Title: PSI 2016/2017 Foundations of Quantum Mechanics (Review) - Lecture 8

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Abstract:

# Quantum Foundations: Lecture 8

## 4) Ontological Models

# Today's Lecture

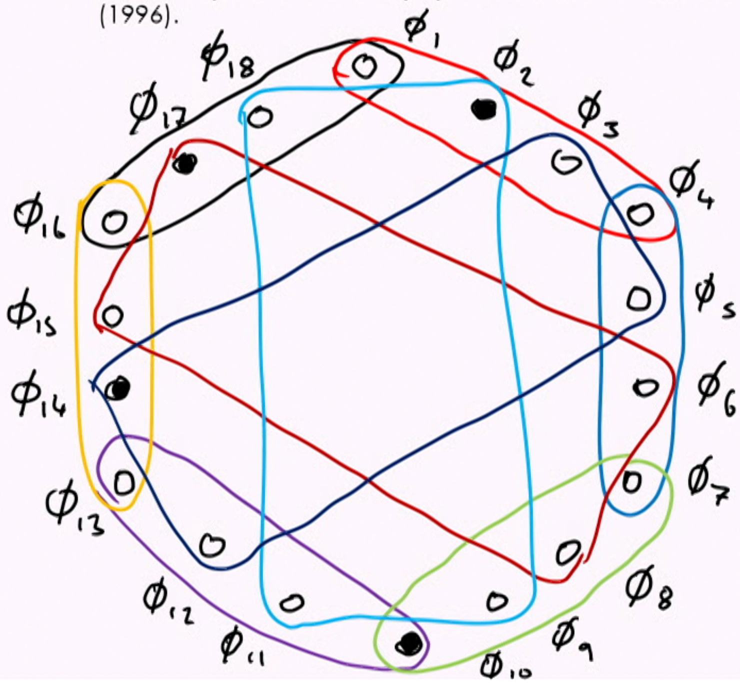
4. Ontological Models
    5. Contextuality
    6. Ψ-ontology

# Review of KS Contextuality

- An ontological model is KS noncontextual if:
  - The measurements are all projective measurements.
  - The model is outcome deterministic: $\Pr(\Pi|M,\lambda) = 0 \text{ or } 1$.
  - The model is measurement noncontextual: $\Pr(\Pi|M,\lambda) = \Pr(\Pi|\lambda)$.
- Equivalently:
  - Every projector gets assigned a value 0 or 1.
  - For every set $\{\Pi_j\}$ of orthogonal projectors s.t. $\sum_j \Pi_j = I$, exactly one projector gets assigned the value 1, the rest are assigned value 0.

# Noncontextuality Inequalities

- People sometimes want to detect contextuality using inequalities like we do for nonlocality in Bell's theorem.
- Example: 18 ray proof in 4 dimensions – A. Cabello, J. Estebaranz, G. Garcia-Alcaine, Phys. Lett. A 212:183 (1996).




- Each grouped set of vertices is a basis, one vector should get value 1, the rest 0.
- But however you try to do this there is always one basis left over that cannot be filled.
- People then say that, in a noncontextual theory

$$\sum_j \Pr(\phi_j|\lambda) \leq 4$$

- However, this is wrong: any theory must predict a valid probability distribution for every measurement $\Rightarrow$ no noncontextual model exists.

# Noncontextual Sets

- We can make sense of noncontextuality inequalities in the following way.
- Let $\mathcal{M} = \{M_1, M_2, \dots, M_n\}$ be a finite set of orthonormal bases.
- If $|\phi\rangle$ is an outcome in $M \in \mathcal{M}$, define

$$\Gamma_\phi^M = \{\lambda \mid Pr(\phi \mid M, \lambda) = 1\}$$

- Define the noncontextual set for $|\phi\rangle$ as

$$\Gamma_\phi = \bigcap_{\{M \in \mathcal{M} \mid |\phi\rangle \in M\}} \Gamma_\phi^M$$

This is the set of ontic states that always assign $|\phi\rangle\langle\phi|$ probability 1 regardless of the basis it appears in

i.e. the set of ontic states that give the outcome $|\phi\rangle$ noncontextually.

