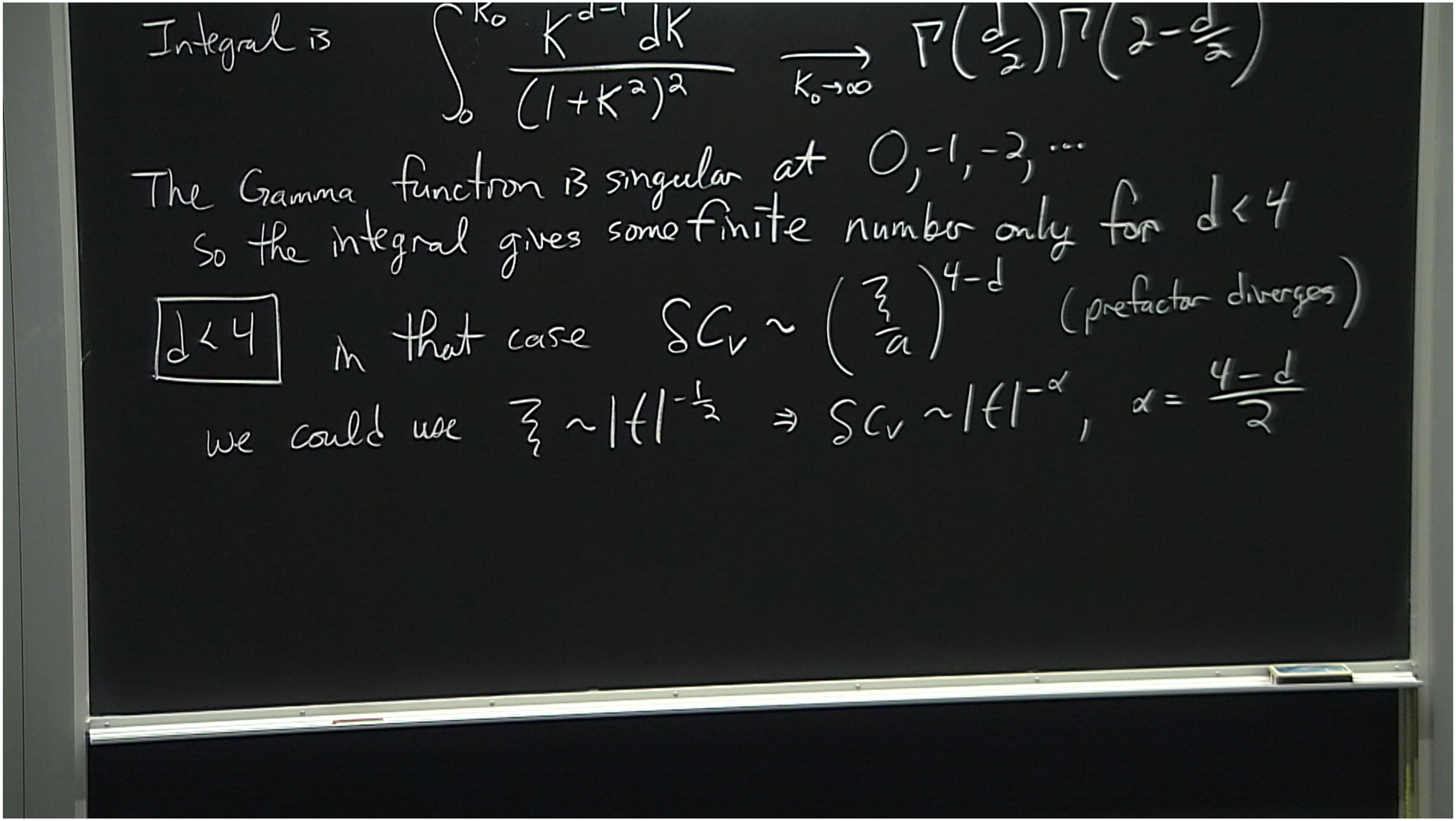
Integral is $\int_0^{K_0} \frac{K^{d-1}\,dK}{(1+K^2)^2} \xrightarrow[K_0\to\infty]{} \Gamma\left(\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)$

The Gamma function is singular at $0, -1, -2, \cdots$

So the integral gives some finite number only for $d < 4$

$\boxed{d<4}$ in that case $\delta C_V \sim \left(\frac{\xi}{a}\right)^{4-d}$ (prefactor diverges)

we could use $\xi \sim |t|^{-\frac{1}{2}} \Rightarrow \delta C_V \sim |t|^{-\alpha}$, $\alpha = \frac{4-d}{2}$



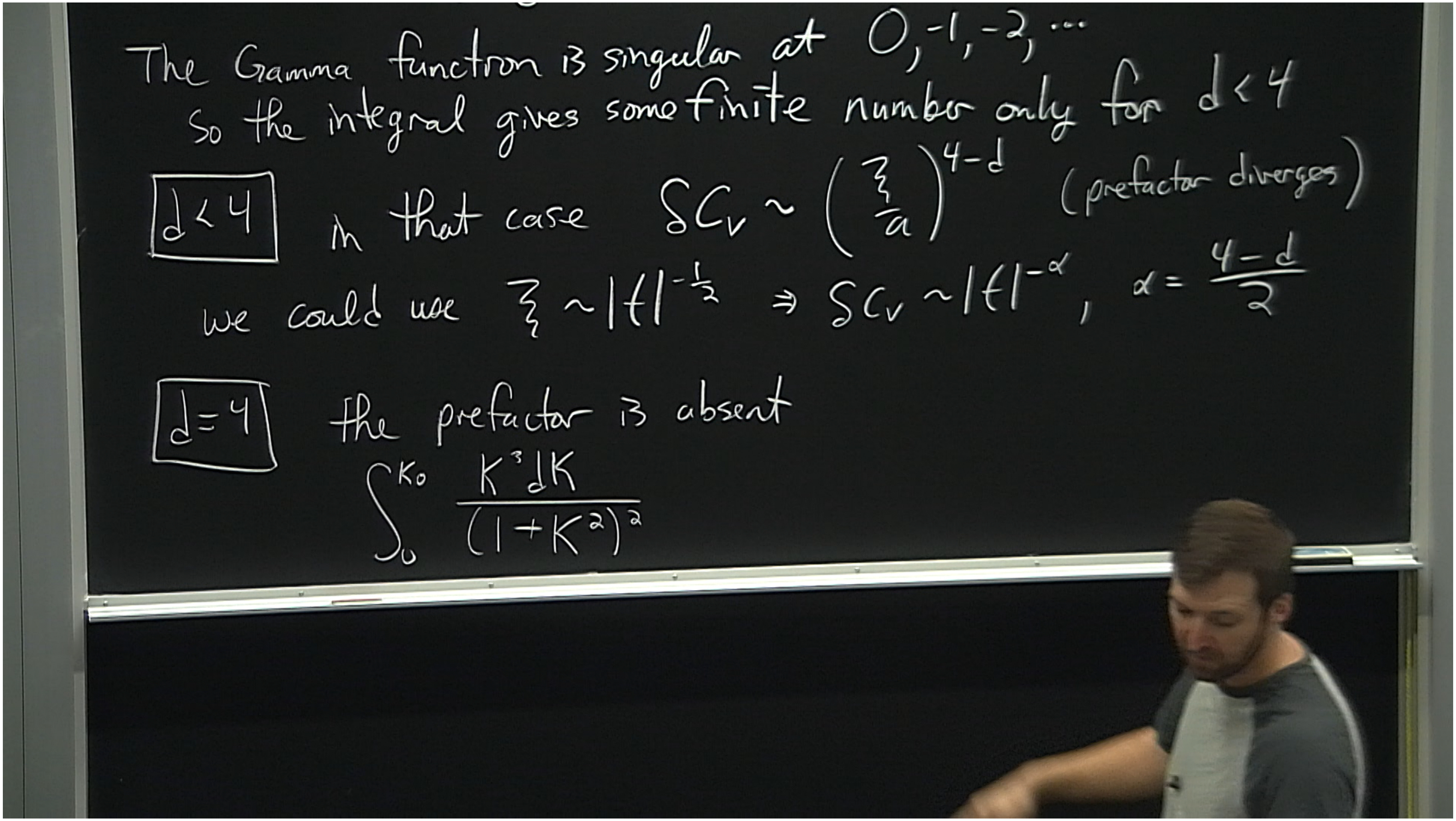
Integral is $\int_0^{K_0} \frac{K^{d-1}\,dK}{(1+K^2)^2} \xrightarrow[K_0\to\infty]{} \Gamma\left(\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)$

The Gamma function is singular at $0, -1, -2, \cdots$

So the integral gives some finite number only for $d<4$

$\boxed{d<4}$ In that case $\delta C_V \sim \left(\frac{\xi}{a}\right)^{4-d}$ (prefactor diverges)

we could use $\xi \sim |t|^{-\frac{1}{2}} \Rightarrow \delta C_V \sim |t|^{-\alpha}$, $\alpha = \frac{4-d}{2}$



The Gamma function is singular at $0, -1, -2, \cdots$

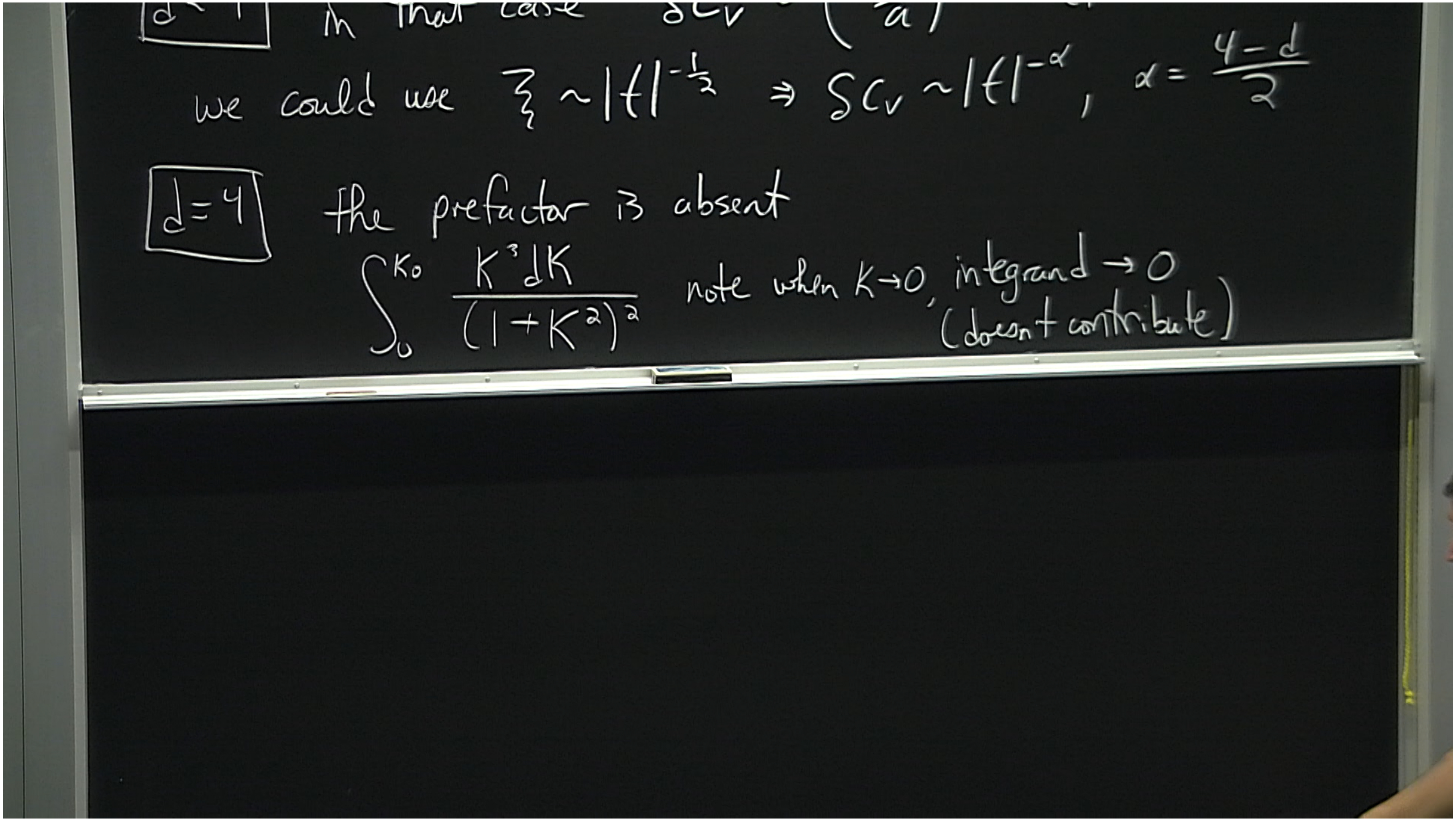
So the integral gives some finite number only for $d < 4$

$\boxed{d<4}$ in that case $\delta C_v \sim \left(\frac{\xi}{a}\right)^{4-d}$ (prefactor diverges)

we could use $\xi \sim |t|^{-\frac{1}{2}} \Rightarrow \delta C_v \sim |t|^{-\alpha}$, $\alpha = \frac{4-d}{2}$

$\boxed{d=4}$ the prefactor is absent

$$\int_0^{K_0} \frac{K^3 dK}{(1+K^2)^2}$$



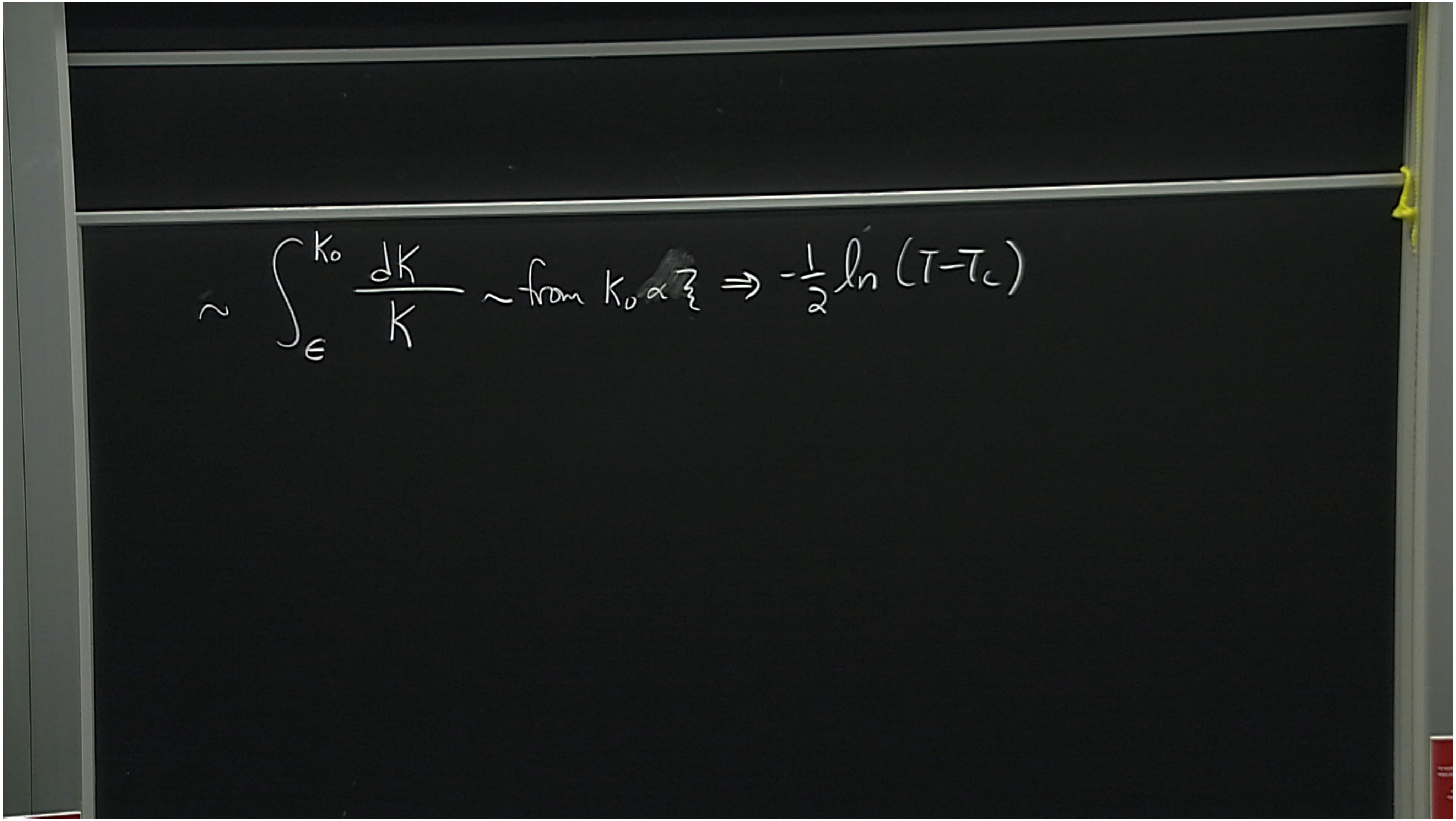
$\boxed{d<4}$ in that case $\delta C_v \sim \left(\frac{\xi}{a}\right)$

we could use $\xi \sim |t|^{-\frac{1}{2}} \Rightarrow \delta C_v \sim |t|^{-\alpha}, \quad \alpha = \frac{4-d}{2}$

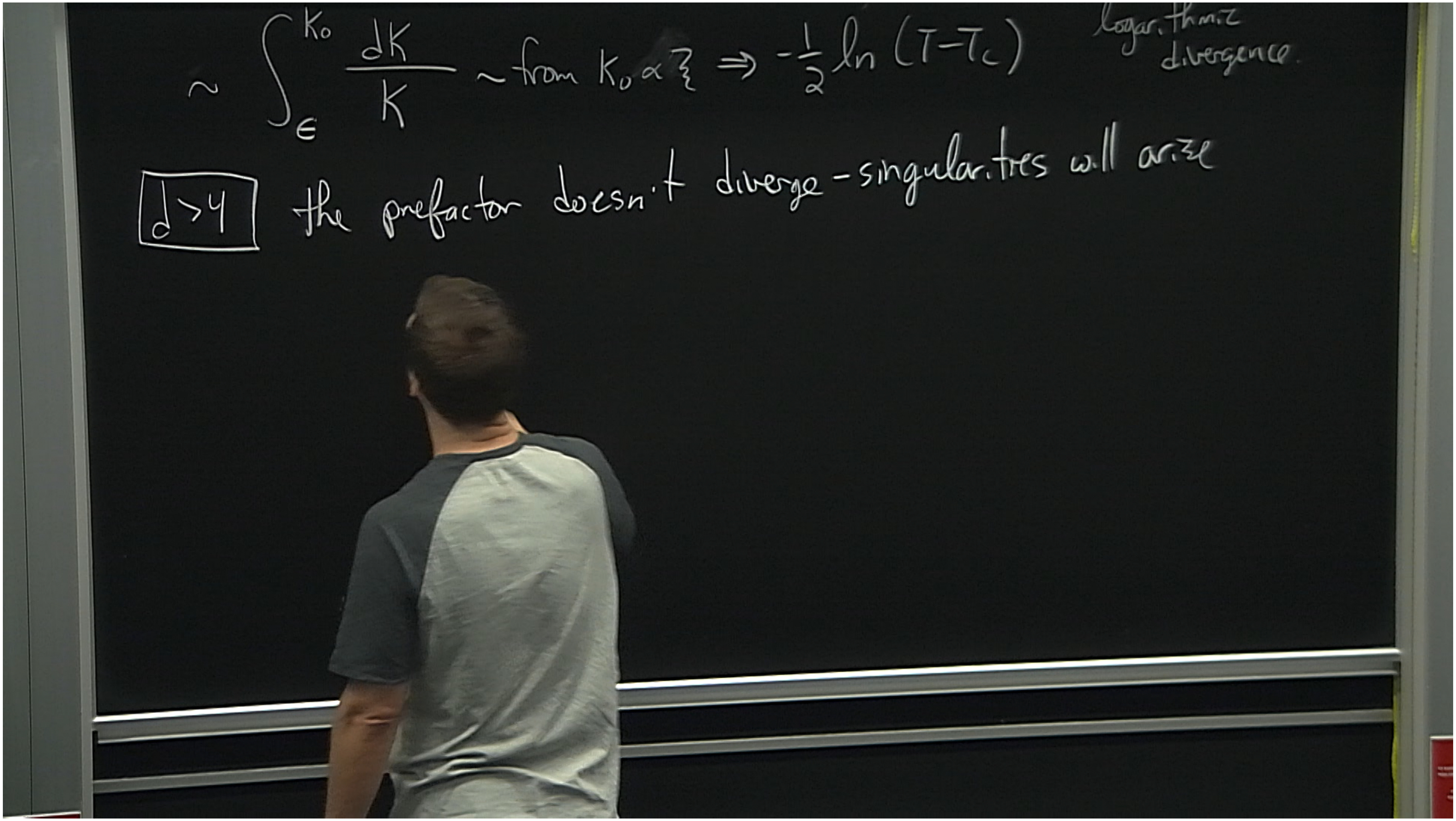
$\boxed{d=4}$ the prefactor is absent

$$\int_0^{K_0} \frac{K^3 dK}{(1+K^2)^2}$$

note when $K \to 0$, integrand $\to 0$
(doesn't contribute)



$\sim \int_{\epsilon}^{K_0} \frac{dK}{K} \sim$ from $K_0 \propto \xi \Rightarrow -\frac{1}{2} \ln (T-T_c)$



$\sim \int_{\epsilon}^{K_0} \frac{dK}{K} \sim$ from $K_0 \propto \xi \Rightarrow -\frac{1}{2} \ln (T-T_c)$ logarithmic divergence.

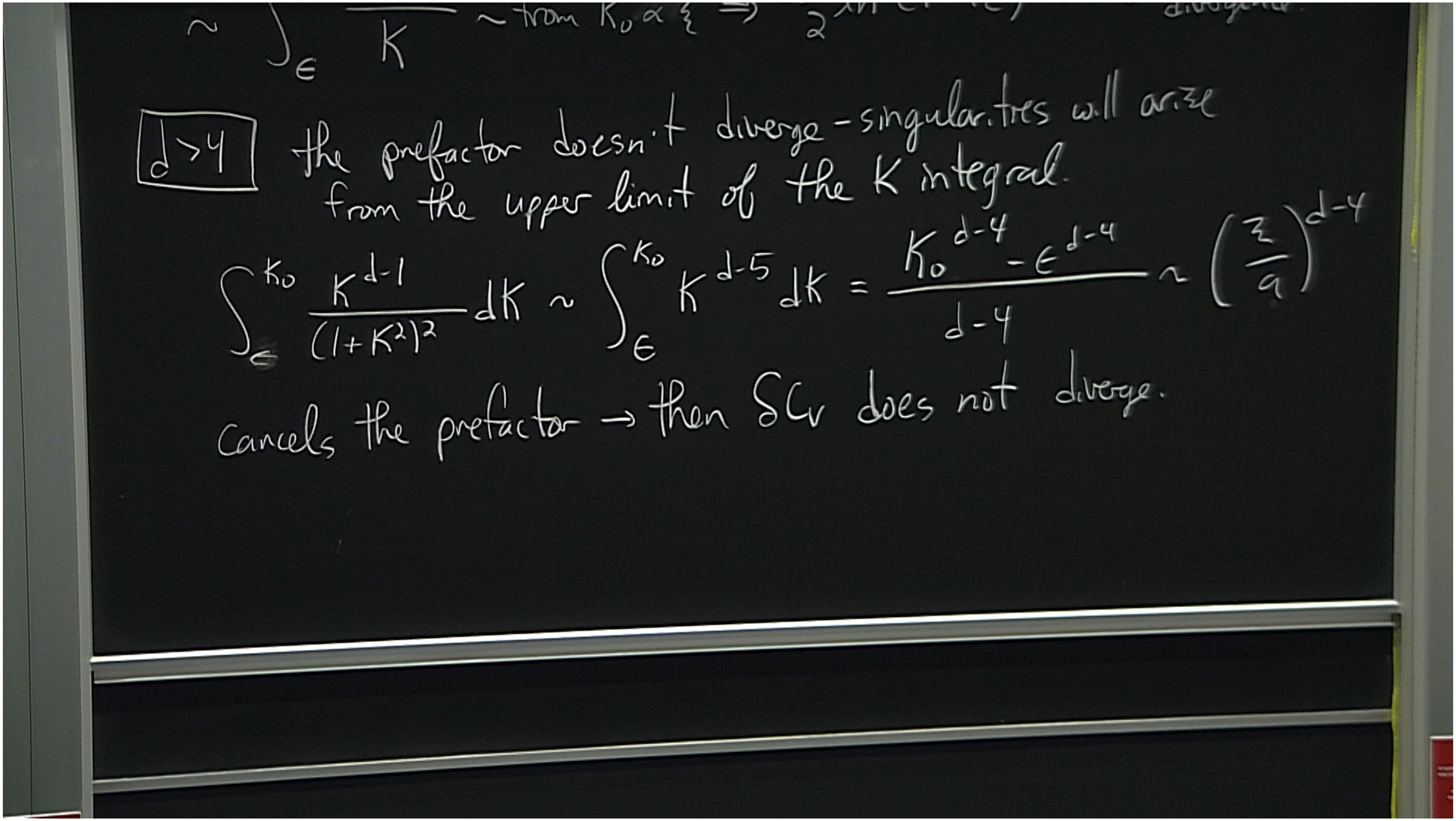
$\boxed{d>4}$ the prefactor doesn't diverge - singularities will arise

$$\sim \int_{\epsilon}^{K_0} \frac{dK}{K} \sim \text{from } K_0 \propto \xi \Rightarrow -\frac{1}{2}\ln(T-T_c)$$

logarithmic divergence.

