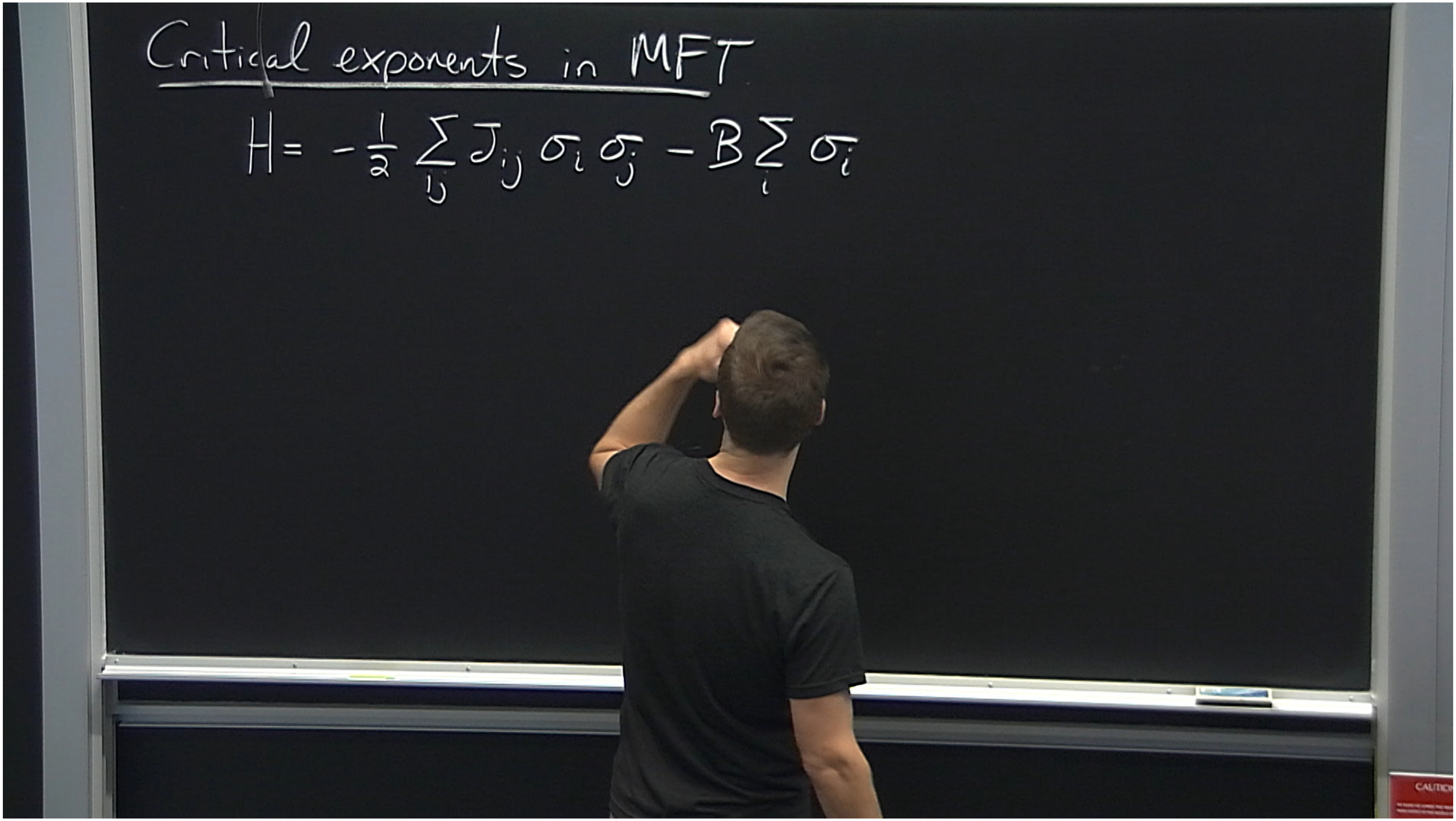
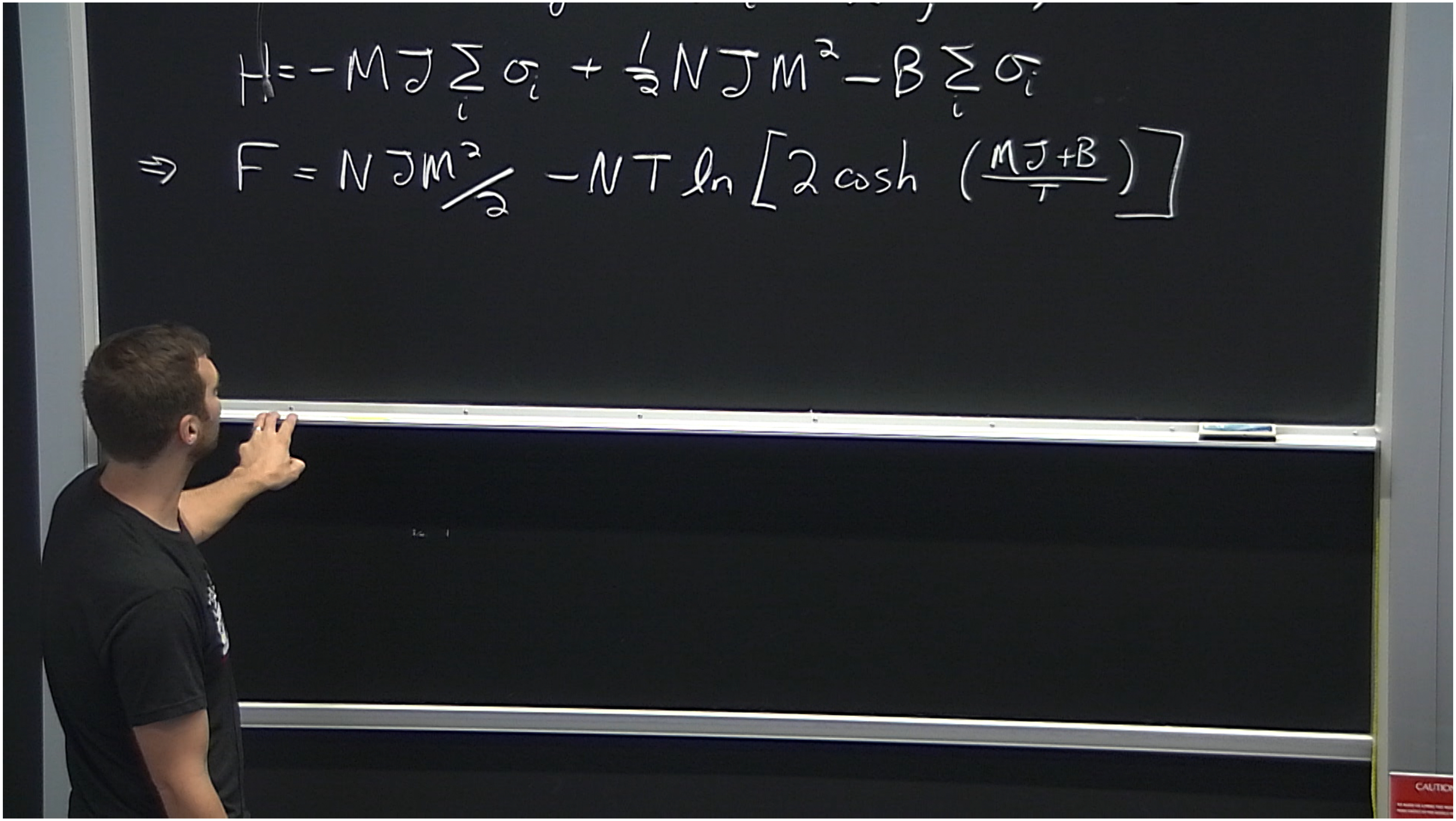
Title: 2016/2017 Statistical Mechanics 2 - Roger Melko - Lecture 3

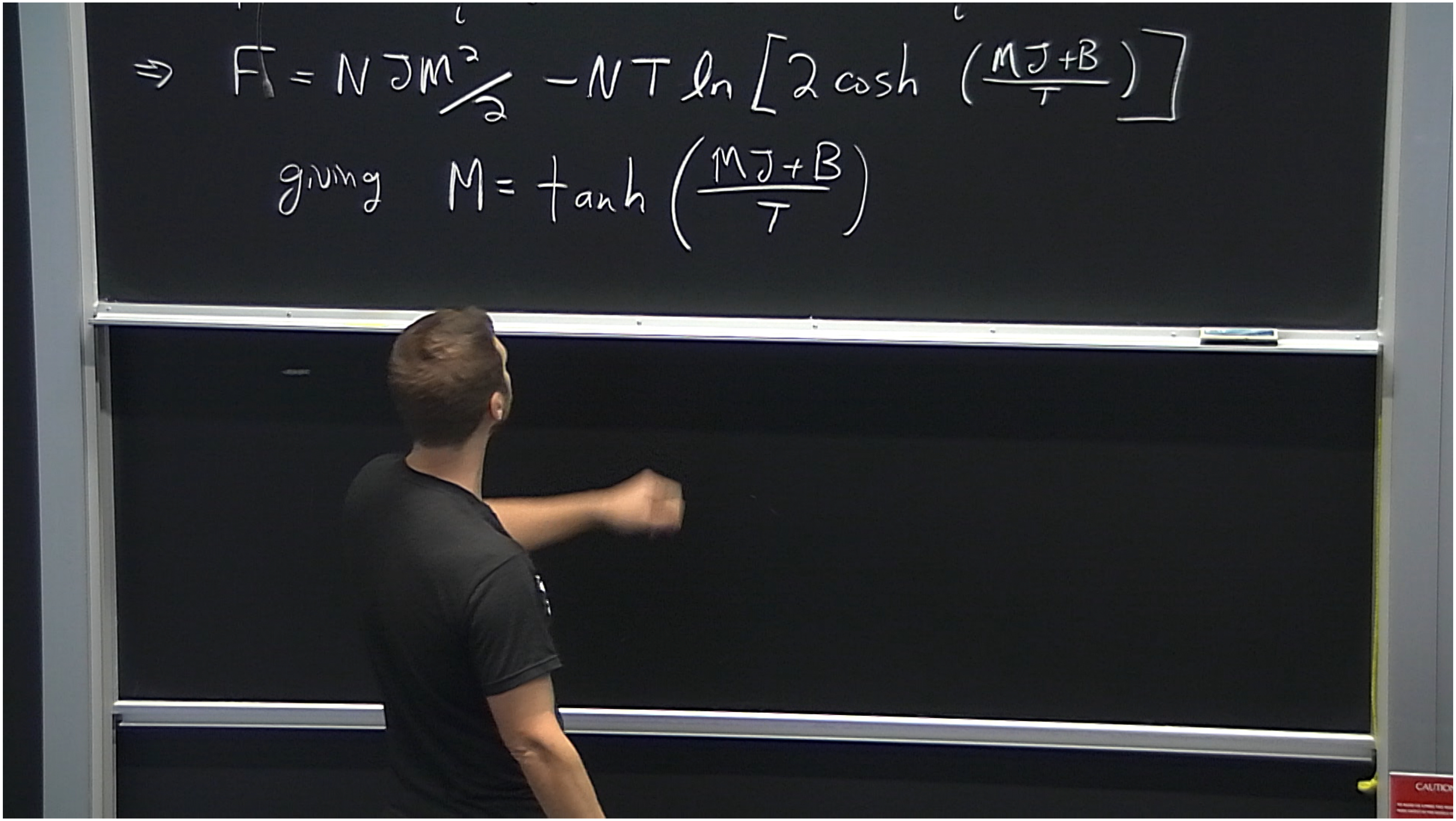
Date: Jan 11, 2017 10:30 AM

URL: http://pirsa.org/17010049

Abstract:


Critical exponents in MFT
$H = -\frac{1}{2}\sum_{ij} J_{ij}\sigma_i\sigma_j - B\sum_i \sigma_i$


$H = -MJ\sum_i \sigma_i + \frac{1}{2}NJm^2 - B\sum_i \sigma_i$
$\Rightarrow \quad F = NJm^2/2 - NT \ln\left[2\cosh\left(\frac{MJ+B}{T}\right)\right]$

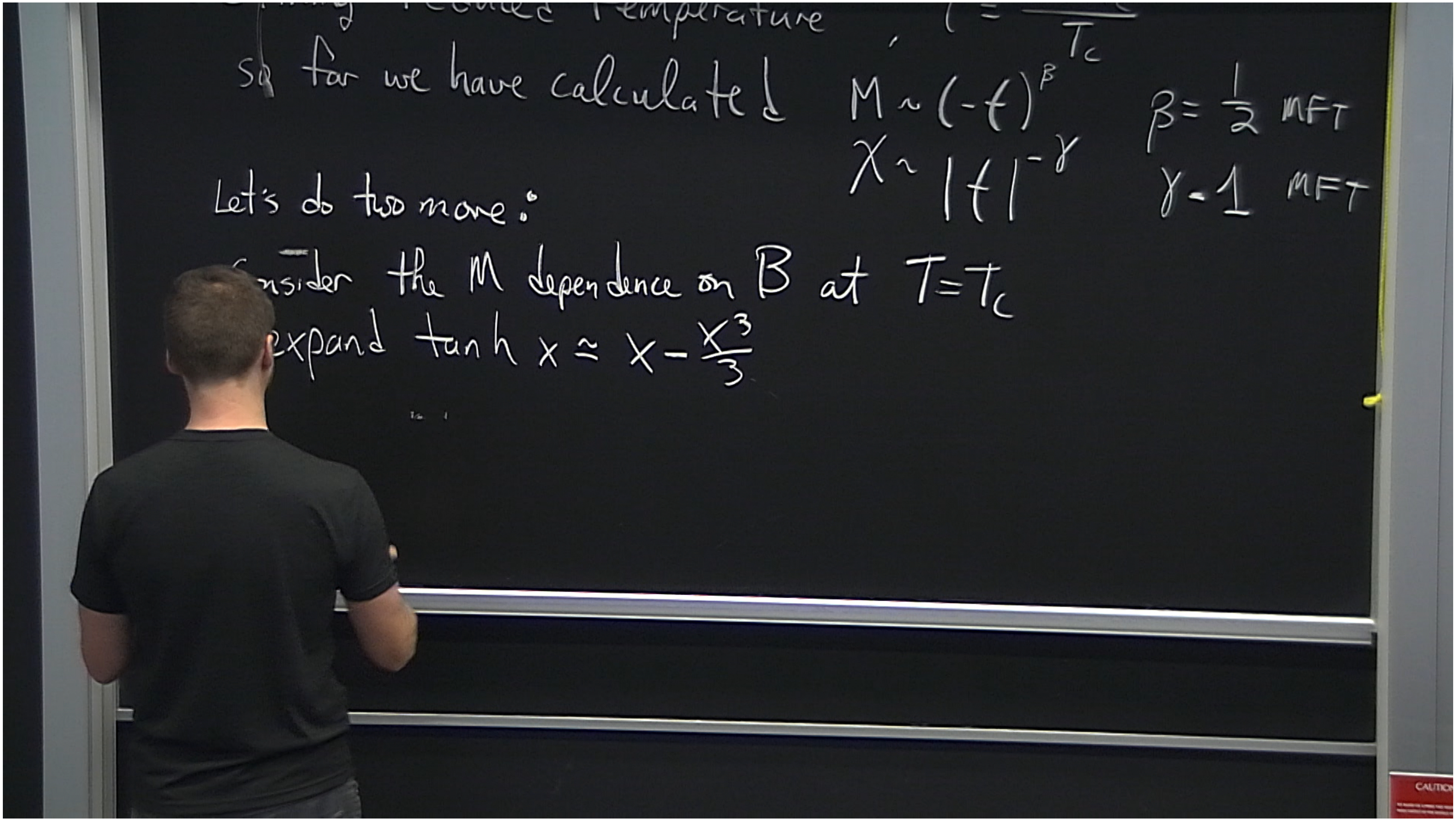

$\Rightarrow \quad F = NJM^2/2 - NT \ln\left[2\cosh\left(\frac{MJ+B}{T}\right)\right]$
giving $\quad M = \tanh\left(\frac{MJ+B}{T}\right)$
CAUTION

giving $M = \tanh\left(\frac{MJ+B}{T}\right)$

Defining "reduced temperature", $t = \frac{T-T_c}{T_c}$

so far we have calculated $M \sim (-t)^{\beta}$ $\quad \beta = \frac{1}{2}$ MFT

$\chi \sim |t|^{-\gamma}$ $\quad \gamma = 1$ MFT



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Let's do two more:
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expand $\tanh x \simeq x - \frac{x^3}{3}$

