Title: PSI 2016/2017 Foundations of Quantum Mechanics (Review) - Lecture 10

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URL: http://pirsa.org/17010042

Abstract:

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In what sense

$$\lim_{\hbar \to 0} QM = CM$$

Path integral argument

$$P_t(x_1 \to x_2) = |\langle x_2 | e^{-iHt} | x_1 \rangle|^2$$

$$= \left| \int_{x_1}^{x_2} \mathcal{D}[x(t)]\, e^{\frac{i}{\hbar} S[x(t), x'(t)]} \right|^2$$

Ehrenfests thm

$$\frac{d\langle \hat{x} \rangle}{dt} = -\frac{1}{\hbar} \langle [\hat{x}, \hat{H}] \rangle = \left\langle \frac{\partial \hat{H}}{\partial p} \right\rangle$$

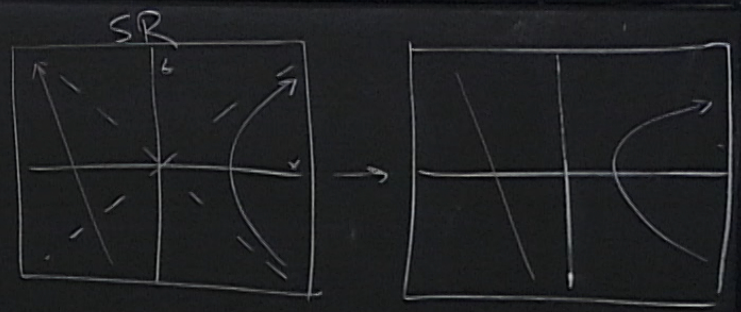
$$\frac{d\langle \hat{p} \rangle}{dt} = -\frac{1}{\hbar} \langle [\hat{p}, \hat{H}] \rangle = \left\langle \frac{\partial \hat{H}}{\partial x} \right\rangle$$

Ehrenfests thm

$$\frac{d\langle \hat{x} \rangle}{dt} = -\frac{1}{\hbar}\langle [\hat{x}, \hat{H}] \rangle = \left\langle \frac{\partial \hat{H}}{\partial p} \right\rangle$$

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- Galilean kinematics $\xleftarrow{c \to \infty}$ Relativistic kinematics

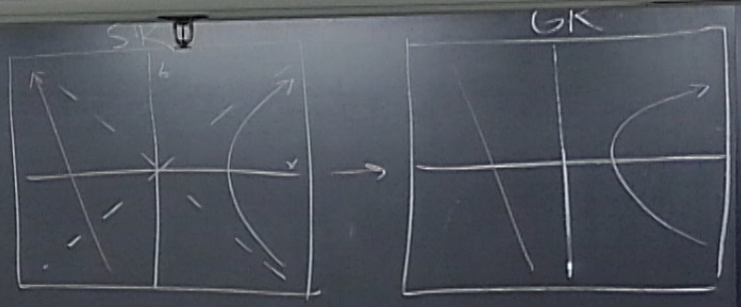


Ehrenfest's thm

$$\frac{d\langle \hat{x} \rangle}{dt} = -\frac{i}{\hbar}\langle [\hat{x}, \hat{H}] \rangle = \left\langle \frac{\partial \hat{H}}{\partial p} \right\rangle$$

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o Galilean kinematics $\xleftarrow{c \to \infty}$ Relativistic kinematics



