Title: PSI 2016/2017 Foundations of Quantum Mechanics (Review) - Lecture 9

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Abstract:

Quantum Foundations: Lecture 9

4) Ontological Models

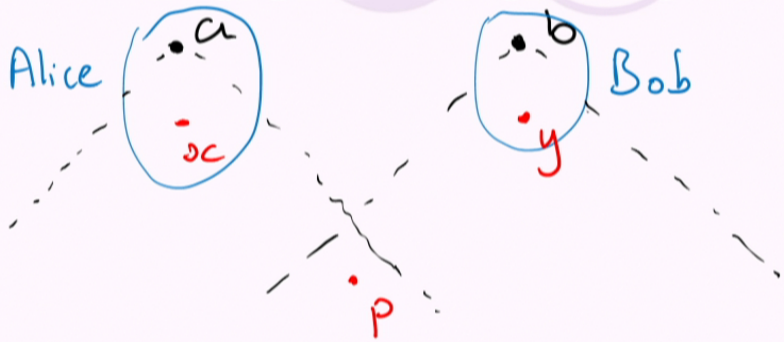
Today's Lecture

4. Ontological Models

7. Bell's Theorem
8. The Colbeck-Renner Theorem

4.7) Bell’s Theorem

Basic Framework



- We are interested in experiments of the form shown on the left with operational predictions

$$\mathrm{Prob}(a,b|x,y,P)$$

- As before, we assume there may be additional parameters $\lambda \in \Lambda$ and so

$$\mathrm{Prob}(a,b|x,y,P) = \int_\Lambda d\lambda \, \Pr(a,b|x,y,\lambda,P) \Pr(\lambda|x,y,P)$$

- We impose the measurement independence assumption: $\Pr(\lambda|x,y,P) = \Pr(\lambda|P)$

$$\Rightarrow \mathrm{Prob}(a,b|x,y,P) = \int_\Lambda d\lambda \, \Pr(a,b|x,y,\lambda,P) \Pr(\lambda|P)$$

- Note: People do not usually impose λ-mediation, but in any case there is often a single fixed P, so $\mathrm{Prob}(a,b|x,y) = \int_\Lambda d\lambda \, \Pr(a,b|x,y,\lambda) \Pr(\lambda)$

Two Bell's Theorems

- The mathematics of Bell's theorem is fairly straightforward, but there is a lot of disagreement about its assumptions and what it means.
- Howard Wiseman has identified two common takes on Bell's theorem - J. Phys. A: Math. Theor. 47:424001 (2014)
  - "Operationalist":

    Assumptions: outcome determinism and parameter independence.

    Conclusion: nature is not deterministic – "unperformed measurements have no results"
  - "Realist":

    Assumption: Local causality.

    Conclusion: nature is nonlocal.

“Operationalist” Assumptions

- For an operationalist locality = no superluminal signaling.
  - This holds at the operational level: $\mathrm{Prob}(b|x,y) = \mathrm{Prob}(b|y)$.
  - However, suppose $\Pr(b|x,y,\lambda) \neq \Pr(b|y,\lambda)$.
  - Then, if Bob were somehow able to observe the exact ontic state λ, Alice could send him a superluminal message.
  - Therefore, even though we can't directly observe them, nature is sending superluminal signals in such a theory.
- To rule this out, we make the assumption of parameter independence:

$$\Pr(b|x,y,\lambda) = \Pr(b|y,\lambda) \quad \text{and} \quad \Pr(a|x,y,\lambda) = \Pr(a|x,\lambda).$$