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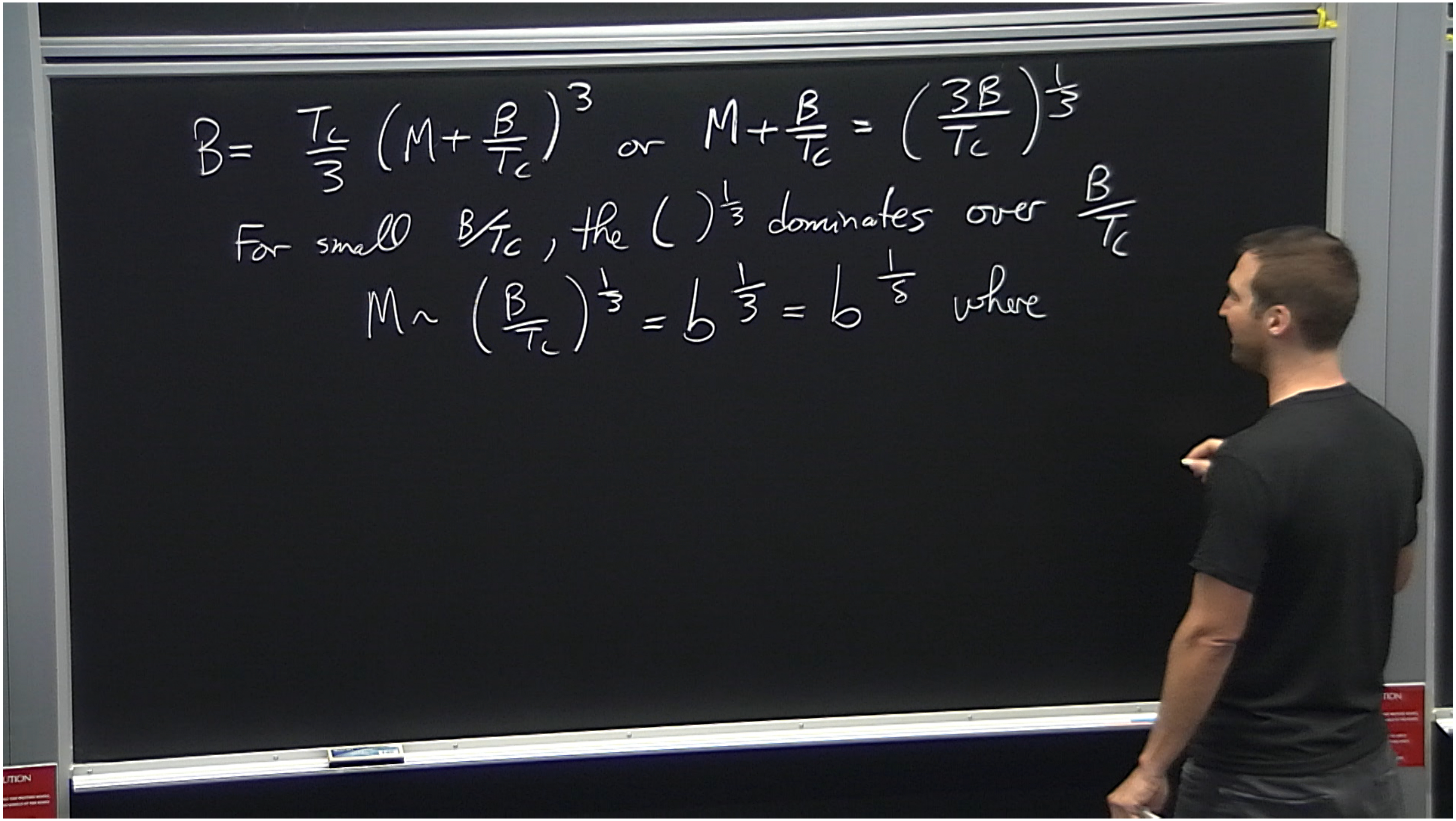
Let's do two more:

Consider the M dependence on B at $T = T_c$

expand $\tanh x \simeq x - \frac{x^3}{3}$

$$M = \frac{MJ + B}{T_c} - \frac{1}{3}\left(\frac{MJ+B}{T_c}\right)^3 = \frac{MJ}{T_c} + \frac{B}{T_c} - \frac{1}{3}\left(\frac{MJ+B}{T_c}\right)^3$$

$$B = \frac{T_c}{3}\left(M + \frac{B}{T_c}\right)^3 \quad \text{or} \quad M + \frac{B}{T_c} = \left(\frac{3B}{T_c}\right)^{\frac{1}{3}}$$


$B = \frac{T_c}{3}\left(M + \frac{B}{T_c}\right)^3$ or $M + \frac{B}{T_c} = \left(\frac{3B}{T_c}\right)^{\frac{1}{3}}$
For small B/T_c, the $(\ )^{\frac{1}{3}}$ dominates over $\frac{B}{T_c}$
$M \sim \left(\frac{B}{T_c}\right)^{\frac{1}{3}} = b^{\frac{1}{3}} = b^{\frac{1}{\delta}}$ where

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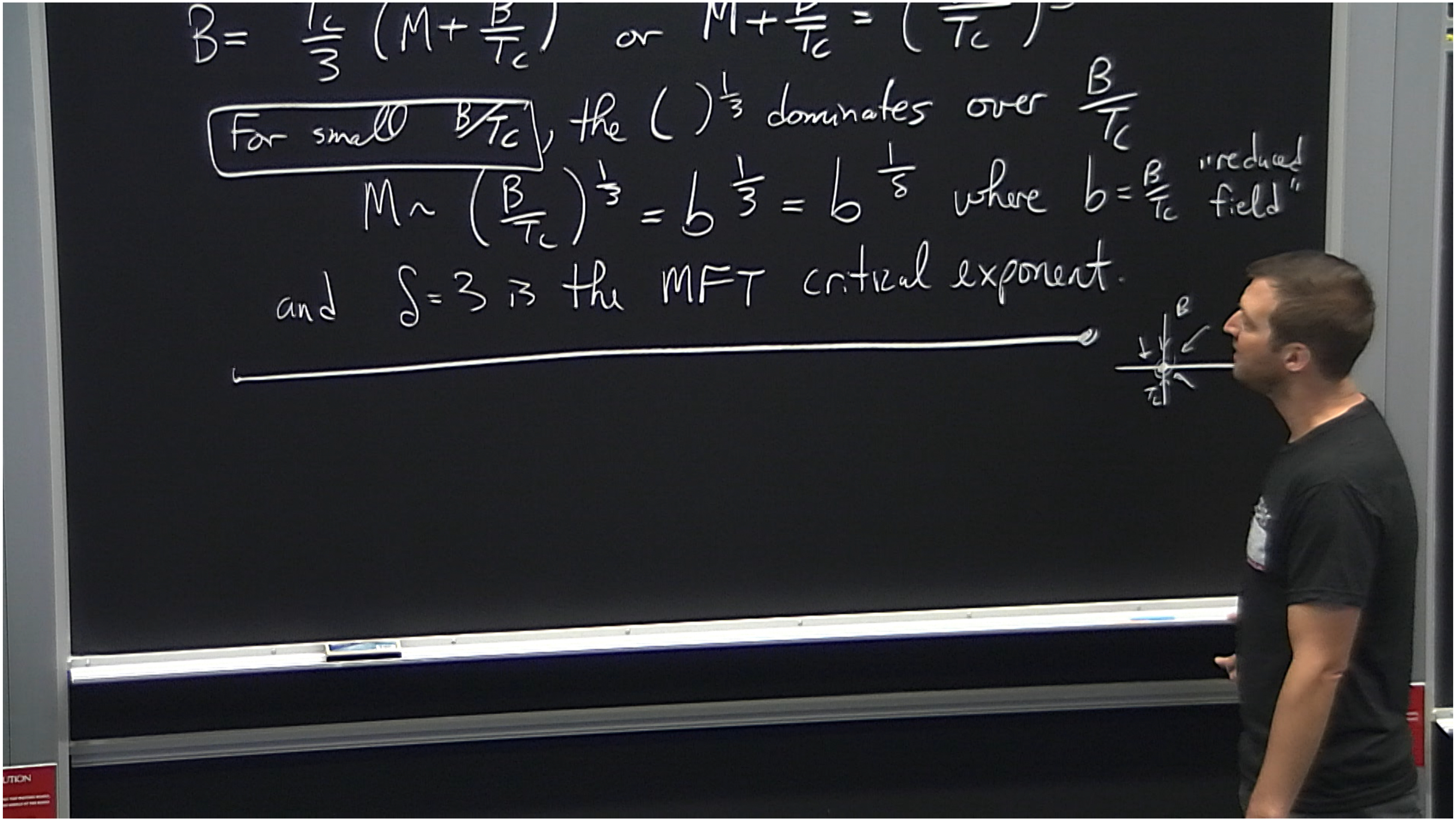
and $\delta = 3$ is the MFT critical exponent.