Initial value problem

$(\vec{X}_0, \vec{P}_0) \longrightarrow (\vec{X}_t, \vec{P}_t)$

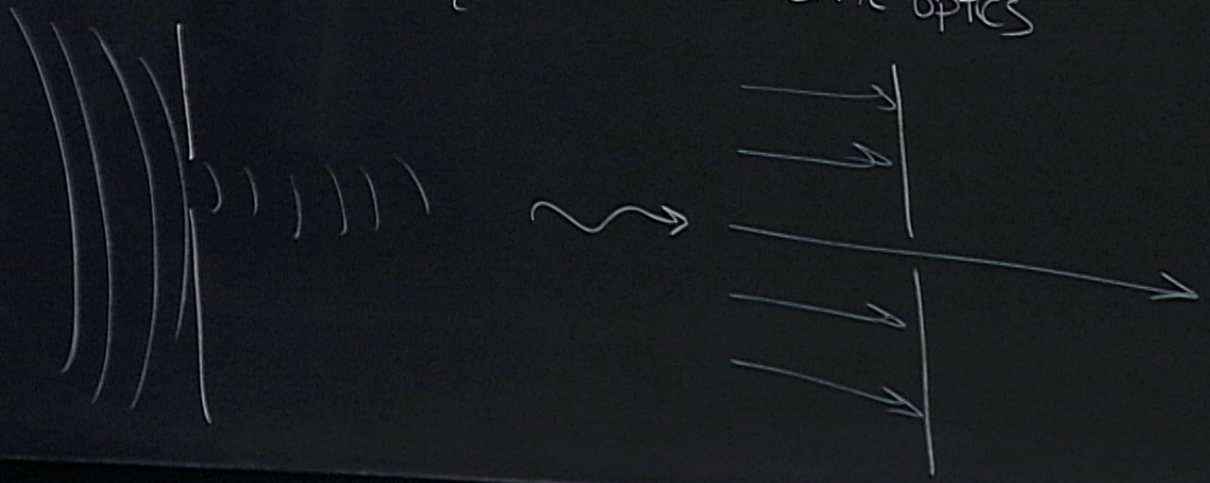
$\mathbb{R}^3 \times \mathbb{R}^3$

⊗ SR → GK is nice b/c non-singular

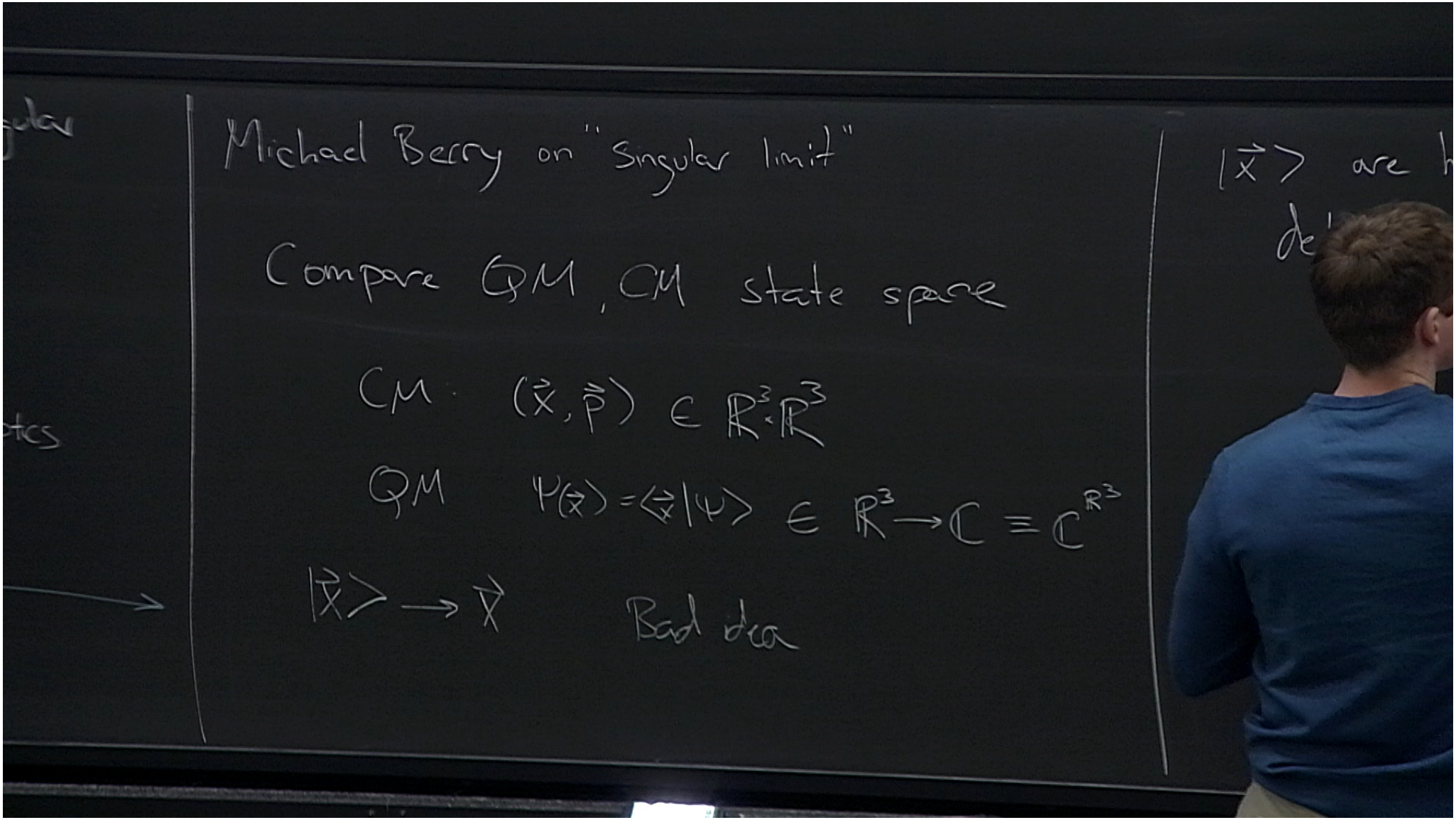
Many other limits are singular

Breakdown in state space

Ex: Maxwell's eq. ⟶ Geometric optics

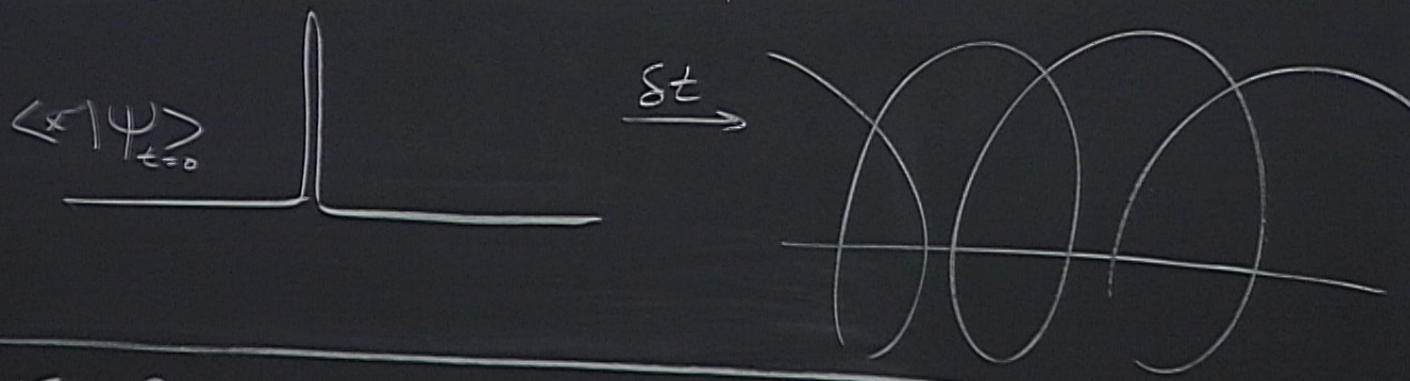


Michael Be


Michael Berry on "singular limit"
Compare QM, CM state space
CM $(\vec{x}, \vec{p}) \in \mathbb{R}^3 \times \mathbb{R}^3$
QM $\psi(\vec{x}) = \langle \vec{x} | \psi \rangle \in \mathbb{R}^3 \to \mathbb{C} \equiv \mathbb{C}^{\mathbb{R}^3}$
$|\vec{x}\rangle \to \vec{x}$ Bad idea
$|\vec{x}\rangle$ are

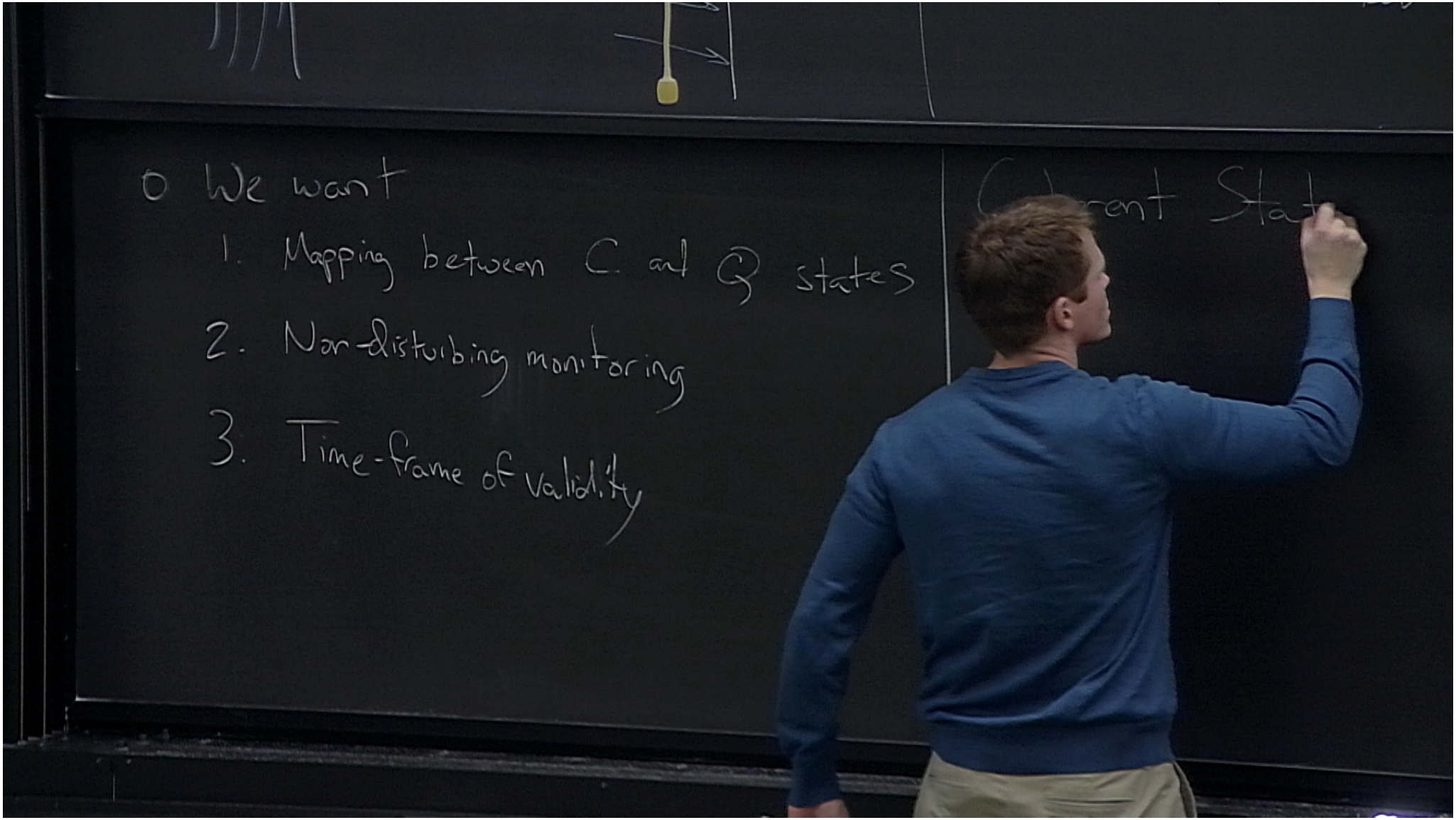
$|\vec{x}\rangle$ are highly singular

delocalize immediately



C^{R^3}

Ehrenfest: $\langle \vec{x} \rangle(t)$, $\langle \vec{p} \rangle(t)$


We want
1. Mapping between C and Q states
2. Non-disturbing monitoring
3. Time-frame of validity

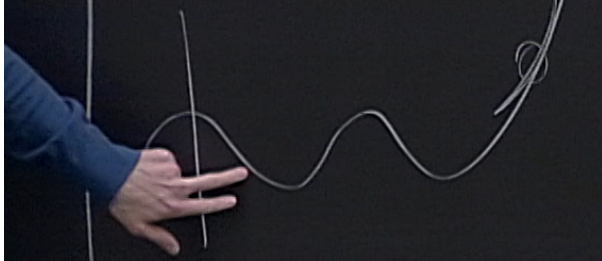
$|x\rangle \rightarrow \hat{x}$ Bad idea

$\langle \hat{x} \rangle(t)$, $\langle \hat{p} \rangle(t)$

Coherent States

Δ = Check

Why SHO important?



$$\hat{H}(\hat{x},\hat{p}) = h_0 I + h_x \hat{x} + h_p \hat{p} + h_{xx} \hat{x}^2 + h_{pp} \hat{p}^2 + \ldots$$

$|\vec{x}\rangle \rightarrow \vec{x}$ Bad idea

Ehrenfest: $\langle \vec{x} \rangle(t)$, $\langle \vec{p} \rangle(t)$

Coherent States

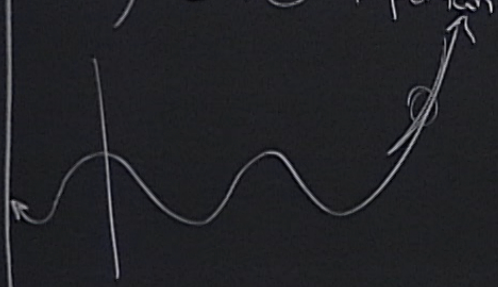
Δ = Check

Start w/ ground state

$$\psi_0^{(x)} = \underbrace{\langle x}_{\text{position state}} \underbrace{| \alpha = 0, \sigma \rangle}_{\text{Coherent}} \propto e^{-\frac{x^2}{4\sigma^2}} \qquad w/\ \sigma = \sqrt{\frac{\hbar}{2m\omega}}$$

$\alpha = (x, p)$

Why SHO important?



$$H = \frac{1}{2} m\omega^2 x^2 + \frac{1}{2} \frac{p^2}{m}$$

$$\hat{H}(\hat{x}, \hat{p}) = h_0 I + h_x \hat{x} + h_p \hat{p} + h_{xx} \hat{x}^2 + h_{pp} \hat{p}^2 + \ldots$$

$$\hat{D}_{\alpha(x,p)} \equiv e^{i(x\hat{p} + p\hat{x})} \sim e^{(\alpha \hat{a}^\dagger - \alpha^* \hat{a})}$$