- This is not enough to prove Bell's theorem, so there is an additional assumption of outcome determinism:

$$\Pr(a,b|x,y,\lambda) = 0 \ \text{or} \ 1$$

“Operationalist” Bell’s Theorem

- By outcome determinism, each λ specifies two functions:

$$a = f(x, y) \qquad\qquad b = g(x, y)$$

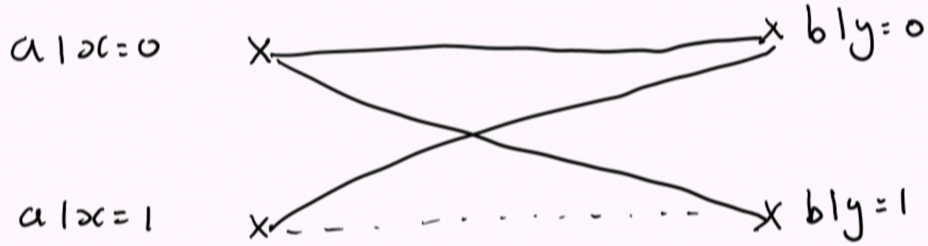
- By parameter independence, actually:

$$a = f(x) \qquad\qquad b = g(y)$$

- As far as the predictions are concerned, specifying λ is equivalent to specifying what a will be for each x and independently specifying what b will be for each y.
- Assume a and b take values ± 1, and x and y take values $0{,}1$. We have to specify four numbers:

$$(a|x = 0), (a|x = 1), (b|y = 0), (b|y = 1)$$

A frustrated network



— = we want these to be equal

-... = we want these to be different.

⊙ On any assignment of values ± 1 to the nodes, at least one link is satisfied and at most three are satisfied.

$$1 \leq \Pr(a=b|0,0,\lambda) + \Pr(a=b|0,1,\lambda) + \Pr(a=b|1,0,\lambda) + \Pr(a\neq b|1,1,\lambda) \leq 3$$

where $\Pr(a=b|i,j,\lambda) = \Pr(a=b|x=i, y=j, \lambda)$

⊙ If we average this w.r.t $\Pr(\lambda)$ we will get

$$1 \leq \Pr(a=b|0,0) + \Pr(a=b|0,1) + \Pr(a=b|1,0) + \Pr(a\neq b|1,1) \leq 3 \qquad (1)$$

Another frustrated network

— = we want these to be equal

-·-·- = we want these to be different.

⊙ By the same reasoning:

$$1 \leq Pr(a \neq b|0,0) + Pr(a \neq b|0,1) + Pr(a \neq b|1,0) + Pr(a = b|1,1) \leq 3$$

$$\Rightarrow -3 \leq -Pr(a \neq b|0,0) - Pr(a \neq b|0,1) - Pr(a \neq b|1,0) - Pr(a = b|1,1) \leq -1 \quad (2)$$

⊙ Now, since $a,b = \pm 1$ $\quad E(ab|i,j) = Pr(a=b|i,j) - Pr(a \neq b|i,j)$

$\therefore$ (1) + (2) gives $\boxed{-2 \leq E(ab|0,0) + E(ab|0,1) + E(ab|1,0) - E(ab|1,1) \leq 2}$

This is the Bell-CHSH inequality

Quantum Violation

- Suppose Alice and Bob share a singlet state

$$|\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B\right)$$

- Let $\sigma_{\vec{n}} = n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3$ with $\vec{n}$ a unit vector

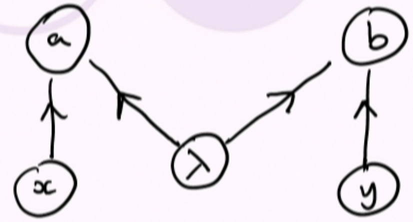
Then $_{AB}\langle\Psi^-|\sigma_{\vec{n}} \otimes \sigma_{\vec{m}}|\Psi^-\rangle = -\vec{n}\cdot\vec{m}$

- If we choose

$a\|x=0$	outome of σ_3	measurement
$a\|x=1$	" " σ_1	"
$b\|y=0$	" " $-\frac{1}{\sqrt{2}}(\sigma_1+\sigma_3)$	"
$b\|y=1$	" " $\frac{1}{\sqrt{2}}(\sigma_1-\sigma_3)$	"

Then we get $E(ab|0,0) + E(ab|0,1) + E(ab|1,0) - E(ab|1,1) = 4\frac{1}{\sqrt{2}} = 2\sqrt{2} > 2$