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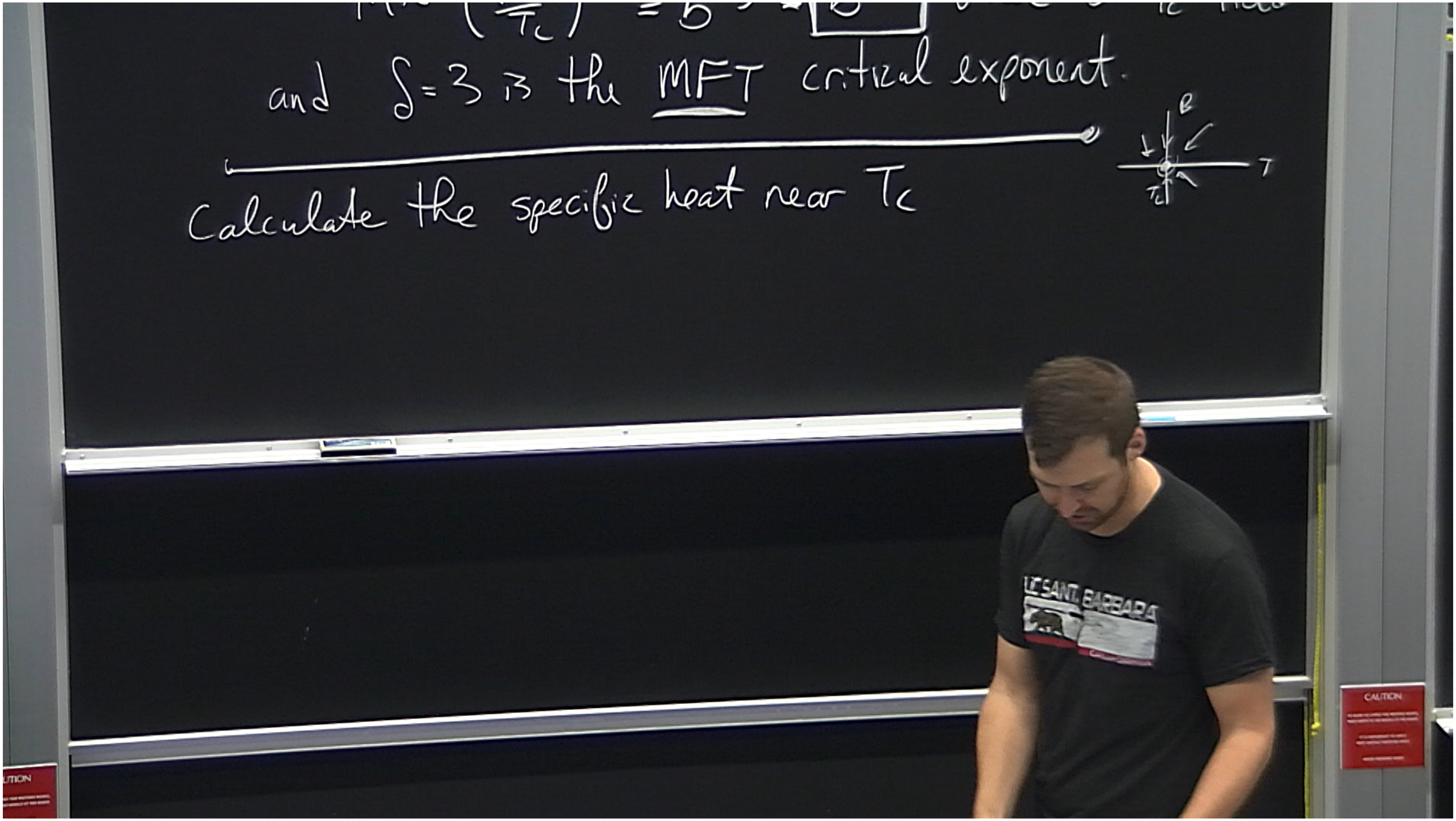
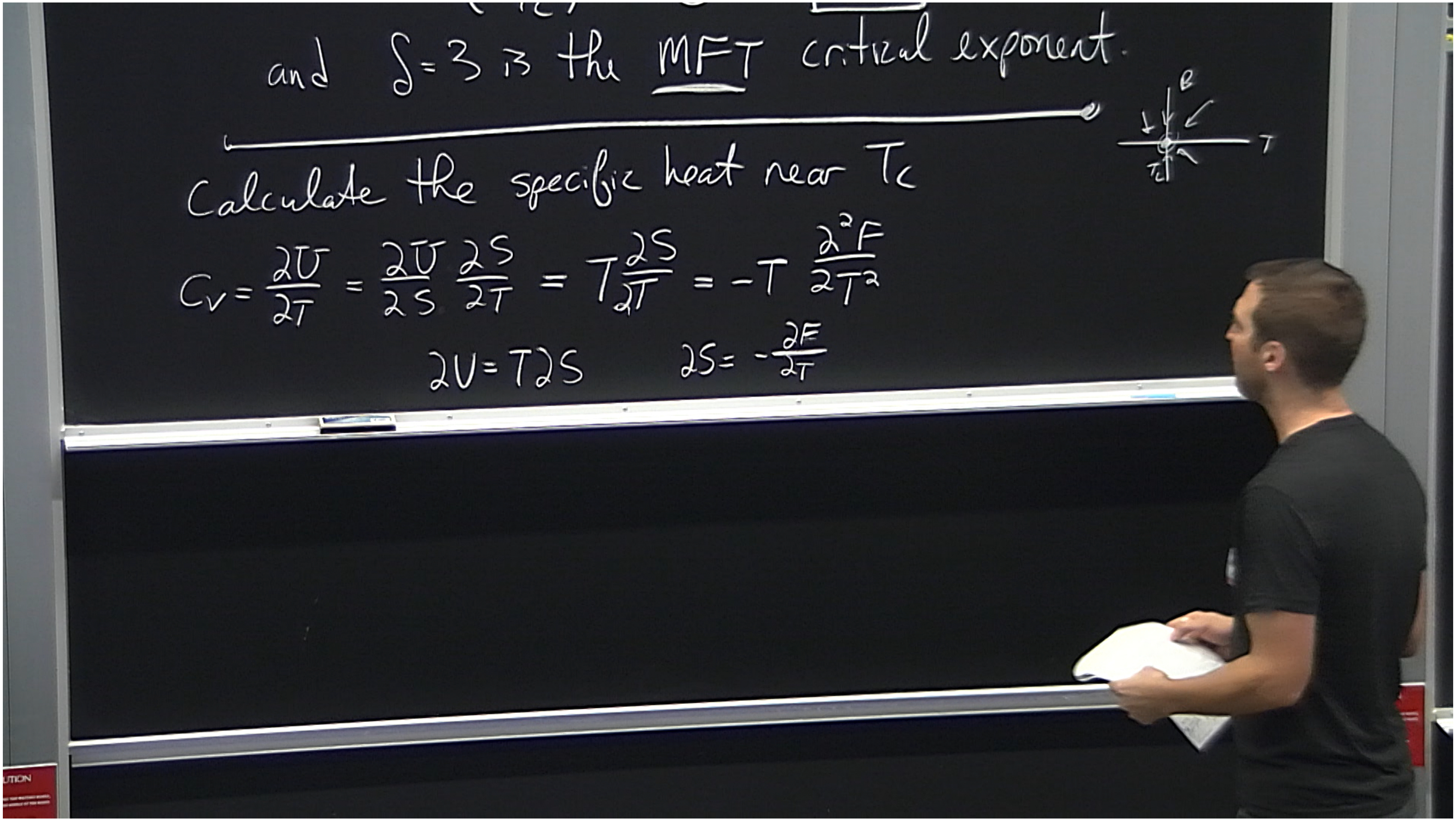
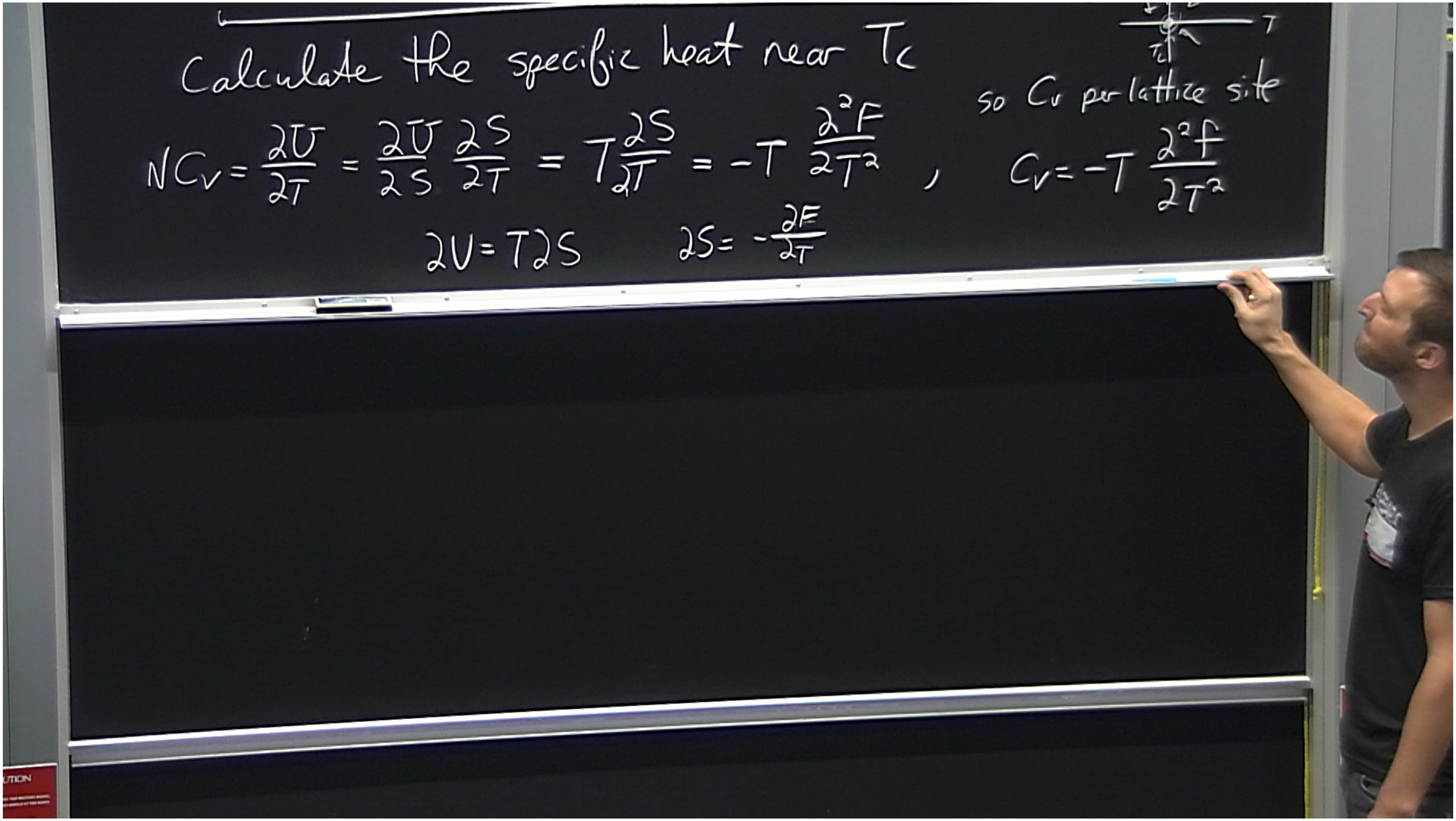
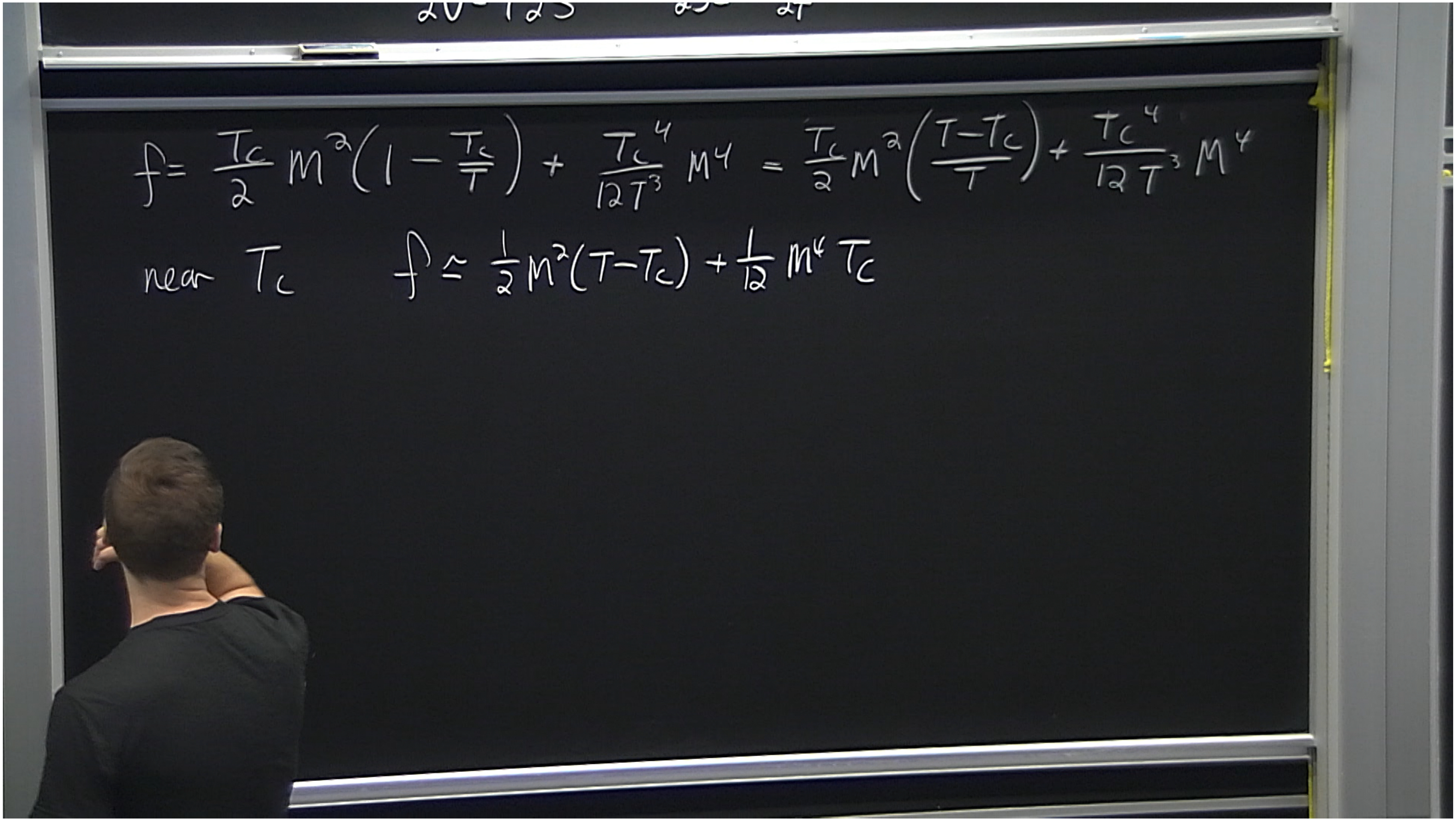
B
T_c



and $\delta = 3$ is the MFT critical exponent.
Calculate the specific heat near T_c
B
T
T_c
CAUTION


and $\delta = 3$ is the MFT critical exponent.
Calculate the specific heat near T_c
$C_V = \frac{\partial U}{\partial T} = \frac{\partial U}{\partial S}\frac{\partial S}{\partial T} = T\frac{\partial S}{\partial T} = -T\frac{\partial^2 F}{\partial T^2}$
$\partial U = T\partial S$
$\partial S = -\frac{\partial F}{\partial T}$
B
T
T_c


Calculate the specific heat near T_c
$NC_v = \frac{\partial \bar{U}}{\partial T} = \frac{\partial \bar{U}}{\partial S}\frac{\partial S}{\partial T} = T\frac{\partial S}{\partial T} = -T\frac{\partial^2 F}{\partial T^2}$,
so C_v per lattice site
$C_v = -T\frac{\partial^2 f}{\partial T^2}$
$\partial U = T\partial S$
$\partial S = -\frac{\partial F}{\partial T}$


$f = \frac{T_c}{2} M^2 \left(1 - \frac{T_c}{T}\right) + \frac{T_c^4}{12T^3} M^4 = \frac{T_c}{2} M^2 \left(\frac{T - T_c}{T}\right) + \frac{T_c^4}{12T^3} M^4$
near T_c $f \simeq \frac{1}{2} M^2 (T - T_c) + \frac{1}{12} M^4 T_c$

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near T_c $\quad f \simeq \frac{1}{2} M^2 (T - T_c) + \frac{1}{12} M^4 T_c$

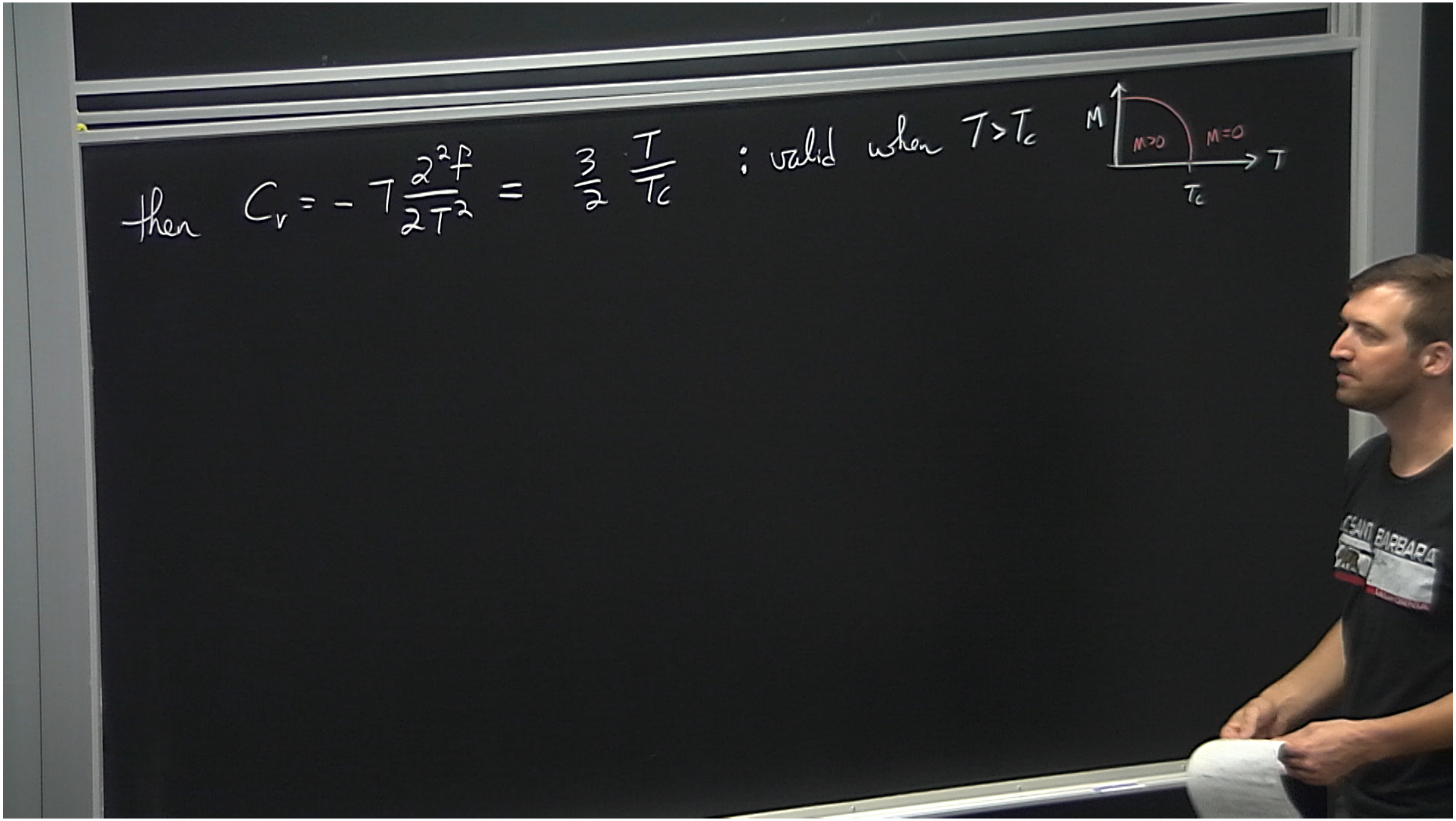
M is determined by minimizing f:

$$\frac{\partial f}{\partial M} = 0 = M(T - T_c) + \frac{1}{3} M^3 T_c$$

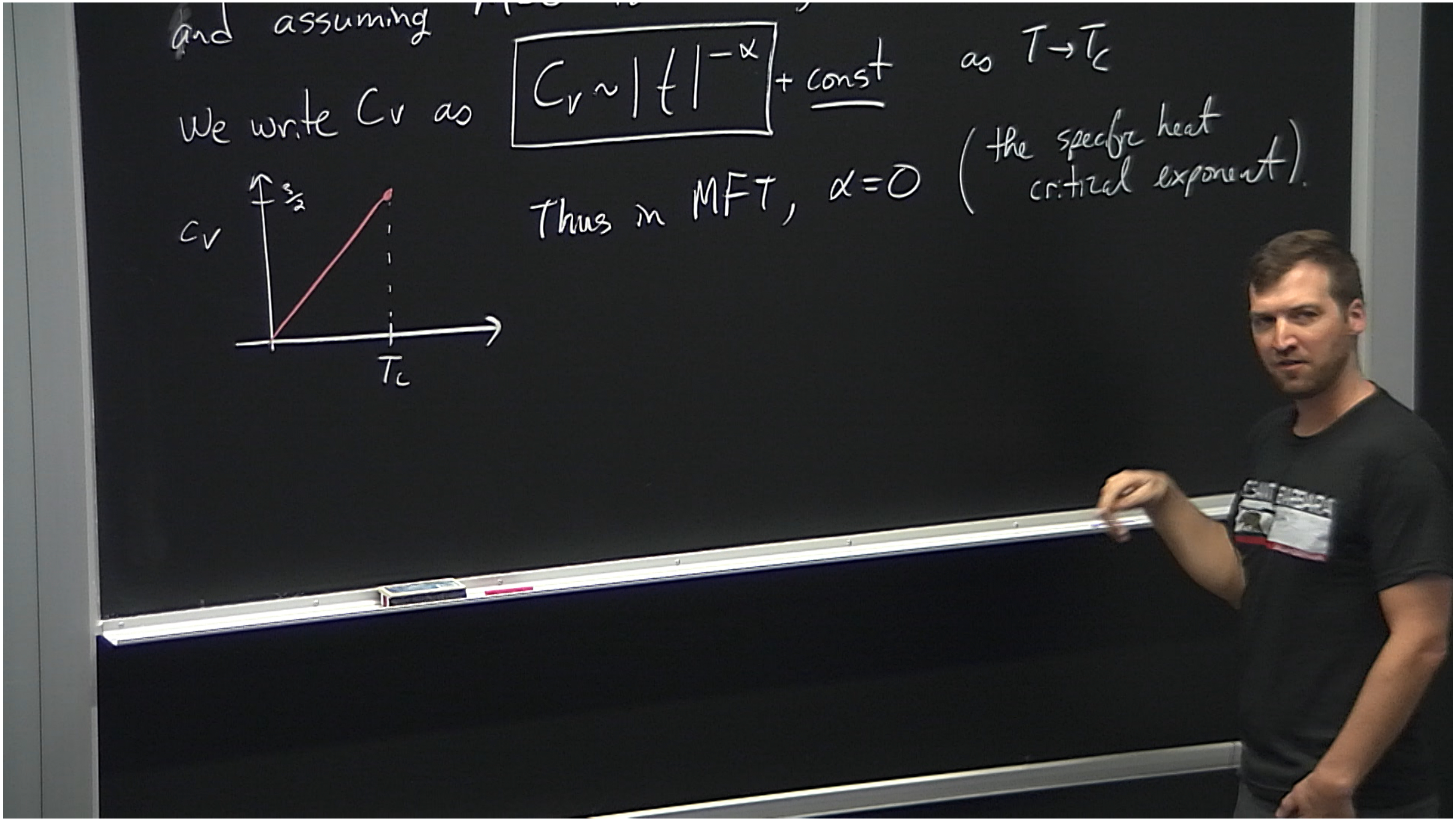
$$M^2 = 3\,\frac{T_c - T}{T_c} \quad \text{or} \quad M = \sqrt{3\left(1 - \frac{T}{T_c}\right)}$$

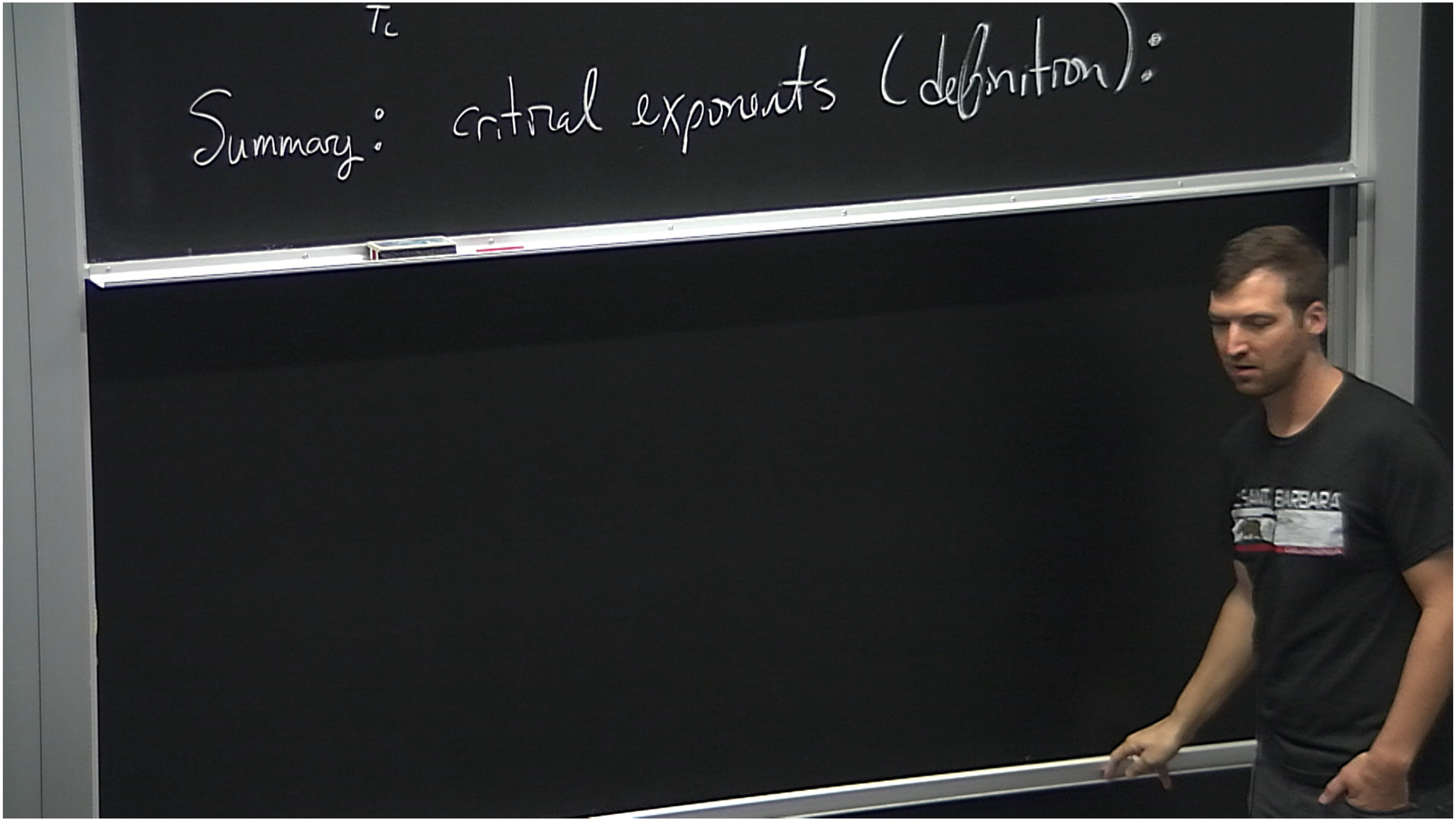
plug into $f = \frac{1}{2}\, 3\, \frac{T_c - T}{T_c} (T - T_c) + \frac{9}{12} \frac{(T_c - T)^2}{T_c}$

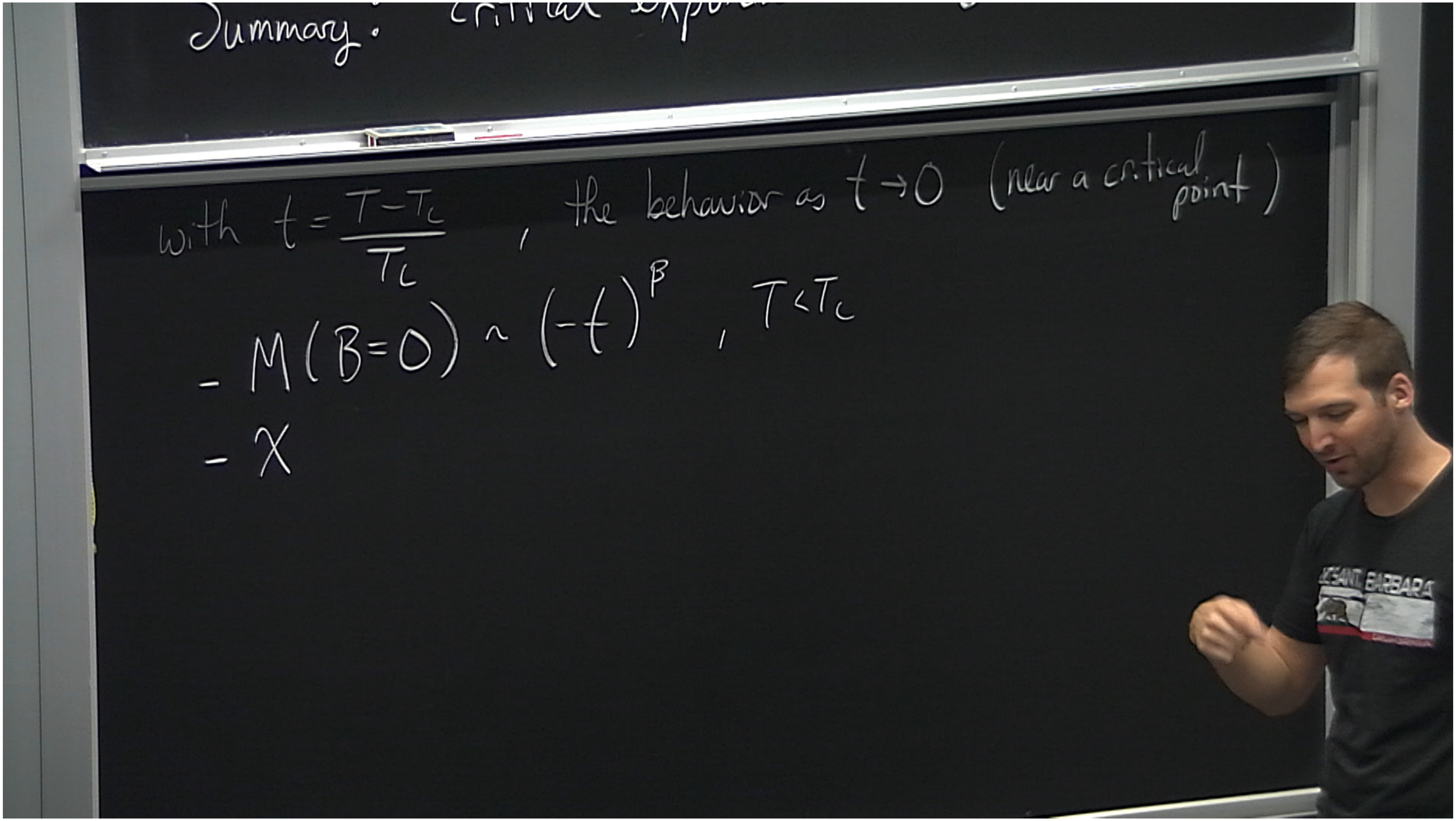
$$= -\frac{3}{4T_c} (T - T_c)^2$$



then $C_v = -T\frac{\partial^2 f}{\partial T^2} = \frac{3}{2}\frac{T}{T_c}$: valid when $T > T_c$
M
M>0
M=0
T
T_c


and assuming
We write C_V as $\boxed{C_V \sim |t|^{-\alpha}}$ + const as $T \to T_c$
Thus in MFT, $\alpha = 0$ (the specific heat critical exponent).
C_V
$\frac{3}{2}$
T_c