# Noncontextual Sets

- In a KS noncontextual model, we would have

$$|\langle\phi|\psi\rangle|^2 = \int_\Lambda d\lambda \, Pr(\phi|M,\lambda) Pr(\lambda|\psi) = \int_{\Gamma_\phi} d\lambda \, Pr(\phi|M,\lambda) Pr(\lambda|\psi)$$

$$= \int_{\Gamma_\phi} d\lambda \, Pr(\lambda|\psi) \underset{\text{def}}{\equiv} Pr(\Gamma_\phi|\psi)$$

- This is actually equivalent to KS noncontextuality (up to measure-zero issues)
- In a KS contextual model $Pr(\Gamma_\phi|\psi) \leq |\langle\phi|\psi\rangle|^2$

  but $Pr(\Gamma_\phi|\psi)$ still makes sense.

  It measures the proportion of the probability of obtaining outcome $|\phi\rangle$ that is accounted for by ontic states that are noncontextual for $|\phi\rangle$.

# Noncontextuality Inequalities Revisited

- Now if $\langle\phi_1|\phi_2\rangle = 0$ then $\Gamma_{\phi_1}$ and $\Gamma_{\phi_2}$ are disjoint.

  Why? $\exists$ a basis $M$ that includes $|\phi_1\rangle$ and $|\phi_2\rangle$

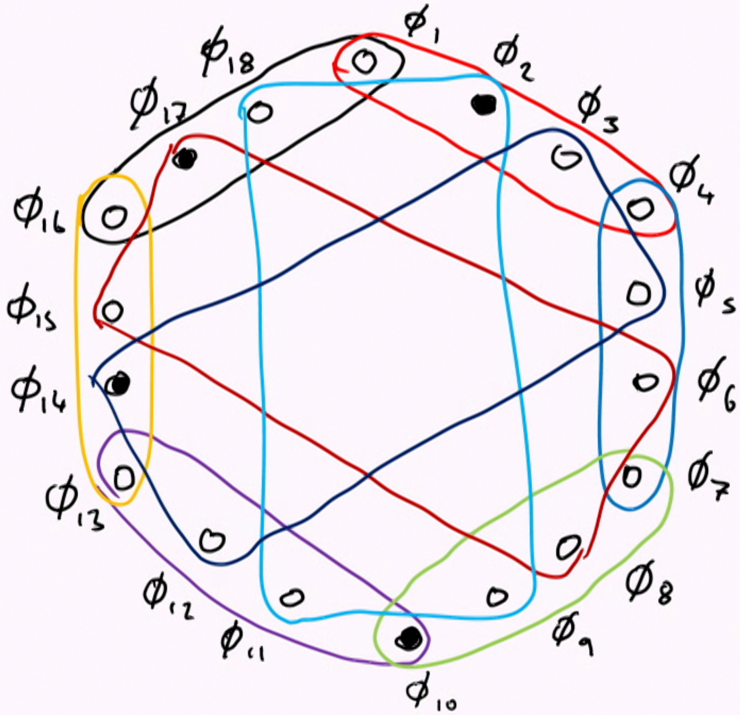
  $\Gamma^M_{\phi_1}$ and $\Gamma^M_{\phi_2}$ are disjoint because $Pr(\phi_1|M,\lambda) + Pr(\phi_2|M,\lambda) \leq 1$

  But $\Gamma^M_{\phi_1} \subseteq \Gamma_{\phi_1}$ and $\Gamma^M_{\phi_2} \subseteq \Gamma_{\phi_2}$

- If $M = \{|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_d\rangle\}$ is an orthonormal basis then any $\lambda$ can be in <u>at most</u> 1 of $\Gamma_{\phi_1}, \Gamma_{\phi_2}, \ldots, \Gamma_{\phi_d}$

- But it doesn't <u>have</u> to be in any of them. It could be in a nondeterministic or contextual state instead.

# Noncontextuality Inequalities Revisited




We can now make sense of the inequality for the 18-ray proof

$$\sum_{j=1}^{18} \Pr(\Gamma_{\phi_j} | \lambda) \leq 4 \quad \text{for all } \lambda$$

$$\therefore \sum_{j=1}^{18} \int_\Lambda d\lambda \Pr(\Gamma_{\phi_j} | \lambda) \Pr(\lambda | \psi) \leq 4$$

$$\Rightarrow \sum_{j=1}^{18} \Pr(\Gamma_{\phi_j} | \psi) \leq 4$$

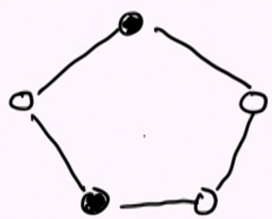
And if $\sum_{j=1}^{18} |\langle \phi_j | \psi \rangle|^2 > 4$ we have detected contextuality.

# CSW Noncontextuality Inequalities

- Cabello, Severini and Winter (CSW) introduced a class of noncontextuality inequalities based on graph theory. Phys. Rev. Lett. 112: 040401 (2014).
- Consider a graph $G = (V, E)$

e.g.