"Realist" Assumptions



$\rightarrow$ = can be a direct cause of

- Let λ be a complete description of all the physical properties in a region that screens off (x, a) from (y, b) i.e. any non future-directed timelike curve from x or a to y or b passes through the region.
- Then we have a causal structure as depicted on the right:

$$Pr(a, b \mid x, y, \lambda) = Pr(a \mid x, \lambda)\, Pr(b \mid y, \lambda)$$

- This is Bell's local causality assumption. (Realist proof is Hwk problem)

Connection between the assumptions

- Parameter independence is:

$$\Pr(b|x,y,\lambda) = \Pr(b|y,\lambda) \quad \text{and} \quad \Pr(a|x,y,\lambda) = \Pr(a|x,\lambda)$$

- If we add an extra assumption called outcome independence:

$$\Pr(b|a,x,y,\lambda) = \Pr(b|x,y,\lambda) \quad \text{and} \quad \Pr(a|b,x,y,\lambda) = \Pr(a|x,y,\lambda)$$

- Then together these are equivalent to local causality:

$$\begin{aligned}\Pr(a,b|x,y,\lambda) &= \Pr(b|a,x,y,\lambda)\Pr(a|x,y,\lambda) \\ &= \Pr(b|x,y,\lambda)\Pr(a|x,y,\lambda) \quad \text{by outcome independence} \\ &= \Pr(b|y,\lambda)\Pr(a|x,\lambda) \quad \text{by parameter independence}\end{aligned}$$

- Outcome determinism implies outcome independence:
  - If both a and b are just functions of x, y and λ, then learning a doesn't tell you anything new about b if you already know x, y and λ.

“Operationalists” vs. “Realists”

- I tend to side with the “realists” in this debate, but it is fairly clear that the debate is unresolvable.

- Since both “operationalists” and “realists” believe that locality requires parameter independence, the real question we should be asking is:

  - What can we prove from parameter independence alone?

  - This leads us to…

4.8) The Colbeck-Renner Theorem

See R. Colbeck, R. Renner, arXiv:1208.4123 (2012) (accessible account)

R. Colbeck, R. Renner, Nature Communications 2, 411 (2011) (original paper)

Maximally Entangled Special Case

- Theorem: Let $|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |0\rangle_A + |1\rangle_A \otimes |1\rangle_A\right)$ and let x and y vary over all possible orthonormal basis measurements. Any model satisfying parameter independence must also satisfy

$$\Pr(a \mid x, \lambda) = \Pr(a \mid x) = \frac{1}{2} \quad \forall \lambda \text{ s.t. } \Pr(\lambda) > 0$$

(up to measure zero sets)

- "Operationalists" say:

  See, not only does a local model have to be nondeterministic, it has to make the exact same predictions as quantum mechanics even if you condition on λ, at least in this case. It might as well just be quantum mechanics.

- "Realists" say:

  We expect that any reasonable theory underlying quantum mechanics will violate parameter independence, not just local causality.

Chained Bell Measurements

- We will consider a case where Alice and Bob each have a choice of N orthonormal basis measurements
- For convenience we let

$$x = 0, 2, 4, \ldots, 2N-2 \quad \text{and } a,b = 0,1$$
$$y = 1, 3, 5, \quad , 2N-1$$

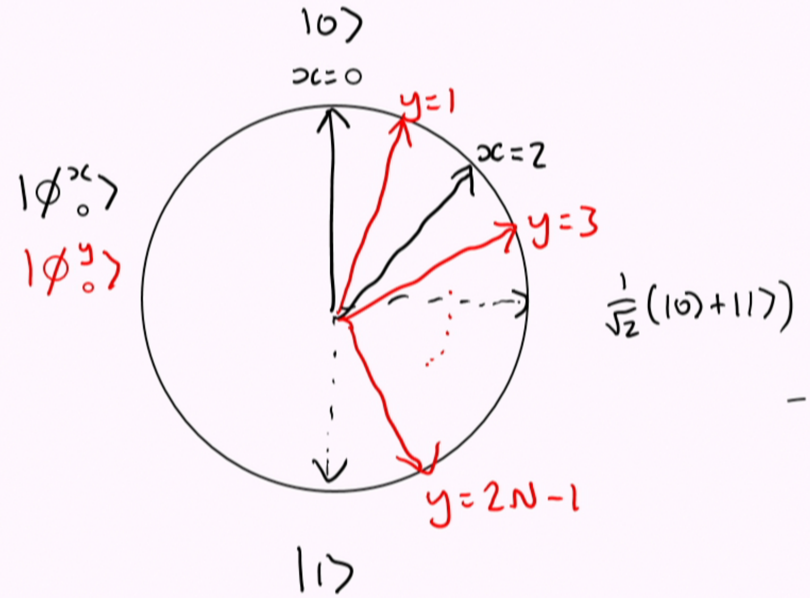
- Let:

$$|\phi^j_k\rangle = \cos\left(\frac{\theta^j_k}{2}\right)|0\rangle + \sin\left(\frac{\theta^j_k}{2}\right)|1\rangle$$

where $\theta^j_k = \left(\frac{j}{2N} + k\right)\pi$

$\langle \phi^x_0 | \phi^x_1 \rangle = 0$ for all x (here $k = a$)

$\langle \phi^y_0 | \phi^y_1 \rangle = 0$ for all y (here $k = b$)



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Variational Distance

- Let $p(a)$, $q(a)$ be two probability distributions over the same finite variable a.
- The variational distance between $p(a)$ and $q(a)$ is

$$D(p,q) = \frac{1}{2}\sum_a |p(a) - q(a)|$$

It is a metric on probability distributions

- Theorem: Let $Pr(a,a')$ be a joint probability distribution such that

$$p(a) = \sum_{a'} Pr(a,a') \qquad q(a') = \sum_a Pr(a,a')$$

Then $D(p,q) \leq Pr(a \neq a')$

Proof of Main Theorem

- Consider again our correlation measure, except this time conditioned on λ.

$$I_N(\lambda) = \Pr(a=b \mid x=0, y=2N-1, \lambda) + \sum_{|x-y|=1} \Pr(a \neq b \mid x, y, \lambda)$$

- If we define $\tilde{a}$ s.t. $a=0 \Leftrightarrow \tilde{a}=1$ and $a=1 \Leftrightarrow \tilde{a}=0$ then

$$I_N(\lambda) = \Pr(\tilde{a} \neq b \mid x=0, y=2N-1, \lambda) + \sum_{|x-y|=1} \Pr(a \neq b \mid x, y, \lambda)$$

- Then, by our variational distance theorem:

$$I_N(\lambda) \geq D\left[\Pr(\tilde{a} \mid x=0, y=2N-1, \lambda), \Pr(b \mid x=0, y=2N-1, \lambda)\right] + \sum_{|x-y|=1} D\left[\Pr(a \mid x, y, \lambda), \Pr(b \mid x, y, \lambda)\right]$$

- By parameter independence:

$$I_N(\lambda) \geq D\left[\Pr(\tilde{a} \mid x=0, \lambda), \Pr(b \mid y=2N-1, \lambda)\right] + \sum_{|x-y|=1} D\left[\Pr(a \mid x, \lambda), \Pr(b \mid y, \lambda)\right]$$

General Statement of the Colbeck-Renner Theorem

- We have proven a theorem for the maximally entangled state $|\Phi^+\rangle_{AB}$, but it would be nice to have a similar result for any state $|\psi\rangle_{AB}$.
- Note that, for any state $|\psi\rangle_{AB}$, there exist orthonormal bases $\{|\chi_j\rangle_A\}$ and $\{|\eta_j\rangle_B\}$ such that $|\psi\rangle_{AB} = \sum_j \sqrt{p_j}\, |\chi_j\rangle_A |\eta_j\rangle_B$

  This is called the Schmidt decomposition.
- Colbeck Renner Theorem: For any state $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ there exists a state $|\Gamma\rangle_{A'B'} \in \mathcal{H}_{A'B'}$ such that a parameter independent model for all local POVMS $\{E_j^{AA'}\}$ and $\{F_k^{BB'}\}$

$$\Pr(|\chi_j\rangle_A | M, \lambda) = \mathrm{Prob}(|\chi_j\rangle | M) \quad \text{for the basis measurement } \{|\chi_j\rangle_A\}$$