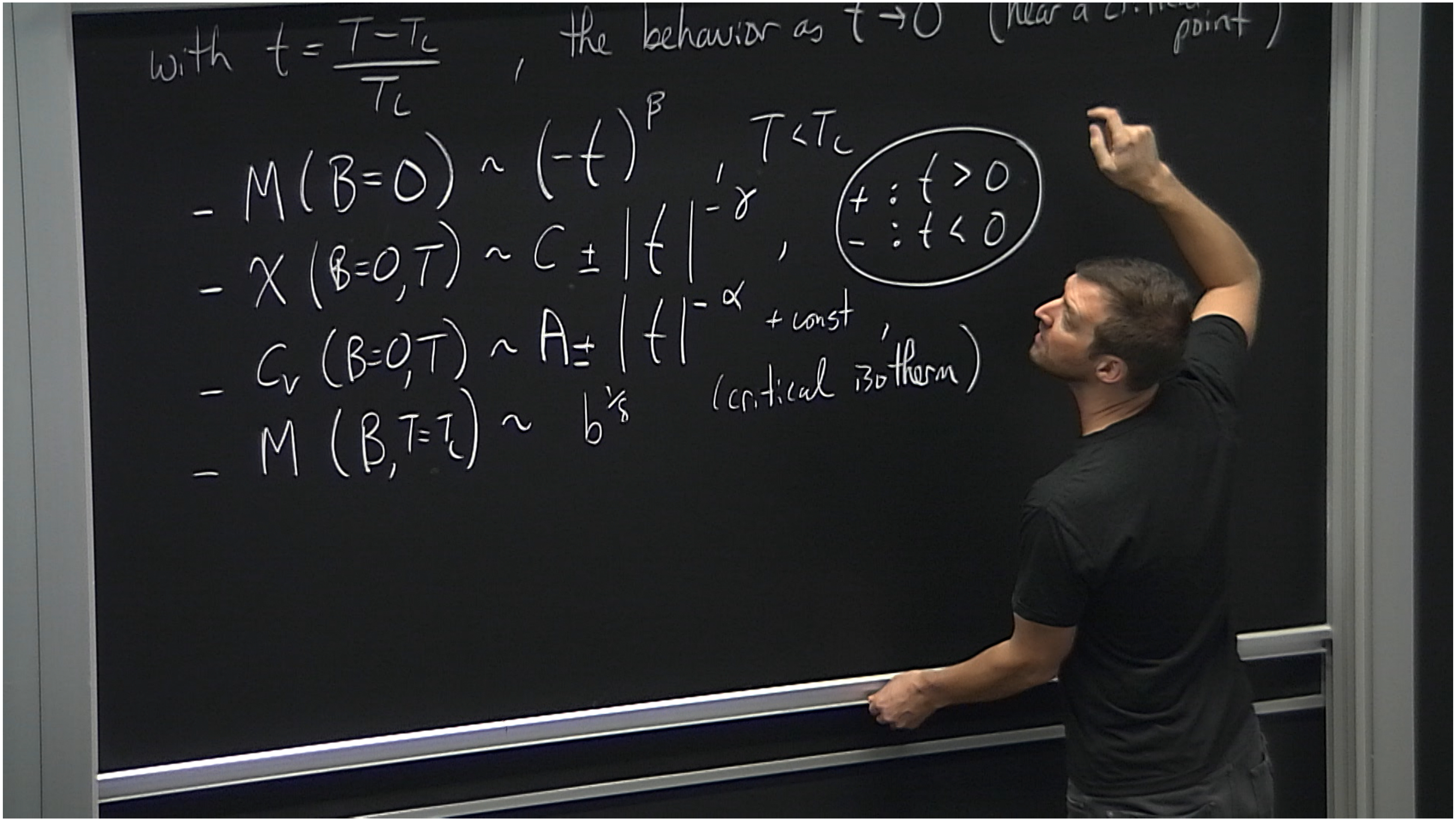

T_c
Summary: critical exponents (definition):

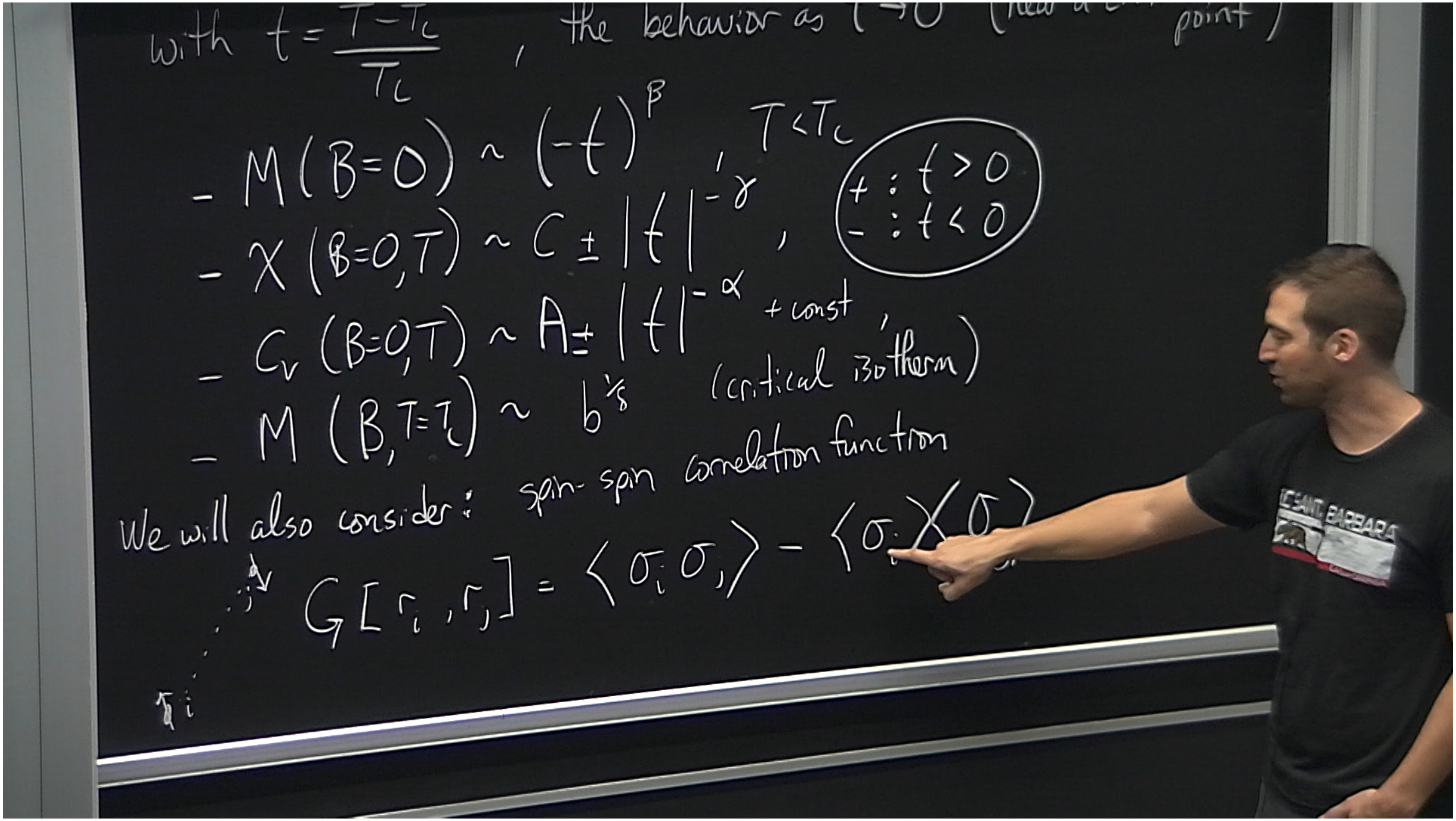

Summary: criti
with $t = \frac{T - T_c}{T_c}$, the behavior as $t \to 0$ (near a critical point)
- $M(B=0) \sim (-t)^{\beta}$, $T < T_c$
- χ

Summary: critical expon

with $t = \frac{T - T_c}{T_c}$, the behavior as $t \to 0$ (near a critical point)

- $M(B=0) \sim (-t)^{\beta}$, $T < T_c$
- $\chi(B=0,T) \sim C_{\pm} |t|^{-\gamma}$, $+ : t > 0$, $- : t < 0$


with t = (T − T_c)/T_c , the behavior as t → 0 (near a
point)
− M(B=0) ~ (−t)^β , T<T_c
− χ(B=0,T) ~ C± |t|^−γ ,
− C_V(B=0,T) ~ A± |t|^−α + const ,
− M(B,T=T_c) ~ b^(1/δ) (critical isotherm)
+ : t > 0
− : t < 0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point)
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$+ : t>0$
$- : t<0$
$- \; C_V(B=0,T) \sim A_{\pm} |t|^{-\alpha} + \text{const},$
$- \; M(B,T=T_c) \sim b^{1/\delta}$ (critical isotherm)
We will also consider: spin-spin correlation function
$G[r_i, r_j] = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle$
j
i

we will see that

$$t \neq 0 \qquad G[r_i - r_j, T] \sim \frac{e^{-|r_i - r_j|/\xi}}{|r_i - r_j|^{\tau}}$$

"correlation length"

depends on dimension
e.g. d-2

calculated $M \sim (-t)$ $\beta = \frac{1}{2}$ MF

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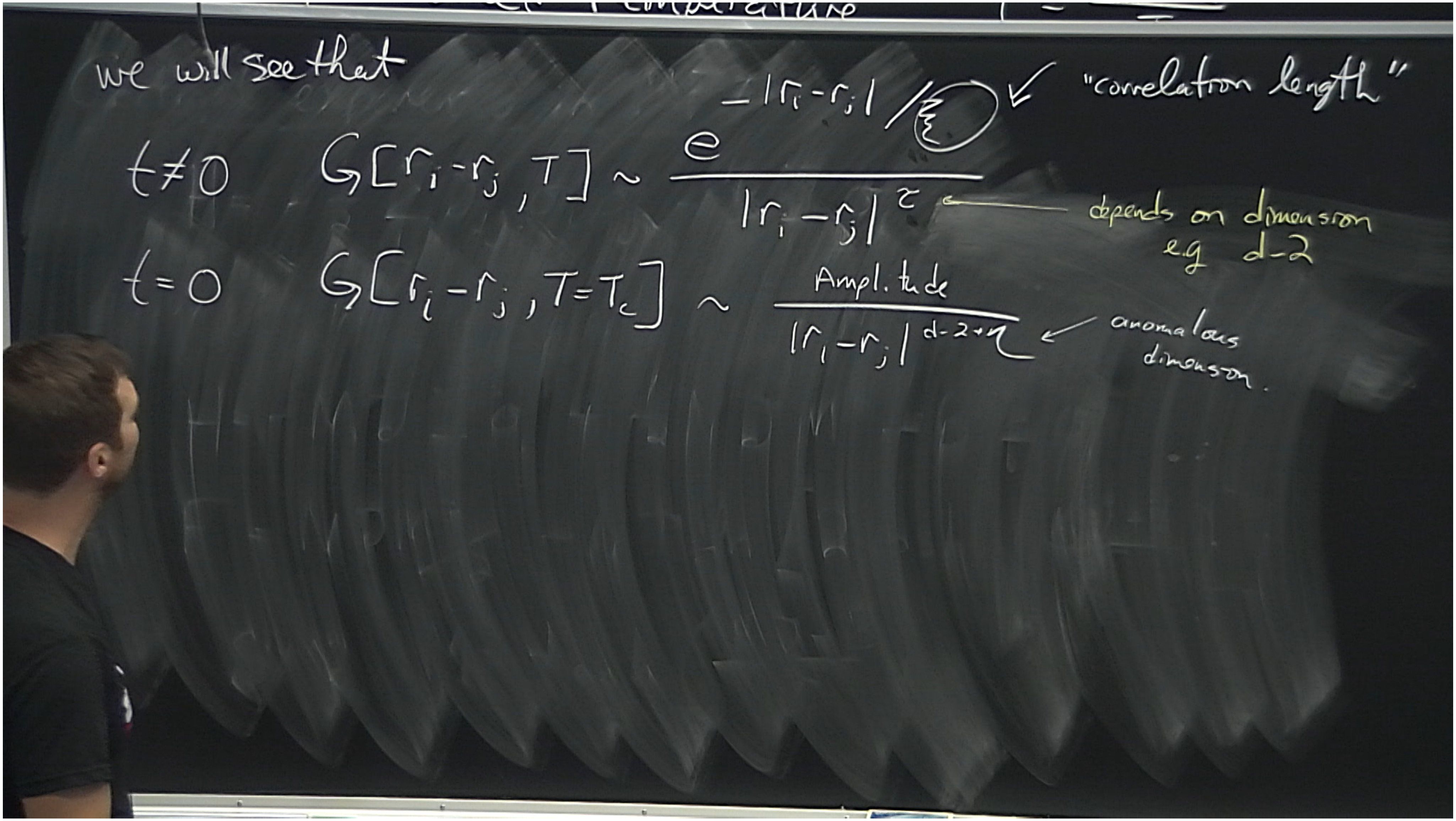
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"correlation length"
depends on dimension e.g. d-2
$t = 0 \quad G[r_i - r_j, T = T_c] \sim \frac{\text{Amplitude}}{|r_i - r_j|^{d-2+\eta}}$
anomalous dimension.

$|r_i - r_j|^{d}$ ← depends on dimension e.g. $d-2$

$t=0 \quad G[r_i - r_j, T=T_c] \sim \dfrac{\text{Amplitude}}{|r_i - r_j|^{d-2+\eta}}$ ← anomalous dimension.

Correlation length $\xi(T)$

is diverging at T_c

$$\xi(T) \sim K\,|t|^{-\nu} \quad : \xi \longrightarrow \infty \text{ as } T \to T_c$$