$\boxed{d>4}$ the prefactor doesn't diverge - singularities will arise from the upper limit of the K integral.

$$\int_{\epsilon}^{K_0} \frac{K^{d-1}}{(1+K^2)^2}\, dK \sim \int_{\epsilon}^{K_0} K^{d-5}\, dK = \frac{K_0^{\,d-4} - \epsilon^{d-4}}{d-4} \sim \left(\frac{\xi}{a}\right)^{d-4}$$

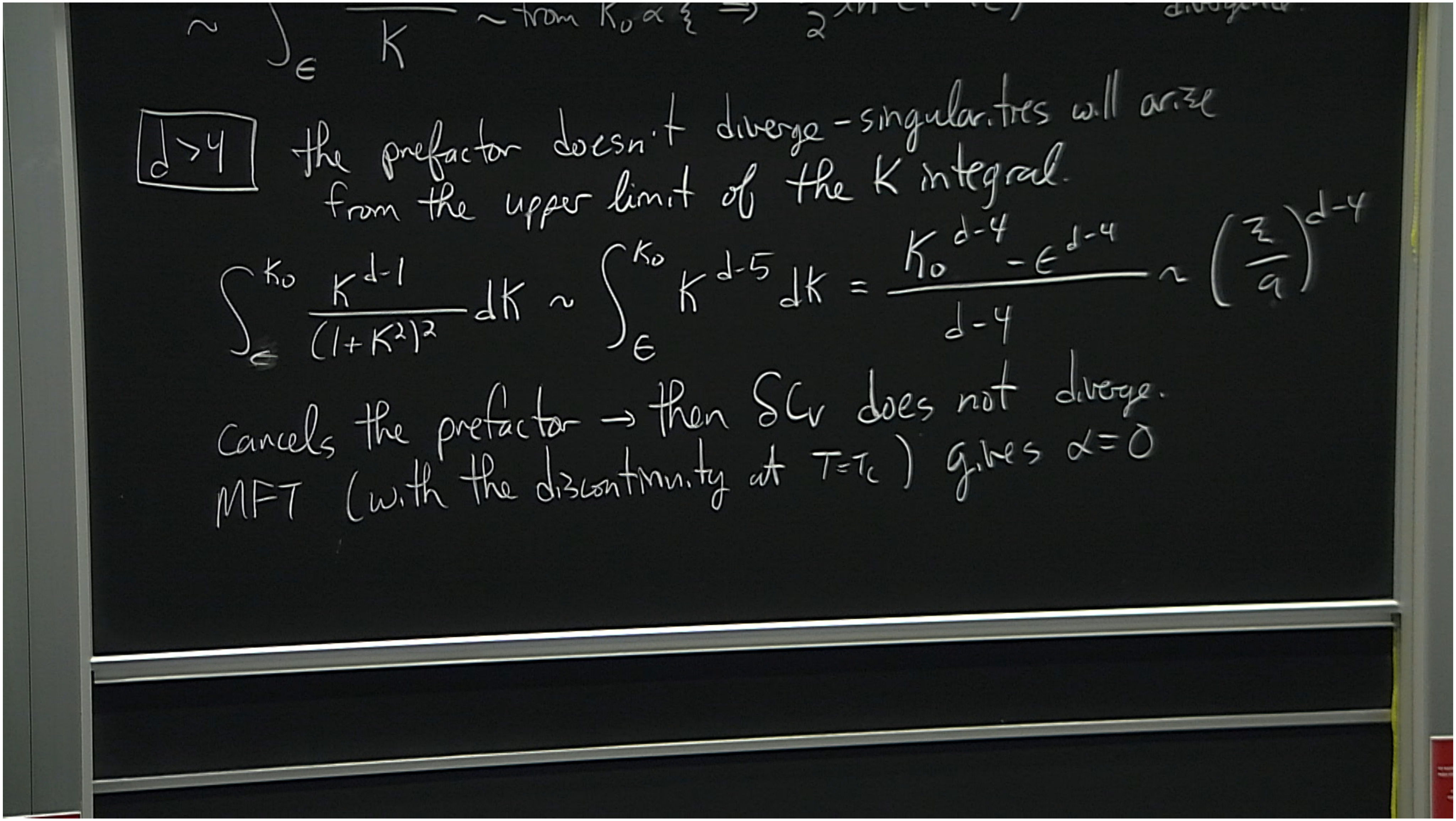


$\sim \int_\epsilon \frac{}{K}$ ~ from $K_0 \propto \xi \Rightarrow$ 2 divergence.

$\boxed{d>4}$ the prefactor doesn't diverge - singularities will arise from the upper limit of the K integral.

$$\int_\epsilon^{K_0} \frac{K^{d-1}}{(1+K^2)^2} dK \sim \int_\epsilon^{K_0} K^{d-5} dK = \frac{K_0^{d-4} - \epsilon^{d-4}}{d-4} \sim \left(\frac{\xi}{a}\right)^{d-4}$$

cancels the prefactor → then δC_v does not diverge.



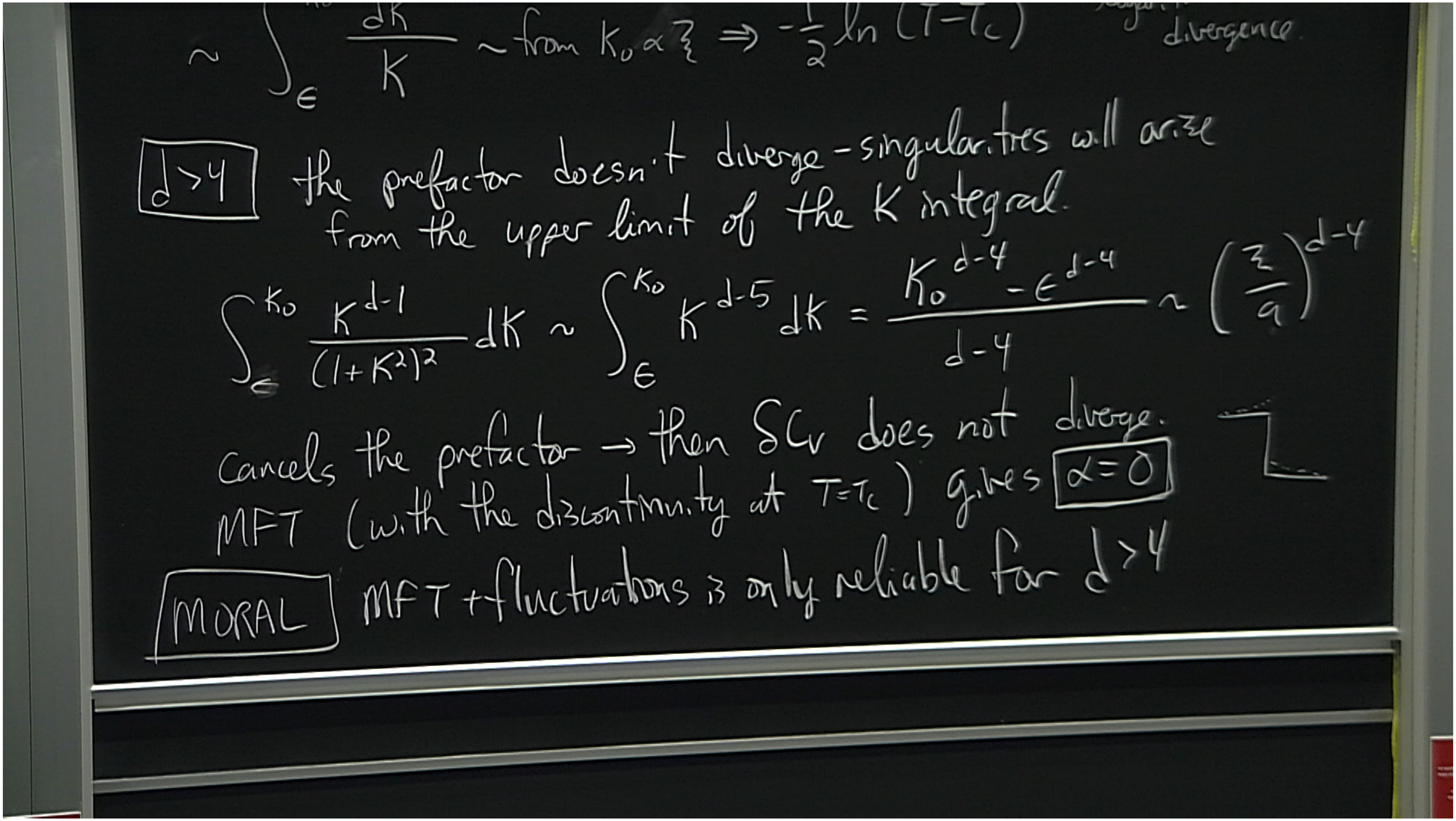
$\sim \int_\epsilon \frac{\quad}{K}$ ~ from $K_0 \propto \xi \Rightarrow \frac{\quad}{2} \ln(T-T_c)$ divergence.

$\boxed{d>4}$ the prefactor doesn't diverge - singularities will arise from the upper limit of the K integral.

$$\int_\epsilon^{K_0} \frac{K^{d-1}}{(1+K^2)^2} dK \sim \int_\epsilon^{K_0} K^{d-5} dK = \frac{K_0^{d-4} - \epsilon^{d-4}}{d-4} \sim \left(\frac{\xi}{a}\right)^{d-4}$$

cancels the prefactor → then δC_V does not diverge.

MFT (with the discontinuity at $T=T_c$) gives $\alpha = 0$



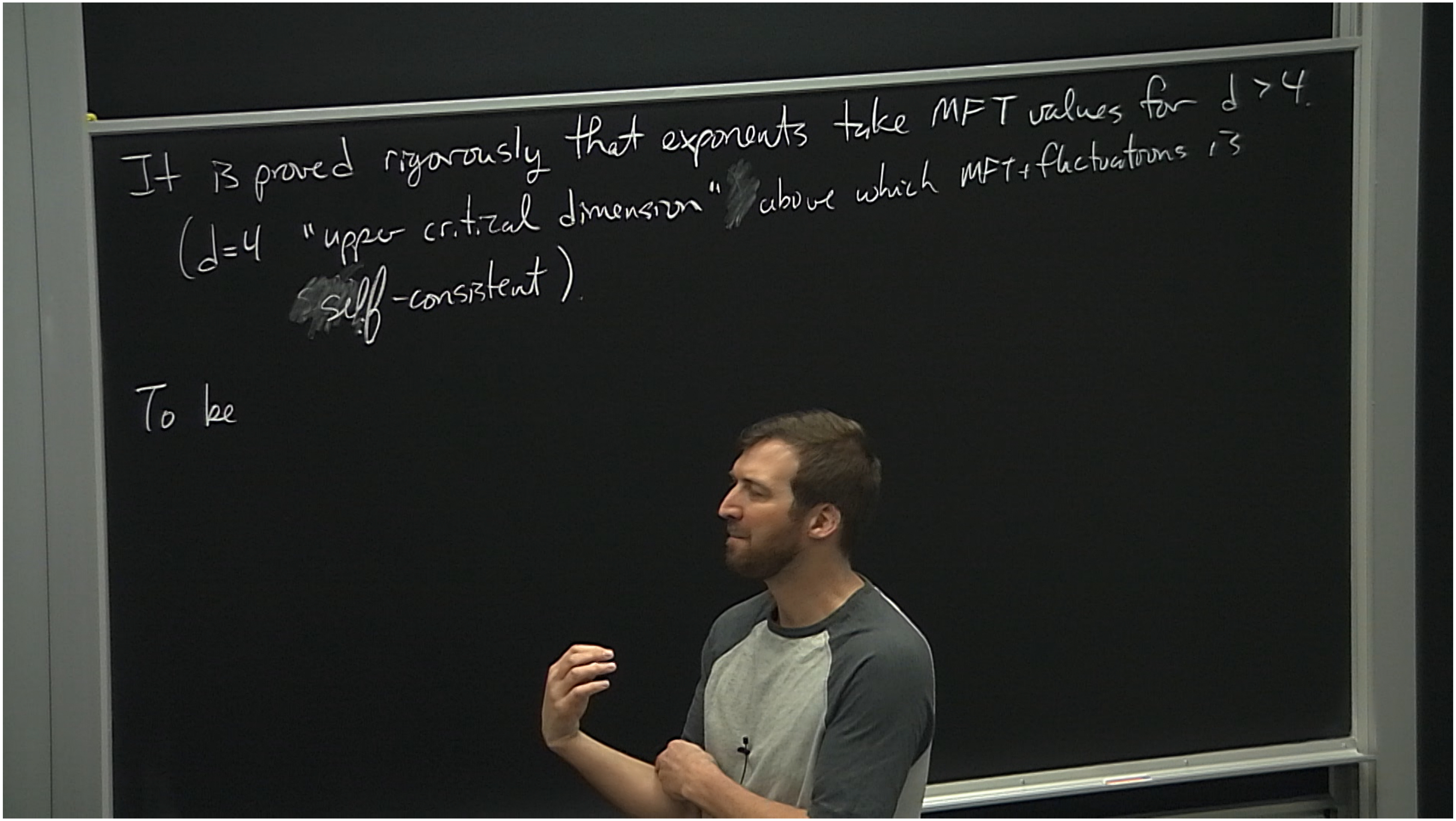
$$\sim \int_{\epsilon} \frac{dK}{K} \sim \text{from } K_0 \propto \xi \Rightarrow -\frac{1}{2}\ln(T-T_c)$$ divergence.

$\boxed{d>4}$ the prefactor doesn't diverge - singularities will arise from the upper limit of the K integral.

$$\int_{\epsilon}^{K_0} \frac{K^{d-1}}{(1+K^2)^2}dK \sim \int_{\epsilon}^{K_0} K^{d-5}\,dK = \frac{K_0^{\,d-4} - \epsilon^{d-4}}{d-4} \sim \left(\frac{\xi}{a}\right)^{d-4}$$

cancels the prefactor → then δC_v does not diverge. MFT (with the discontinuity at $T=T_c$) gives $\boxed{\alpha=0}$

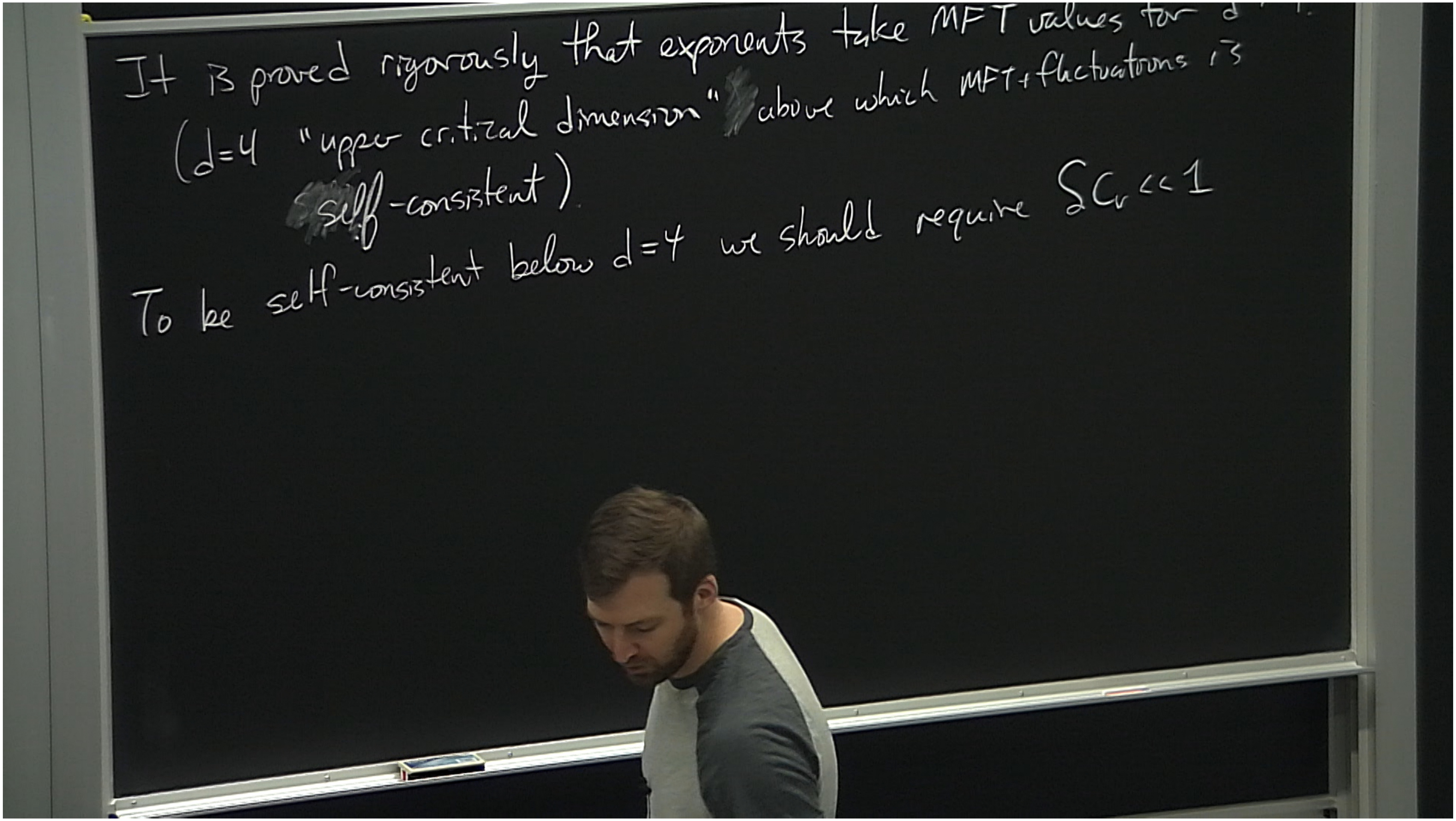
$\boxed{\text{MORAL}}$ MFT + fluctuations is only reliable for $d>4$



It is proved rigorously that exponents take MFT values for $d > 4$.

($d=4$ "upper critical dimension" above which MFT+fluctuations is self-consistent).

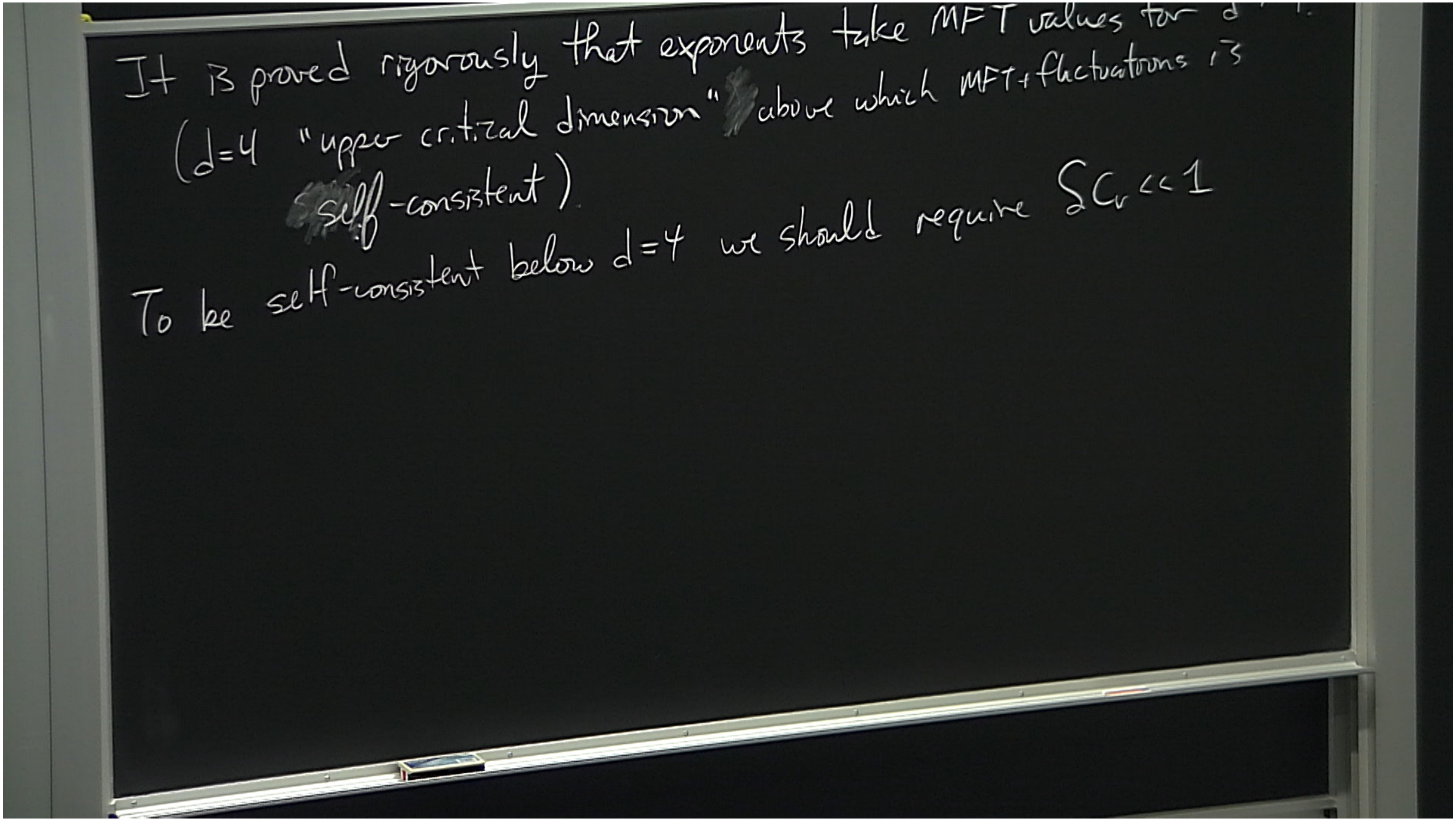
To be



It is proved rigorously that exponents take MFT values for d

(d=4 "upper critical dimension" above which MFT+fluctuations is self-consistent).

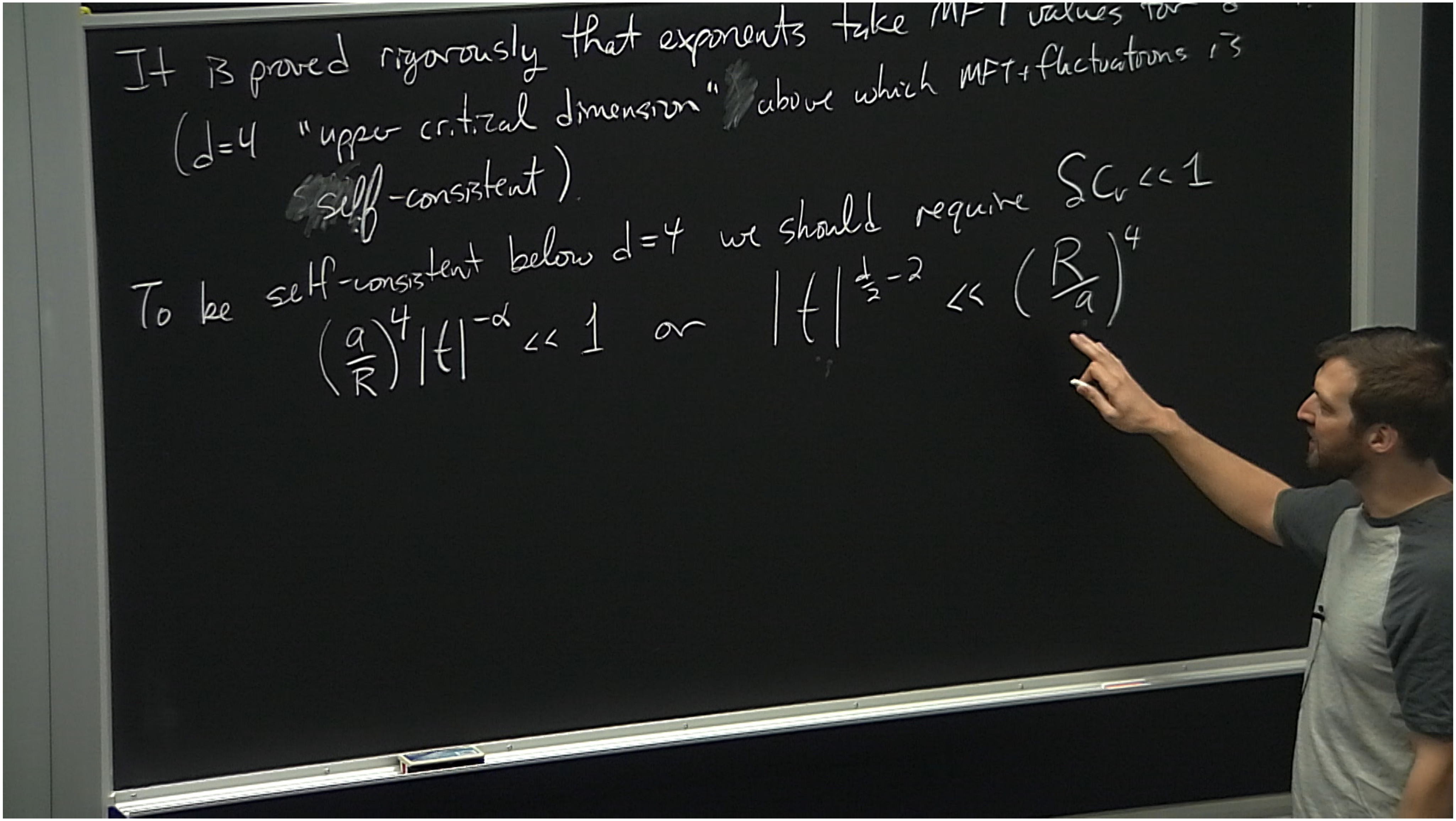
To be self-consistent below d=4 we should require $\delta G_r \ll 1$



It is proved rigorously that exponents take MFT values for $d > 4$.