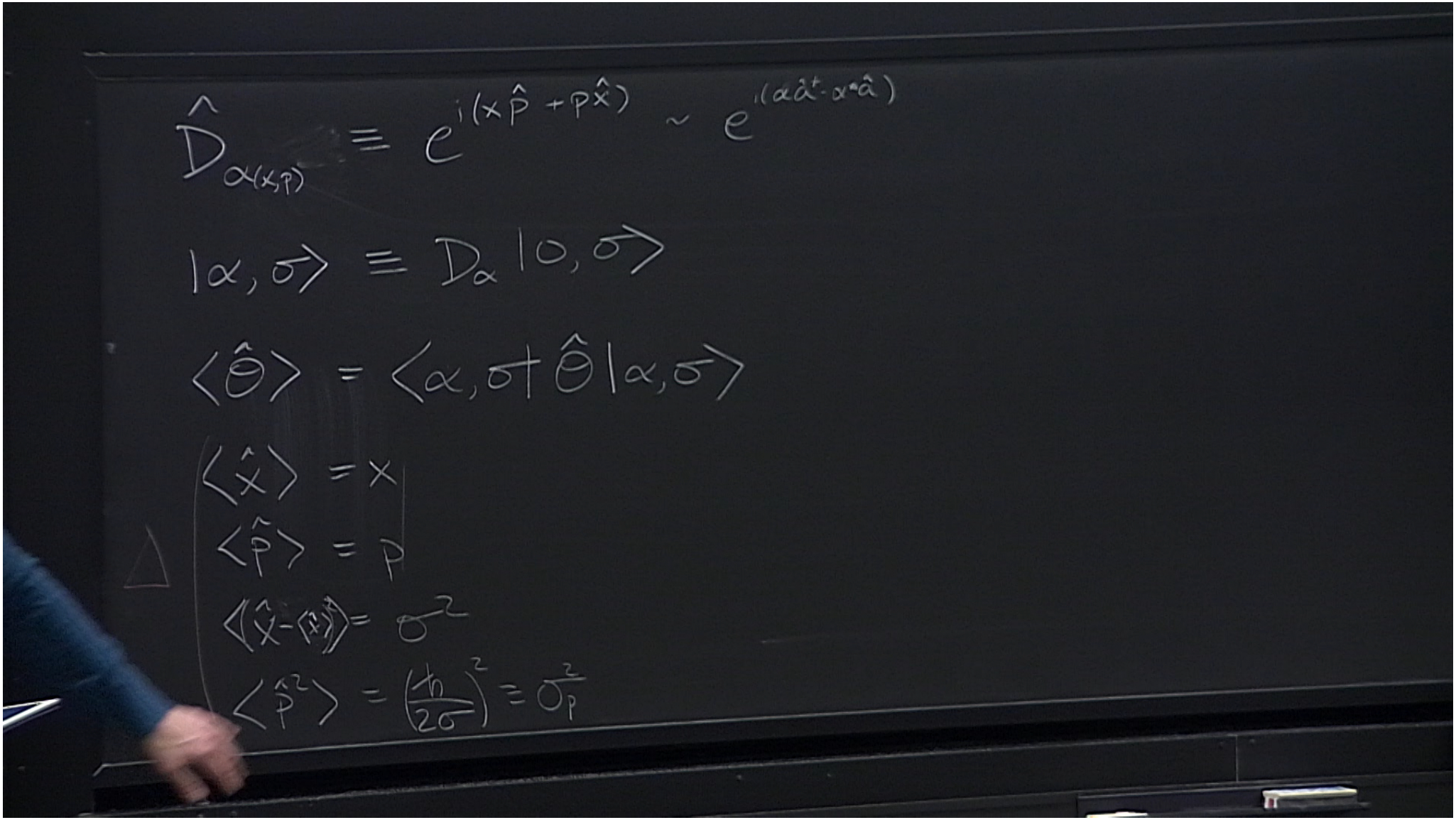
$$|\alpha, \sigma\rangle \equiv D_\alpha |0, \sigma\rangle$$