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critical exponents (universal) : $\alpha, \beta, \gamma, \delta, \nu, \eta$

$$M = \frac{MJ+B}{T_c} - \frac{1}{3}\left(\frac{MJ+B}{T_c}\right) = \underbrace{\frac{MJ}{T_c}}_{M} + \frac{B}{T_c} - \frac{1}{3}\left(\frac{MJ+B}{T_c}\right)$$

critical exponents (universal): $\alpha, \beta, \gamma, \delta, \nu, \eta$

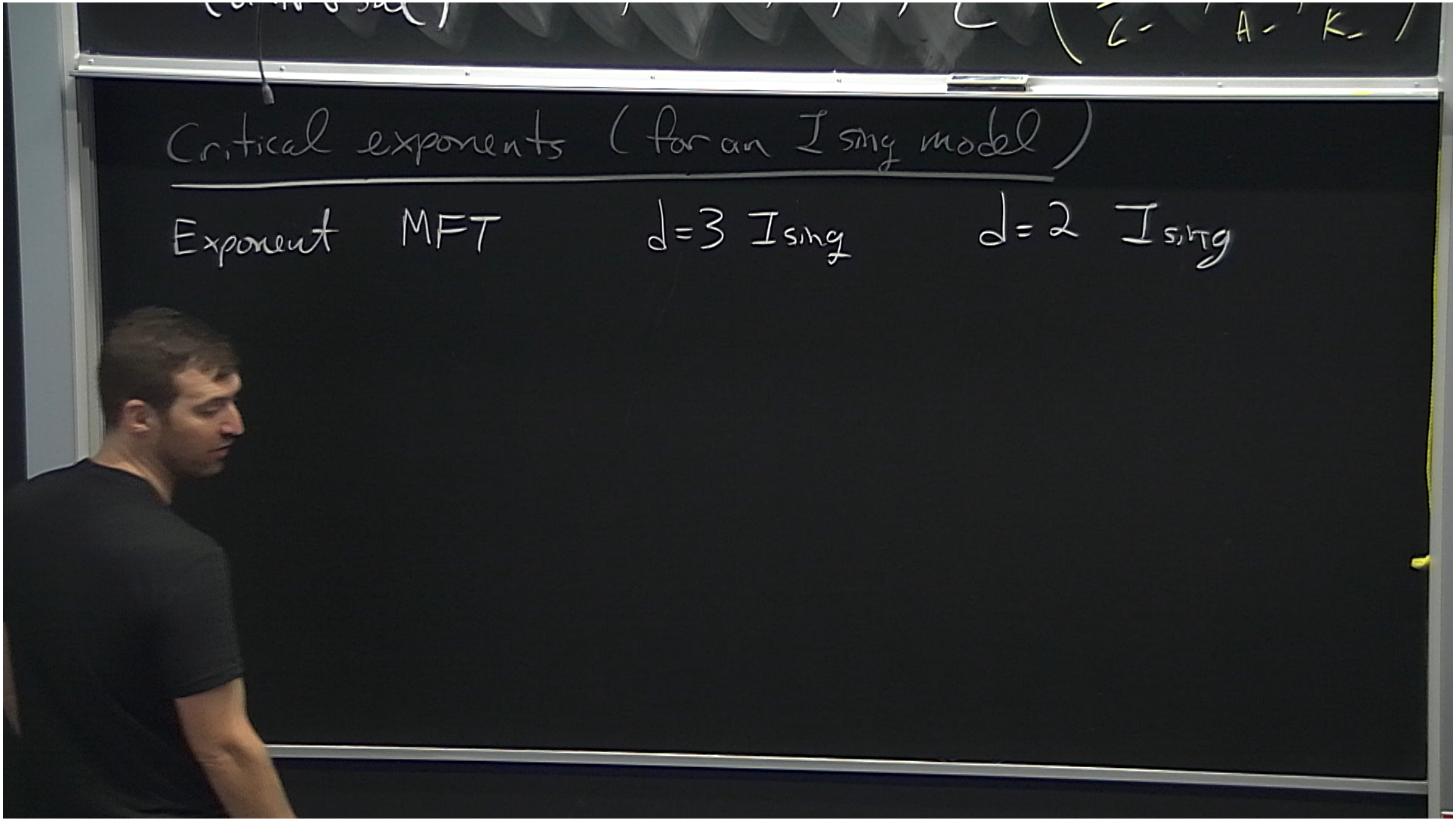
$\Big($ also amplitude ratios $\dfrac{C_+}{C_-}, \dfrac{A_+}{A_-}, \dfrac{K_+}{K_-}$

Correlation length $\xi(T)$

is diverging at T_c

$$\xi(T) \sim K\,|t|^{-\nu} \;:\; \xi \longrightarrow \infty \text{ as } T \to T_c$$

Critical exponents (universal) : $\alpha, \beta, \gamma, \delta, \nu, \eta$ $\left(\text{also amplitude ratios } \frac{C_+}{C_-}, \frac{A_+}{A_-}, \frac{K_+}{K_-}\right)$


Critical exponents (for an Ising model)
Exponent MFT d=3 Ising d=2 Ising

Critical exponents (for an Ising model)

Exponent	MFT	d=3 Ising	d=2 Ising
α	0		
β	$\frac{1}{2}$		
γ	1		
δ	3		
η	0		
ν			

Critical exponents (for an Ising model)

Exponent	MFT	$d=3$ Ising	$d=2$ Ising
α	0	0.110	0
β	$\frac{1}{2}$	0.325	$\frac{1}{8} = 0.125$
γ	1	1.241	$\frac{7}{4} = 1.75$
δ	3	4.82	15
η	0	0.031	$\frac{1}{4} = 0.25$
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these are exact for $d \geq 4$

Critical exponents (for an Ising model) — one universality class

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Phenomenological Landau Theory

Note: the general form of the free energy can be written down without a specific Hamiltonian.

→ we need an order parameter.

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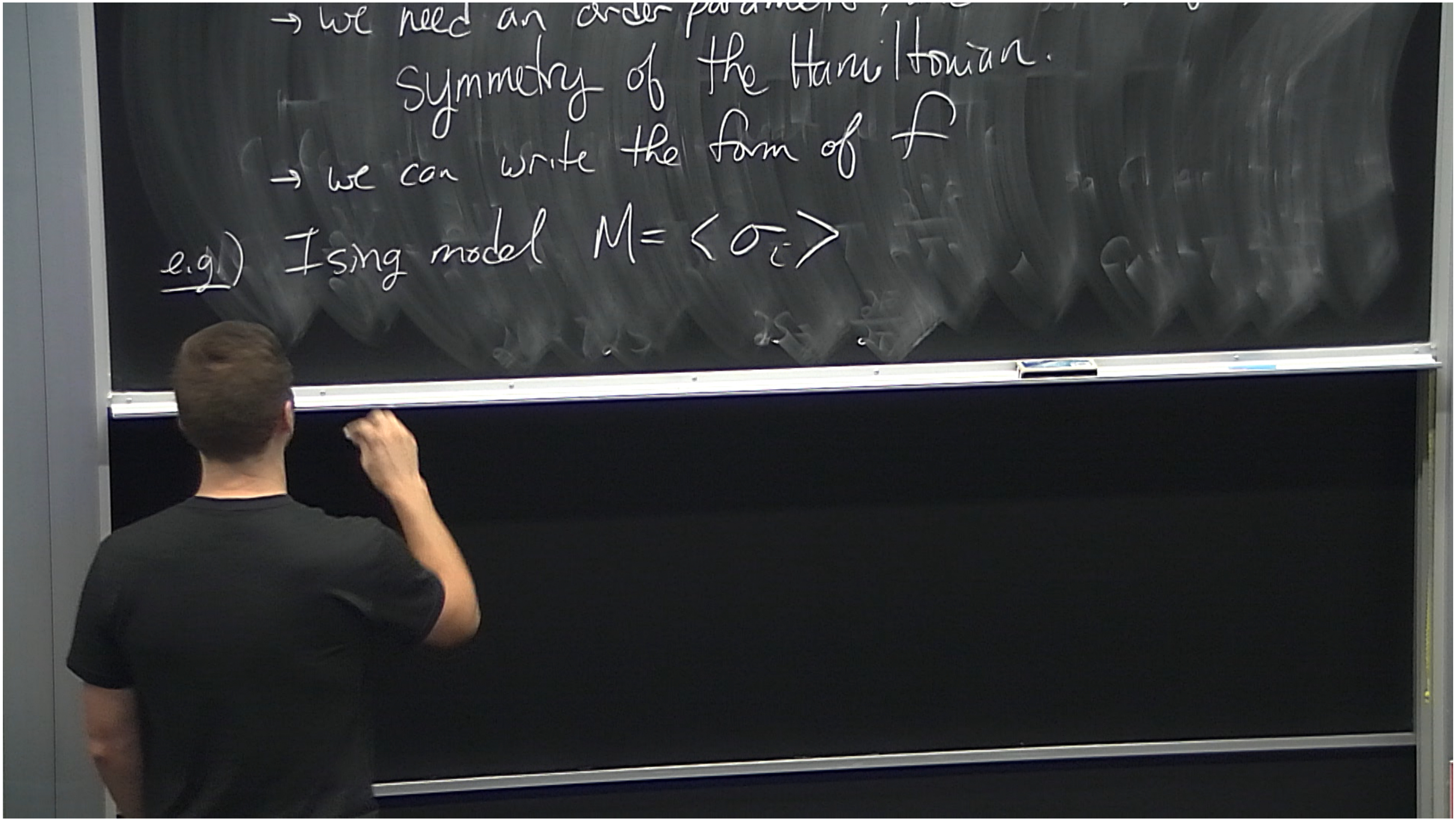
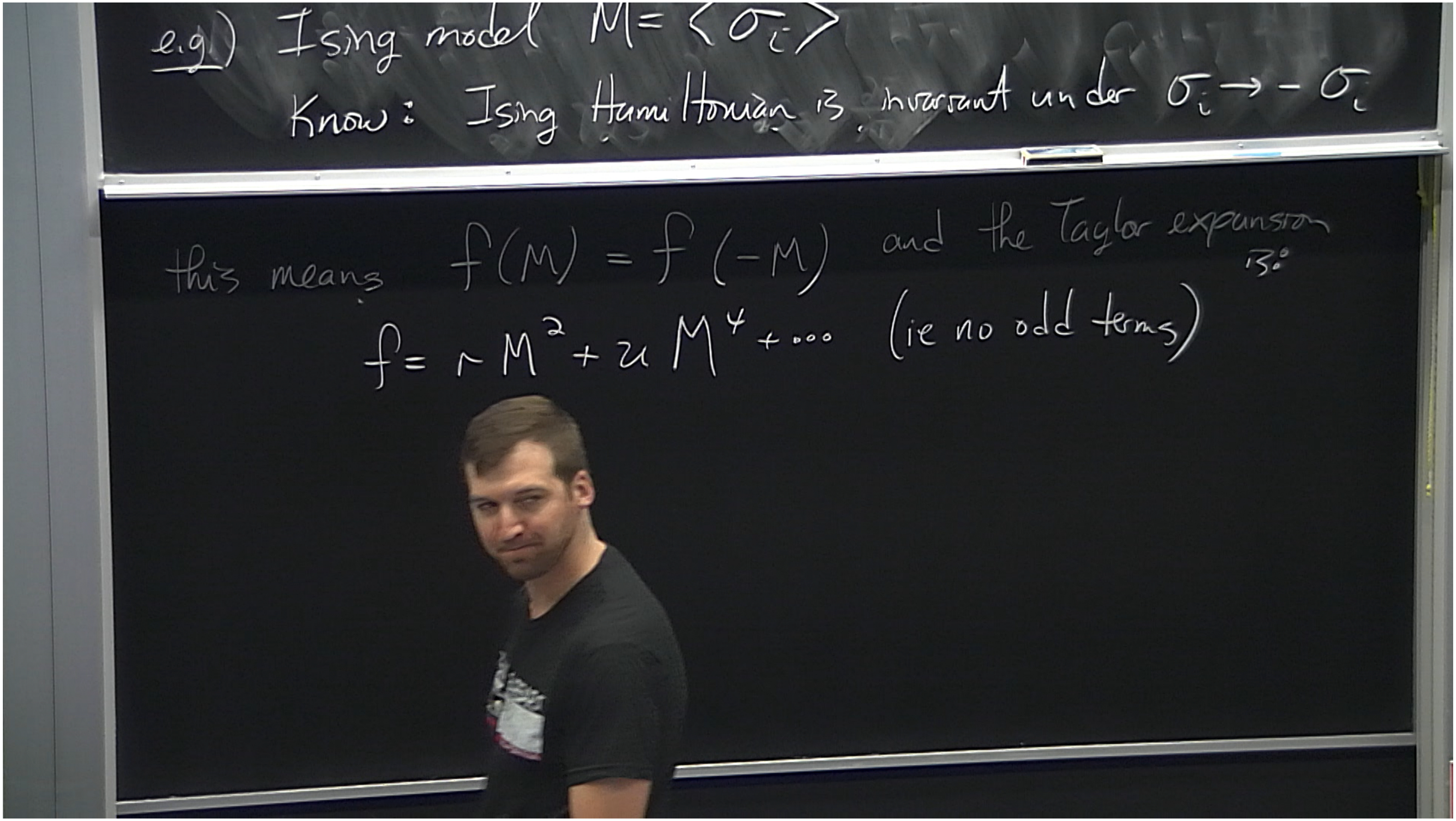
→ we need an order parameter, and knowledge of the symmetry of the Hamiltonian.


→ we need an order
symmetry of the Hamiltonian.
→ we can write the form of f
e.g.) Ising model M = ⟨σ_i⟩


e.g.) Ising model $M = \langle \sigma_i \rangle$
Know: Ising Hamiltonian is invariant under $\sigma_i \to -\sigma_i$
this means $f(M) = f(-M)$ and the Taylor expansion is:
$f = r M^2 + u M^4 + \cdots$ (ie no odd terms)

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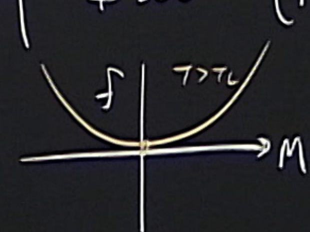
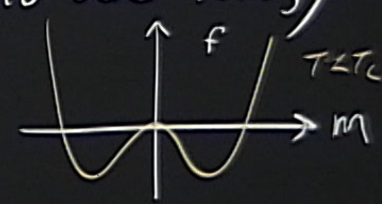
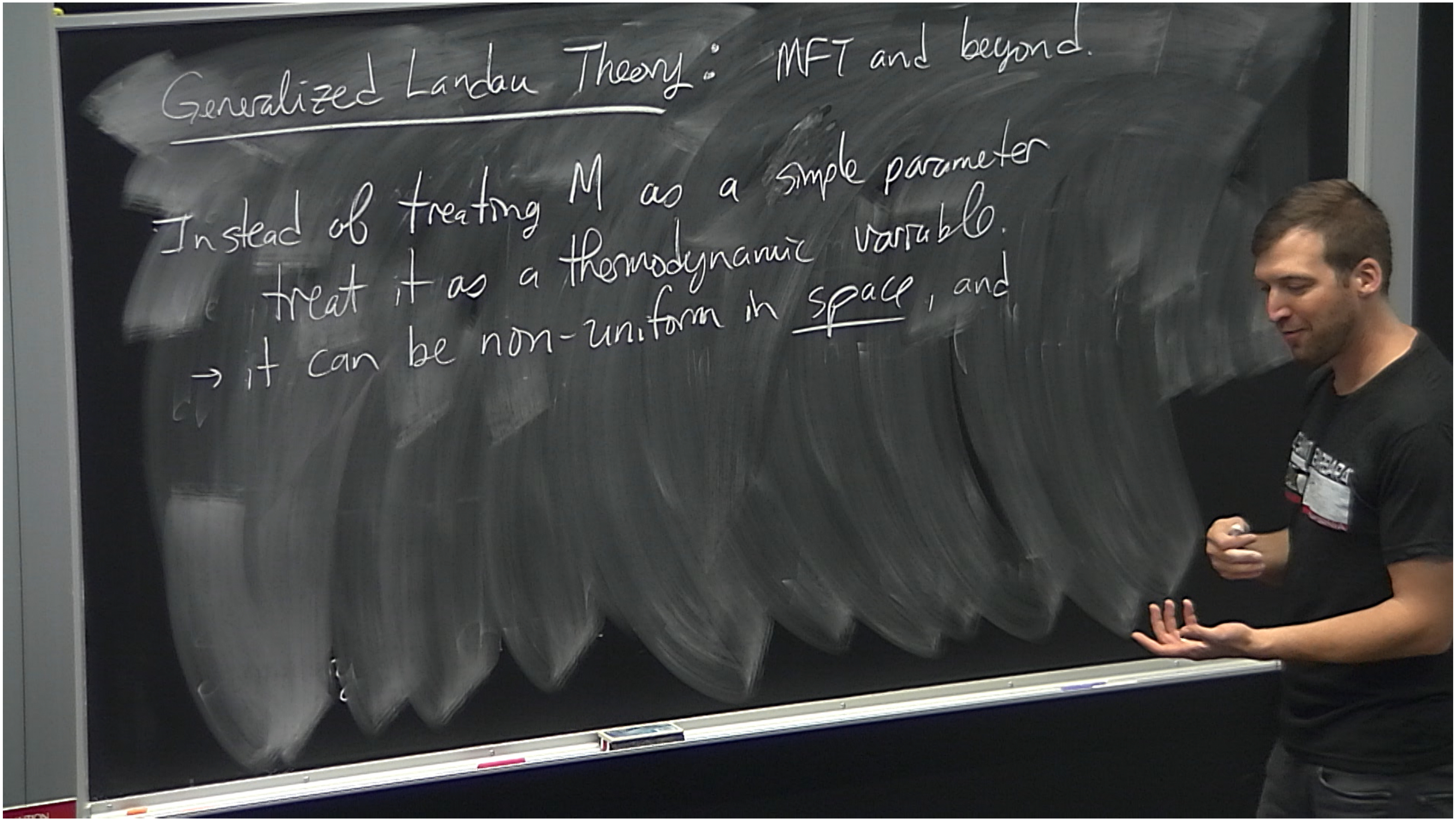
$$f = r M^2 + u M^4 + \cdots \quad \text{(ie no odd terms)}$$