($d=4$ "upper critical dimension" above which MFT + fluctuations is self-consistent)

To be self-consistent below $d=4$ we should require $SG_r \ll 1$

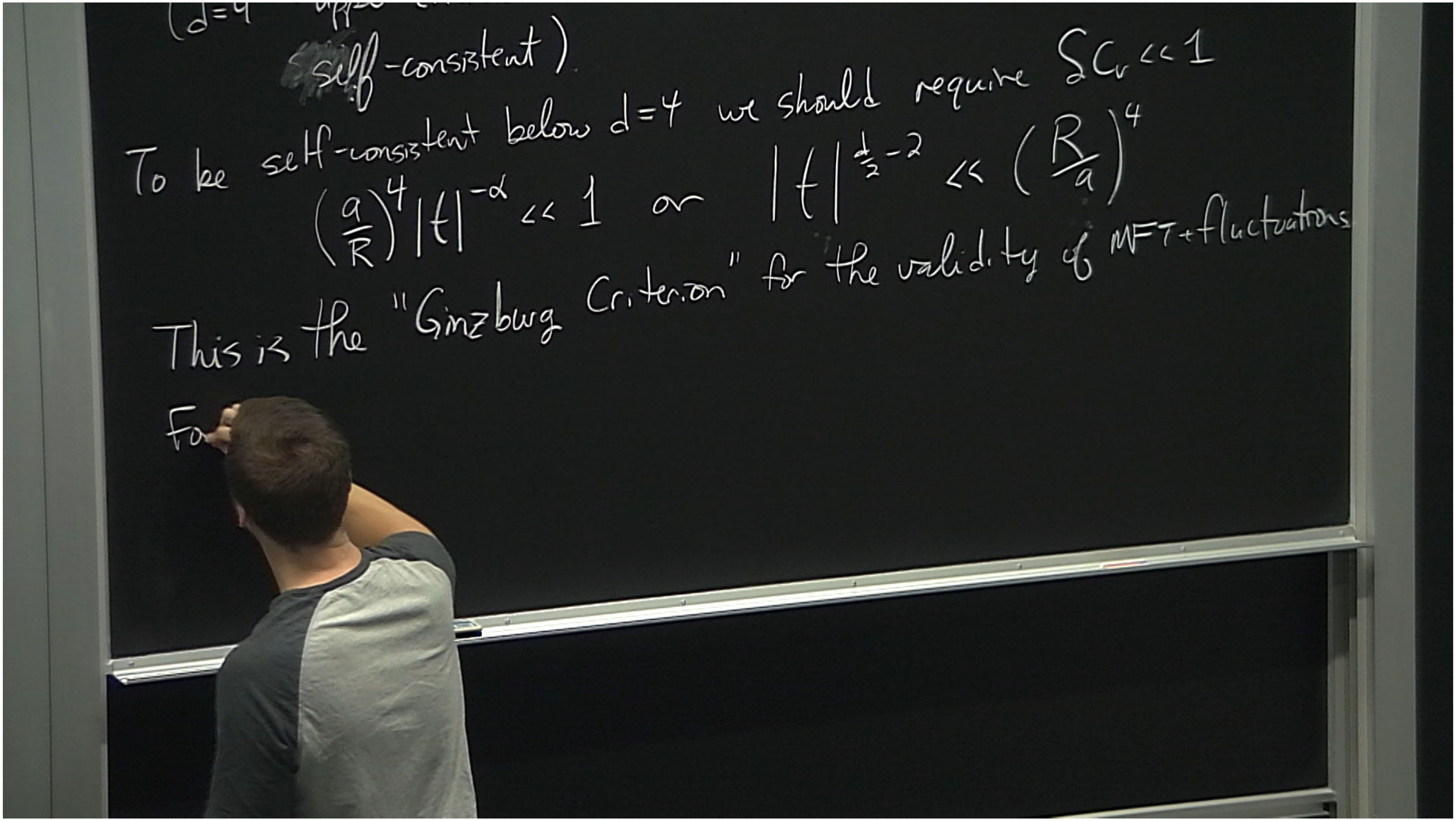


It is proved rigorously that exponents take MFT values for

(d=4 "upper critical dimension" above which MFT+fluctuations is self-consistent).

To be self-consistent below d=4 we should require $\delta C_V << 1$

$$\left(\frac{a}{R}\right)^4 |t|^{-\alpha} << 1 \quad \text{or} \quad |t|^{\frac{d}{2}-2} << \left(\frac{R}{a}\right)^4$$



(d=4 upp

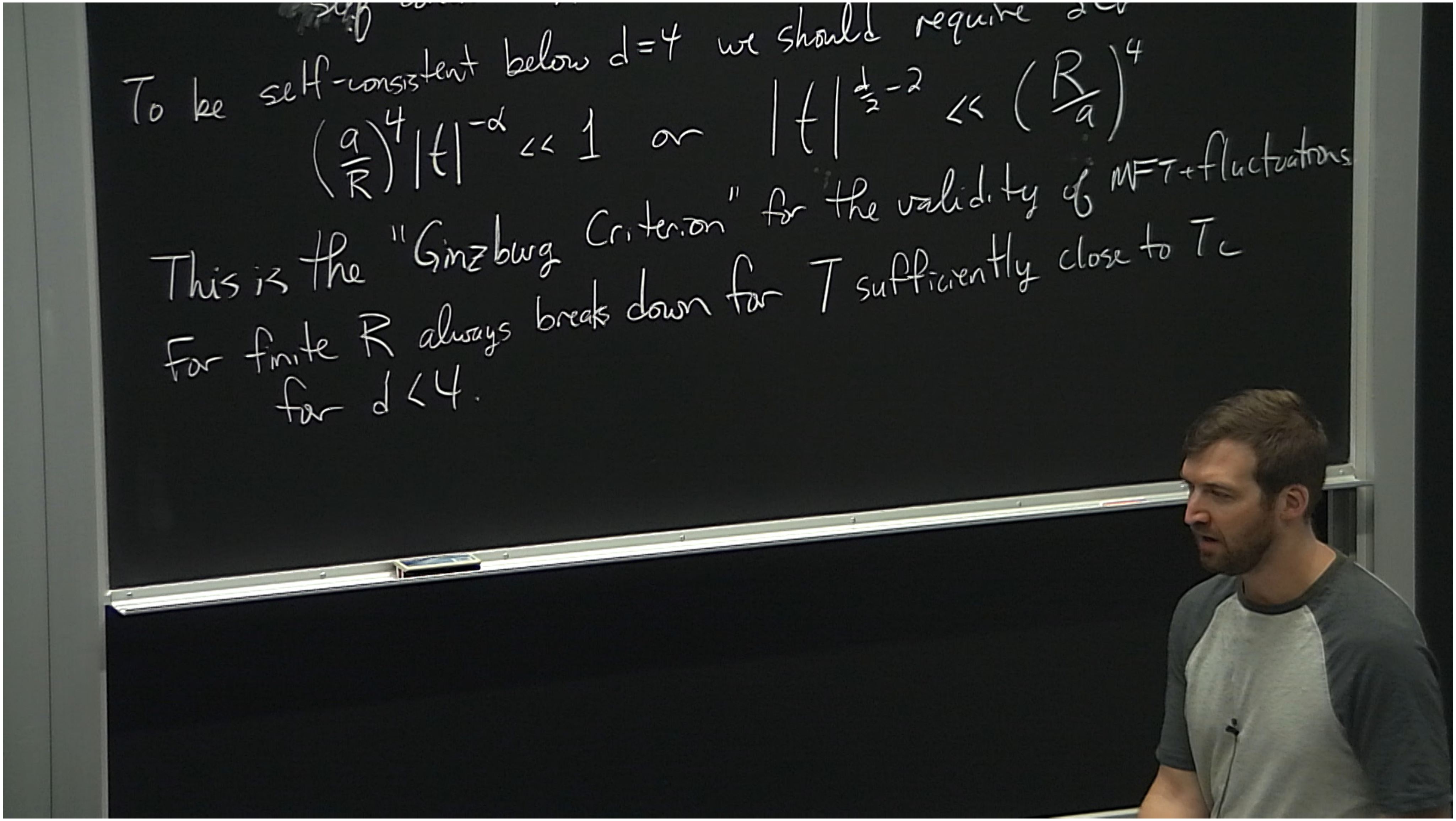
self-consistent)

To be self-consistent below d=4 we should require $\delta C_v \ll 1$

$$\left(\frac{a}{R}\right)^4 |t|^{-\alpha} \ll 1 \quad \text{or} \quad |t|^{\frac{d}{2}-2} \ll \left(\frac{R}{a}\right)^4$$

This is the "Ginzburg Criterion" for the validity of MFT+fluctuations

Fo



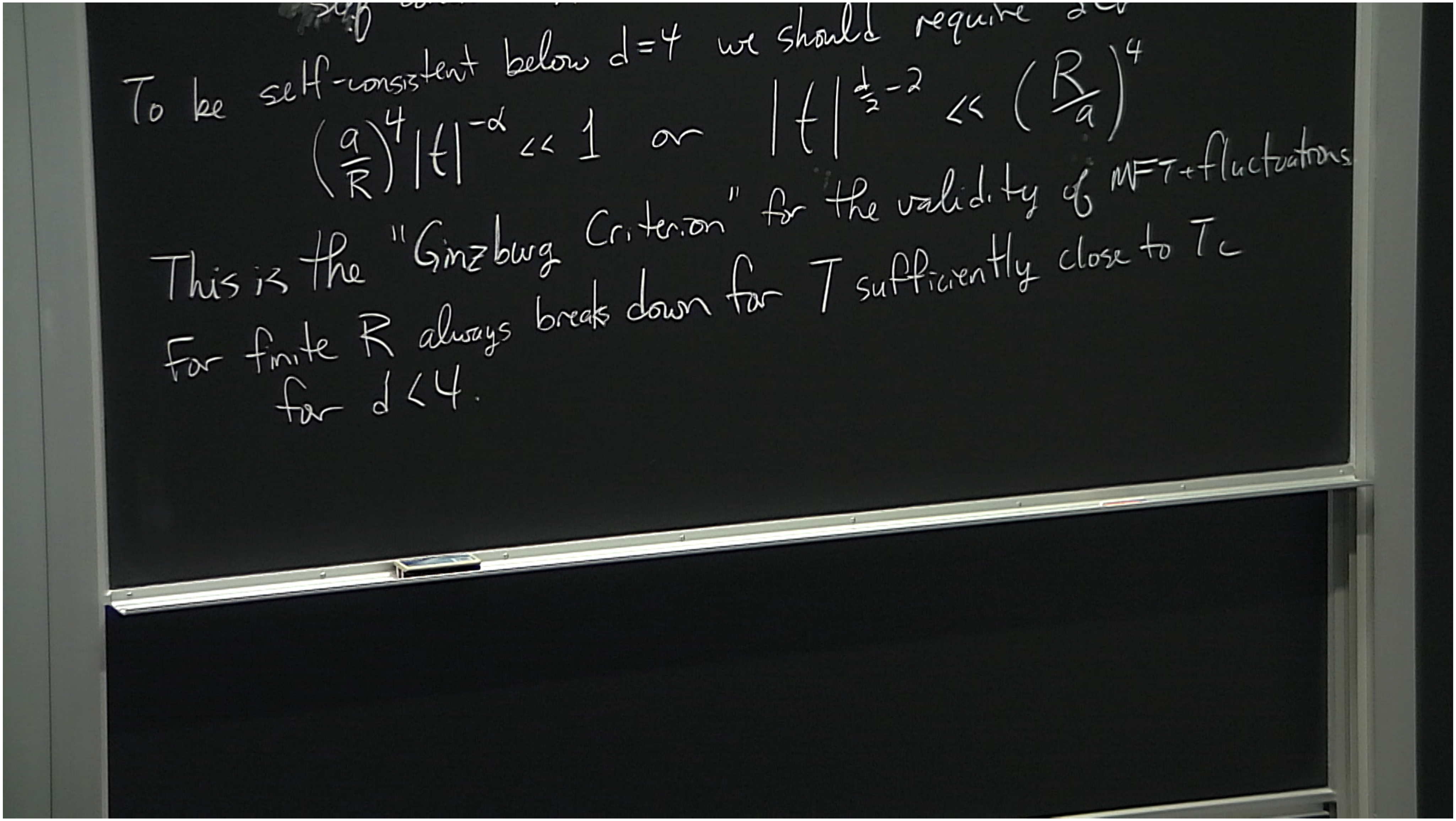
we should require

To be self-consistent below $d=4$ we should require

$$\left(\frac{a}{R}\right)^4 |t|^{-\alpha} \ll 1 \quad \text{or} \quad |t|^{\frac{d}{2}-2} \ll \left(\frac{R}{a}\right)^4$$

This is the "Ginzburg Criterion" for the validity of MFT+fluctuations

For finite R always breaks down for T sufficiently close to T_c for $d<4$.

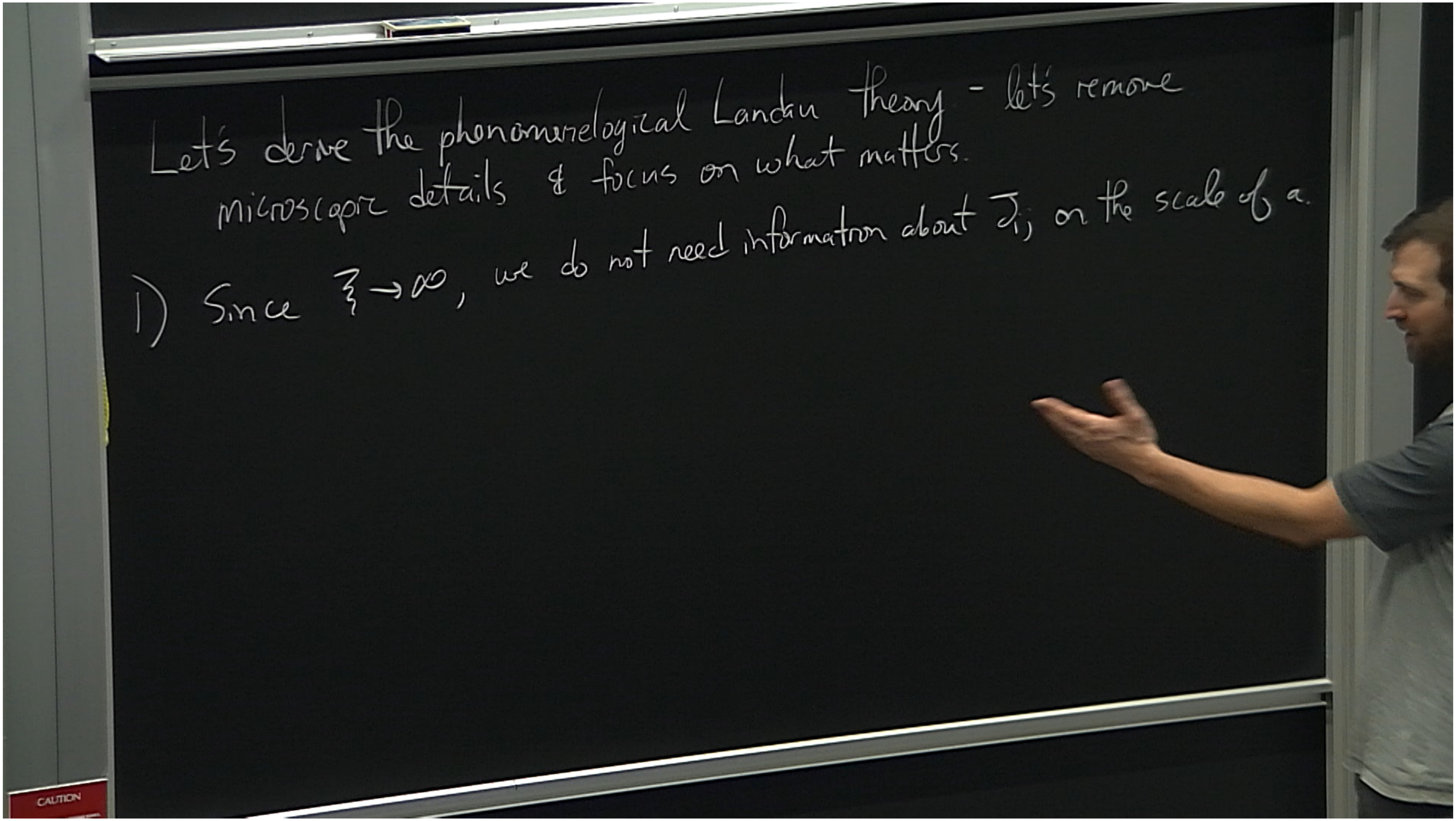


To be self-consistent below $d=4$ we should require

$$\left(\frac{a}{R}\right)^4 |t|^{-\alpha} \ll 1 \quad \text{or} \quad |t|^{\frac{d}{2}-2} \ll \left(\frac{R}{a}\right)^4$$

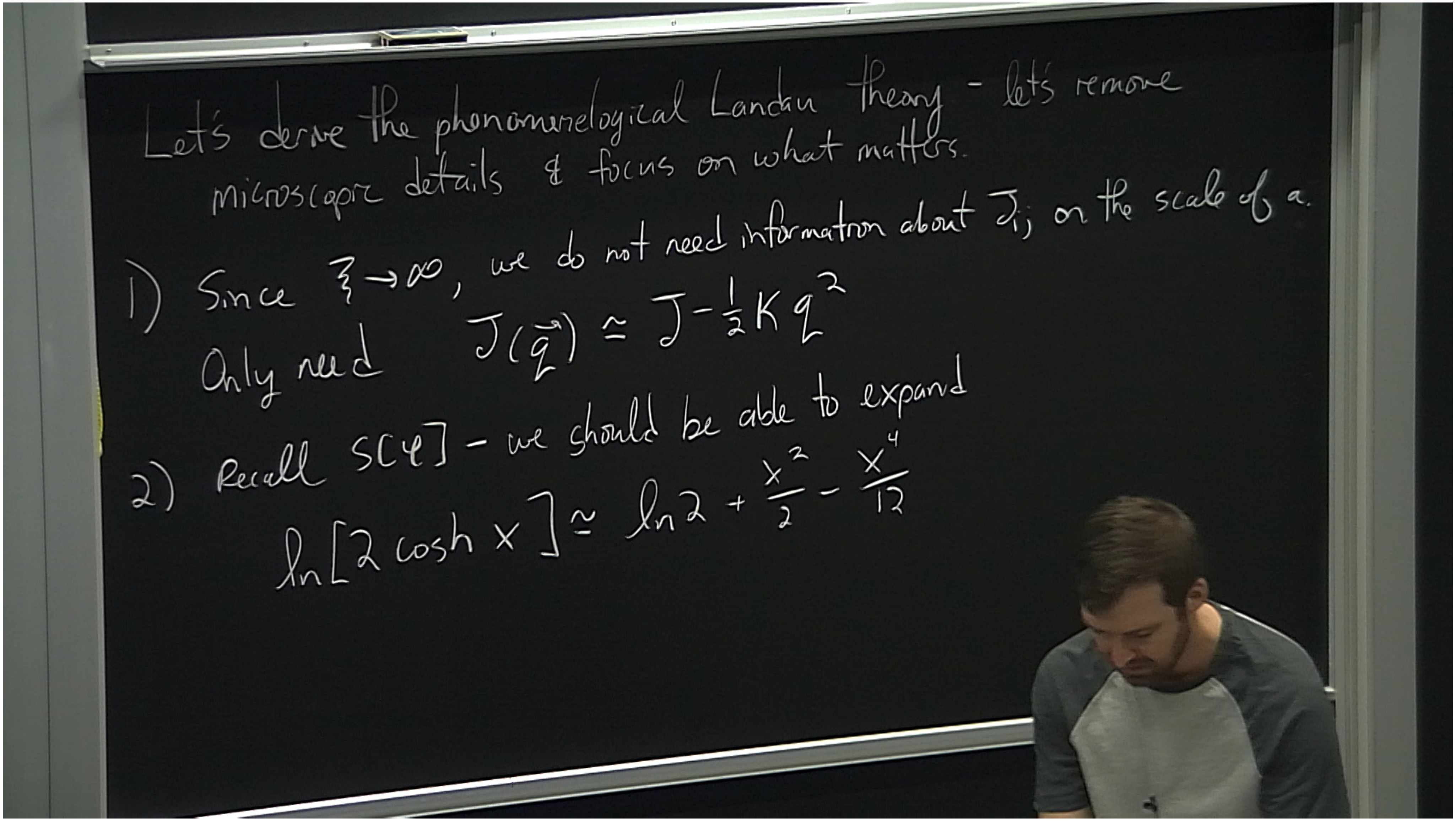
This is the "Ginzburg Criterion" for the validity of MFT + fluctuations

For finite R always breaks down for T sufficiently close to T_c for $d<4$.



Let's derive the phenomenological Landau theory - let's remove microscopic details & focus on what matters.

1) Since $\xi \to \infty$, we do not need information about J_{ij} on the scale of a.



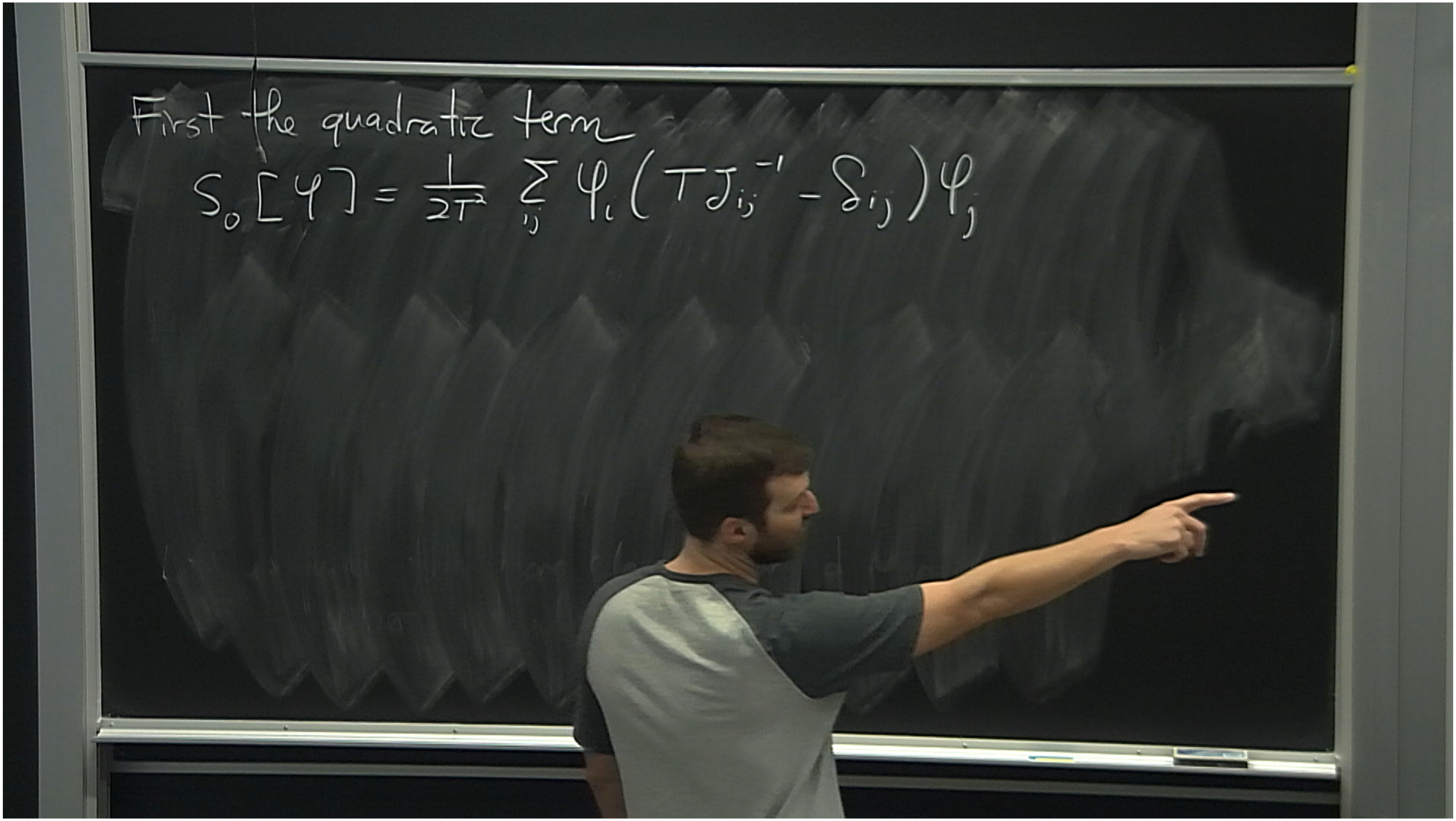
Let's derive the phenomenological Landau theory - let's remove microscopic details & focus on what matters.

1) Since $\xi \to \infty$, we do not need information about J_{ij} on the scale of a.