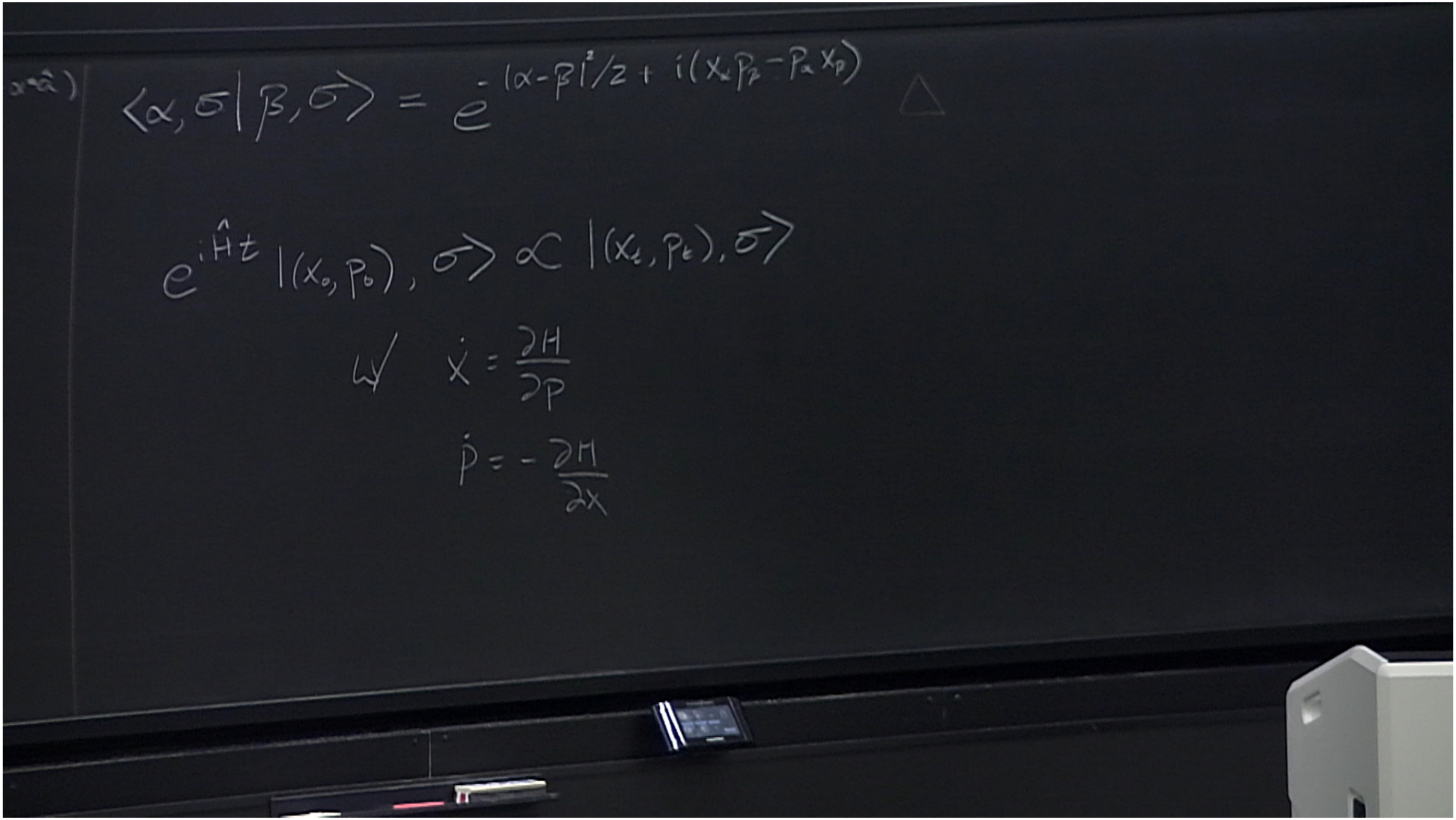
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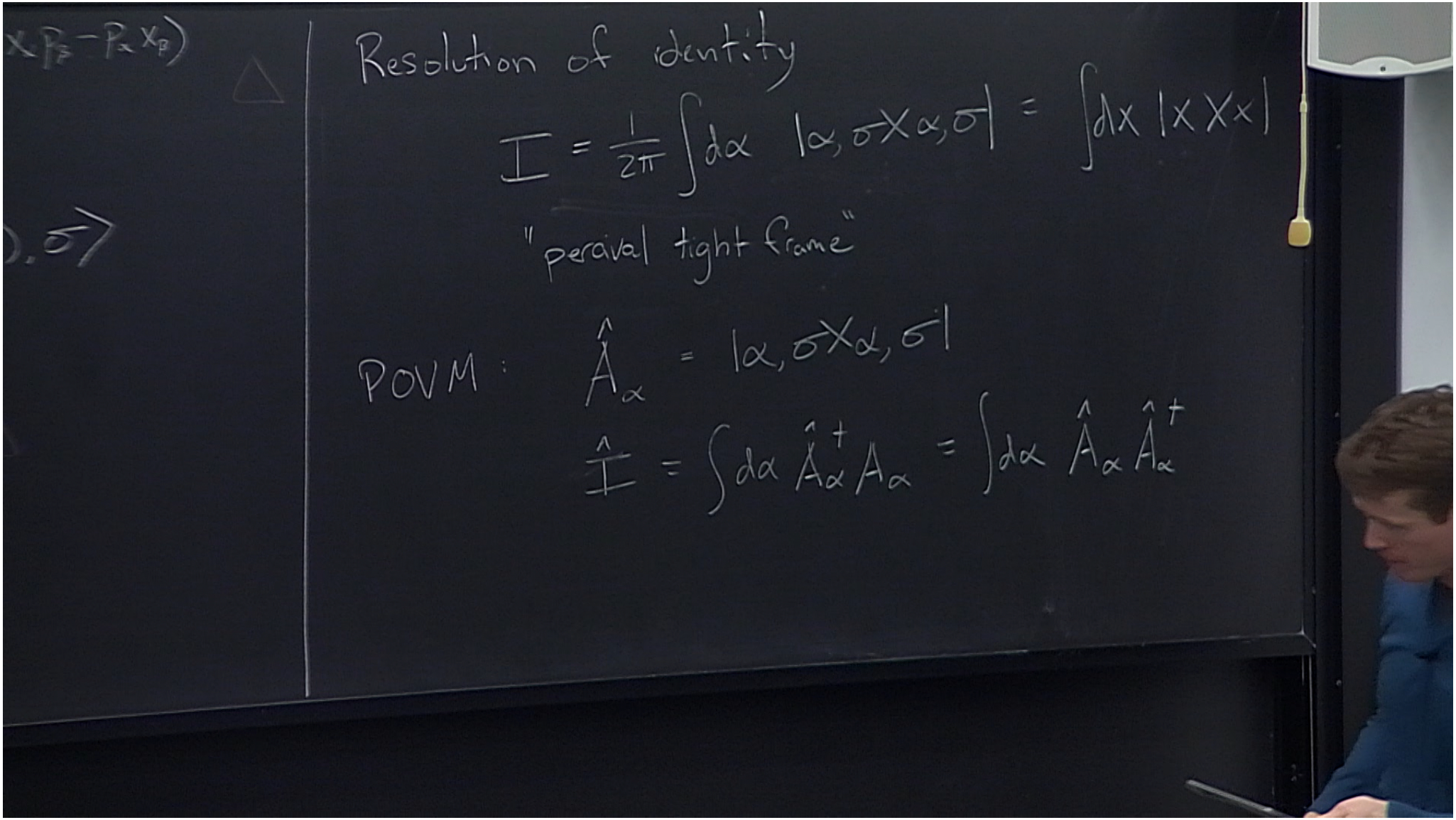
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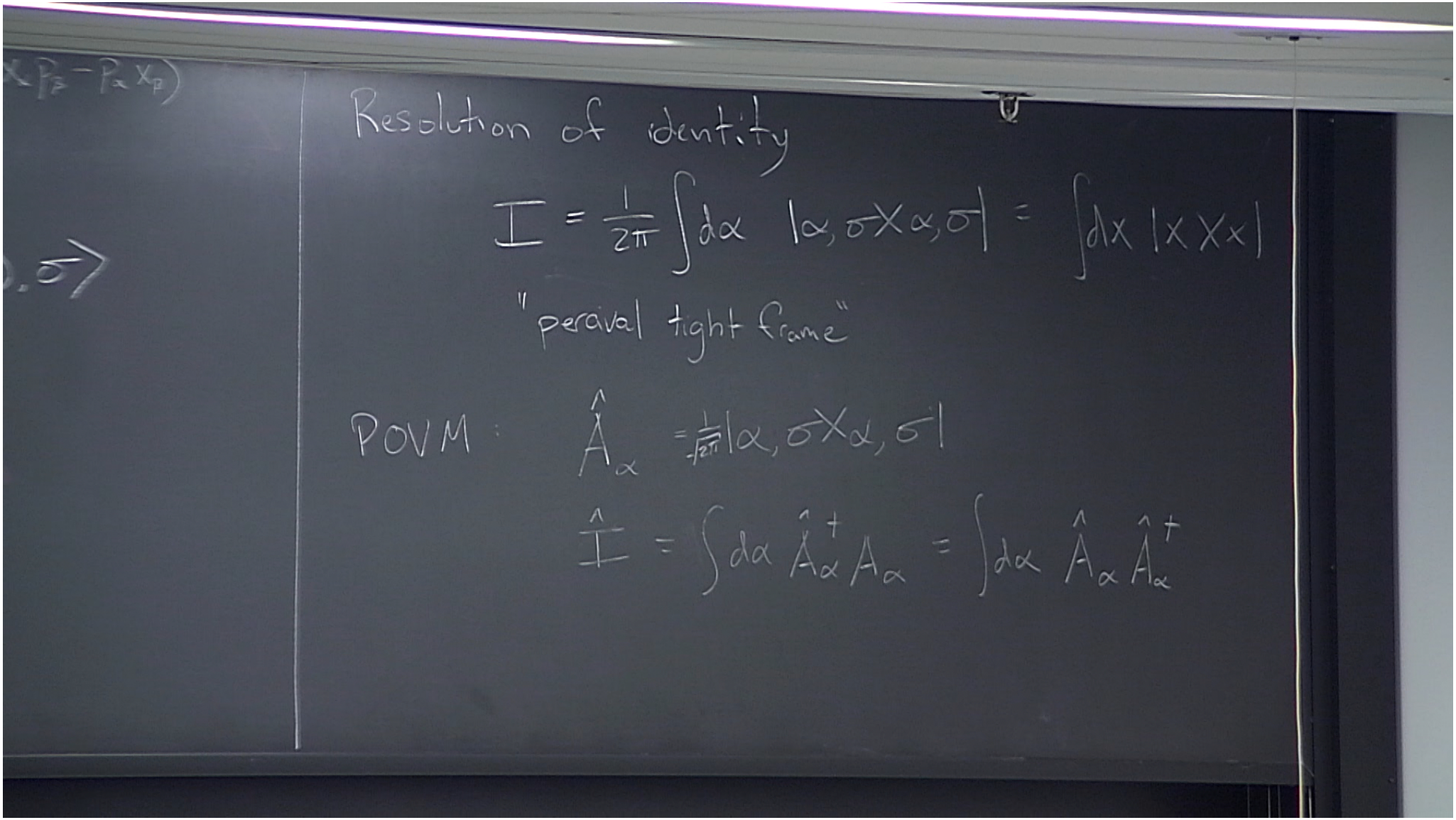
$$\langle\hat{\theta}\rangle = \langle\alpha, \sigma|\hat{\theta}|\alpha, \sigma\rangle$$

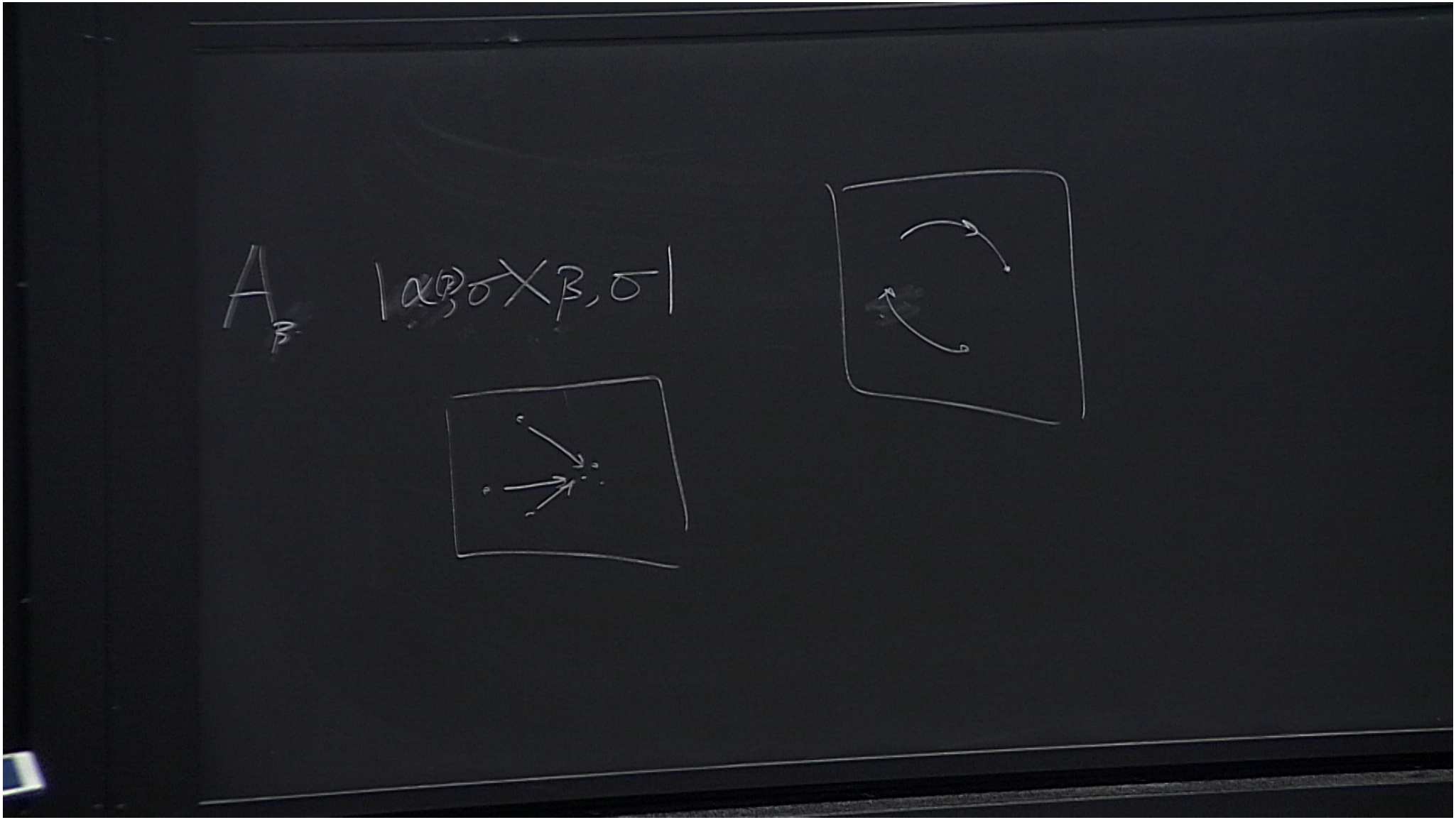
$$\Delta \begin{cases} \langle\hat{x}\rangle = x \\ \langle\hat{p}\rangle = p \\ \langle\hat{x}^2\rangle = \sigma^2 \\ \langle\hat{p}^2\rangle = \left(\frac{\hbar}{2\sigma}\right)^2 \equiv \sigma_p^2 \end{cases}$$