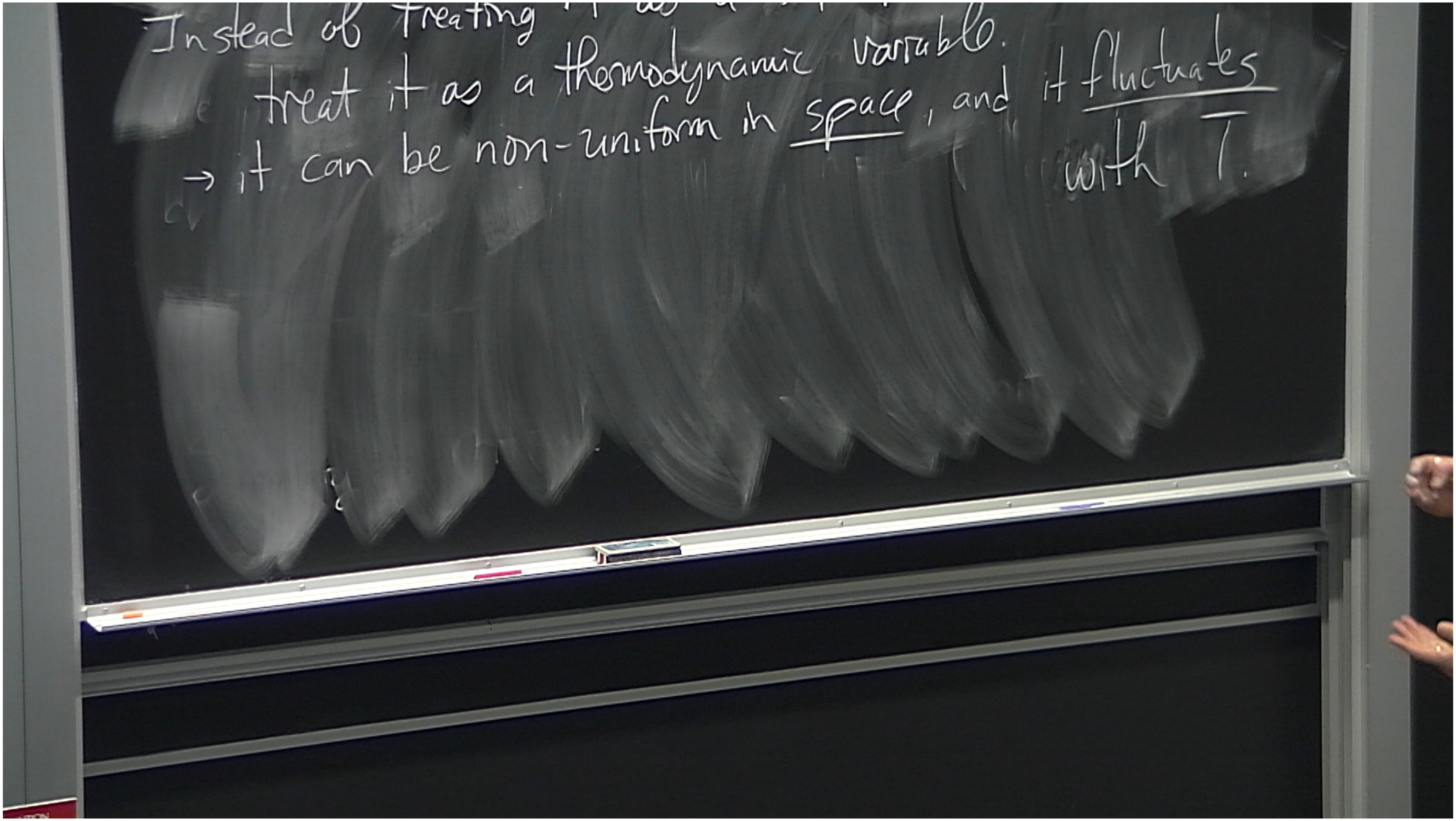
and require "stability" e.g.

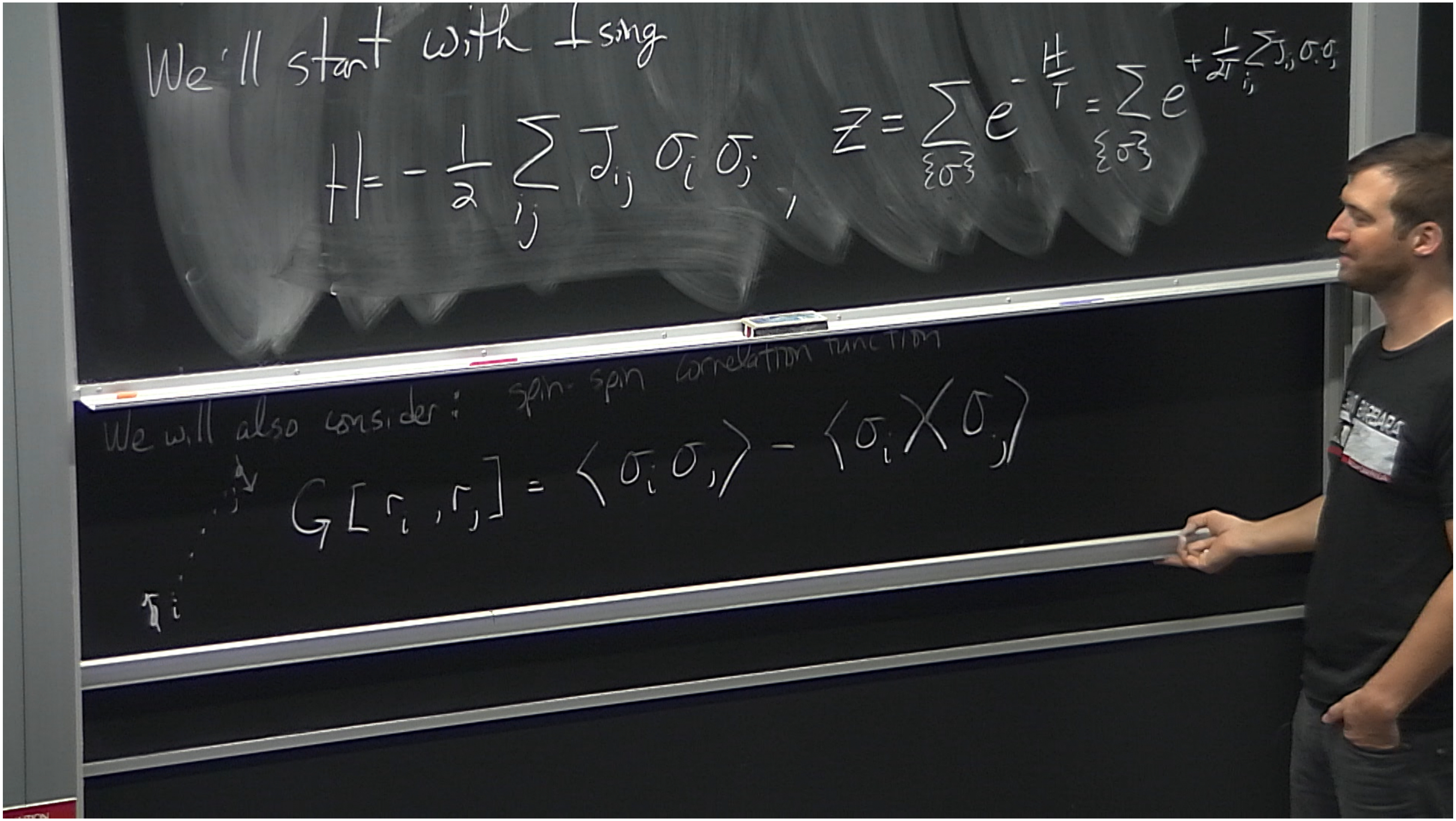




- "postulate" that r changes sign at T_c
- and require <u>that $u > 0$</u>.


Generalized Landau Theory: MFT and beyond.
Instead of treating M as a simple parameter
treat it as a thermodynamic variable.
→ it can be non-uniform in space, and


Instead of treating
treat it as a thermodynamic variable.
→ it can be non-uniform in space, and it fluctuates with T.


We'll start with Ising
$H = -\frac{1}{2}\sum_{ij} J_{ij}\sigma_i\sigma_j$, $Z = \sum_{\{\sigma\}} e^{-\frac{H}{T}} = \sum_{\{\sigma\}} e^{+\frac{1}{2T}\sum_{ij} J_{ij}\sigma_i\sigma_j}$
We will also consider: spin-spin correlation function
$G[r_i, r_j] = \langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle$

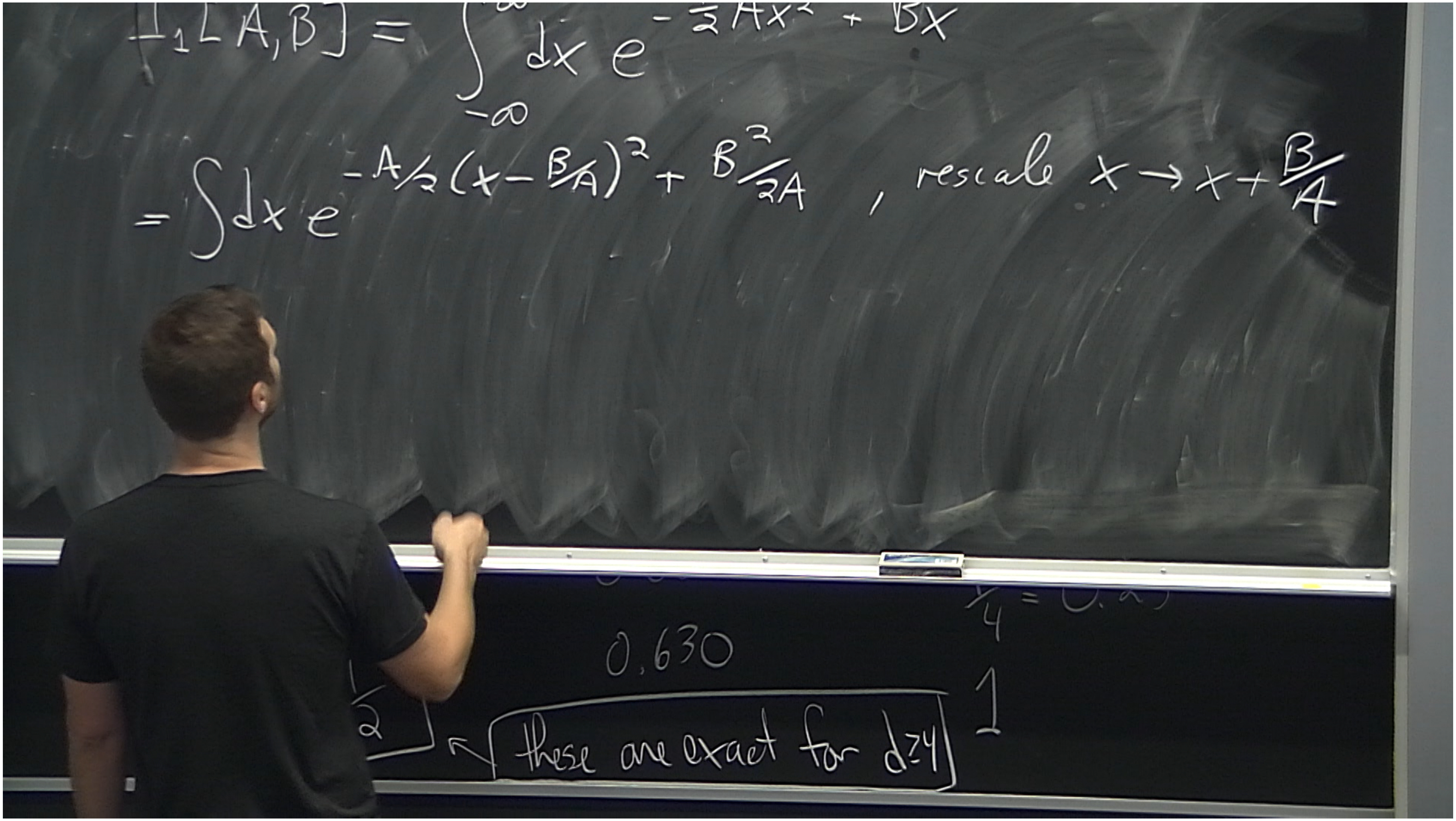
We will use the Hubbard-Stratonovich identity.