Only need $J(\vec{q}) \simeq J - \frac{1}{2} K q^2$

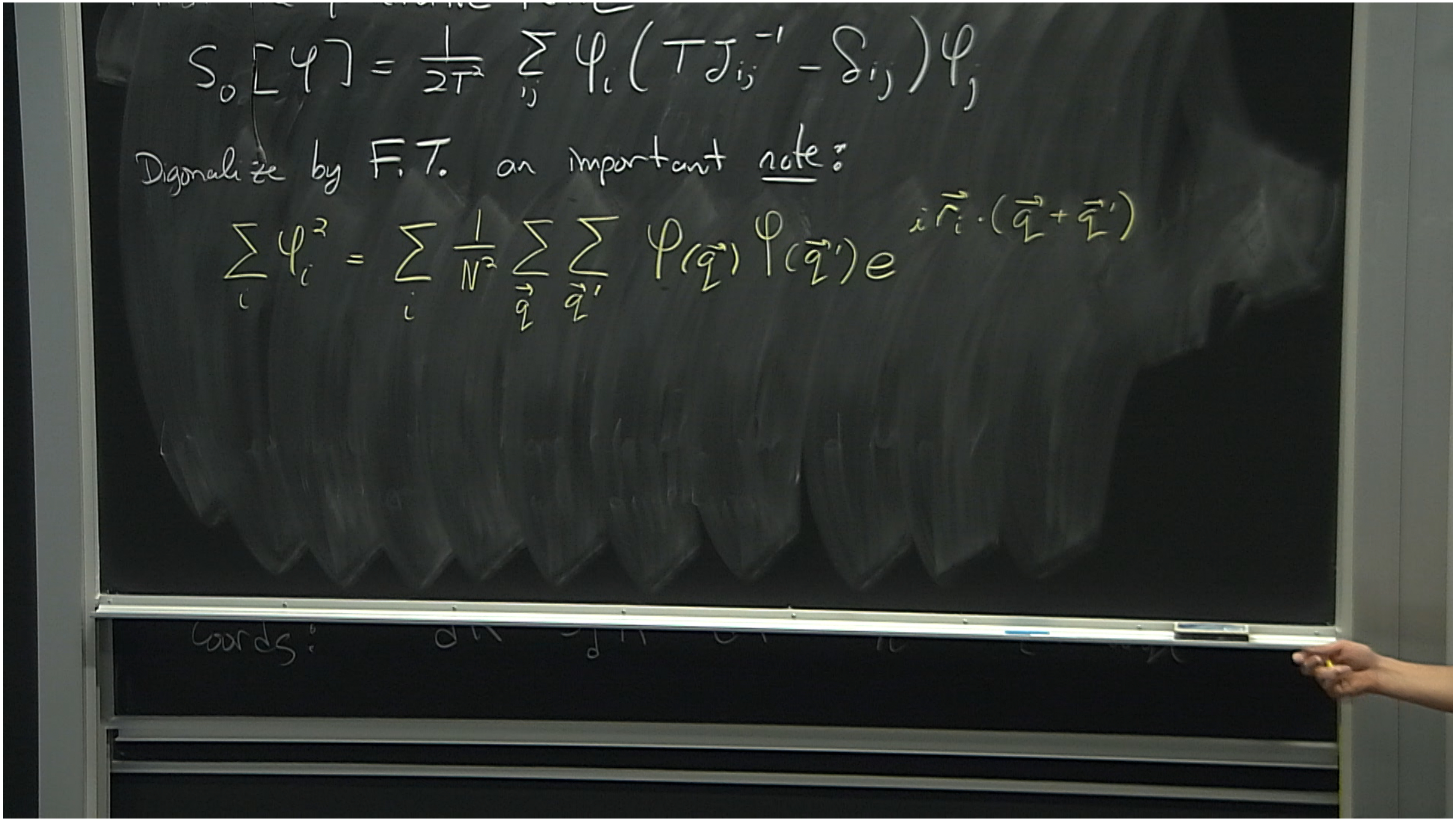
2) Recall $S[\varphi]$ - we should be able to expand

$$\ln[2 \cosh x] \simeq \ln 2 + \frac{x^2}{2} - \frac{x^4}{12}$$



First the quadratic term

$$S_0[\varphi] = \frac{1}{2T^2}\sum_{ij}\varphi_i\left(T J_{ij}^{-1} - \delta_{ij}\right)\varphi_j$$

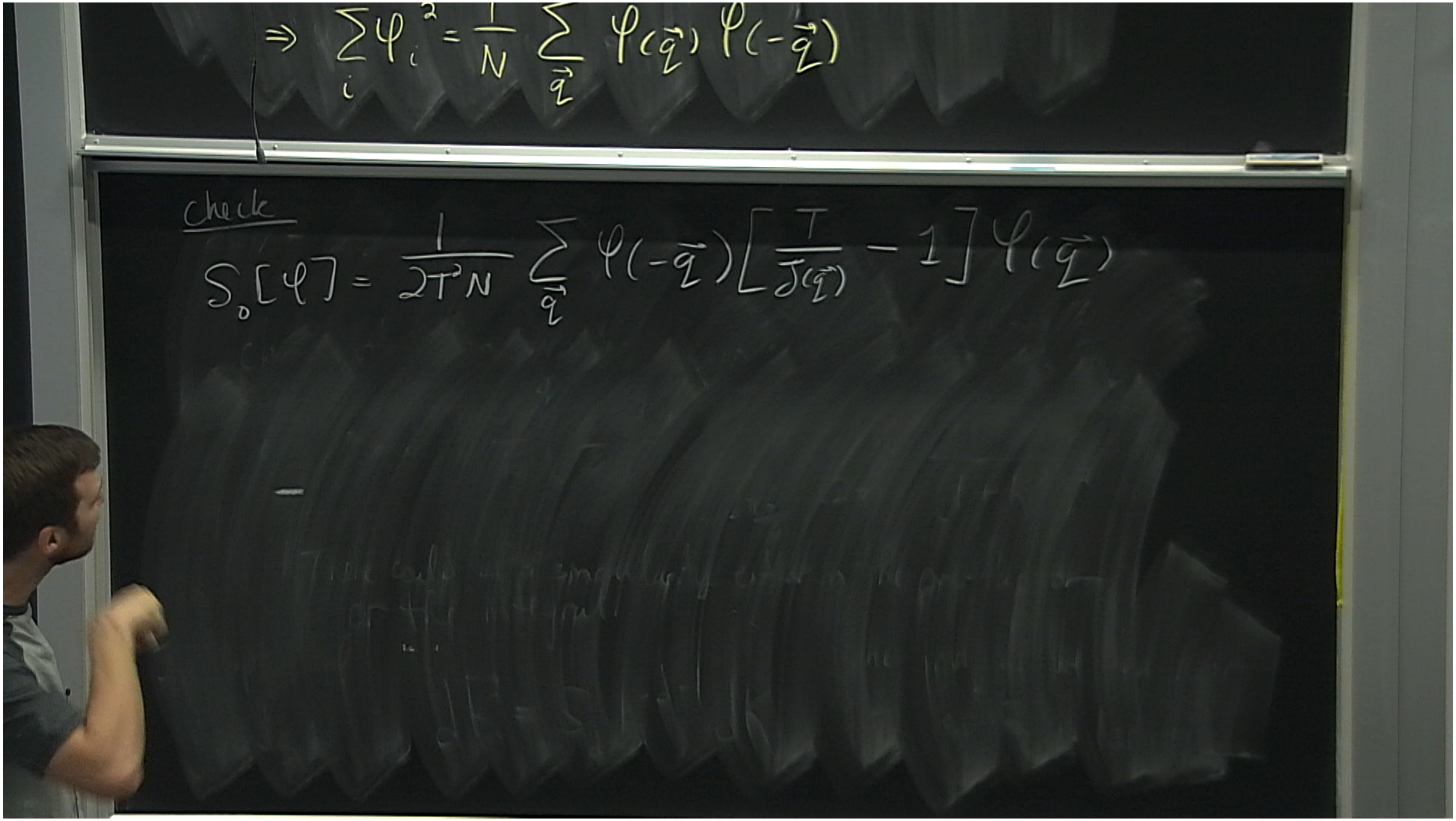


$$S_0[\varphi] = \frac{1}{2T^2}\sum_{ij}\varphi_i\left(TJ_{ij}^{-1}-\delta_{ij}\right)\varphi_j$$

Digonalize by F.T. an important <u>note</u>:

$$\sum_i \varphi_i^2 = \sum_i \frac{1}{N^2}\sum_{\vec{q}}\sum_{\vec{q}\,'}\varphi(\vec{q})\,\varphi(\vec{q}\,')\,e^{i\vec{r}_i\cdot(\vec{q}+\vec{q}\,')}$$

Coords:



$$\Rightarrow \sum_i \varphi_i^2 = \frac{1}{N}\sum_{\vec{q}} \varphi(\vec{q})\,\varphi(-\vec{q})$$

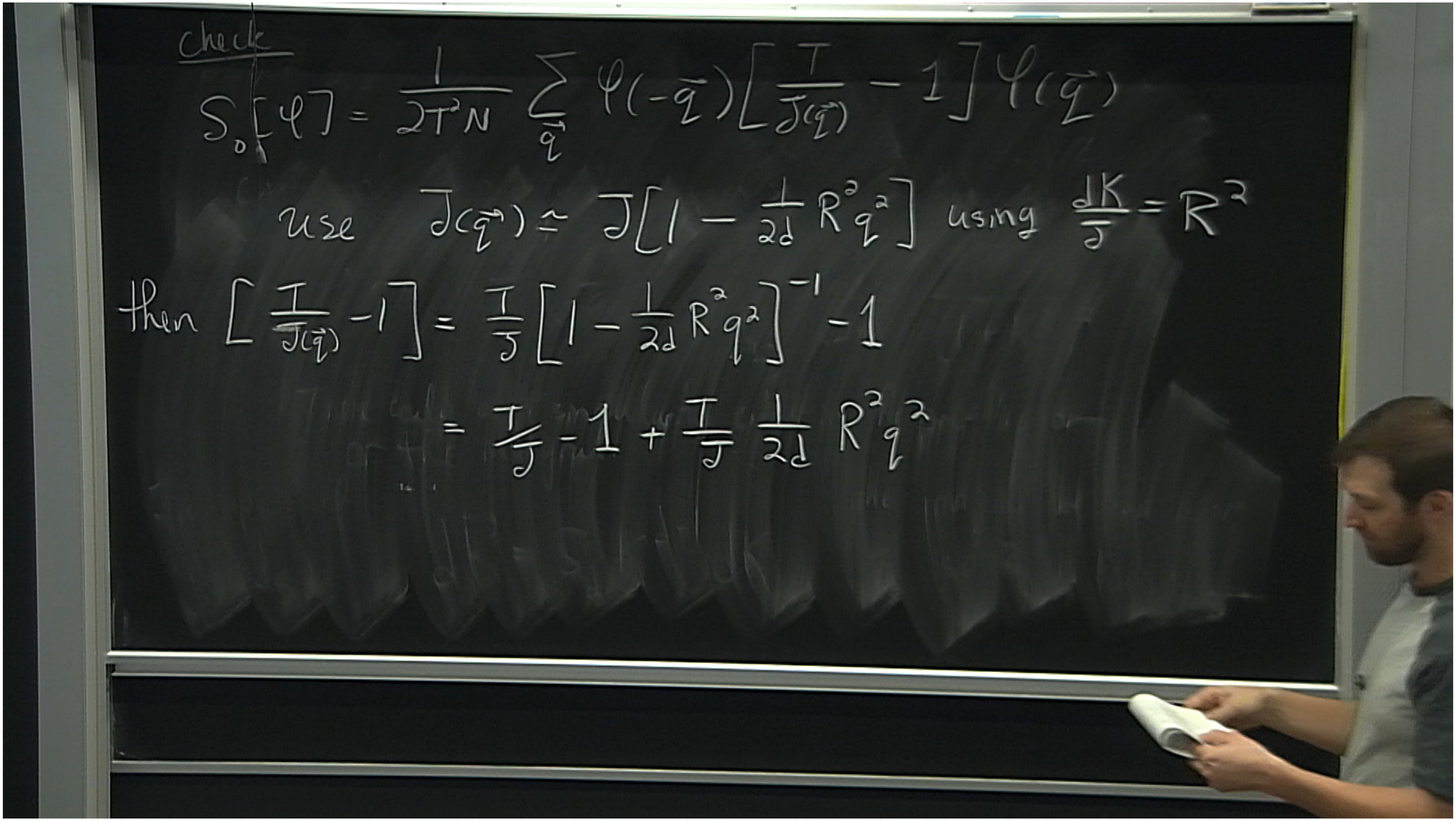
check

$$S_0[\varphi] = \frac{1}{2T^2N}\sum_{\vec{q}} \varphi(-\vec{q})\left[\frac{T}{J(\vec{q})} - 1\right]\varphi(\vec{q})$$

$$\Rightarrow \sum_i \varphi_i^2 = \frac{1}{N} \sum_{\vec{q}} \varphi(\vec{q})\, \varphi(-\vec{q})$$

check

$$S_0[\varphi] = \frac{1}{2T^2 N} \sum_{\vec{q}} \varphi(-\vec{q}) \left[\frac{T}{J(\vec{q})} - 1 \right] \varphi(\vec{q})$$



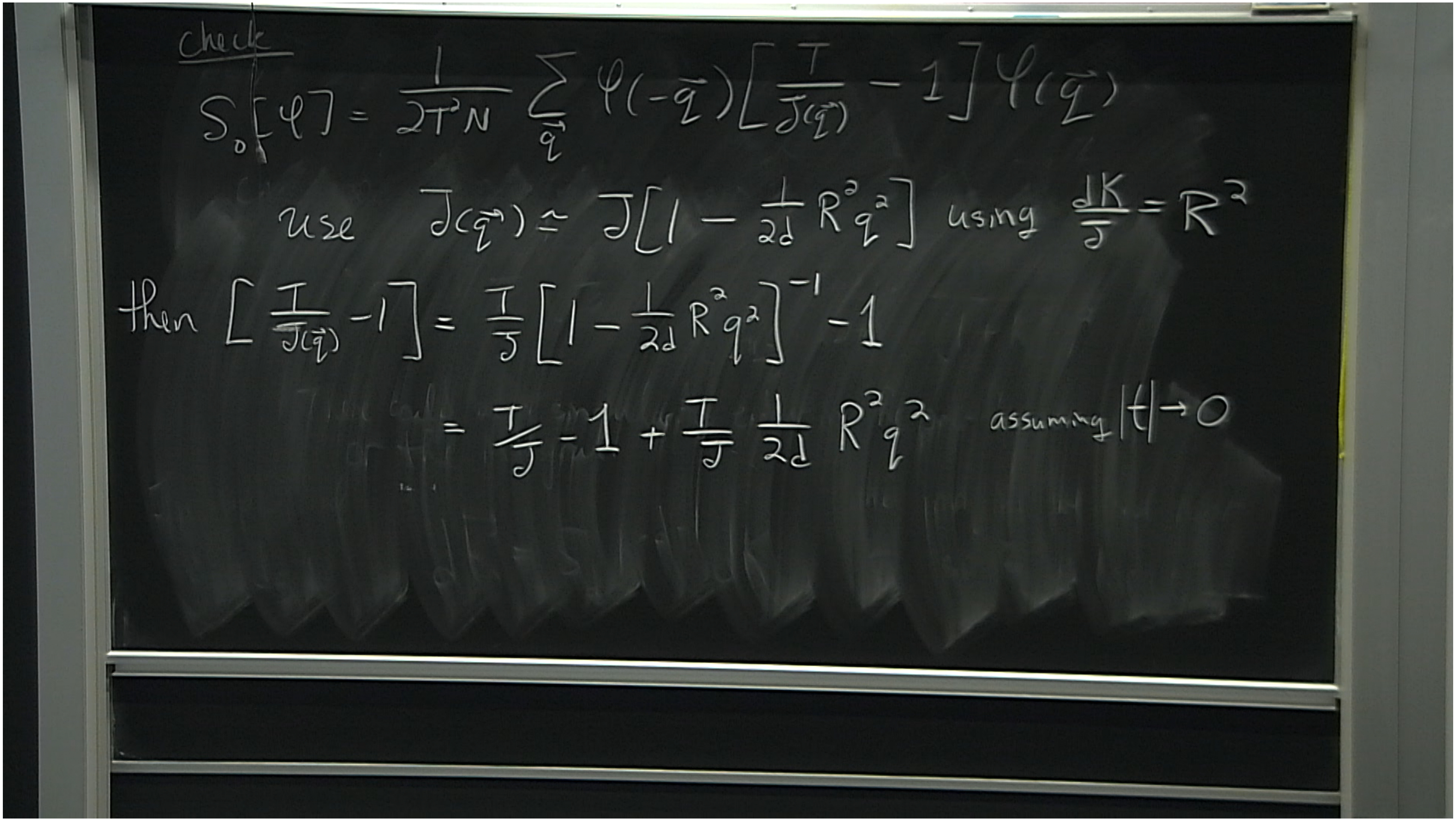
Check

$$S_0[\varphi] = \frac{1}{2T^2N}\sum_{\vec{q}} \varphi(-\vec{q})\left[\frac{T}{J(\vec{q})} - 1\right]\varphi(\vec{q})$$

use $J(\vec{q}) \simeq J\left[1 - \frac{1}{2d}R^2q^2\right]$ using $\frac{dK}{J} = R^2$

then $\left[\frac{T}{J(\vec{q})} - 1\right] = \frac{T}{J}\left[1 - \frac{1}{2d}R^2q^2\right]^{-1} - 1$

$$= \frac{T}{J} - 1 + \frac{T}{J}\frac{1}{2d}R^2q^2$$



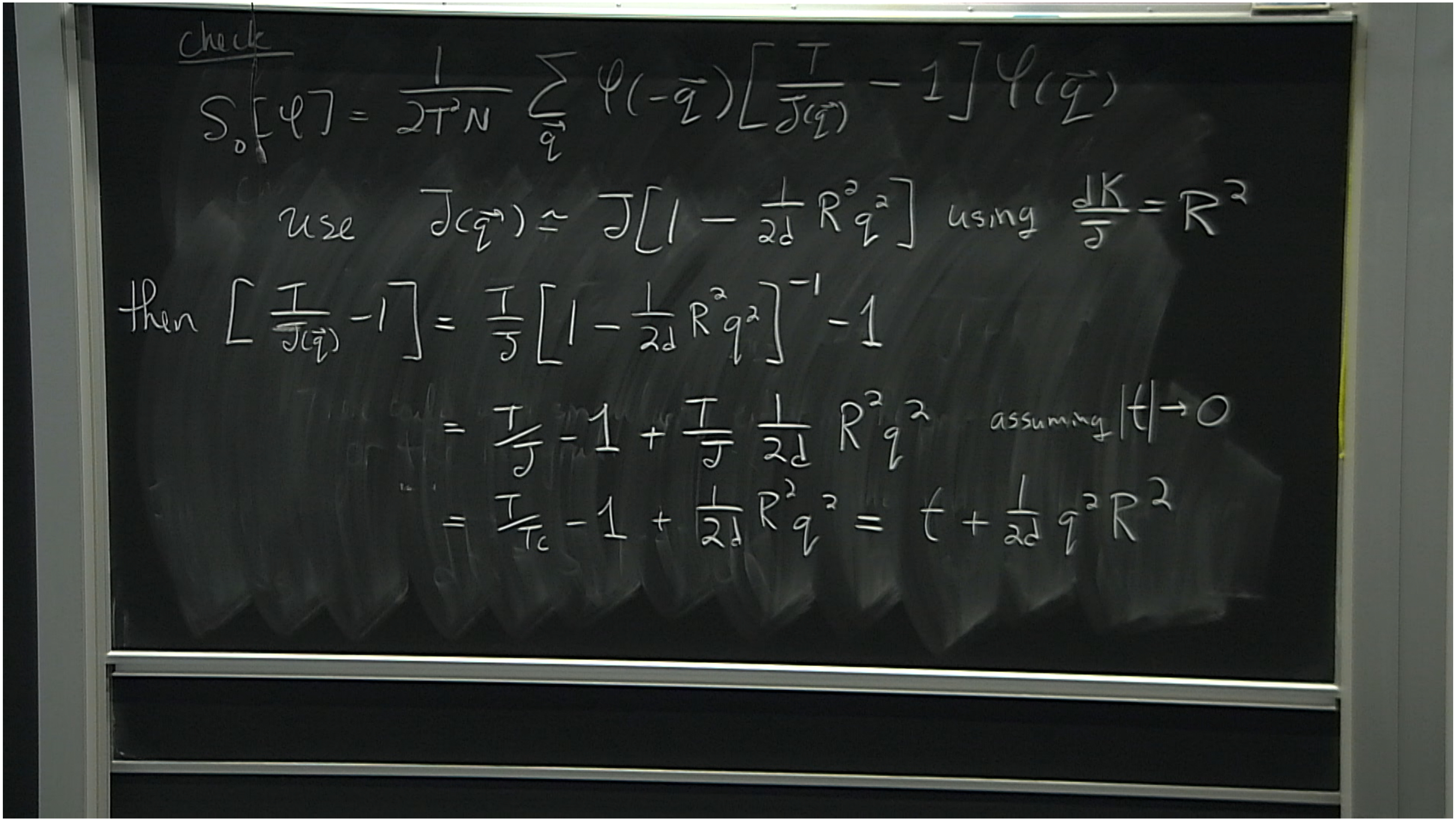
check

$$S_0[\varphi] = \frac{1}{2T^2N}\sum_{\vec{q}} \varphi(-\vec{q})\left[\frac{T}{J(\vec{q})} - 1\right]\varphi(\vec{q})$$

use $J(\vec{q}) \simeq J\left[1 - \frac{1}{2d}R^2q^2\right]$ using $\frac{dK}{J} = R^2$

then $\left[\frac{T}{J(\vec{q})} - 1\right] = \frac{T}{J}\left[1 - \frac{1}{2d}R^2q^2\right]^{-1} - 1$

$$= \frac{T}{J} - 1 + \frac{T}{J}\frac{1}{2d}R^2q^2$$ assuming $|t| \to 0$



Check

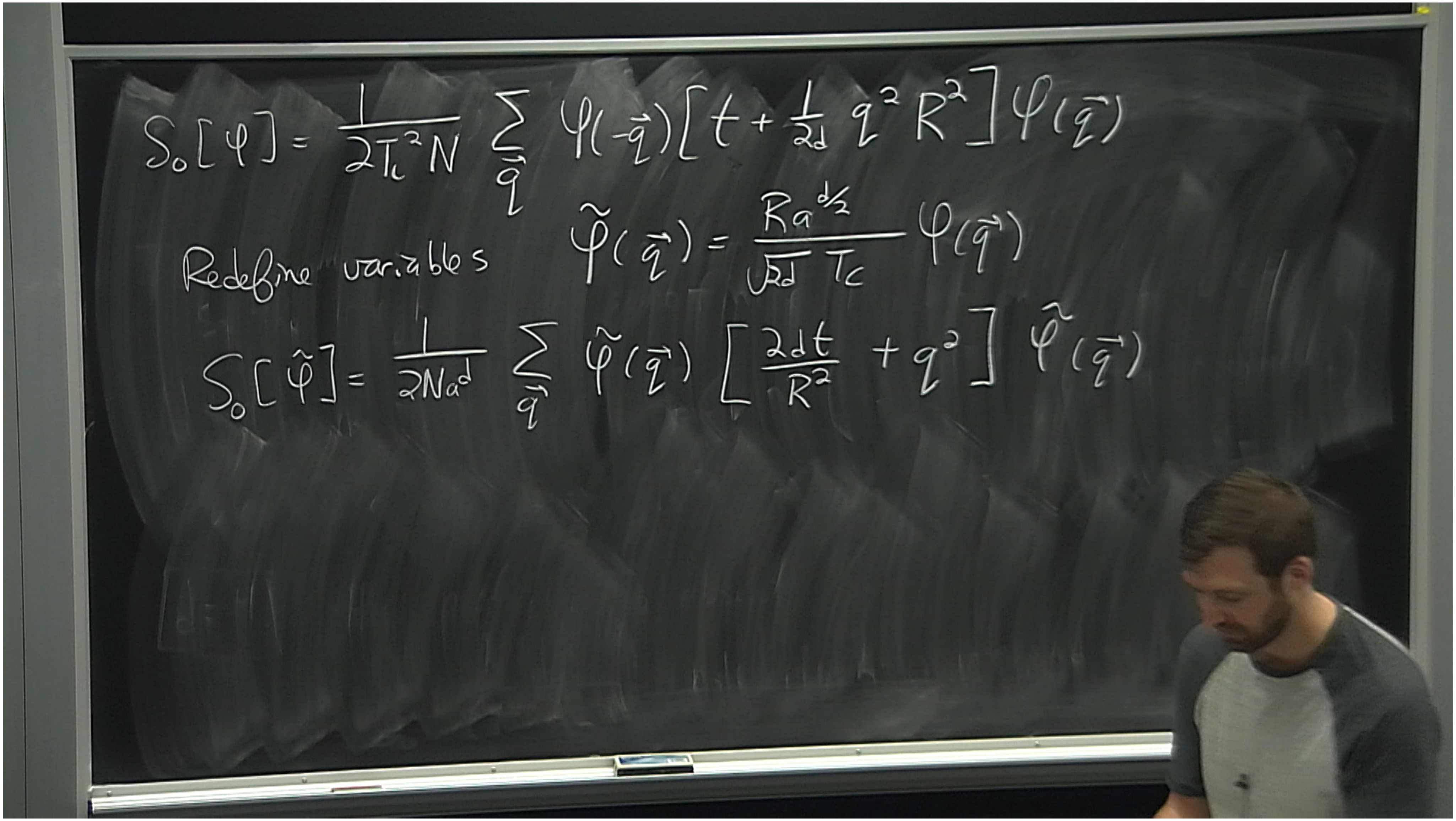
$$S_0[\varphi] = \frac{1}{2T^2 N} \sum_{\vec{q}} \varphi(-\vec{q}) \left[\frac{T}{J(\vec{q})} - 1 \right] \varphi(\vec{q})$$

use $J(\vec{q}) \simeq J\left[1 - \frac{1}{2d} R^2 q^2\right]$ using $\frac{dK}{J} = R^2$

then $\left[\frac{T}{J(\vec{q})} - 1 \right] = \frac{T}{J} \left[1 - \frac{1}{2d} R^2 q^2 \right]^{-1} - 1$

$$= \frac{T}{J} - 1 + \frac{T}{J} \frac{1}{2d} R^2 q^2 \quad \text{assuming } |t| \to 0$$

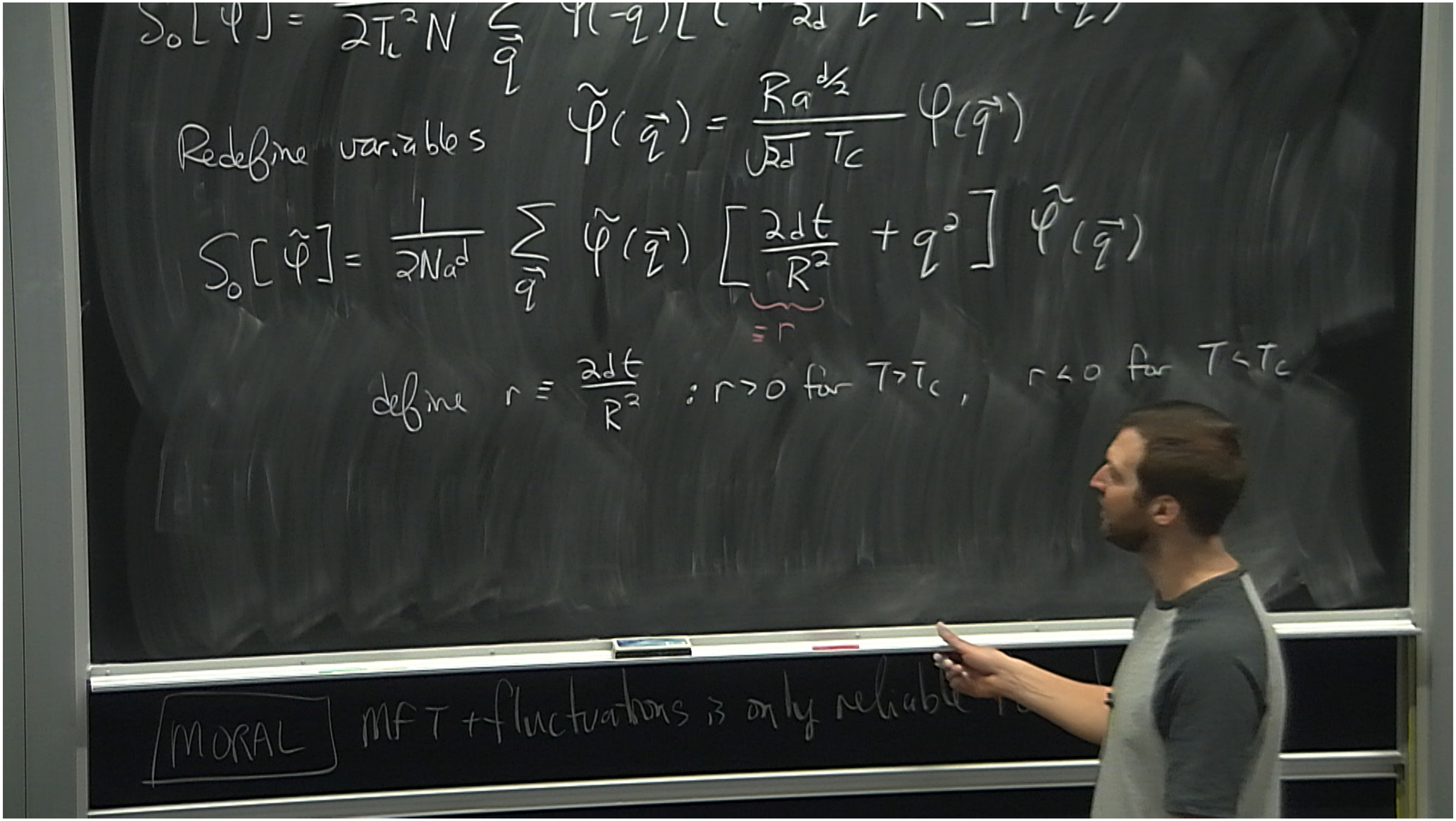
$$= \frac{T}{T_c} - 1 + \frac{1}{2d} R^2 q^2 = t + \frac{1}{2d} q^2 R^2$$



$$S_0[\varphi] = \frac{1}{2T_c^2 N}\sum_{\vec{q}} \varphi(-\vec{q})\left[t + \frac{1}{2d}\, q^2 R^2\right]\varphi(\vec{q})$$