$\hat{D}_{\alpha(x,p)} \equiv e^{i(x\hat{p} + p\hat{x})} \sim e^{i(\alpha\hat{a}^\dagger - \alpha^*\hat{a})}$
$|\alpha, \sigma\rangle \equiv D_\alpha |0, \sigma\rangle$
$\langle\hat{\theta}\rangle = \langle\alpha, \sigma|\hat{\theta}|\alpha, \sigma\rangle$
$\langle\hat{x}\rangle = x$
$\langle\hat{p}\rangle = p$
$\langle(\hat{x} - \langle\hat{x}\rangle)^2\rangle = \sigma^2$
$\langle\hat{p}^2\rangle = \left(\frac{\hbar}{2\sigma}\right)^2 \equiv \sigma_p^2$


$\langle \alpha, \sigma | \beta, \sigma \rangle = e^{-|\alpha-\beta|^2/2 + i(x_\alpha p_\beta - p_\alpha x_\beta)}$
$e^{i\hat{H}t} |(x_0, p_0), \sigma\rangle \propto |(x_t, p_t), \sigma\rangle$
$\dot{x} = \frac{\partial H}{\partial p}$
$\dot{p} = -\frac{\partial H}{\partial x}$


Resolution of identity
$I = \frac{1}{2\pi}\int d\alpha\ |\alpha,\sigma\rangle\langle\alpha,\sigma| = \int dx\ |x\rangle\langle x|$
"peraival tight frame"
POVM: $\hat{A}_\alpha = |\alpha,\sigma\rangle\langle\alpha,\sigma|$
$\hat{I} = \int d\alpha\ \hat{A}_\alpha^\dagger A_\alpha = \int d\alpha\ \hat{A}_\alpha \hat{A}_\alpha^\dagger$


Resolution of identity
$\mathbb{I} = \frac{1}{2\pi}\int d\alpha\ |\alpha,\sigma\rangle\langle\alpha,\sigma| = \int dx\ |x\rangle\langle x|$
"peraval tight frame"
POVM: $\hat{A}_\alpha = \frac{1}{\sqrt{2\pi}}|\alpha,\sigma\rangle\langle\alpha,\sigma|$
$\hat{\mathbb{I}} = \int d\alpha\ \hat{A}_\alpha^\dagger A_\alpha = \int d\alpha\ \hat{A}_\alpha \hat{A}_\alpha^\dagger$



Wigner function

Husimi Q function:

$$Q_\rho^{(\sigma)}(\alpha) \equiv \frac{1}{2\pi}\langle \alpha,\sigma|\rho|\alpha,\sigma\rangle = \mathrm{Tr}[\hat{A}_\alpha \rho]$$

Glauber P function

$$\rho = \int d\alpha\, P_\rho^{(\sigma)}(\alpha) \cdot |\alpha,\sigma\rangle\langle\alpha,\sigma|$$

Both $Q(\alpha)$ and $P(\alpha)$

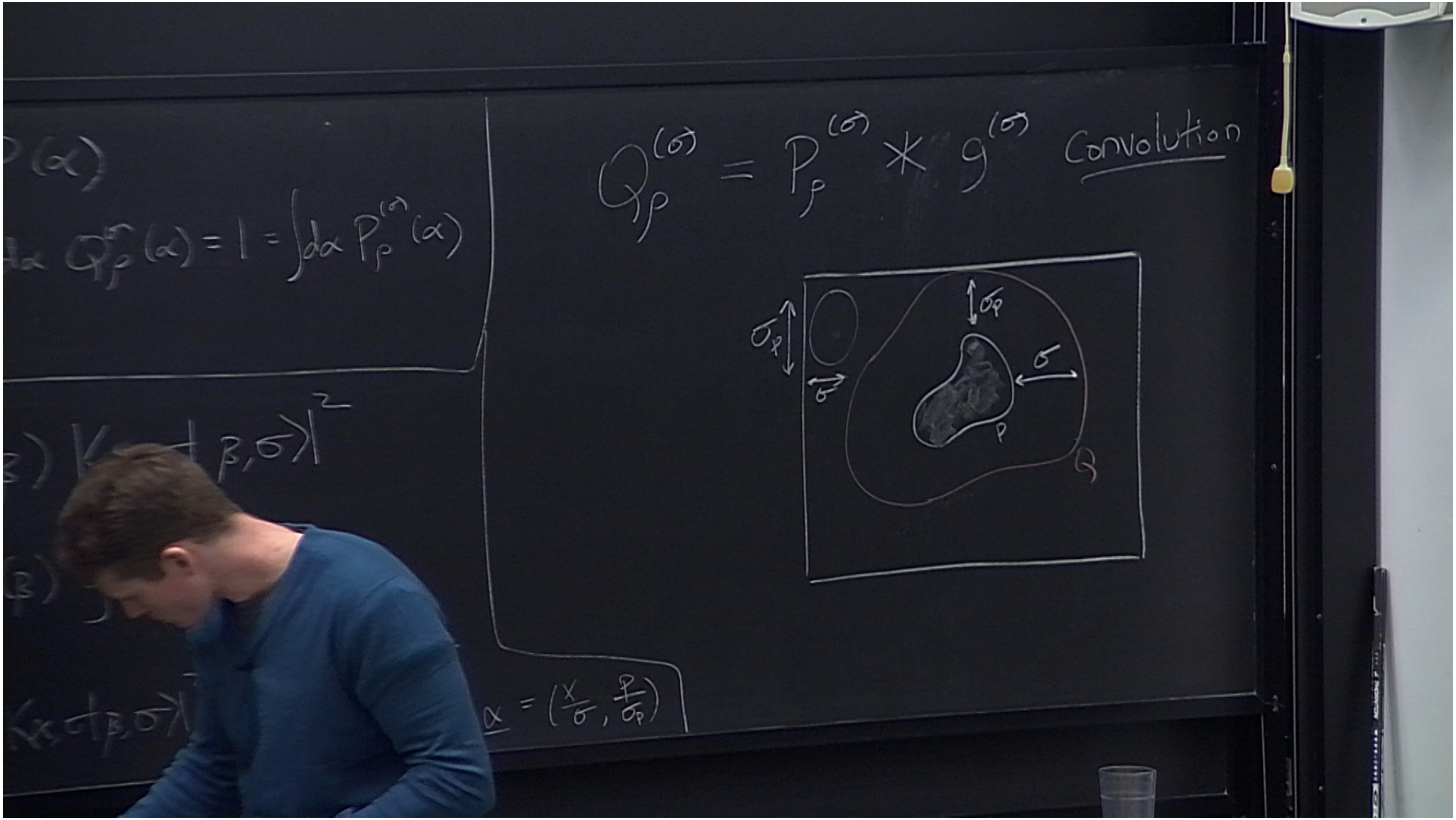
normalized : $\int d\alpha\, Q_\rho^{(\sigma)}(\alpha) = 1 = \int d\alpha\, P_\rho^{(\sigma)}(\alpha)$