$\alpha(G) = 2$



- To each vertex $v \in V$ we assign a pure state $|\phi_v\rangle$
- If the vertices are connected by an edge $(v, v') \in E$ then we demand $\langle \phi_v | \phi_{v'} \rangle = 0$

- The independence number $\alpha(G)$ is the size of the largest set of vertices such that no two vertices are connected by an edge.
- Since $\Gamma_{\phi_v}$ and $\Gamma_{\phi_{v'}}$ are disjoint for orthogonal states, a KS noncontextual model satisfies

$$\sum_{v \in V} \Pr(\Gamma_{\phi_v} | \psi) \leq \alpha(G) \quad \text{for any state } |\psi\rangle.$$

# CSW Noncontextuality Inequalities

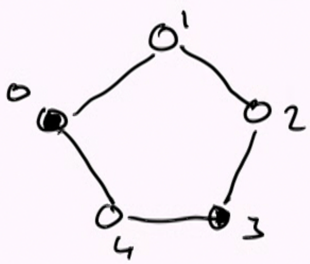
- To determine the maximum possible quantum violation we want to optimize

$$\max_{\{|\phi_v\rangle\}, |\psi\rangle} \sum_{v \in V} |\langle \phi_v | \psi \rangle|^2 = \Theta(G)$$ subject to the orthogonality constraints.

- It turns out that $\Theta(G)$ had been studied in graph theory for other reasons. It is called the Lovasz theta function. In particular, it can be efficiently computed numerically.
- So finding CSW contextuality proofs is equivalent to finding graphs with $\Theta(G) > \alpha(G)$.

# Example: Klyatchko Inequality

- A previously known example is the Klyatchko inequality, based on a 5-cycle




$\alpha(G) = 2$ so $\sum_{j=0}^{4} Pr(\Gamma_{\phi_j} | \psi) \leq 2$

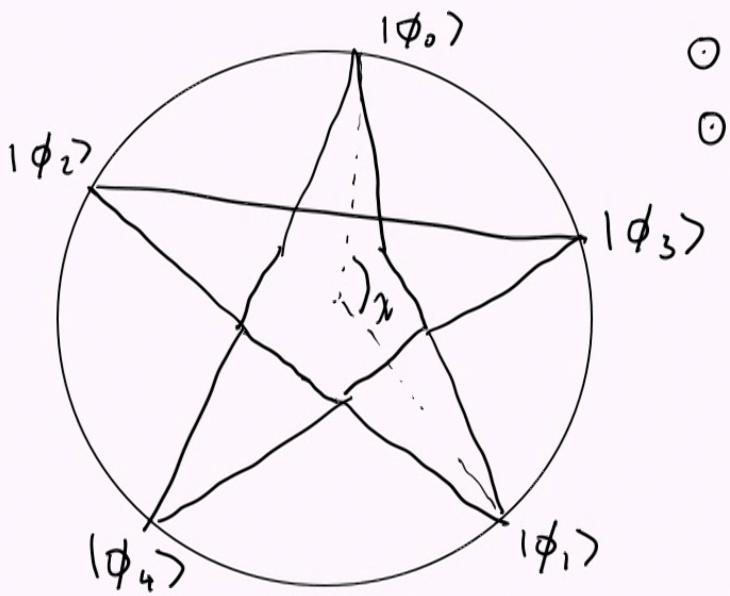
- The maximum quantum violation is found in a 3-d real Hilbert space

$$|\phi_j\rangle = \begin{pmatrix} \sin\chi \cos\eta_j \\ \sin\chi \sin\eta_j \\ \cos\chi \end{pmatrix} \quad |\psi\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{with } \eta_j = \frac{4\pi j}{5} \quad \cos\chi = \frac{1}{\sqrt[4]{5}}$$

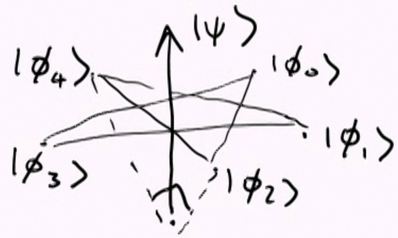
$$\Theta(G) = \max \sum_{j=0}^{4} |\langle \phi_j | \psi \rangle|^2 = 5\cos^2\chi = \frac{5}{\sqrt{5}} = \sqrt{5} \simeq 2.24$$

# Example: Klyatchko Inequality

- Geometrically we can understand the states $|\phi_j\rangle$ as follows
- Consider the following 5 states on the equator of the unit sphere