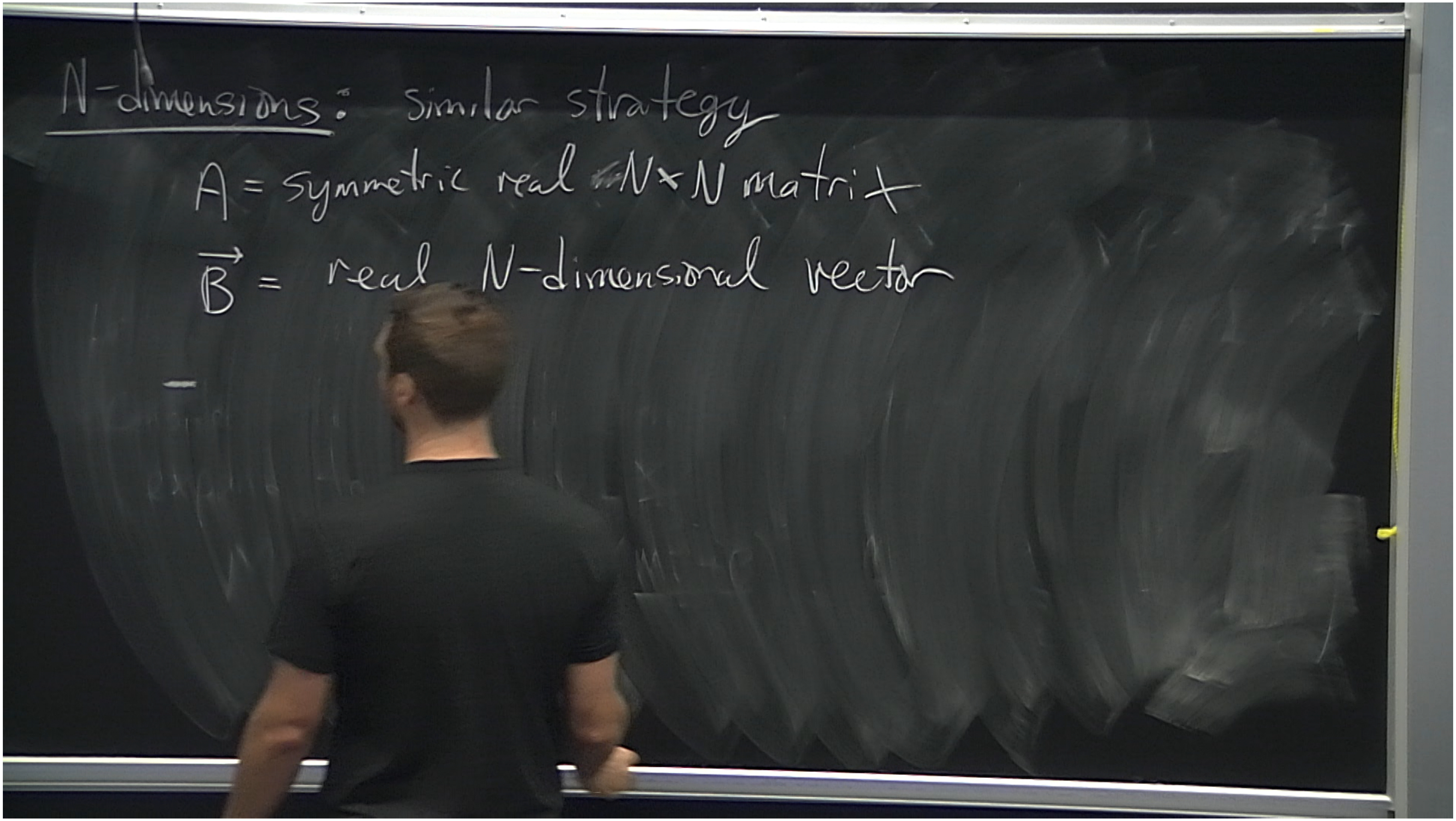
$$\frac{1}{(2\pi)^{N/2}}\int_{-\infty}^{\infty} d\varphi_1 \cdots d\varphi_N \, e^{-\frac{1}{2}\varphi_i A_{ij}\varphi_j + \varphi_i \gamma_i} = \frac{1}{\sqrt{\det A}} \, e^{\frac{1}{2}\gamma_i A^{-1}_{ij}\gamma_j}$$

<u>where:</u> φ_i are real variables, γ_i arbitrary real number

A is symmetric, real matrix, invertible

- Einstein summation rules apply (sum over repeated indices)


$I_1[A,B] = \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}Ax^2 + Bx}$
$= \int dx\, e^{-A/2\,(x-B/A)^2 + B^2/2A}$, rescale $x \to x + B/A$
0.630
1
these are exact for $d \geq 4$


N-dimensions: similar strategy
A = symmetric real N×N matrix
$\vec{B}$ = real N-dimensional vector

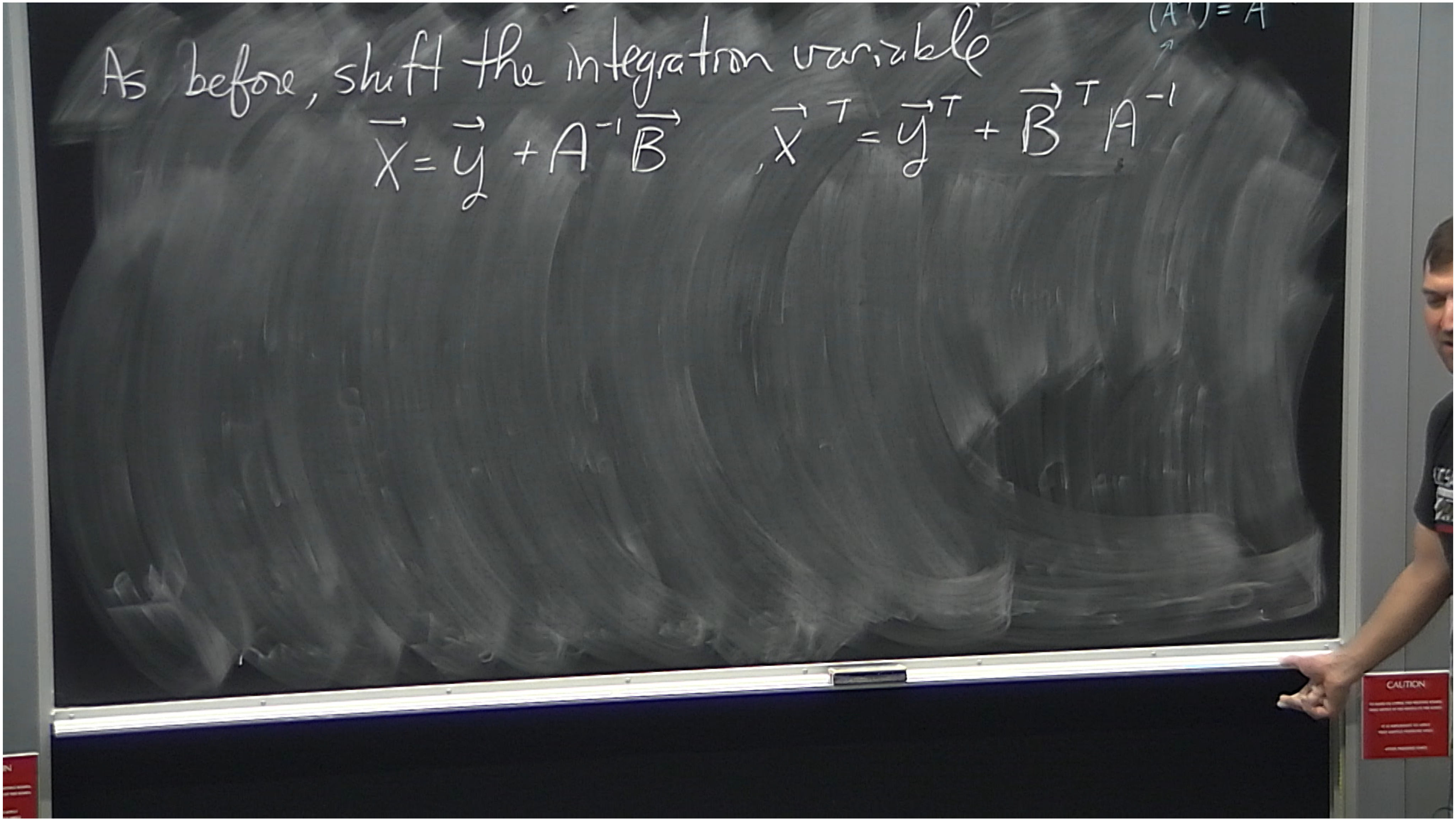
N-dimensions: similar strategy

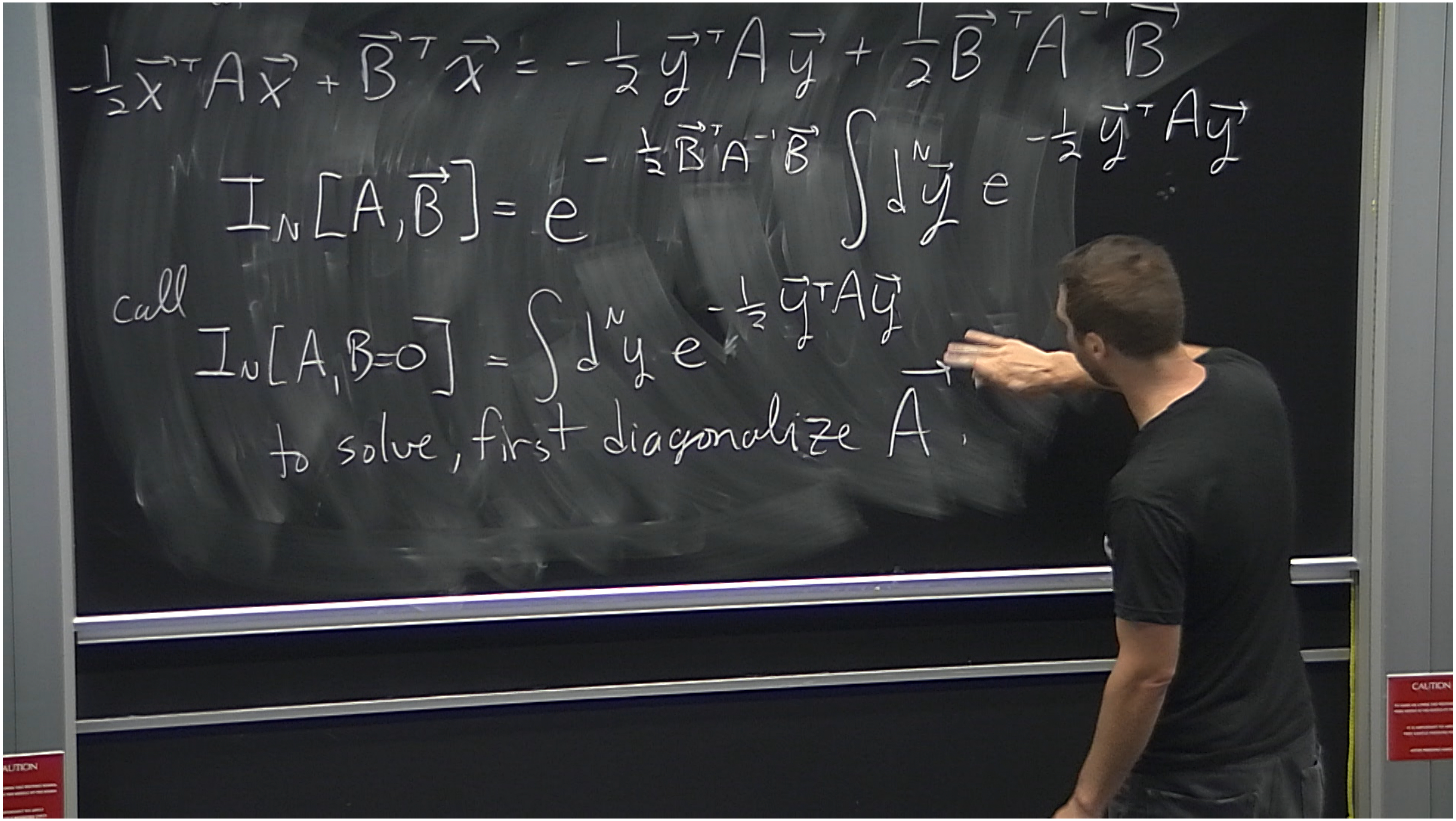
A = symmetric real $N \times N$ matrix

$\vec{B}$ = real N-dimensional vector

$$I_N[A, \vec{B}] = \int_{-\infty}^{\infty} d^N \vec{x}\, e^{-\frac{1}{2} x_i A_{ij} x_j + B_i x_i}$$

$$= \int_{-\infty}^{\infty} d^N \vec{x}\, e^{-\frac{1}{2} \vec{x}^T A \vec{x} + B^T \cdot \vec{x}}$$


As before, shift the integration variable
$\vec{X} = \vec{y} + A^{-1}\vec{B}$, $\vec{X}^T = \vec{y}^T + \vec{B}^T A^{-1}$


$-\frac{1}{2}\vec{x}^T A \vec{x} + \vec{B}^T \vec{x} = -\frac{1}{2}\vec{y}^T A \vec{y} + \frac{1}{2}\vec{B}^T A^{-1} \vec{B}$
$I_N[A,\vec{B}] = e^{-\frac{1}{2}\vec{B}^T A^{-1}\vec{B}} \int d^N\vec{y}\, e^{-\frac{1}{2}\vec{y}^T A \vec{y}}$
call
$I_N[A,B=0] = \int d^N y\, e^{-\frac{1}{2}\vec{y}^T A \vec{y}}$
to solve, first diagonalize $\vec{A}$.

$$-\frac{1}{2}\vec{x}^T A \vec{x} + \vec{B}^T \vec{x} = -\frac{1}{2}\vec{y}^T A \vec{y} + \frac{1}{2}\vec{B}^T A^{-1} \vec{B}$$

$$I_N[A,\vec{B}] = e^{-\frac{1}{2}\vec{B}^T A^{-1}\vec{B}} \int d^N\vec{y}\, e^{-\frac{1}{2}\vec{y}^T A \vec{y}}$$

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$$I_N[A, B=0] = \int d^N y\, e^{-\frac{1}{2}\vec{y}^T A \vec{y}}$$

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a B orthogonally diagonalizable.

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$$O^T A O = \Lambda = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \lambda_N \end{pmatrix} \quad \underline{\text{diagonal}}.$$

Then make another change of variables

$$\vec{y} = O \vec{v} \qquad \Big(\underline{\text{note}}: \; d^N \vec{y} = \underbrace{(\det O)}_{=1 \text{ for orthogonal}} d^N \vec{v}$$

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$$\vec{y}^{\,T} A \vec{y} = \vec{v}\, O^T A O \vec{v}$$

$$= \vec{v}^{\,T} \Lambda \vec{v} = \sum_{i=1}^{N} \lambda_i v_i^2$$

$$I_N[A,\vec{B}=0] = \int d^N\vec{v}\, e^{-\frac{1}{2}\sum_{i=1}^{N} \lambda_i v_i^2}$$

$$= \prod_{i=1}^{N}\left[\int dv_i\, e^{-\frac{1}{2}\lambda_i v_i^2}\right] = \prod_{i=1}^{N}\sqrt{\frac{2\pi}{\lambda_i}}$$

$$= (2\pi)^{N/2}(\det A)^{-\frac{1}{2}}$$

where $(\det A)^{-\frac{1}{2}} = \dfrac{1}{(\lambda_1\lambda_2\cdots\lambda_N)^{\frac{1}{2}}}$

$$\boxed{\begin{aligned} I_N[A,\vec{B}] &= \int d^N\vec{x}\, e^{-\frac{1}{2}\vec{x}^T A\vec{x} + \vec{B}^T\vec{x}} \\ &= (2\pi)^{N/2}(\det A)^{-\frac{1}{2}}\, e^{\frac{1}{2}\vec{B}^T A^{-1}\vec{B}} \end{aligned}}$$

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