Redefine variables $\tilde{\varphi}(\vec{q}) = \frac{R a^{d/2}}{\sqrt{2d}\; T_c}\varphi(\vec{q})$

$$S_0[\hat{\varphi}] = \frac{1}{2Na^d}\sum_{\vec{q}} \tilde{\varphi}(\vec{q})\left[\frac{2dt}{R^2} + q^2\right]\tilde{\varphi}(\vec{q})$$



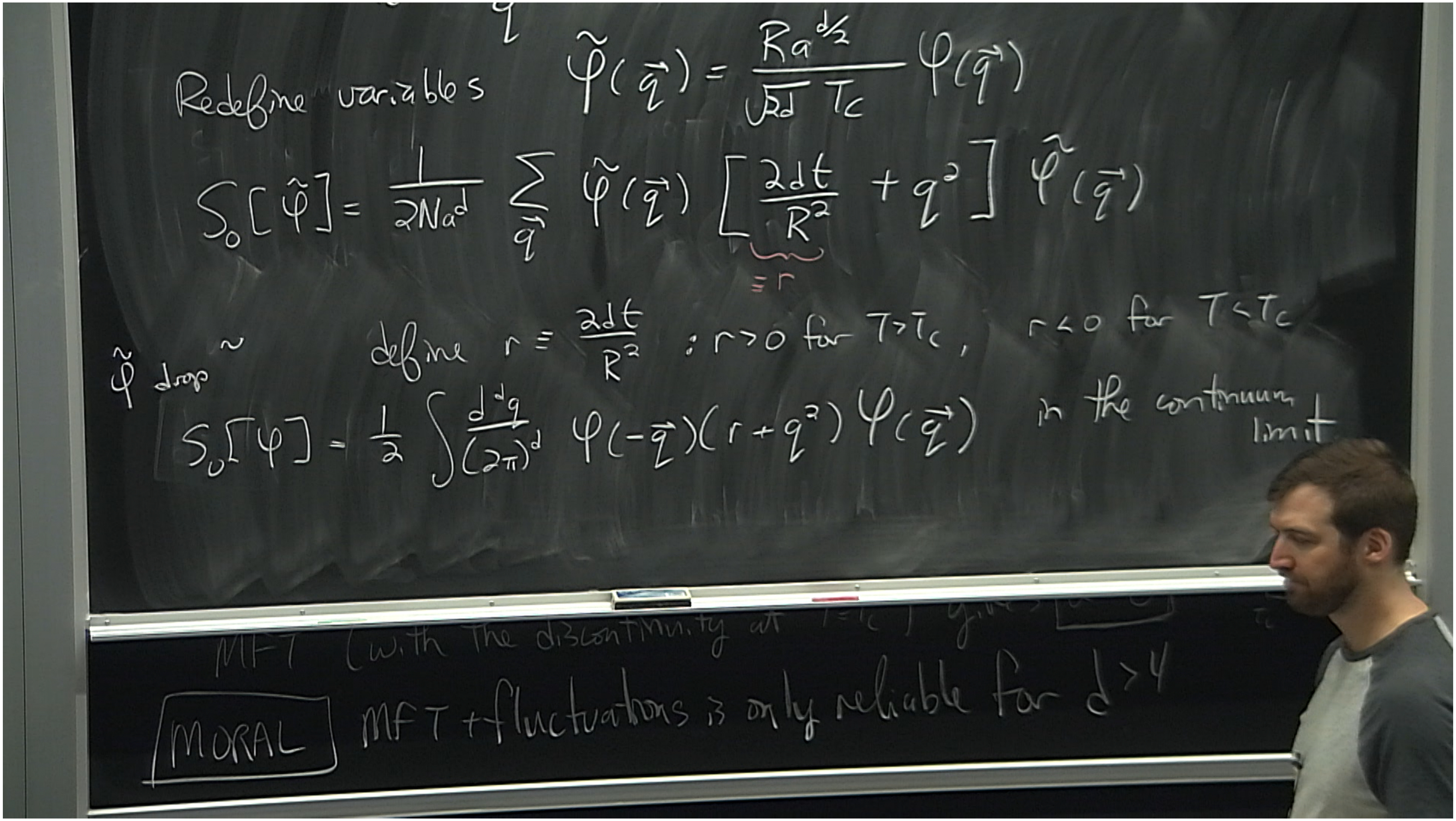
$$S_0[\varphi] = \frac{}{2T_c^2 N} \sum_{\vec{q}} \varphi(-q)[\quad 2d \quad]$$

Redefine variables $\tilde{\varphi}(\vec{q}) = \frac{R a^{d/2}}{\sqrt{2d}\, T_c} \varphi(\vec{q})$

$$S_0[\tilde{\varphi}] = \frac{1}{2Na^d} \sum_{\vec{q}} \tilde{\varphi}(\vec{q}) \Big[\underbrace{\frac{2dt}{R^2}}_{= r} + q^2 \Big] \tilde{\varphi}(\vec{q})$$

define $r \equiv \frac{2dt}{R^2}$; $r > 0$ for $T > T_c$, $r < 0$ for $T < T_c$

$\boxed{\text{MORAL}}$ MFT + fluctuations is only reliable



Redefine variables $\tilde{\varphi}(\vec{q}) = \frac{R a^{d/2}}{\sqrt{2d}\; T_c} \varphi(\vec{q})$

$$S_0[\tilde{\varphi}] = \frac{1}{2Na^d} \sum_{\vec{q}} \tilde{\varphi}(\vec{q}) \Big[\underbrace{\frac{2dt}{R^2}}_{=r} + q^2 \Big] \tilde{\varphi}(\vec{q})$$

define $r = \frac{2dt}{R^2}$; $r > 0$ for $T > T_c$, $r < 0$ for $T < T_c$

$\tilde{\varphi}$ drop ~

$$S_0[\varphi] = \frac{1}{2} \int \frac{d^dq}{(2\pi)^d} \varphi(-\vec{q})(r+q^2)\varphi(\vec{q})$$ in the continuum limit

MFT (with the discontinuity at $T = T_c$) gives

$\boxed{\text{MORAL}}$ MFT + fluctuations is only reliable for $d > 4$

$$\vec{q}$$

Redefine variables $\tilde{\varphi}(\vec{q}) = \frac{R a^{d/2}}{\sqrt{2d}\; T_c} \varphi(\vec{q})$

$$S_0[\tilde{\varphi}] = \frac{1}{2Na^d} \sum_{\vec{q}} \tilde{\varphi}(\vec{q}) \Big[\underbrace{\frac{2dt}{R^2}}_{\equiv r} + q^2 \Big] \tilde{\varphi}(\vec{q})$$

$\tilde{\varphi}$ drop ~

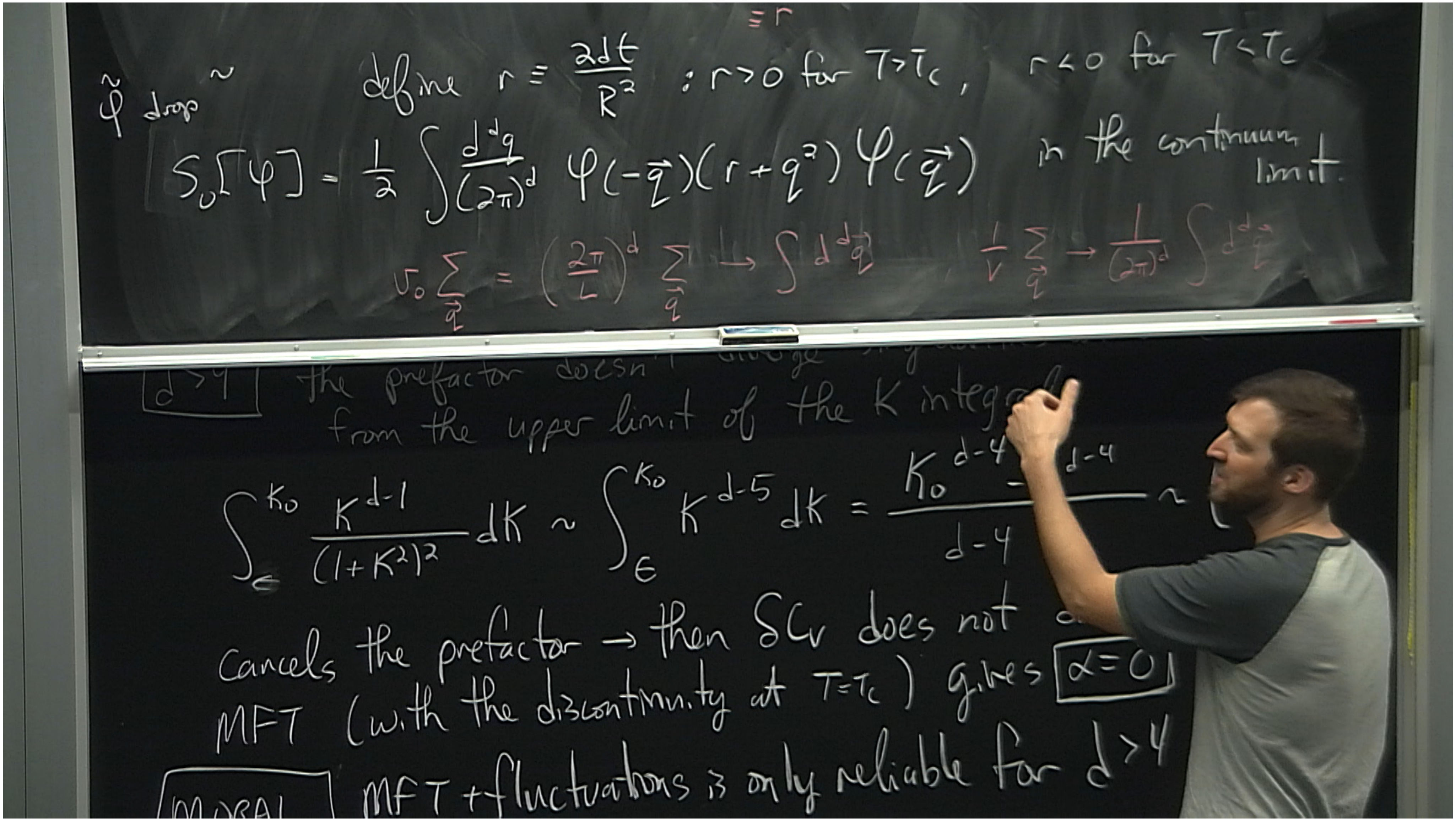
define $r \equiv \frac{2dt}{R^2}$; $r > 0$ for $T > T_c$, $r < 0$ for $T < T_c$

$$S_0[\varphi] = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \varphi(-\vec{q})(r + q^2)\varphi(\vec{q})$$ in the continuum limit.

$$v_0 \sum_{\vec{q}} = \left(\frac{2\pi}{L}\right)^d \sum_{\vec{q}} \to \int d^d\vec{q} \qquad \frac{1}{V}\sum_{\vec{q}} \to \frac{1}{(2\pi)^d}\int d^d\vec{q}$$

MFT (with the discontinuity at $T = T_c$) gives

MORAL MFT + fluctuations is only reliable for $d > 4$



$\equiv r$

$\tilde{\varphi}$ drop ~

define $r \equiv \frac{2dt}{R^2}$: $r>0$ for $T>T_c$, $r<0$ for $T<T_c$

$$S_0[\varphi] = \frac{1}{2}\int \frac{d^dq}{(2\pi)^d}\, \varphi(-\vec{q})(r+q^2)\varphi(\vec{q})$$ in the continuum limit.

$v_0 \sum_{\vec{q}} = \left(\frac{2\pi}{L}\right)^d \sum_{\vec{q}} \longrightarrow \int d^d\vec{q}$, $\frac{1}{V}\sum_{\vec{q}} \to \frac{1}{(2\pi)^d}\int d^d\vec{q}$

$\boxed{d>4}$ the prefactor doesn't diverge

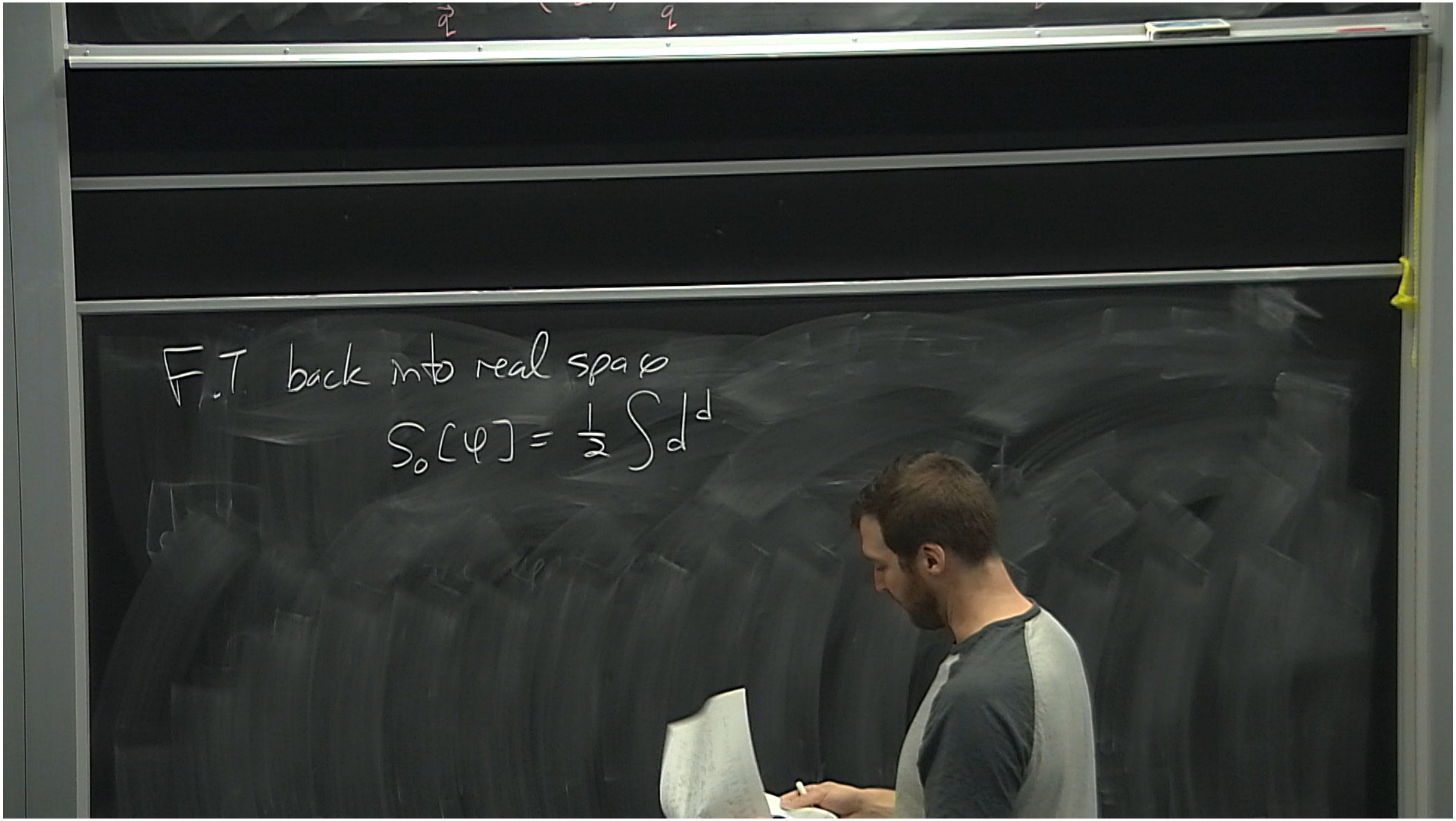
from the upper limit of the K integral

$$\int_{\epsilon}^{K_0} \frac{K^{d-1}}{(1+K^2)^2}\, dK \sim \int_{\epsilon}^{K_0} K^{d-5}\, dK = \frac{K_0^{d-4} - \quad^{d-4}}{d-4} \sim ($$

cancels the prefactor → then δC_V does not

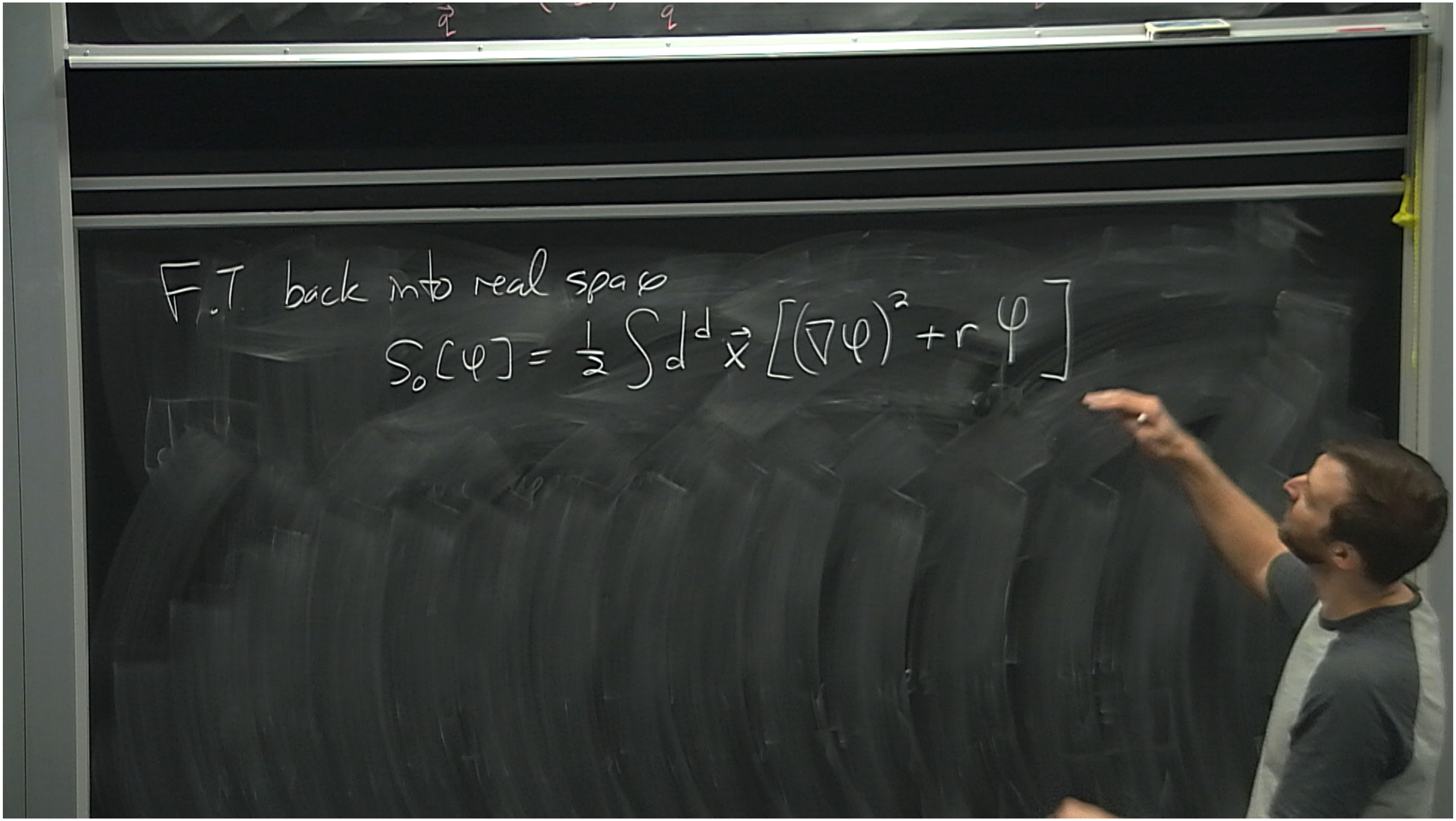
MFT (with the discontinuity at $T=T_c$) gives $\boxed{\alpha = 0}$

MORAL MFT + fluctuations is only reliable for $d>4$



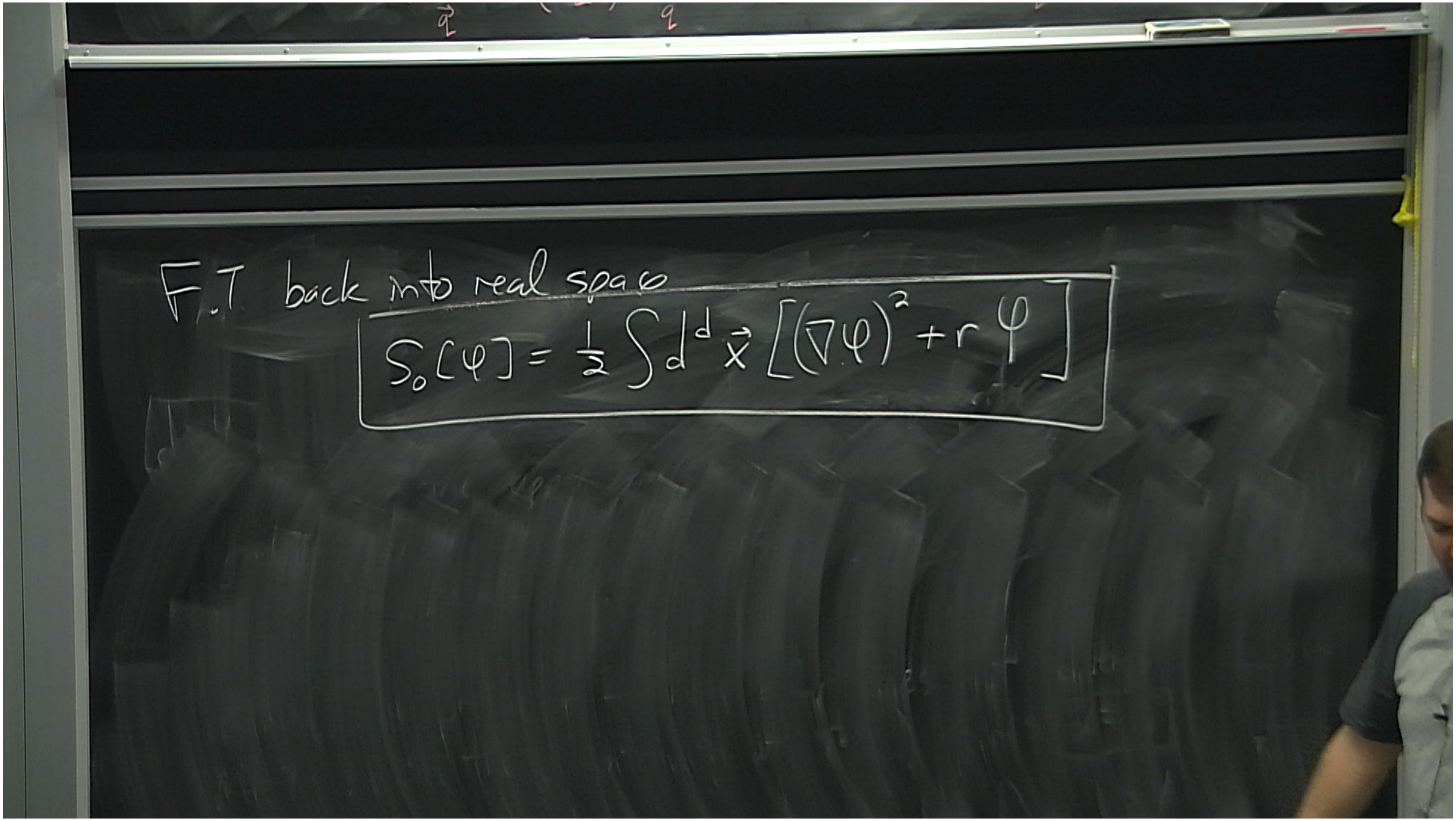
F.T. back into real space

$$S_0[\varphi] = \frac{1}{2}\int d^d$$



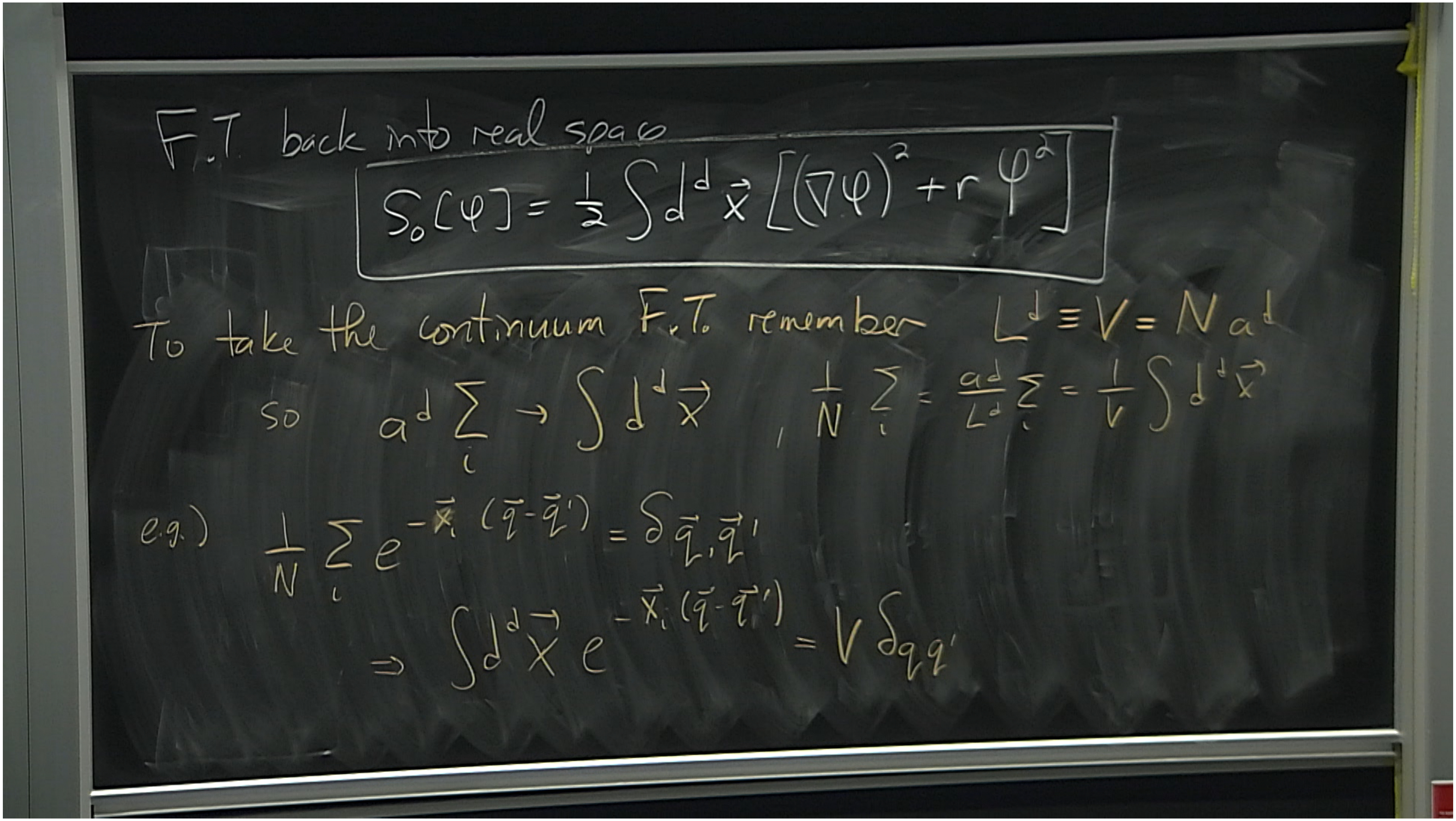
F.T. back into real space

$$S_0[\varphi] = \frac{1}{2}\int d^d\vec{x}\left[(\nabla\varphi)^2 + r\,\varphi\right]$$