- The angle $\chi > 90^\circ$ so these states are not orthogonal.
- However if we raise the pentagram up the surface of the sphere, keeping it parallel to the equator then eventually the angle will hit $90^\circ$



# 3.6) ψ-ontology

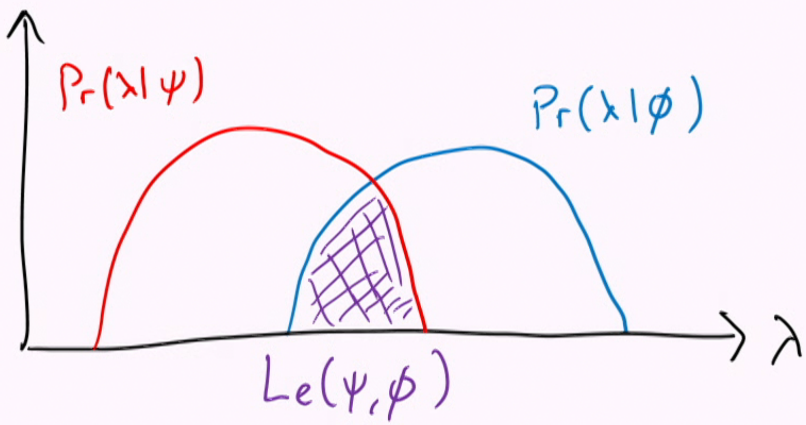
# Reality of the Quantum State

- We now wish to investigate whether the (pure) quantum state has to be part of the ontology as it is in Beltrametti-Bugajski, the Bell model and de Broglie-Bohm theory.
- Our objective is to determine whether the kind of ψ-epistemic explanations that occur in the Spekkens toy theory can work in quantum theory.
- I will use naughty notation $\Pr(\lambda|\psi)$ for epistemic states:
  - We can only prove preparation contextuality for mixed states anyway.
  - What we will prove applies to *any* method of preparing $|\psi\rangle$, so it is best to avoid cluttering notation.

# Definitions

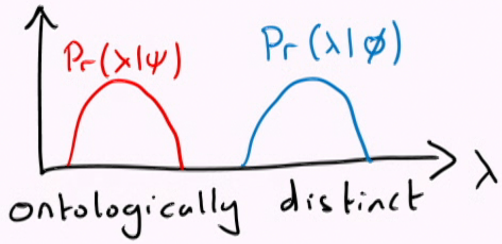
- For two quantum states $|\psi\rangle$ and $|\phi\rangle$, we define their epistemic overlap in an ontological model as:

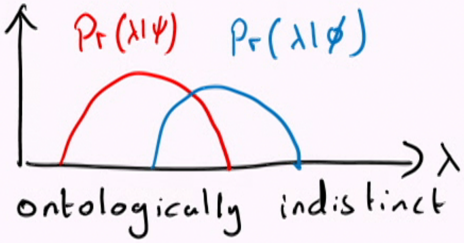
$$L_e(\psi,\phi) = \int_\Lambda d\lambda \; \min[\Pr(\lambda|\psi), \Pr(\lambda|\phi)]$$



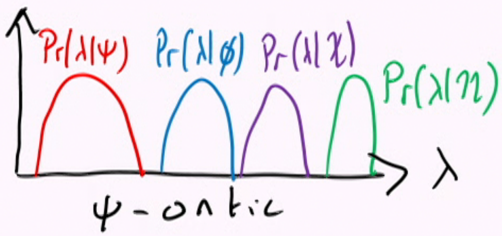
# Definitions

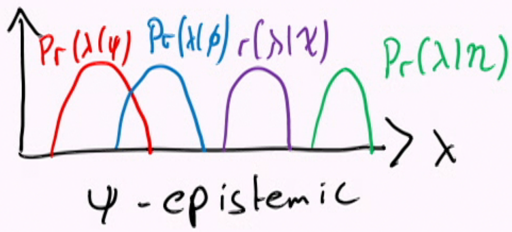
- $|\psi\rangle$ and $|\phi\rangle$ are ontologically distinct in an ontological model if $L_e(\phi,\psi)=0$.