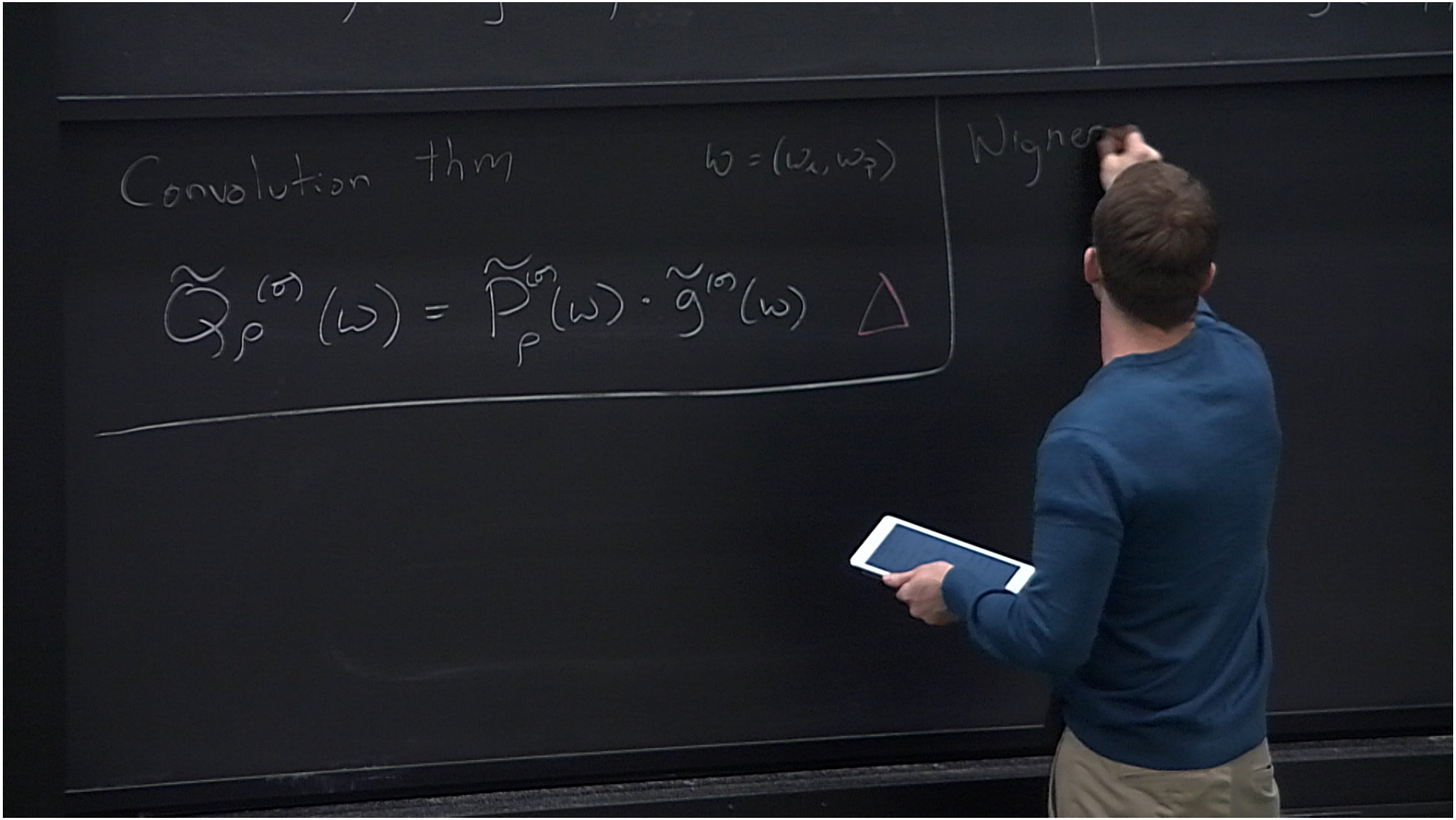
positive

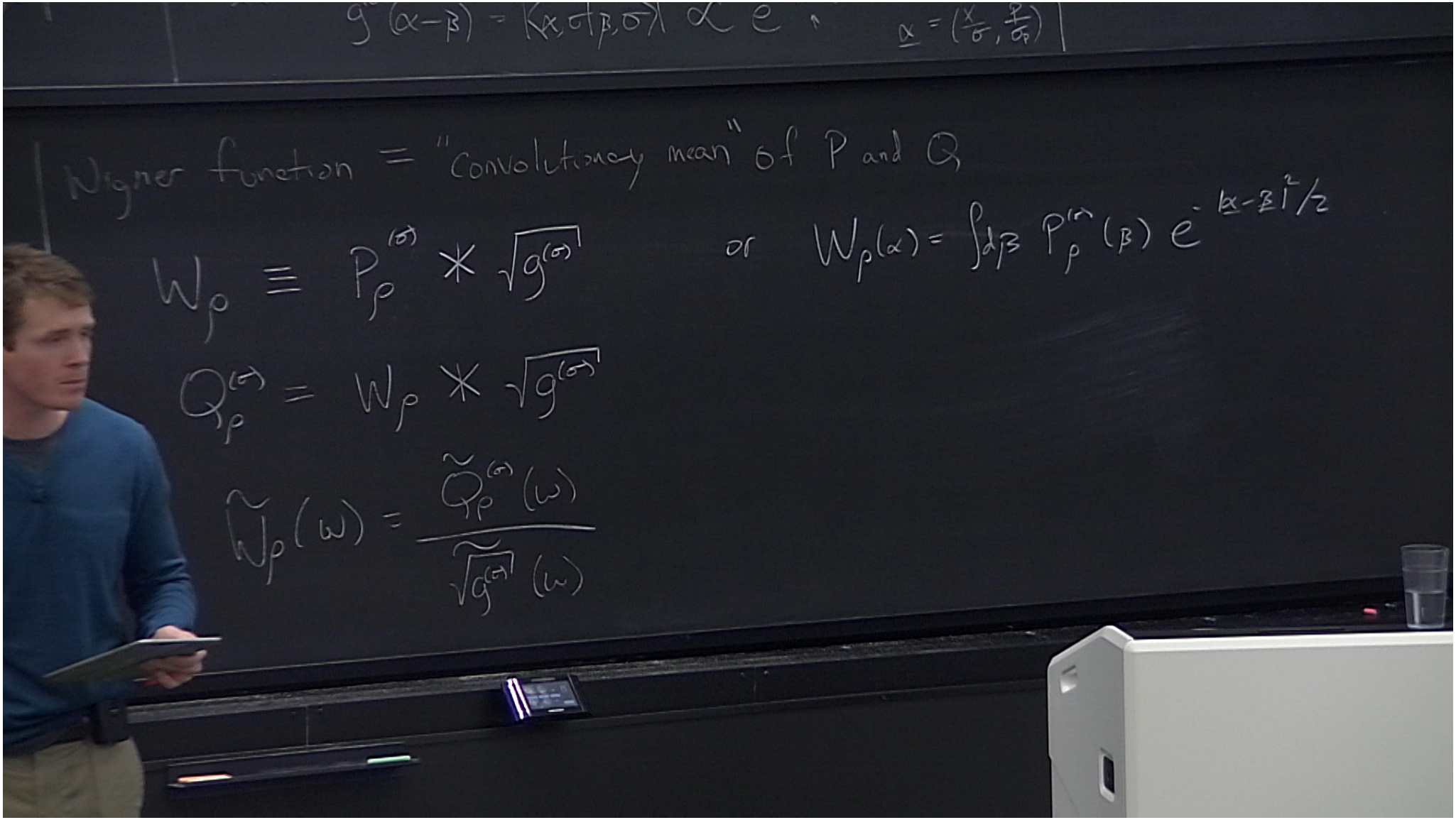
$$Q_\rho^{(\sigma)} = \int d\beta\, P_\rho^{(\sigma)}(\beta)\, |\langle \alpha, \sigma | \beta, \sigma \rangle|^2$$