F.T. back into real space

$$\boxed{S_0[\varphi] = \frac{1}{2}\int d^d\vec{x}\left[(\nabla\varphi)^2 + r\,\varphi\right]}$$



F.T. back into real space

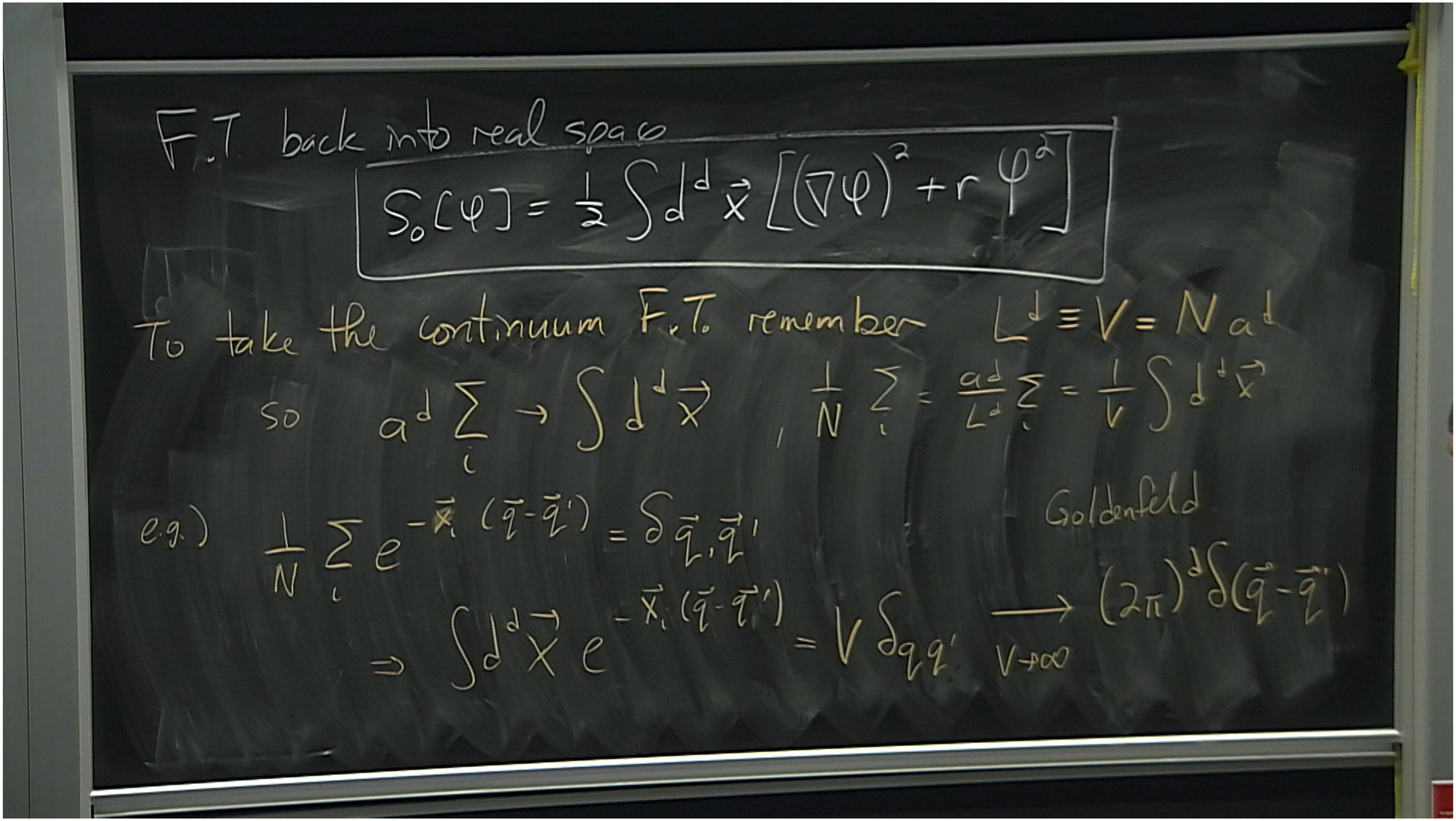
$$\boxed{S_0[\varphi] = \frac{1}{2}\int d^d\vec{x}\left[(\nabla\varphi)^2 + r\varphi^2\right]}$$

To take the continuum F.T. remember $L^d \equiv V = N a^d$

so $a^d\sum_i \to \int d^d\vec{x}$, $\frac{1}{N}\sum_i = \frac{a^d}{L^d}\sum_i = \frac{1}{V}\int d^d\vec{x}$

e.g.) $\frac{1}{N}\sum_i e^{-\vec{x}_i(\vec{q}-\vec{q}')} = \delta_{\vec{q},\vec{q}'}$

$$\Rightarrow \int d^d\vec{x}\, e^{-\vec{x}_i(\vec{q}-\vec{q}')} = V\delta_{qq'}$$



F.T. back into real space

$$\boxed{S_0[\varphi] = \frac{1}{2}\int d^d\vec{x}\,\left[(\nabla\varphi)^2 + r\,\varphi^2\right]}$$

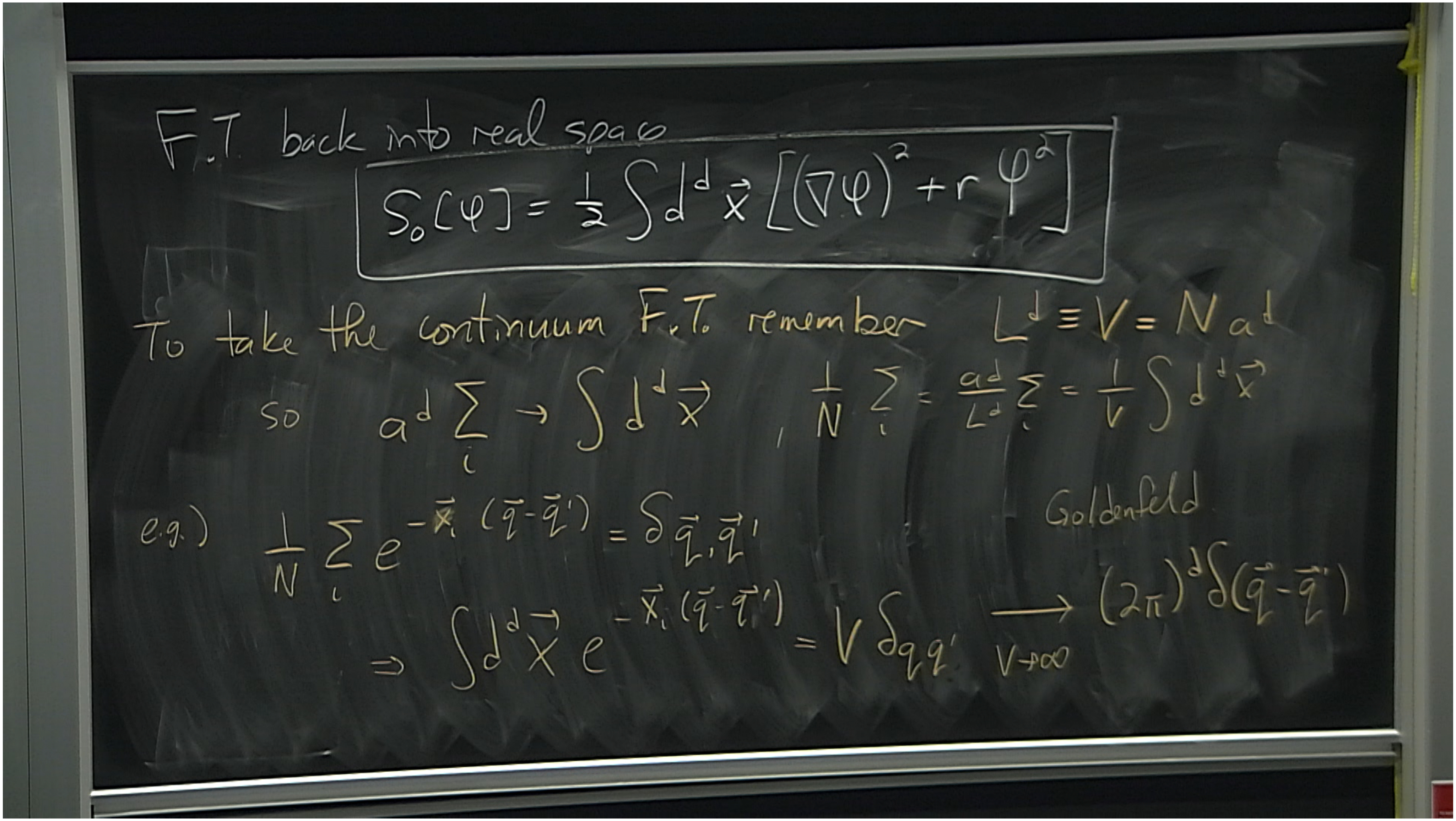
To take the continuum F.T. remember $L^d \equiv V = N a^d$

so $a^d \sum_i \to \int d^d\vec{x}$, $\frac{1}{N}\sum_i = \frac{a^d}{L^d}\sum_i = \frac{1}{V}\int d^d\vec{x}$

Goldenfeld

e.g.) $\frac{1}{N}\sum_i e^{-\vec{x}_i(\vec{q}-\vec{q}\,')} = \delta_{\vec{q},\vec{q}\,'}$

$$\Rightarrow \int d^d\vec{x}\, e^{-\vec{x}_i(\vec{q}-\vec{q}\,')} = V\delta_{qq'} \xrightarrow[V\to\infty]{} (2\pi)^d\delta(\vec{q}-\vec{q}\,')$$



F.T. back into real space

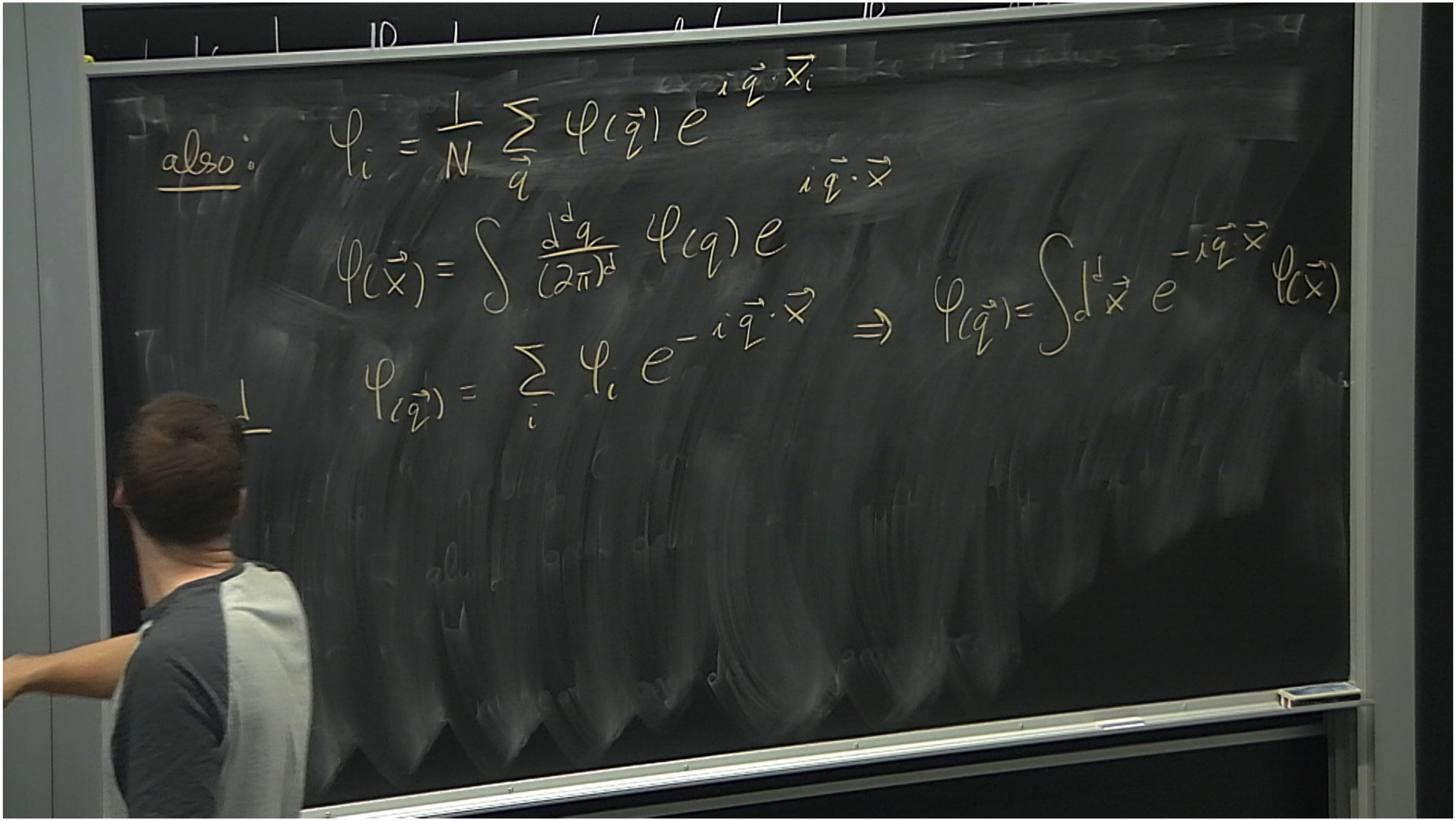
$$\boxed{S_0[\varphi] = \frac{1}{2}\int d^d\vec{x}\,\left[(\nabla\varphi)^2 + r\varphi^2\right]}$$

To take the continuum F.T. remember $L^d \equiv V = N a^d$

so $a^d \sum_i \to \int d^d\vec{x}$, $\frac{1}{N}\sum_i = \frac{a^d}{L^d}\sum_i = \frac{1}{V}\int d^d\vec{x}$

e.g.) $\frac{1}{N}\sum_i e^{-\vec{x}_i(\vec{q}-\vec{q}\,')} = \delta_{\vec{q},\vec{q}\,'}$ Goldenfeld

$$\Rightarrow \int d^d\vec{x}\, e^{-\vec{x}_i(\vec{q}-\vec{q}\,')} = V\delta_{qq'} \xrightarrow[V\to\infty]{} (2\pi)^d\delta(\vec{q}-\vec{q}\,')$$

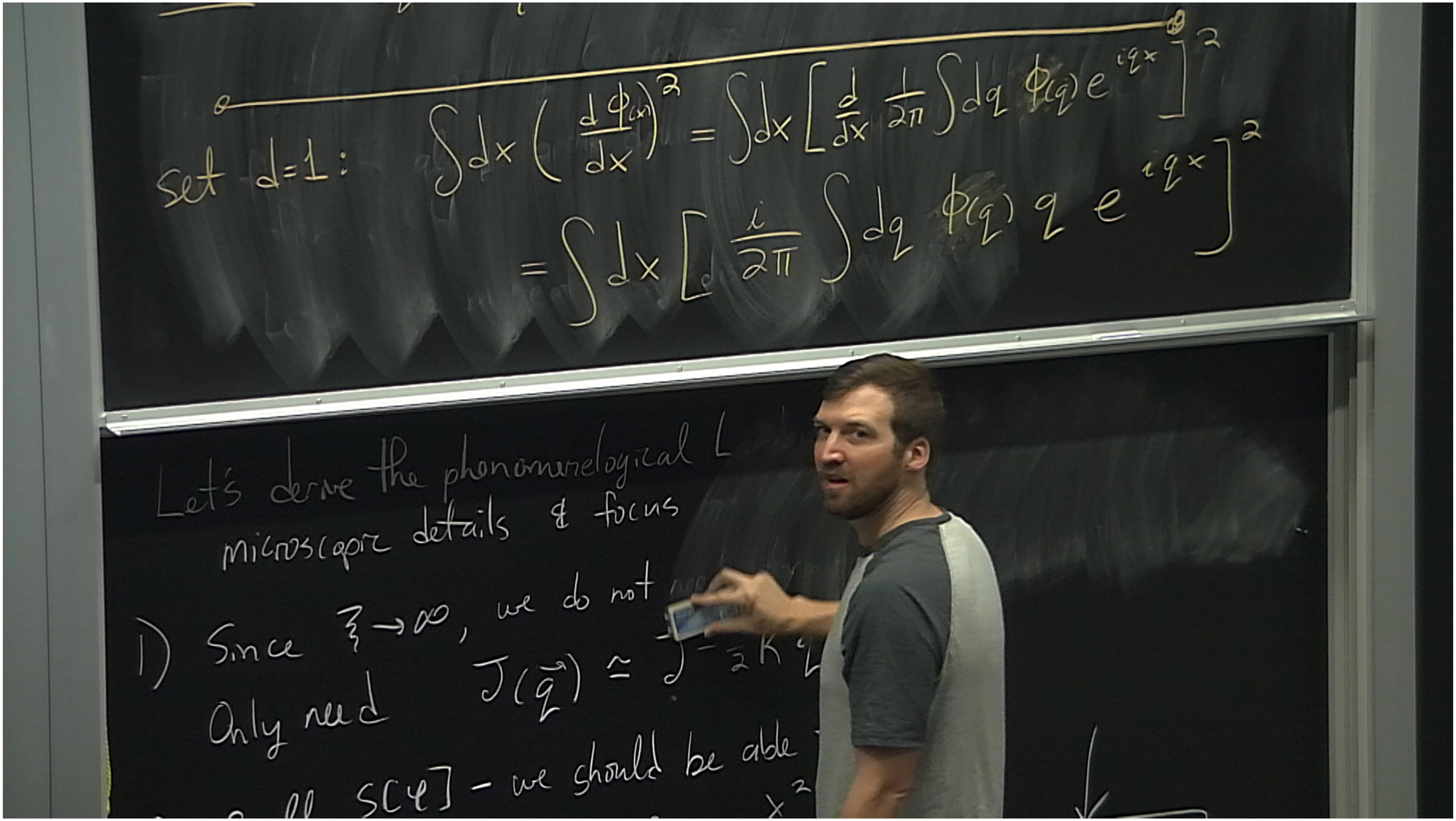


also:

$$\varphi_i = \frac{1}{N} \sum_{\vec{q}} \varphi(\vec{q})\, e^{i\vec{q}\cdot\vec{x}_i}$$

$$\varphi(\vec{x}) = \int \frac{d^dq}{(2\pi)^d}\, \varphi(q)\, e^{i\vec{q}\cdot\vec{x}}$$

$$\varphi(\vec{q}) = \sum_i \varphi_i\, e^{-i\vec{q}\cdot\vec{x}} \Rightarrow \varphi(\vec{q}) = \int d^d\vec{x}\, e^{-i\vec{q}\cdot\vec{x}}\, \varphi(\vec{x})$$



set d=1: $\int dx \left(\frac{d\phi(x)}{dx}\right)^2 = \int dx \left[\frac{d}{dx}\frac{1}{2\pi}\int dq\, \phi(q)\, e^{iqx}\right]^2$