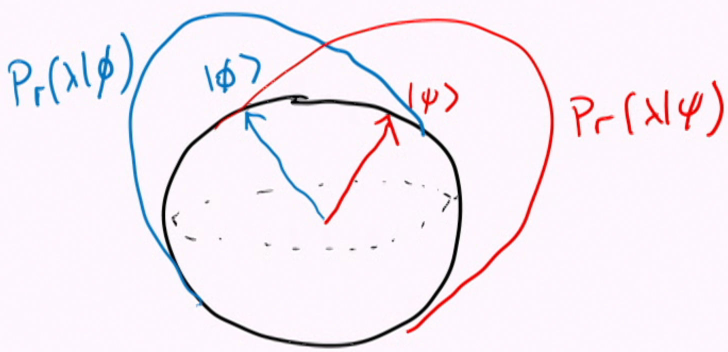

- An ontological model is called $\psi$-ontic if every pair of pure states in the model is ontologically distinct. Otherwise, it is called $\psi$-epistemic.



# $\psi$-epistemic models exist

- $\psi$-epistemic models exist in all finite Hilbert space dimensions.
  - For d=2, the Kochen-Specker model is $\psi$-epistemic.




  - For d>2, it was proved by Lewis et. al. (Phys. Rev. Lett. 109:150404 (2012)) and Aaronson et. al. (*Phys. Rev. A* 88:032111 (2013)).

# What next for $\psi$-ontology?

- Given that $\psi$-epistemic models exist, is that the end of the story? No.
  - We can try to prove something weaker than $\psi$-ontology, that still makes $\psi$-epistemic explanations implausible:

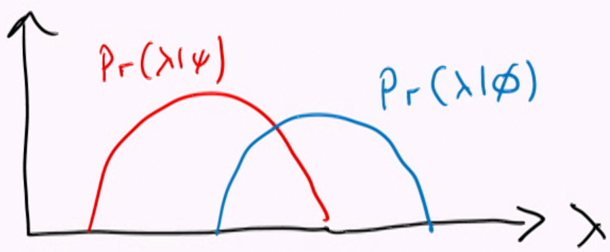
    $\Rightarrow$ non maximal $\psi$-epistemicity

  - We can add additional assumptions to the ontological models framework to prove $\psi$-ontology:

    $\Rightarrow$ Pusey-Barrett-Rudolph (PBR) theorem

# Maximally $\psi$-epistemic models

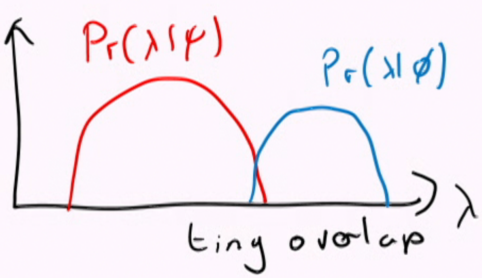
- Consider the $\psi$-epistemic explanation of the indistinguishability of quantum states:




$|\psi\rangle$ and $|\phi\rangle$ cannot be perfectly distinguished because sometimes the ontic state is exactly the same regardless of whether $|\psi\rangle$ or $|\phi\rangle$ was prepared.

- This explanation is rendered implausible if a suitable measure of the overlap of the probability distributions is small compared to a suitable measure of the overlap/indistinguishability of the quantum states.

i.e.




but $|\langle\phi|\psi\rangle|^2$ is large

$\Rightarrow$ this explanation plays almost no role.

# Maximally $\psi$-epistemic models

- We need to be comparing measures of quantum and probability overlap that have a comparable operational meaning.
- We already have the epistemic overlap measure:

$$L_e(\psi,\phi) = \int_\Lambda d\lambda \ \min[\Pr(\lambda|\psi), \Pr(\lambda|\phi)]$$

- This measure has the following interpretation:
  - If the system is prepared in state $|\psi\rangle$ or state $|\phi\rangle$ with 50/50 probability and you don't know which, then if you knew the exact ontic state $\lambda$ your optimal probability of guessing correctly is

$$p = \frac{1}{2}(2 - L_e(\psi,\phi))$$

- The comparable quantum overlap measure is:

$$L_q(\psi,\phi) = 1 - \sqrt{1 - |\langle\phi|\psi\rangle|^2}$$

  - If the system is prepared in state $|\psi\rangle$ or state $|\phi\rangle$ with 50/50 probability and you don't know which, then if you want to guess based on the outcome of a quantum measurement, your optimal probability of guessing correctly is

$$p = \frac{1}{2}(2 - L_e(\psi,\phi))$$

# Ruling out Maximally $\psi$-epistemic models

- First note that $L_e(\psi,\phi) = \int_\Lambda d\lambda \min\{\Pr(\lambda|\psi), \Pr(\lambda|\phi)\} \leq \int_{\Lambda_\phi} d\lambda \Pr(\lambda|\psi)$

  where $\Lambda_\phi := \{\lambda \in \Lambda | \Pr(\lambda|\phi) > 0\}$

- We already showed that $\Lambda_\phi \subseteq \Gamma_\phi^M$ for any measurement M that has $|\phi\rangle$ as an outcome.