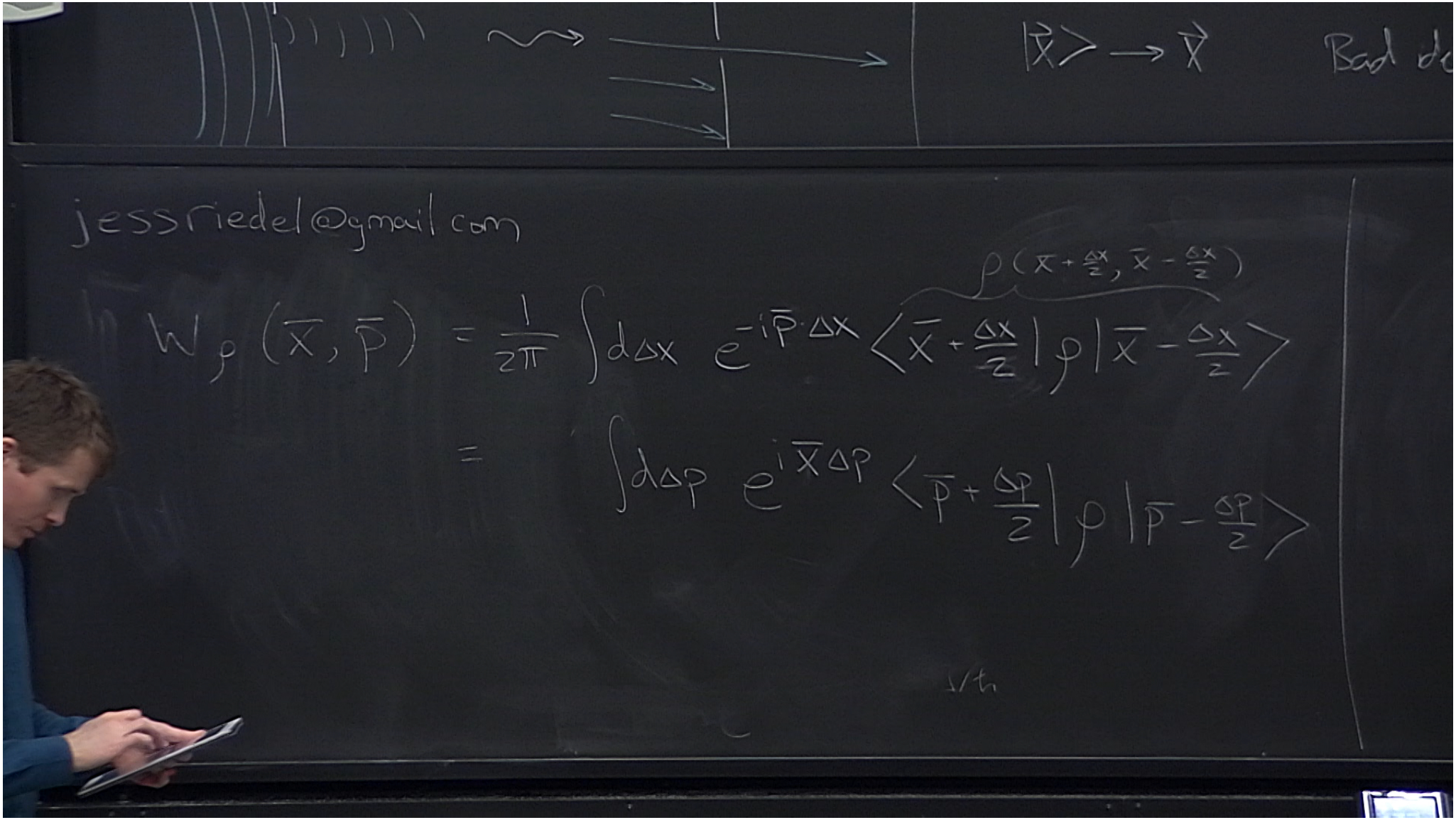
$$= \int d\beta\, P_\rho^{(\sigma)}(\beta)\, g^{(\sigma)}(\alpha - \beta)$$

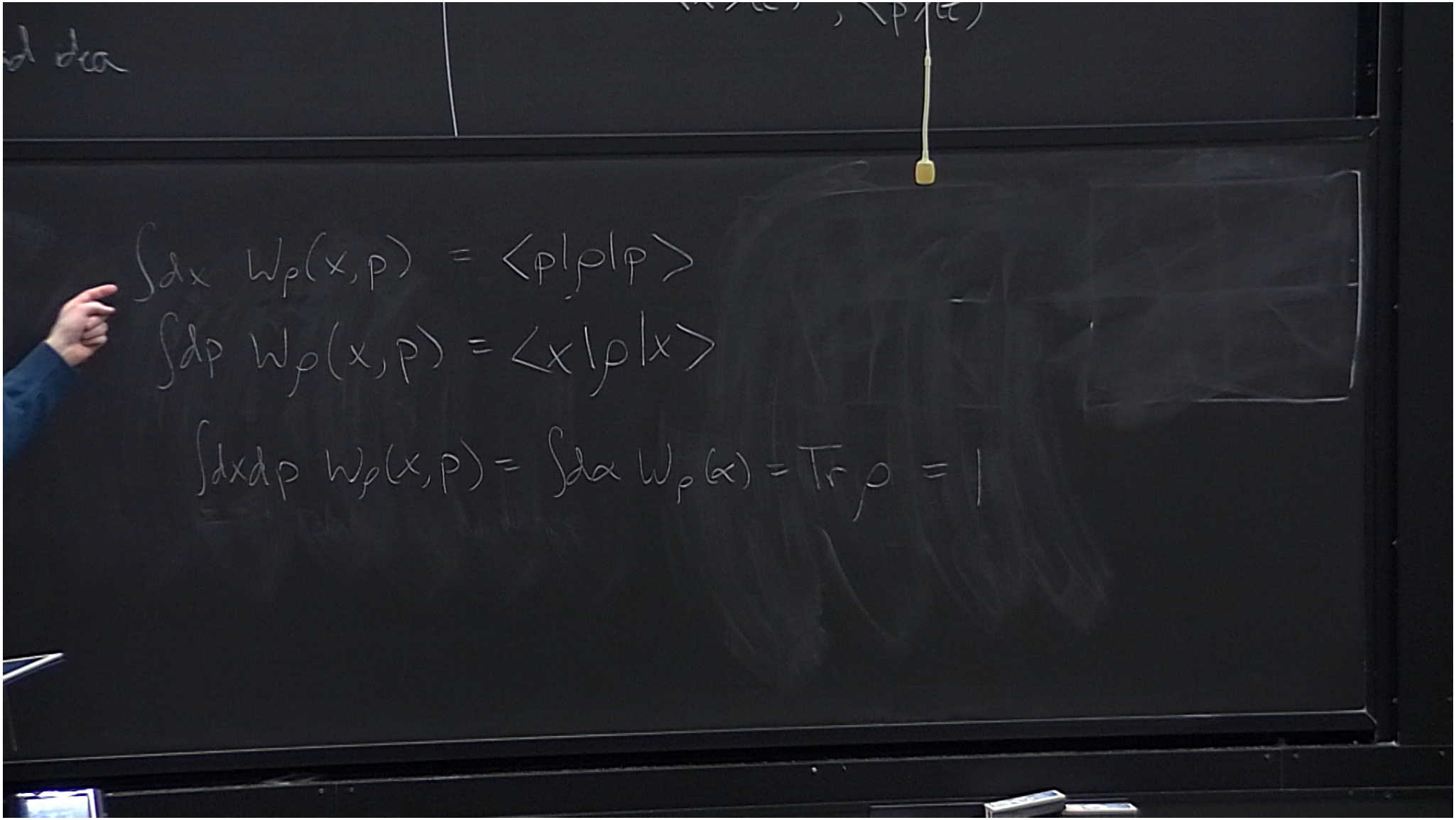
where $g^{(\sigma)}(\alpha - \beta) = |\langle \alpha, \sigma | \beta, \sigma \rangle|^2 \propto e^{-|\alpha - \beta|^2}$ $\quad \underline{\alpha} = \left(\frac{x}{\sigma}, \frac{p}{\sigma_p}\right)$


$Q_p^{(\sigma)} = P_p^{(\sigma)} * g^{(\sigma)}$ Convolution
σ_p
σ_p
σ
σ
P
Q


Convolution thm
$\omega = (\omega_1, \omega_2)$
$\tilde{Q}_p^{(\sigma)}(\omega) = \tilde{P}_p^{(\sigma)}(\omega) \cdot \tilde{g}^{(\sigma)}(\omega)$ Δ


Wigner function = "convolutionary mean" of P and Q
$W_\rho \equiv P_\rho^{(\sigma)} * \sqrt{g^{(\sigma)}}$
or $W_\rho(\alpha) = \int d\beta\, P_\rho^{(\sigma)}(\beta)\, e^{-|\alpha-\beta|^2/2}$
$Q_\rho^{(\sigma)} = W_\rho * \sqrt{g^{(\sigma)}}$
$\tilde{W}_\rho(\omega) = \frac{\tilde{Q}_\rho^{(\sigma)}(\omega)}{\sqrt{\tilde{g}^{(\sigma)}}(\omega)}$


$|\vec{x}\rangle \to \vec{x}$
Bad d
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$W_\rho(\bar{x},\bar{p}) = \frac{1}{2\pi}\int d\Delta x\; e^{-i\bar{p}\cdot\Delta x}\underbrace{\langle \bar{x}+\frac{\Delta x}{2}|\rho|\bar{x}-\frac{\Delta x}{2}\rangle}_{\rho(\bar{x}+\frac{\Delta x}{2},\,\bar{x}-\frac{\Delta x}{2})}$
$= \int d\Delta p\; e^{i\bar{x}\Delta p}\langle \bar{p}+\frac{\Delta p}{2}|\rho|\bar{p}-\frac{\Delta p}{2}\rangle$
1/h

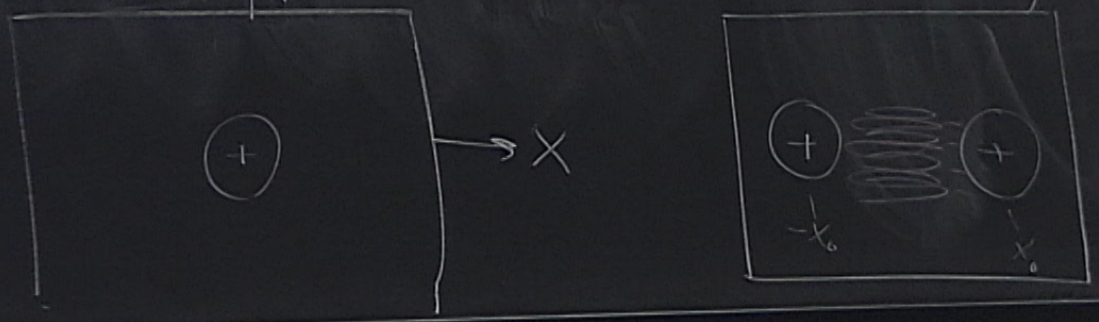

$\int dx\ W_\rho(x,p) = \langle p|\rho|p\rangle$
$\int dp\ W_\rho(x,p) = \langle x|\rho|x\rangle$
$\int dx dp\ W_\rho(x,p) = \int d\alpha\ W_\rho(\alpha) = \mathrm{Tr}\,\rho = 1$