$= \int dx \left[\frac{i}{2\pi}\int dq\, \phi(q)\, q\, e^{iqx}\right]^2$

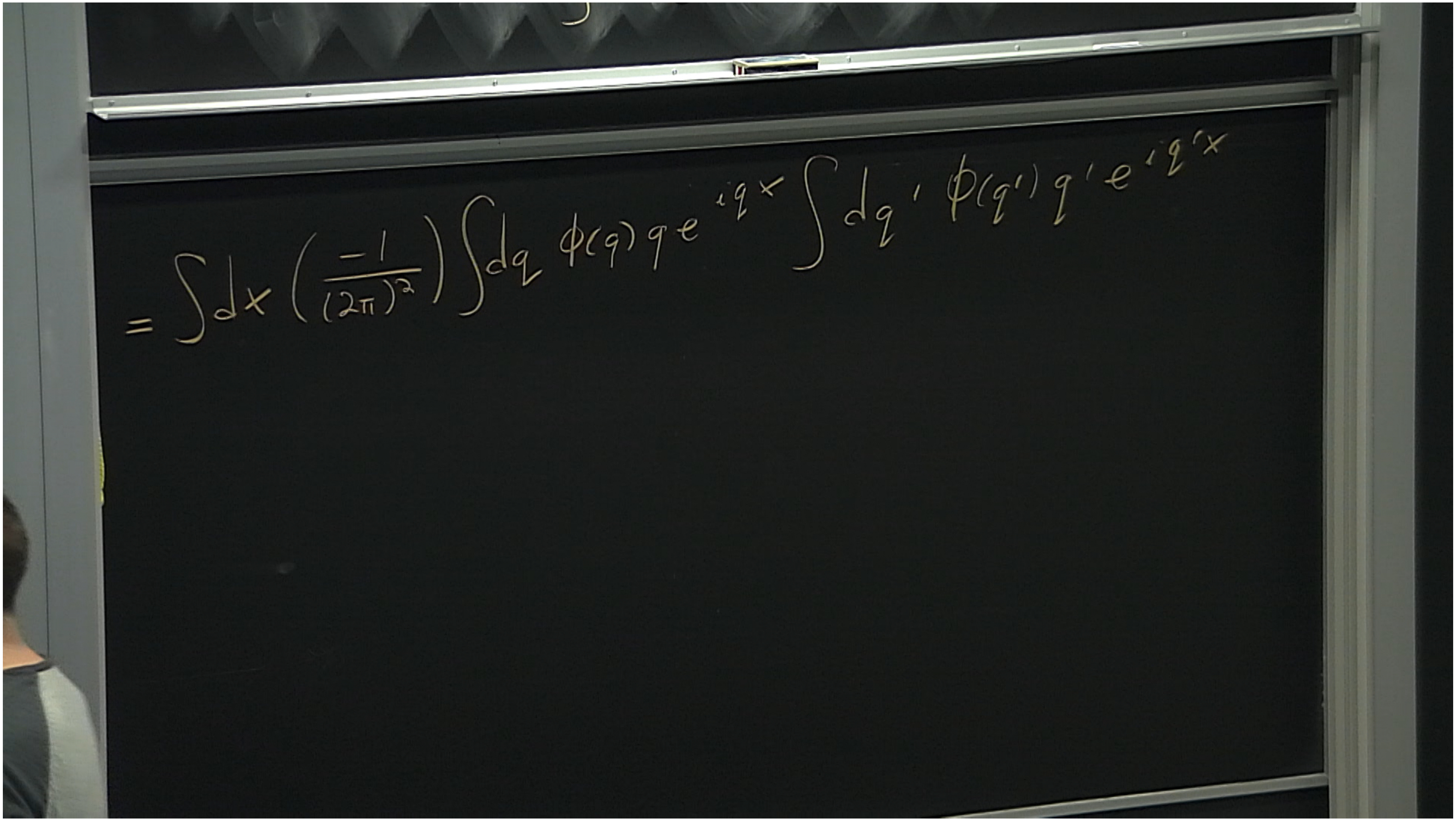
Let's derive the phenomenological L

microscopic details & focus

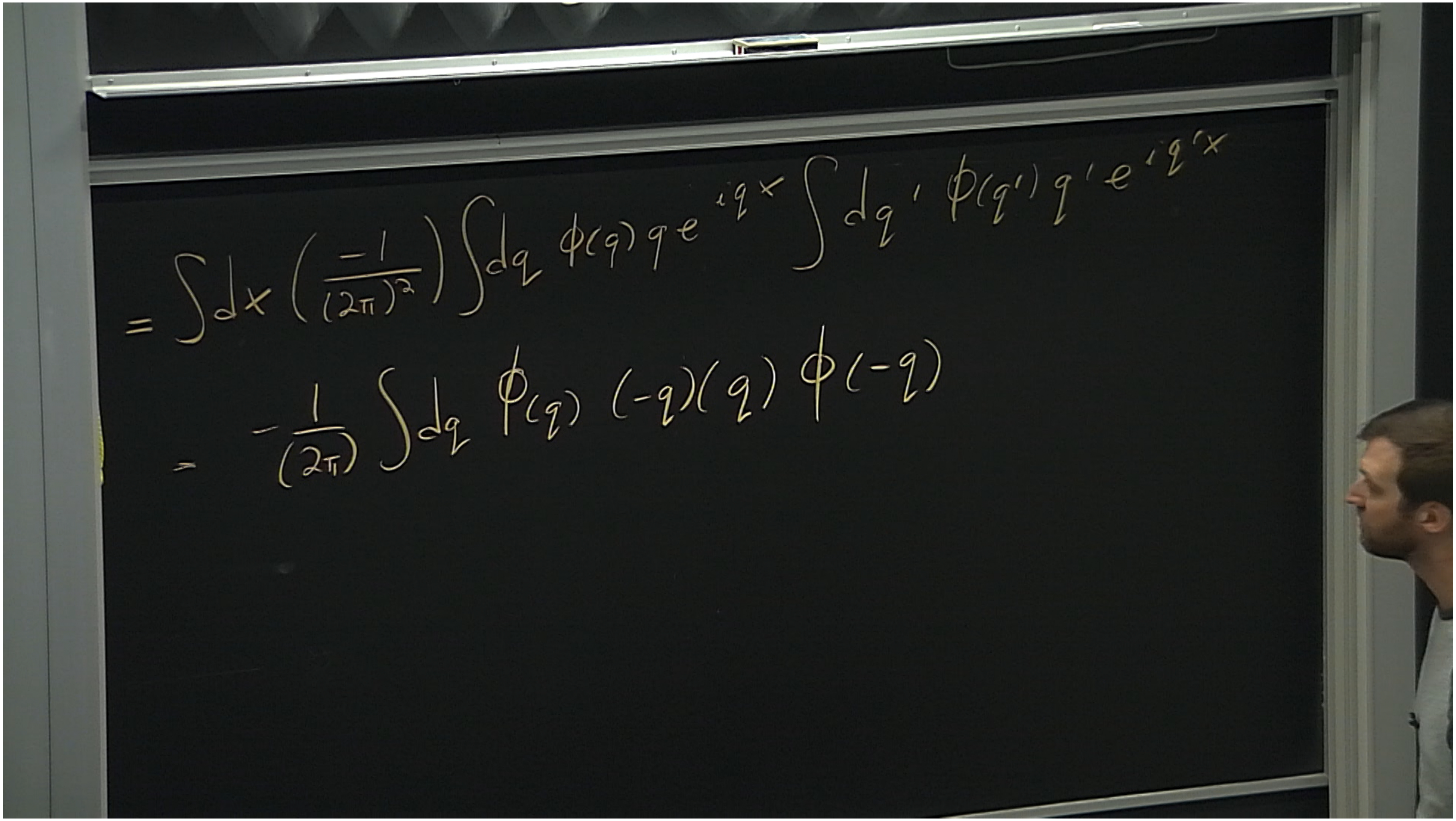
1) Since $\xi \to \infty$, we do not

Only need $J(\vec{q}) \simeq$

$S[\varphi]$ - we should be able

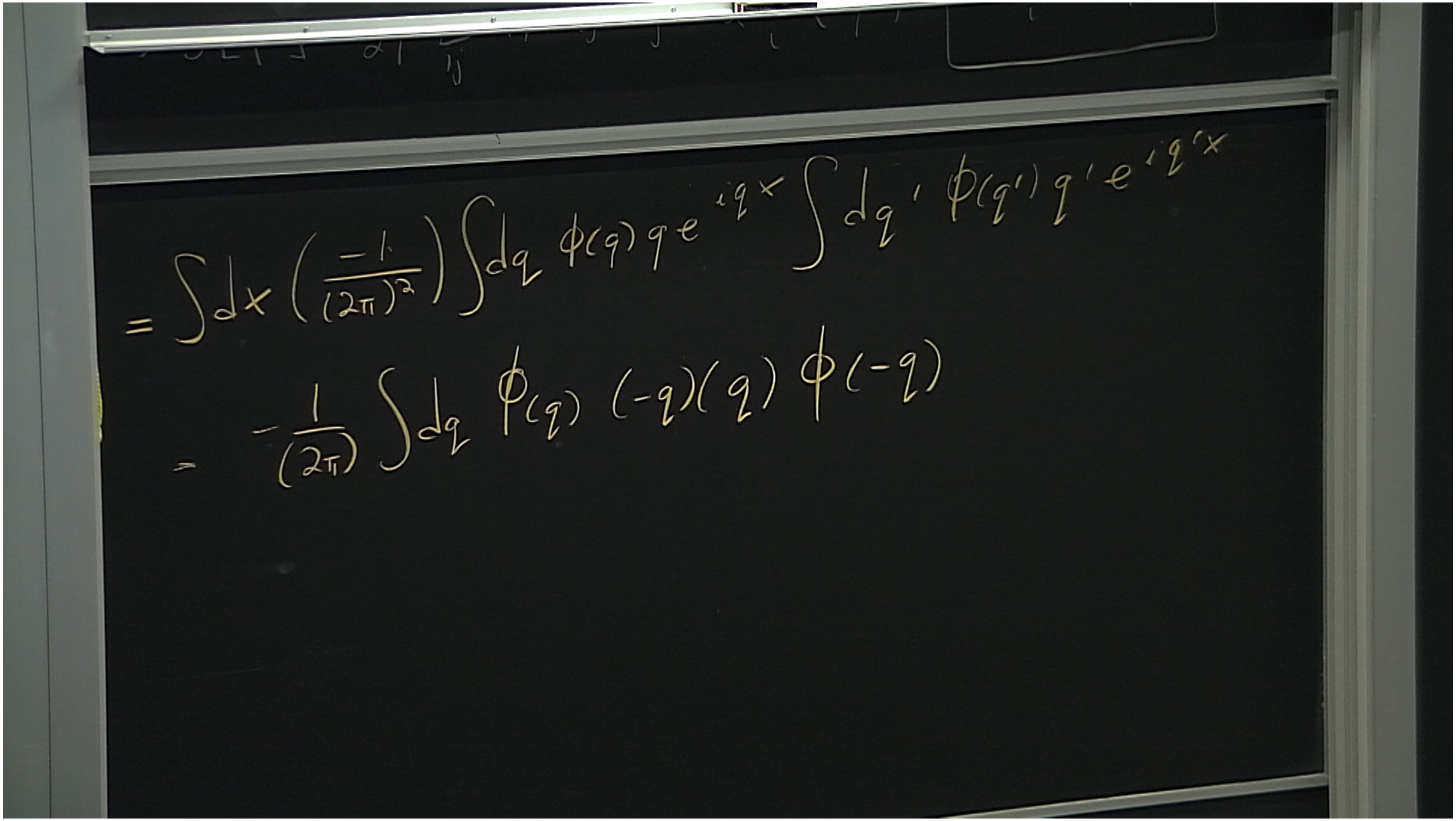


$$= \int dx \left(\frac{-1}{(2\pi)^2}\right) \int dq\, \phi(q)\, q\, e^{iqx} \int dq'\, \phi(q')\, q'\, e^{iq'x}$$



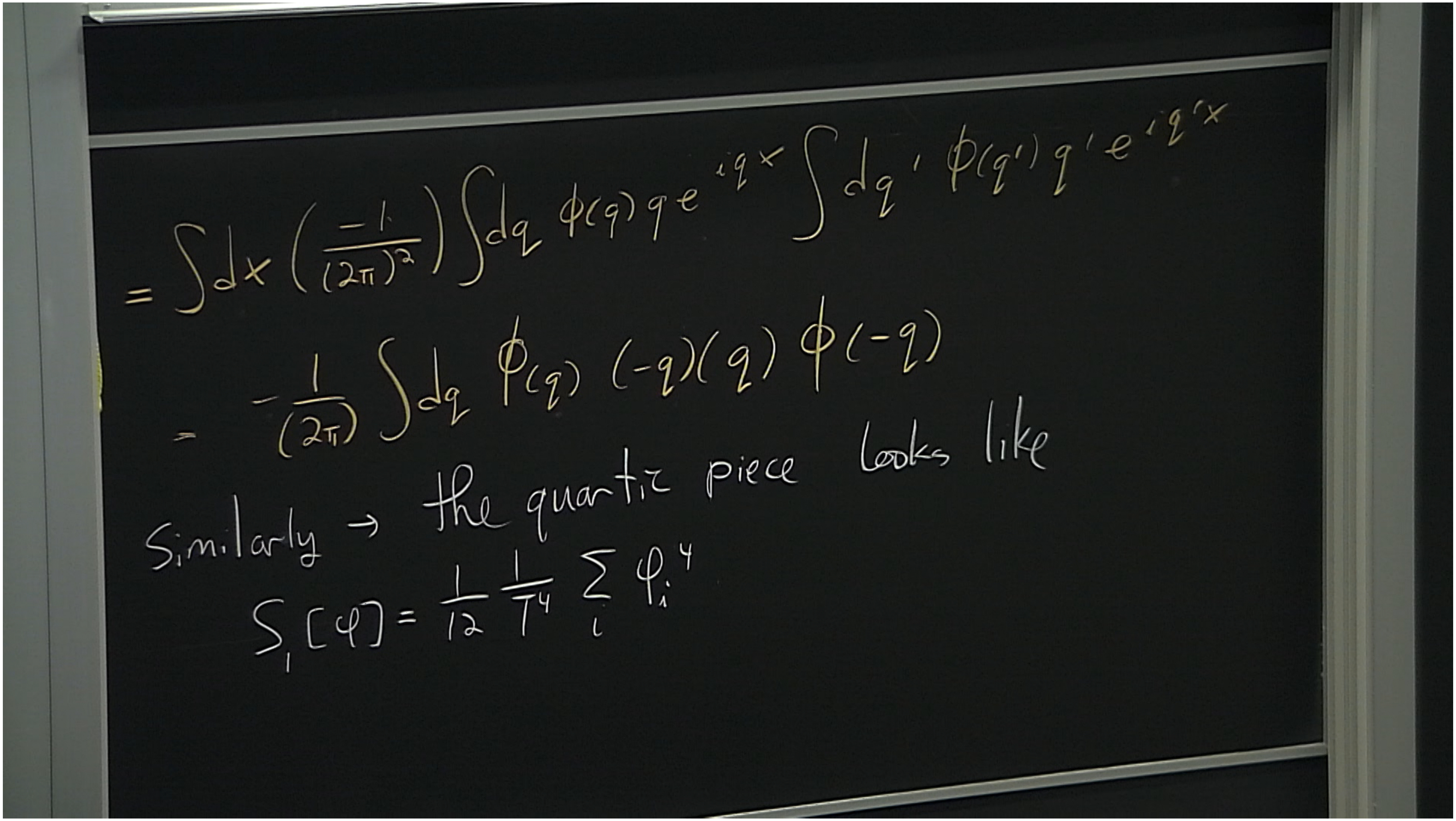
$$= \int dx \left(\frac{-1}{(2\pi)^2}\right) \int dq \, \phi(q) \, q \, e^{iqx} \int dq' \, \phi(q') \, q' \, e^{iq'x}$$

$$= -\frac{1}{(2\pi)} \int dq \, \phi(q) \, (-q)(q) \, \phi(-q)$$



$$= \int dx \left(\frac{-1}{(2\pi)^2}\right) \int dq\, \phi(q)\, q\, e^{iqx} \int dq'\, \phi(q')\, q'\, e^{iq'x}$$

$$= -\frac{1}{(2\pi)} \int dq\, \phi(q)\, (-q)(q)\, \phi(-q)$$



$$= \int dx \left(\frac{-1}{(2\pi)^2}\right) \int dq\, \phi(q)\, q\, e^{iqx} \int dq'\, \phi(q')\, q'\, e^{iq'x}$$

$$= -\frac{1}{(2\pi)} \int dq\, \phi(q)\, (-q)(q)\, \phi(-q)$$

Similarly → the quartic piece looks like

$$S_1[\varphi] = \frac{1}{12}\frac{1}{T^4}\sum_i \varphi_i^4$$

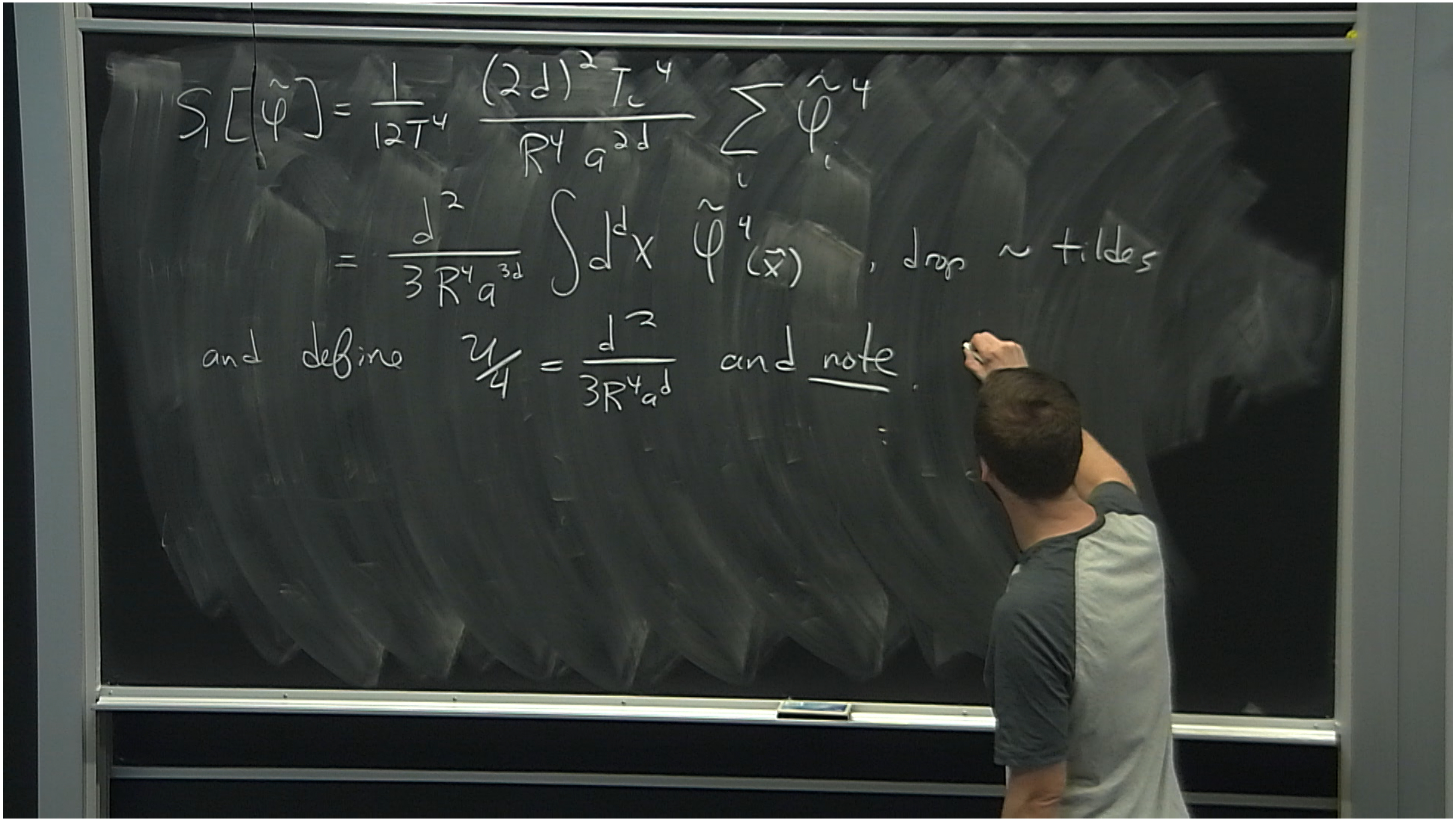
$$= \int dx \left(\frac{-1}{(2\pi)^2}\right) \int dq\, \phi(q)\, q\, e^{iqx} \int dq'\, \phi(q')\, q'\, e^{iq'x}$$

$$= -\frac{1}{(2\pi)} \int dq\, \phi(q)\, (-q)(q)\, \phi(-q)$$

Similarly → the quartic piece looks like

$$S_1[\varphi] = \frac{1}{12}\frac{1}{T^4}\sum_i \varphi_i^4$$ use $\sum_i \to \frac{1}{a^d}\int d^dx$

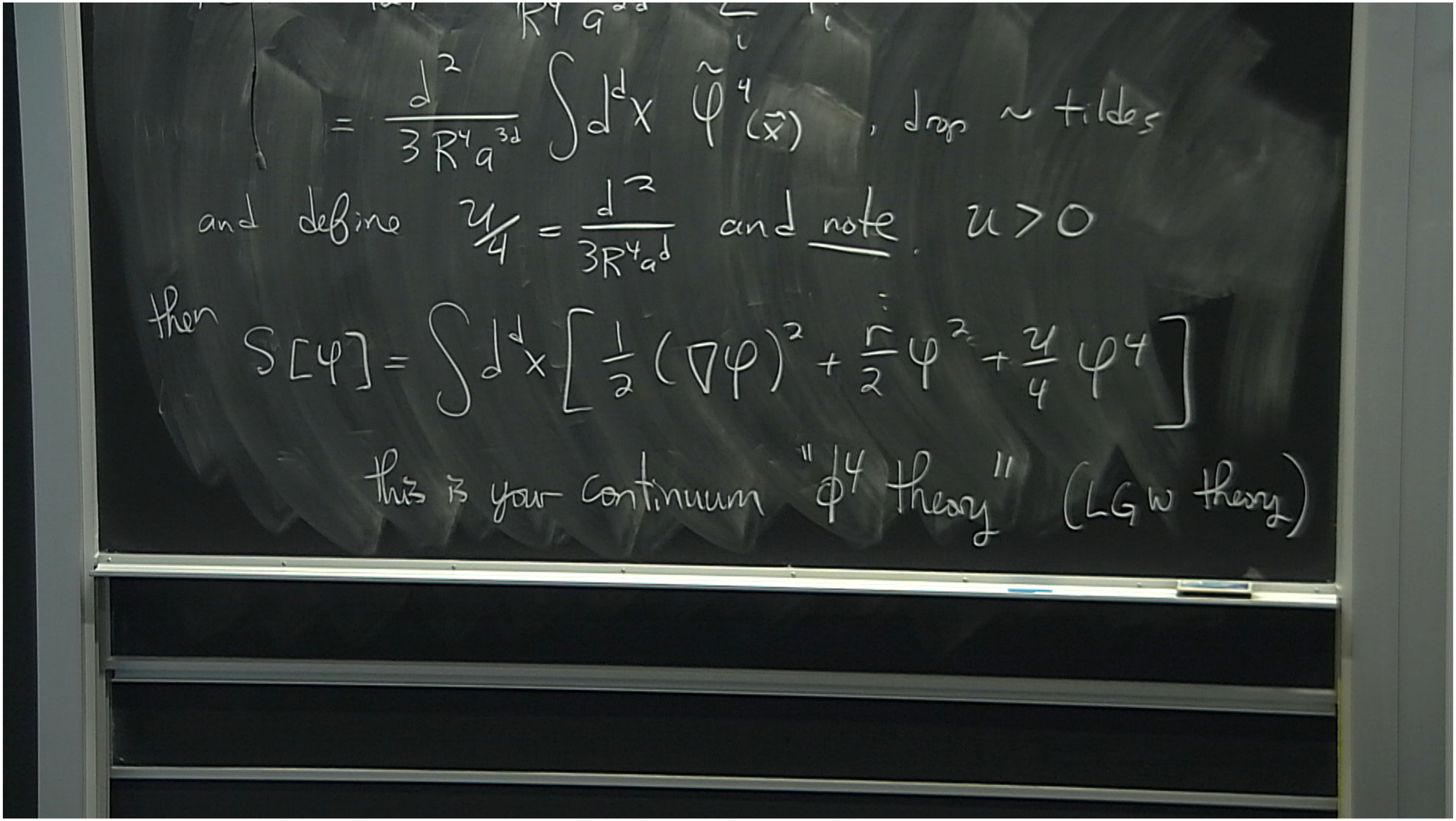
and rescale $\varphi_i = \tilde{\varphi}_i \frac{\sqrt{2d}\,T_c}{R a^{d/2}}$ as before.



$$S_1[\tilde{\varphi}] = \frac{1}{12T^4} \frac{(2d)^2 T_c^{\,4}}{R^4 a^{2d}} \sum_i \tilde{\varphi}_i^{\,4}$$

$$= \frac{d^2}{3R^4 a^{3d}} \int d^dx \, \tilde{\varphi}^4(\vec{x})$$, drop ~ tildes

and define $u/4 = \frac{d^2}{3R^4 a^d}$ and note.

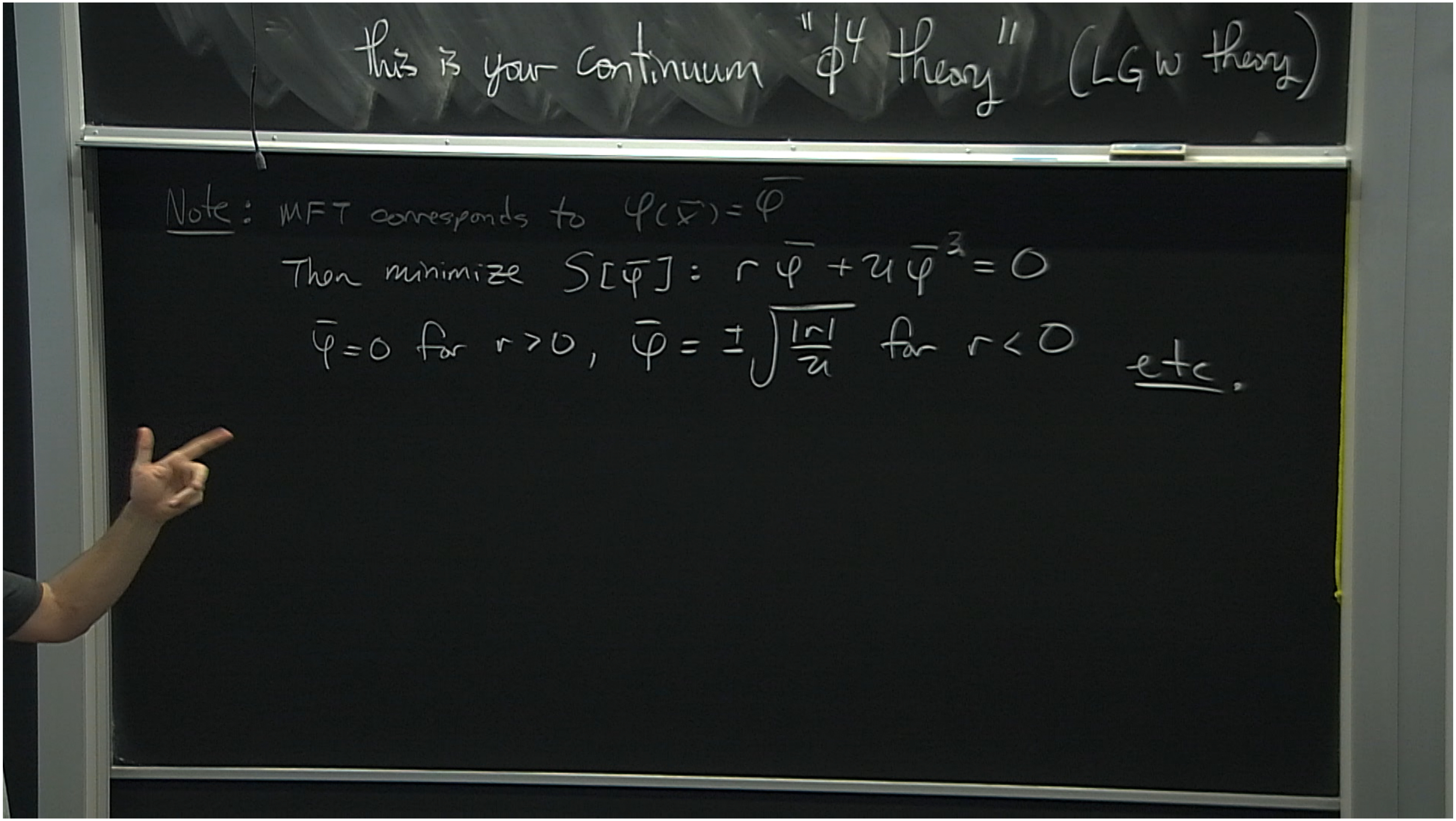


$$= \frac{d^2}{3R^4a^{3d}} \int d^dx\ \tilde{\varphi}^4(\vec{x})$$, drop ~ tildes

and define $\frac{u}{4} = \frac{d^2}{3R^4a^d}$ and <u>note</u>. $u > 0$

then $$S[\varphi] = \int d^dx \left[\frac{1}{2}(\nabla\varphi)^2 + \frac{r}{2}\varphi^2 + \frac{u}{4}\varphi^4 \right]$$

this is your continuum "ϕ^4 theory" (LGW theory)



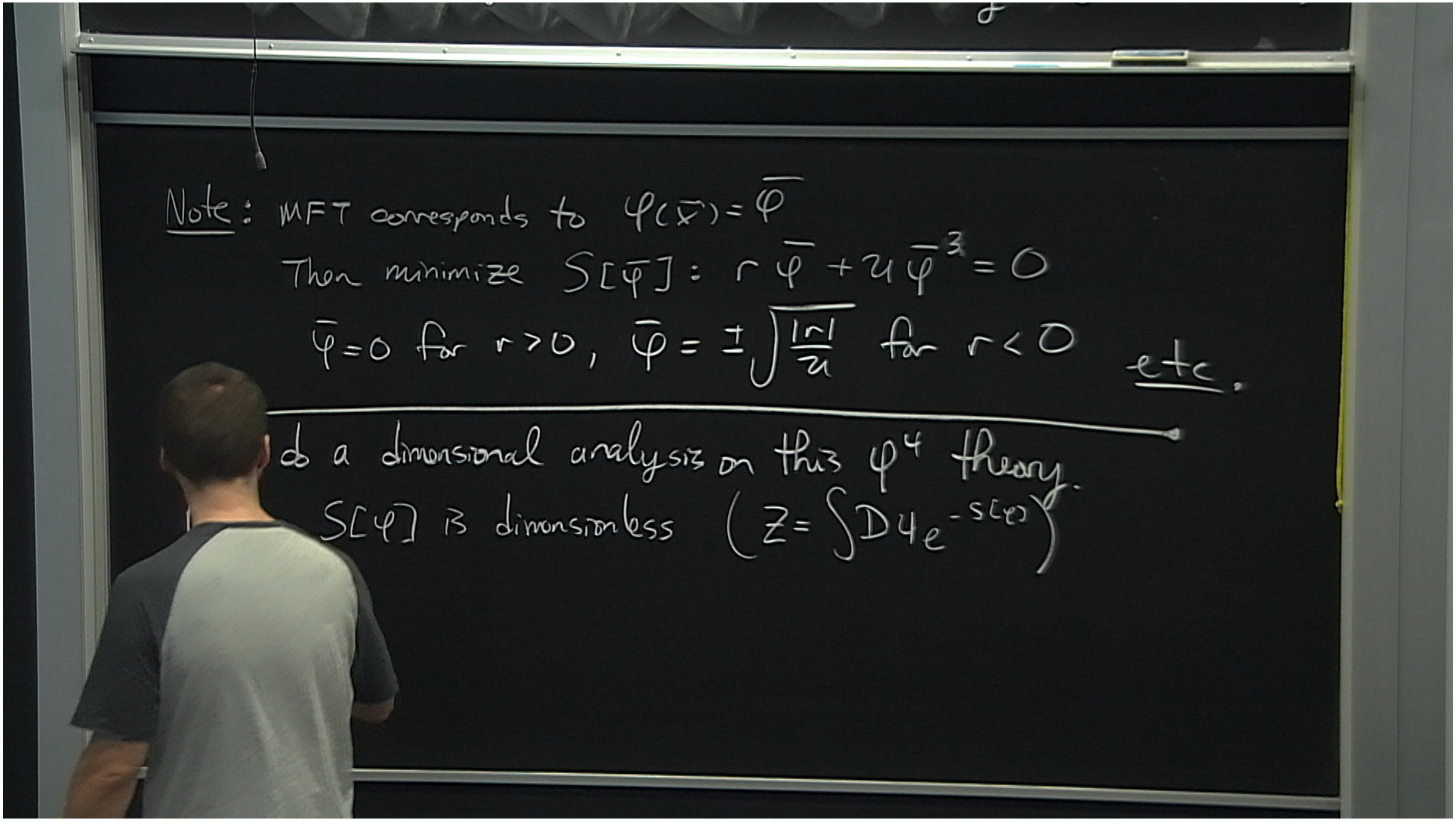
this is your continuum "ϕ^4 theory" (LGW theory)

Note: MFT corresponds to $\varphi(\bar{x}) = \bar{\varphi}$

Then minimize $S[\bar{\varphi}]$: $r\bar{\varphi} + u\bar{\varphi}^3 = 0$

$\bar{\varphi} = 0$ for $r > 0$, $\bar{\varphi} = \pm\sqrt{\frac{|r|}{u}}$ for $r < 0$

etc.