- Since this is true for all such M, we also have

$$\Lambda_\phi \subseteq \Gamma_\phi = \bigcap_{\{M \,|\, |\phi\rangle \in M\}} \Gamma_\phi^M$$

$$\therefore \; L_e(\psi,\phi) \leq \int_{\Gamma_\phi} d\lambda \Pr(\lambda|\psi) = \Pr(\Gamma_\phi|\psi)$$

# Ruling out Maximally $\psi$-epistemic models

- Now if we consider a set of states $\{|\phi_j\rangle\}$ then we will have

$$\sum_j L_e(\psi, \phi_j) \leq \sum_j \Pr(\Gamma_{\phi_j}|\psi) \leftarrow$$ This is precisely what is bounded by a noncontextuality inequality

- We can then compute $\sum_j L_q(\psi, \phi_j)$ for the optimal states in the contextuality inequality. If $\sum_j \Pr(\Gamma_{\phi_j}|\psi) < \sum_j L_q(\psi, \phi_j)$ then maximally $\psi$-epistemic models are ruled out.
- It is better to compare the averages

$$\langle L_e \rangle = \frac{1}{n} \sum_{j=1}^{n} L_e(\psi, \phi_j) \qquad \langle L_q \rangle = \frac{1}{n} \sum_{j=1}^{n} L_q(\psi, \phi_j)$$

If $\langle L_q \rangle$ is large while $\langle L_e \rangle$ is small, the $\psi$-epistemic explanation of indistinguishability is in trouble.

# Ruling out Maximally $\psi$-epistemic models

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# Results from Various Contextuality Inequalities

| | Dimension | No. states | $\langle L_e \rangle$ | $\langle L_q \rangle$ |
|---|---|---|---|---|
| Barrett et. al. | Prime power $d \geq 4$ | $d^2$ | $1/d^2$ | $1 - \sqrt{1 - 1/d}$ |
| Leifer | $d \geq 3$ | $2^{d-1}$ | $1/2^{d-1}$ | $1\sqrt{1 - 1/d}$ |
| Branciard | $d \geq 4$ | $n \geq 2$ | $1/n$ | $1 - \sqrt{1 - \frac{1}{4} n^{-1/(d-2)}}$ |
| Amaral et. al. | $d \geq n_j$ | $n_j \geq ?$ | $n_j^{\delta - 1}$ | $1 - \sqrt{\frac{1}{2}-\epsilon}$ |


J. Barrrett et. al., Phys. Rev. Lett. 112, 250403 (2014)
M. Leifer, Phys. Rev. Lett. 112, 160404 (2014)
C. Branciard, Phys. Rev. Lett. 113, 020409 (2014)
B. Amaral et. al., Phys. Rev. A 92, 062125 (2015)

# Optimizing for $\langle L_q \rangle - \langle L_e \rangle$

| | Optimal Dimension | Optimal No. states | Optimal $\langle L_q \rangle - \langle L_e \rangle$ |
|---|---|---|---|
| Barrett et. al. | $4$ | $16$ | $0.0715$ |
| Leifer | $7$ | $64$ | $0.0586$ |
| Branciard | $4$ | $n \to \infty$ | $0.134$ |
| Amaral et. al. | $d \to \infty$ | $n_j \to \infty$ | $0.293$ |

# Is non maximal $\psi$-epistemicity significant?

- In any ontological model, there are two ways of explaining the indistinguishability of quantum states:
  - The epistemic states overlap.
  - Quantum measurements only reveal coarse-grained information about $\lambda$.
- It is not clear why the second explanation should not play some role in a $\psi$-epistemic theory.
- Therefore, I would say that we want to get $\langle L_q \rangle - \langle L_e \rangle$ as close to $1$ as possible in order to convincingly rule out $\psi$-epistemic models.

# The PBR Theorem

- The PBR Theorem (Nature Physics 8:475-478 (2012)) proves that ontological models have to be $\psi$-ontic under an additional assumption called the Preparation Independence Postulate (PIP).
- The PIP can be broken down into two assumptions:
  - The Cartesian Product Assumption:

  When two systems are prepared independently in a product state $|\psi\rangle_A \otimes |\phi\rangle_B$, the joint ontic state space is $\Lambda_{AB} = \Lambda_A \times \Lambda_B$, i.e. each system has its own ontic state, i.e. the ontic state of the joint system is $\lambda_{AB} = (\lambda_A, \lambda_B)$, where $\lambda_A$ is the ontic state of system $A$ and $\lambda_B$ is the ontic state of system $B$.