$\langle \vec{r} | \psi \rangle \in \mathbb{R}^3 \to \Gamma = \Gamma^{\mathbb{R}^3}$

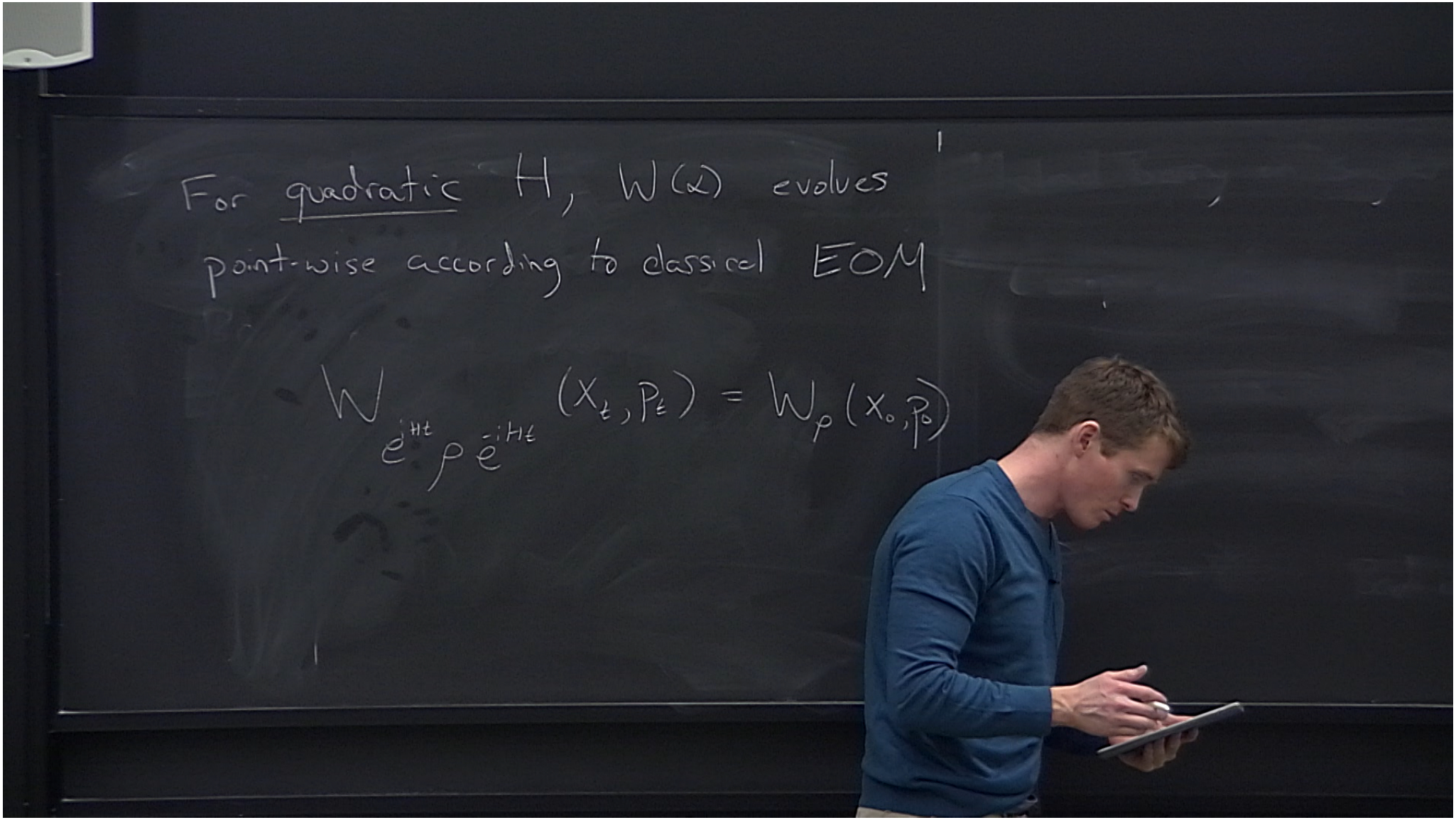
$$\int dx\, W_\rho(x,p) = \langle p|\rho|p\rangle$$

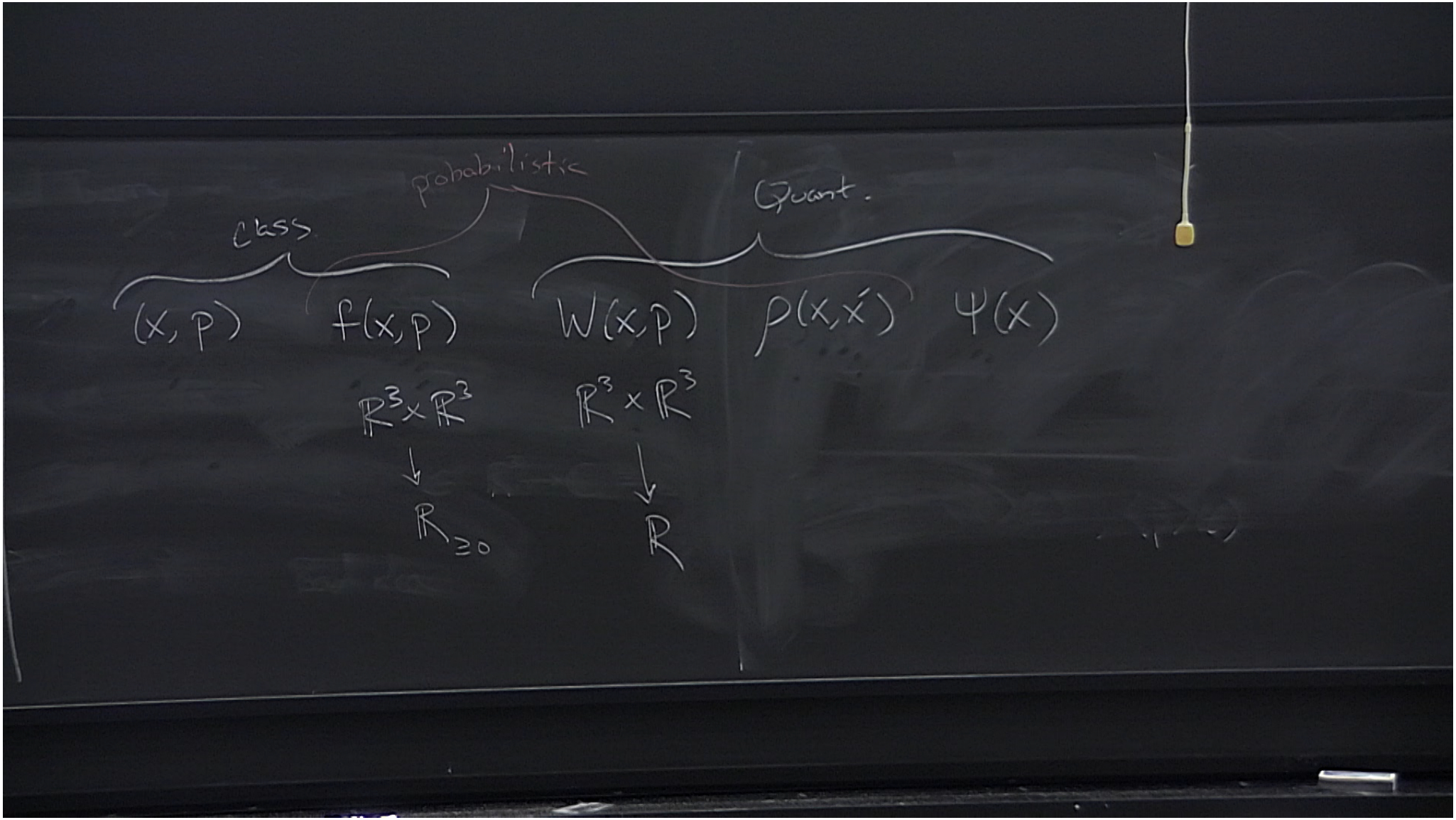
$$\int dp\, W_\rho(x,p) = \langle x|\rho|x\rangle$$

$$\int dx\,dp\, W_\rho(x,p) = \int d\alpha\, W_\rho(\alpha) = \mathrm{Tr}\,\rho = 1$$



$$|\psi\rangle = |(x_0,0),\sigma\rangle + |(-x_0,0),\sigma\rangle$$


For quadratic H, W(x) evolves
point-wise according to classical EOM
$W_{e^{iHt}\rho e^{-iHt}}(x_t, p_t) = W_\rho(x_0, p_0)$


probabilistic
Quant.
class
(x, p)
f(x, p)
W(x, p)
ρ(x, x')
ψ(x)
$\mathbb{R}^3 \times \mathbb{R}^3$
$\mathbb{R}^3 \times \mathbb{R}^3$
$\mathbb{R}_{\geq 0}$
$\mathbb{R}$