Note: MFT corresponds to $\varphi(\vec{x}) = \bar{\varphi}$

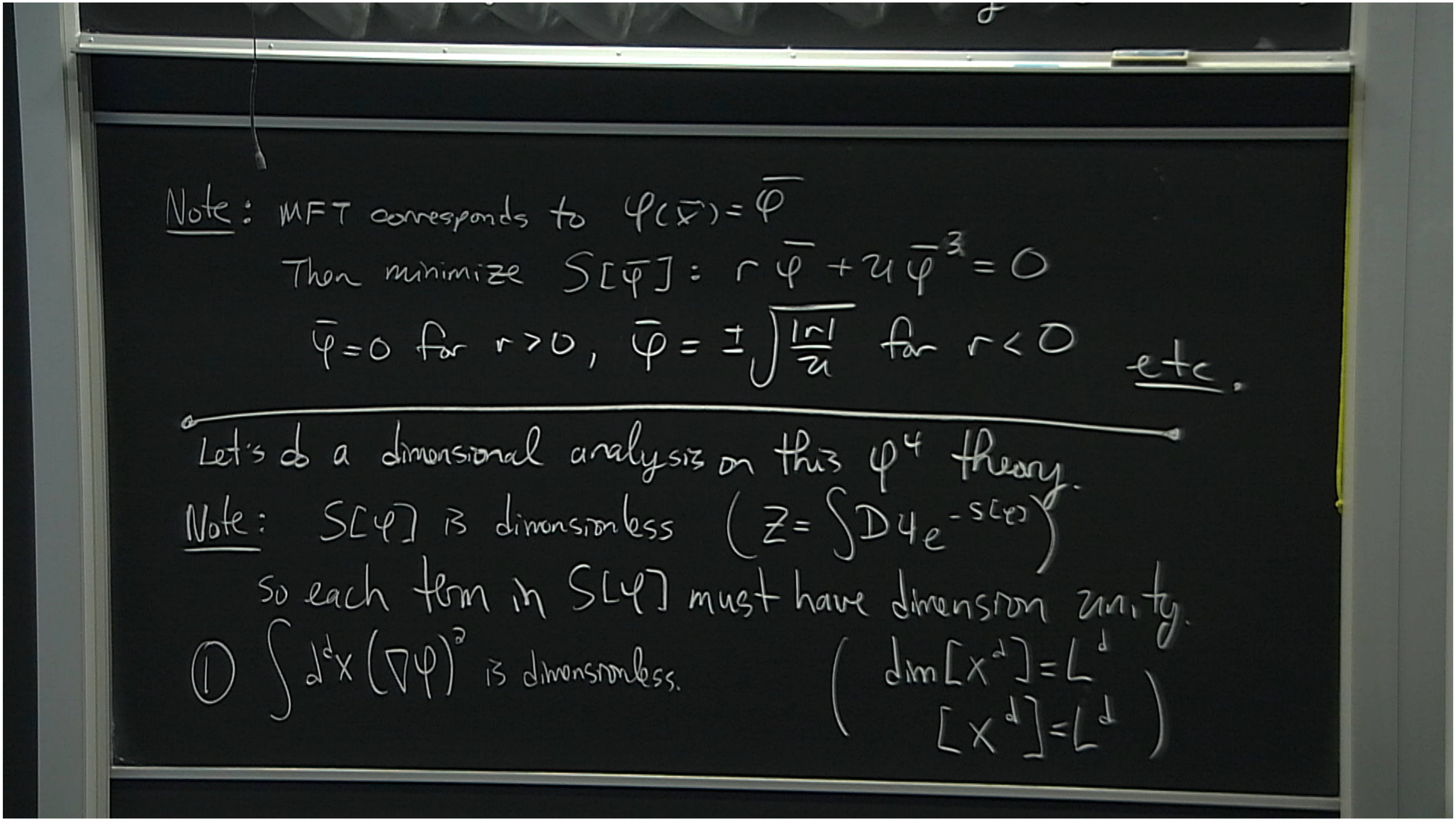
Then minimize $S[\bar{\varphi}]$: $r\bar{\varphi} + u\bar{\varphi}^3 = 0$

$\bar{\varphi} = 0$ for $r > 0$, $\bar{\varphi} = \pm\sqrt{\frac{|r|}{u}}$ for $r < 0$ **etc.**

---

do a dimensional analysis on this φ^4 theory.

$S[\varphi]$ is dimensionless $\left(Z = \int D\varphi\, e^{-S[\varphi]}\right)$



Note: MFT corresponds to $\varphi(\bar{x}) = \bar{\varphi}$

Then minimize $S[\bar{\varphi}]$: $r\bar{\varphi} + u\bar{\varphi}^3 = 0$

$\bar{\varphi} = 0$ for $r > 0$, $\bar{\varphi} = \pm\sqrt{\frac{|r|}{u}}$ for $r < 0$

etc.

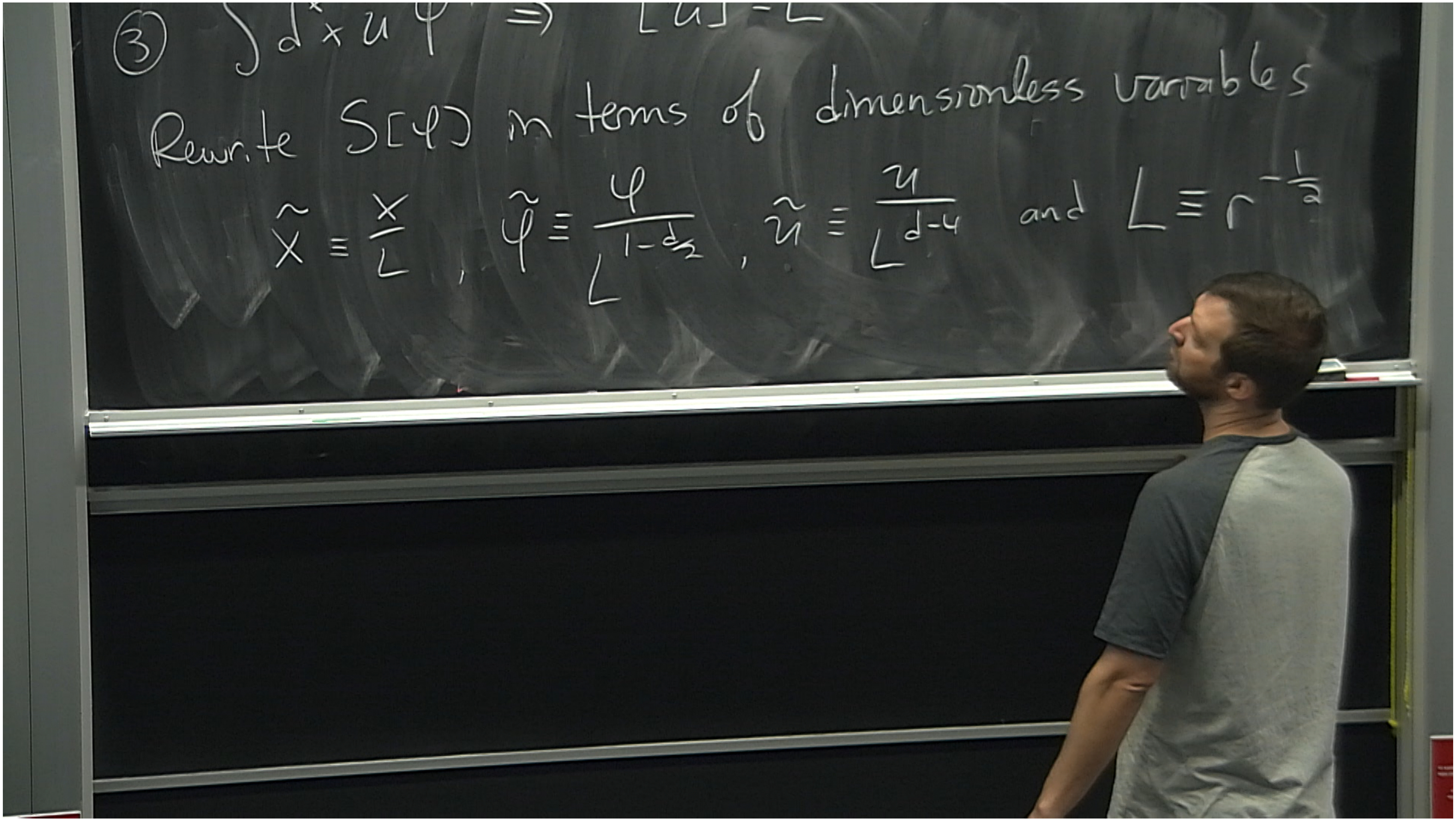
Let's do a dimensional analysis on this φ^4 theory.

Note: $S[\varphi]$ is dimensionless $\left(Z = \int D\varphi \, e^{-S[\varphi]} \right)$

so each term in $S[\varphi]$ must have dimension unity.

① $\int d^dx \, (\nabla\varphi)^2$ is dimensionless.

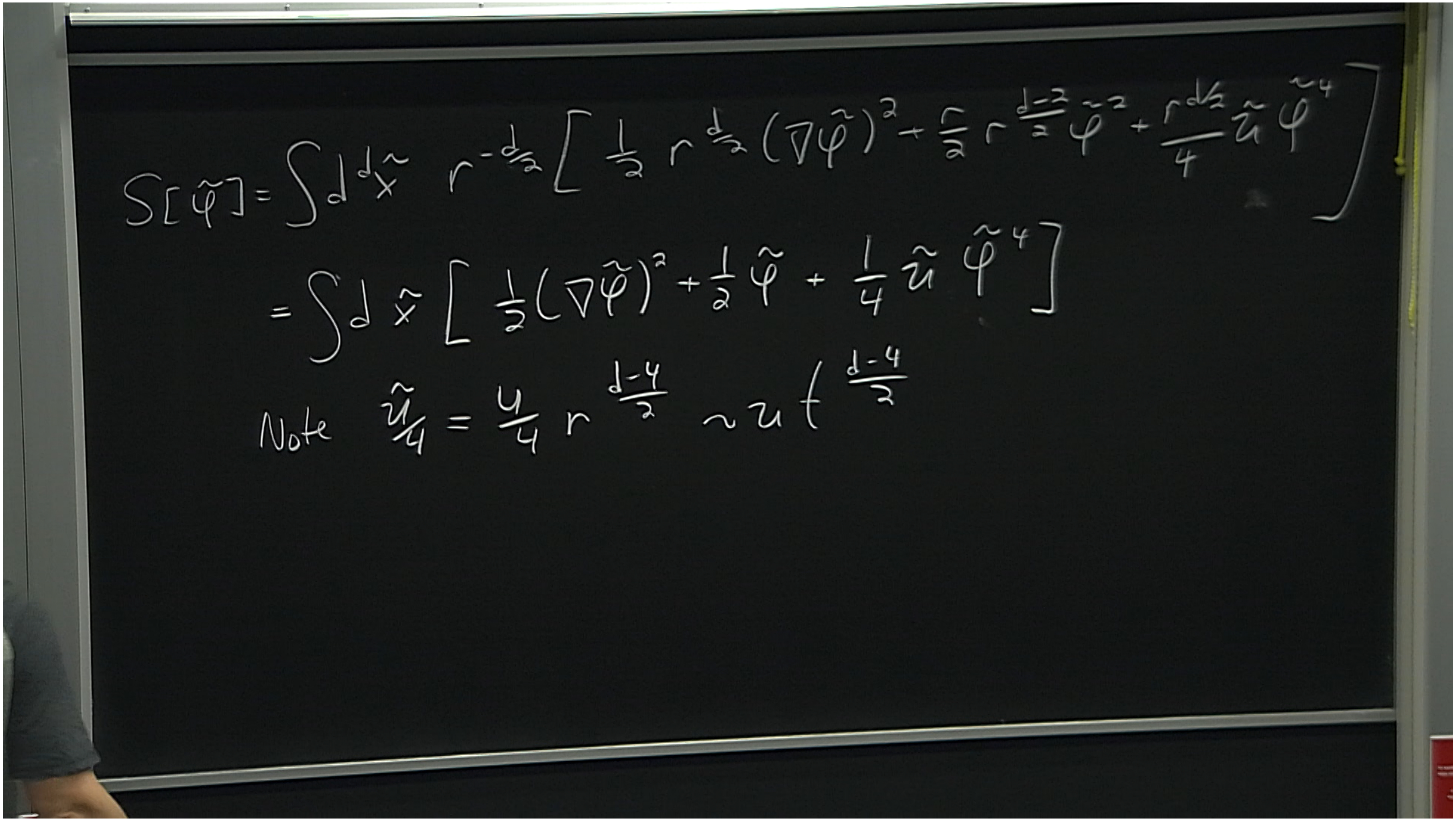
$\left(\begin{array}{l} \dim[x^d] = L^d \\ [x^d] = L^d \end{array} \right)$



③ $\int d x\, u \varphi \Rightarrow [u] = L$

Rewrite $S[\varphi]$ in terms of dimensionless variables

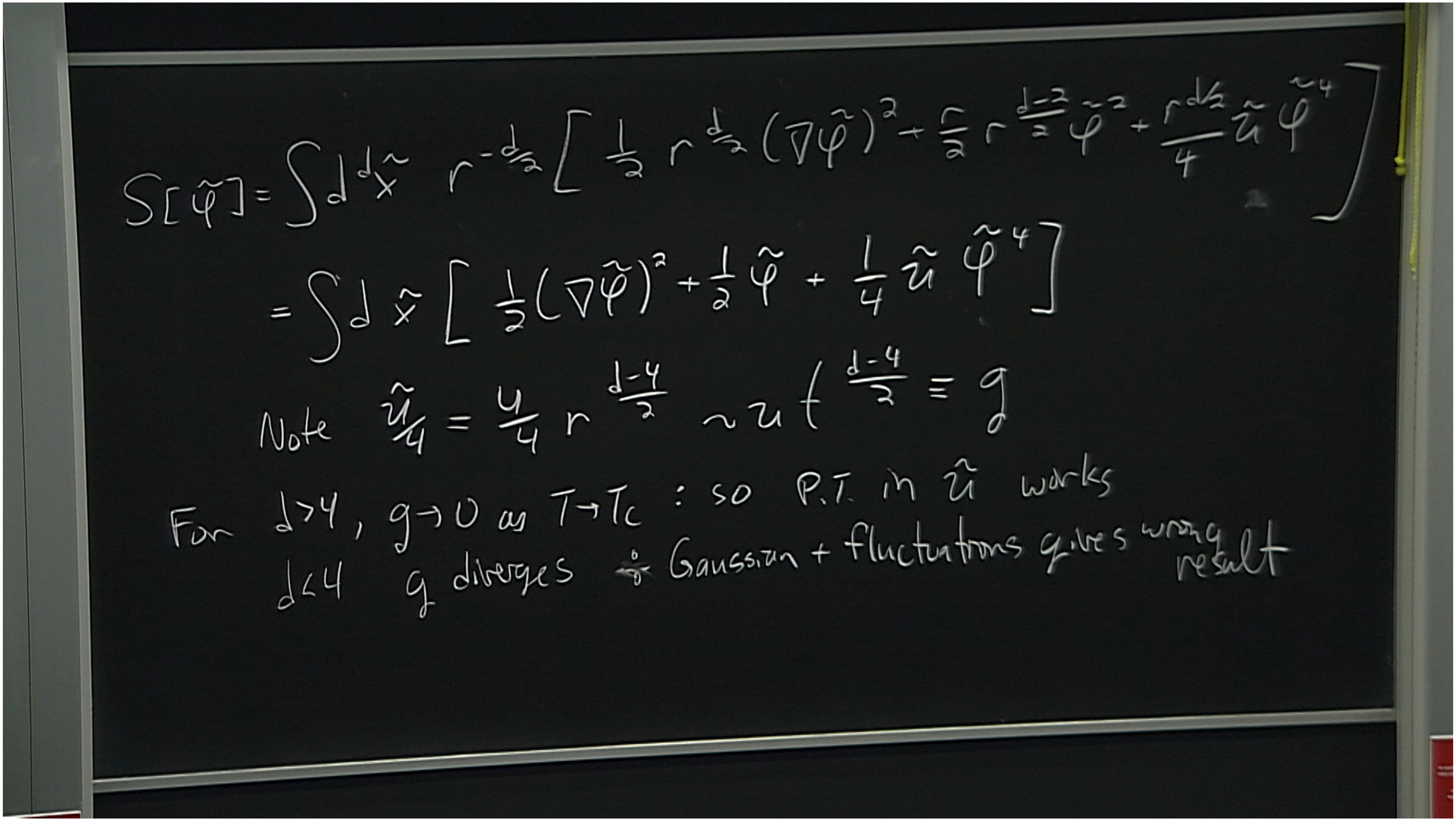
$$\tilde{x} \equiv \frac{x}{L}, \quad \tilde{\varphi} \equiv \frac{\varphi}{L^{1-d/2}}, \quad \tilde{u} \equiv \frac{u}{L^{d-4}} \quad \text{and} \quad L \equiv r^{-\frac{1}{2}}$$



$$S[\tilde{\varphi}] = \int d^d\tilde{x} \; r^{-\frac{d}{2}}\left[\frac{1}{2} r^{\frac{d}{2}}(\nabla\tilde{\varphi})^2 + \frac{r}{2} r^{\frac{d-2}{2}}\tilde{\varphi}^2 + \frac{r^{d/2}}{4}\tilde{u}\,\tilde{\varphi}^4\right]$$

$$= \int d\tilde{x}\left[\frac{1}{2}(\nabla\tilde{\varphi})^2 + \frac{1}{2}\tilde{\varphi} + \frac{1}{4}\tilde{u}\,\tilde{\varphi}^4\right]$$

Note $\tilde{u}/4 = \frac{u}{4} r^{\frac{d-4}{2}} \sim u\, t^{\frac{d-4}{2}}$



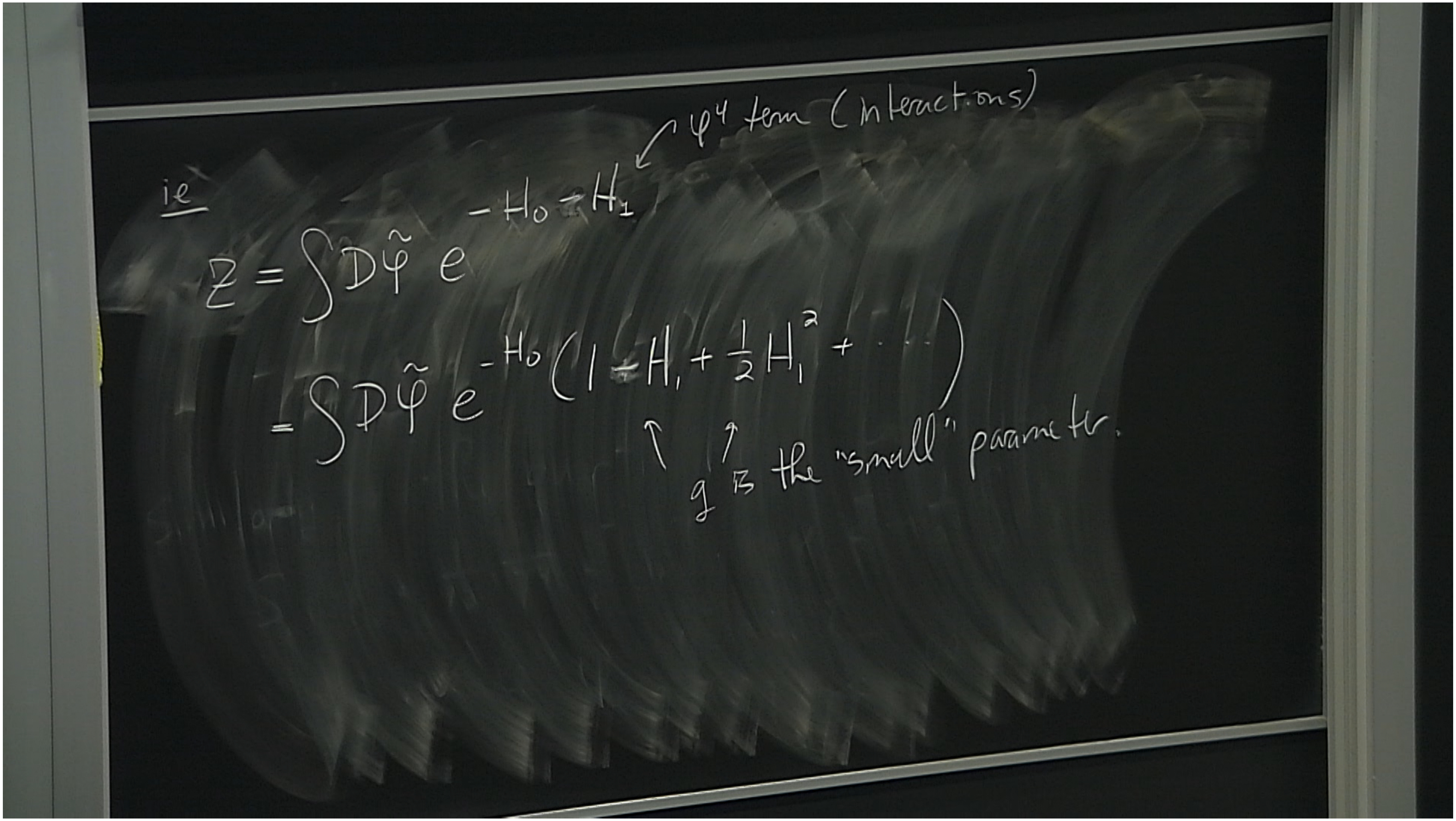
$$S[\tilde{\varphi}] = \int d^d\tilde{x}\; r^{-\frac{d}{2}} \left[\frac{1}{2} r^{\frac{d}{2}} (\nabla\tilde{\varphi})^2 + \frac{r}{2} r^{\frac{d-2}{2}} \tilde{\varphi}^2 + \frac{r^{d/2}}{4} \tilde{u}\, \tilde{\varphi}^4 \right]$$

$$= \int d\tilde{x} \left[\frac{1}{2}(\nabla\tilde{\varphi})^2 + \frac{1}{2}\tilde{\varphi} + \frac{1}{4}\tilde{u}\,\tilde{\varphi}^4 \right]$$

Note $\tilde{u}/4 = \frac{u}{4} r^{\frac{d-4}{2}} \sim u\, t^{\frac{d-4}{2}} \equiv g$

For $d > 4$, $g \to 0$ as $T \to T_c$: so P.T. in $\hat{u}$ works

$d < 4$ g diverges ÷ Gaussian + fluctuations gives wrong result



ie

φ^4 term (interactions)

$$Z = \int D\tilde{\varphi}\, e^{-H_0 - H_1}$$

$$= \int D\tilde{\varphi}\, e^{-H_0}\left(1 - H_1 + \frac{1}{2}H_1^2 + \cdots\right)$$

g is the "small" parameter.