  - The No Correlation Assumption:

  The epistemic state corresponding to $|\psi\rangle_A \otimes |\phi\rangle_B$ is:

$$\Pr(\lambda_A, \lambda_B | \psi_A, \phi_B) = \Pr(\lambda_A | \psi_A) \Pr(\lambda_B | \phi_B)$$

# Comments on the PIP

- In general, a joint system with two subsystems might have global ontic properties that do not reduce to properties of the individual subsystems.
  - In a $\psi$-ontic model with entangled states this would be the case: $|\psi\rangle_{AB}$ is not a property of either subsystem.
  - So, in general, we need $\Lambda_{AB} = \Lambda_A \times \Lambda_B \times \Lambda_{\text{global}}$.
  - All we really require from the Cartesian Product Assumption is that $\Lambda_{\text{global}}$ plays no role in determining measurement outcomes when we prepare a product state, e.g. for product states $\lambda_g \in \Lambda_{\text{global}}$ always takes the same specific value.
  - Then, the No Correlation Assumption should be read as applying to the marginal on $\Lambda_A \times \Lambda_B$.

# The PBR Theorem

- Theorem: An ontological model of quantum theory that satisfies the PIP must be $\psi$-ontic.
  - We will prove the more restricted claim that the two states:

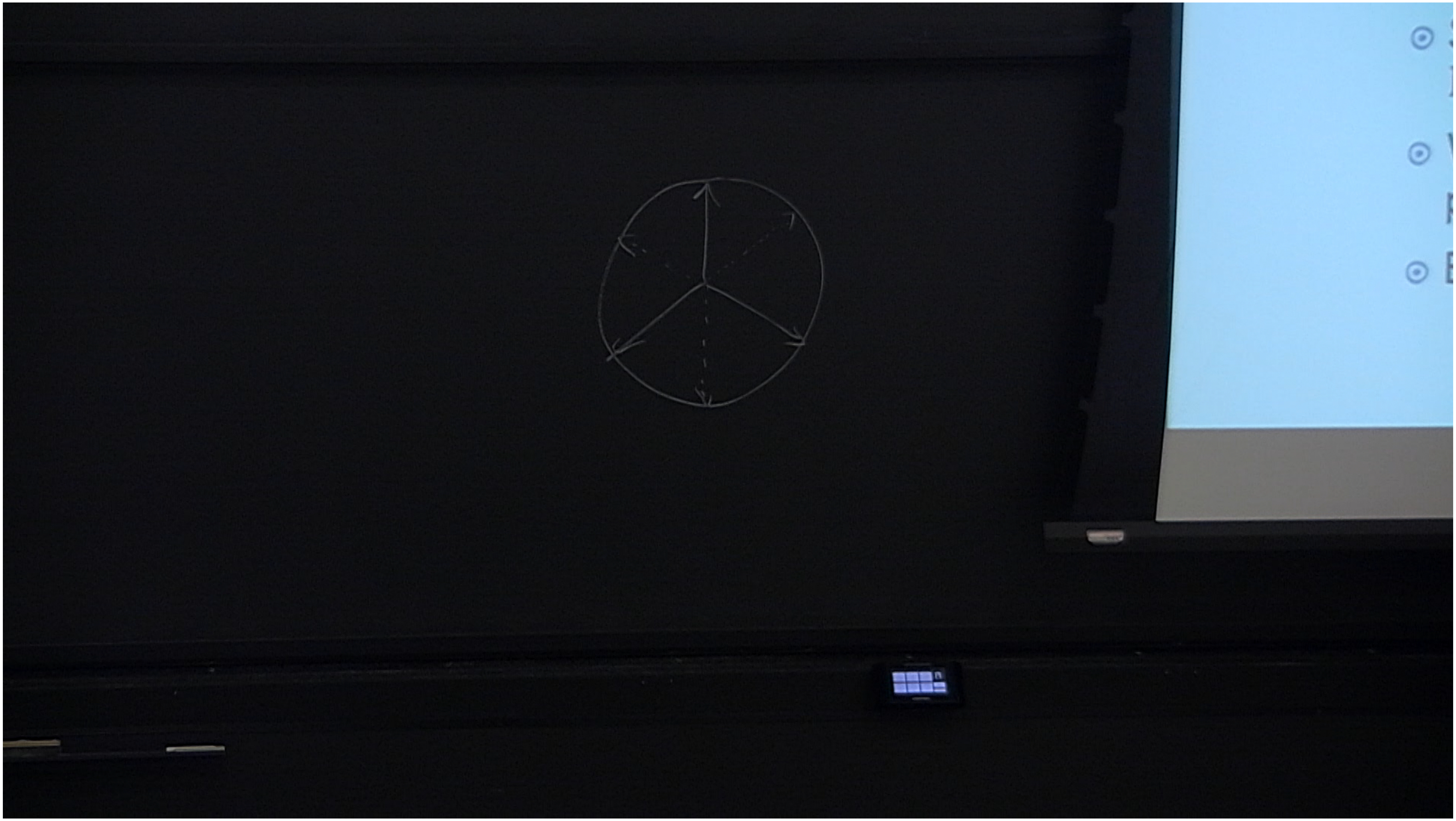
$$|0\rangle \qquad \text{and} \qquad |+\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

  must be ontologically distinct.
  - The simplest known proof is by C. Moseley (arXiv:1401.0026) and is not too much harder.
- We start by introducing the n-way epistemic overlap:

$$L_e(\psi_1, \psi_2, \dots, \psi_n) = \int_\Lambda d\lambda \; \min_j\{\Pr(\lambda|\psi_j)\}$$

# Antidistinguishability

- A set $\{|\psi_j\rangle\}_{j=1}^{n}$ of $n$ quantum states in is called antidistinguishable if there exists an $n$-outcome measurement $\{E\_j\}_{j=1}^{n}$ such that
$$\langle\psi_j|E_j|\psi_j\rangle = 0 \quad \text{for all} \quad j = 1,2,\ldots,n$$
- Lemma: If a set of states is antidistinguishable, then, in any ontological model $L_e(\psi_1, \psi_2, \ldots, \psi_n) = 0$.
- Proof outline:
  - Suppose there is a $\lambda$ assigned nonzero probability by all of $\Pr(\lambda|\psi_1), \Pr(\lambda|\psi_2), \ldots, \Pr(\lambda|\psi_n)$.
  - We require $\Pr(E_j|\lambda) = 0$ for all $E_j$ in order to reproduce the quantum predictions.
  - But $\sum_{j=1}^{n} \Pr(E_j|\lambda) = 1$, so no such $\lambda$ can exist.



# Ontological Distinctness of $|0\rangle$ and $|+\rangle$

- We first show that the following four states are antidistinguishable:

$$|\psi_1\rangle_{AB} = |0\rangle_A \otimes |0\rangle_B$$
$$|\psi_2\rangle_{AB} = |0\rangle_A \otimes |+\rangle_B$$
$$|\psi_3\rangle_{AB} = |+\rangle_A \otimes |0\rangle_B$$
$$|\psi_4\rangle_{AB} = |+\rangle_A \otimes |+\rangle_B$$

- Consider the four states:

$$|\phi_1\rangle_{AB} = \frac{1}{\sqrt{2}}(\,|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B\,)$$
$$|\phi_2\rangle_{AB} = \frac{1}{\sqrt{2}}(\,|0\rangle_A \otimes |-\rangle_B + |1\rangle_A \otimes |+\rangle_B\,)$$
$$|\phi_3\rangle_{AB} = \frac{1}{\sqrt{2}}(\,|+\rangle_A \otimes |1\rangle_B + |-\rangle_A \otimes |0\rangle_B\,)$$
$$|\phi_4\rangle_{AB} = \frac{1}{\sqrt{2}}(\,|+\rangle_A \otimes |-\rangle_B + |-\rangle_A \otimes |+\rangle_B\,)$$

- where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. These states form an orthonormal basis, and $\langle\phi_j|\psi_j\rangle = 0$, so a measurement in this basis antidistinguishes the $|\psi_j\rangle$'s.

# Prospects for $\psi$-ontology theorems

- The PBR theorem renders $\psi$-epistemic explanations implausible within the ontological models framework.
- The upshot of overlap bounds is more ambiguous.
- Apart from fundamental interest, $\psi$-ontology theorems are interesting because they imply most of the other known no-go theorems.
  - From this point of view, the extra assumptions needed for PBR are not ideal.
- It is still possible that:
  - Better overlap bounds could be obtained.
  - $\psi$-epistemic models are impossible for infinite dimensional Hilbert Spaces.
  - $\psi$-epistemic models are impossible for POVMs (We already know that the Kochen-Specker model cannot be extended to POVMs).