Title: PSI 2016/2017 Foundations of Quantum Mechanics (Review) - Lecture 5

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Abstract:

Today's Lecture

3. The Generalized Formalism

    6. Completely Positive, Trace Preserving Maps
    7. Positive Operator Valued Measures
    8. The Lindblad Equation

Review of Last Lecture

- We showed that the state of a system S with Hilbert space $\mathcal{H}_S$ should be described by a density operator $\rho_S \in \mathcal{L}(\mathcal{H}_S)$

  which is positive $\quad {}_S\langle\psi|\rho_S|\psi\rangle_S \geq 0 \quad \forall\, |\psi\rangle_S \in \mathcal{H}_S$

  and has unit trace $\quad \mathrm{Tr}(\rho_S) = 1$

- From the larger church point of view, we showed that the finite-time dynamics of a system interacting with its environment is described by

$$\mathcal{E}(\rho_S) = \sum_j M^{(j)} \rho_S M^{(j)\dagger}$$

where the $M^{(j)}$'s are any set of operators satisfying

$$\sum_j M^{(j)\dagger} M^{(j)} = I_S$$

The View from the Smaller Church

- According to the smaller church, dynamics should be any mapping of states to states that leads to well-defined probabilities for all observables at the output.
- This turns out to be remarkably subtle.
- Firstly, we will allow the output Hilbert space $\mathcal{H}_B$ to be different from the input Hilbert space $\mathcal{H}_A$

  We may add a new subsystem or discard part of the system during the dynamics.
- So we need some sort of map $\mathcal{E}_{B|A}$ from $\mathcal{L}(\mathcal{H}_A)$ to $\mathcal{L}(\mathcal{H}_B)$ that maps density operators to density operators.

The View from the Smaller Church

- We will demand that $\mathcal{E}_{B|A}$ is linear. Why?

  If we prepare ρ_A with probability p
  or σ_A with probability $(1-p)$

  Then $\mathcal{E}_{B|A}(p\rho_A + (1-p)\sigma_A) = p\,\mathcal{E}_{B|A}(\rho_A) + (1-p)\,\mathcal{E}_{B|A}(\sigma_A)$

- Strictly speaking, this only means that $\mathcal{E}_{B|A}$ has to be affine, i.e. acts linearly on positive linear combinations.
- But you can always extend an affine map to a linear one just by defining $\mathcal{E}_{B|A}(-\rho_A) = -\mathcal{E}_{B|A}(\rho_A)$
- So, we will have a linear operator from linear operators to linear operators

  $\mathcal{E}_{B|A} \in \mathcal{L}(\mathcal{L}(\mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_B))$ Sometimes called a superoperator.

The View from the Smaller Space

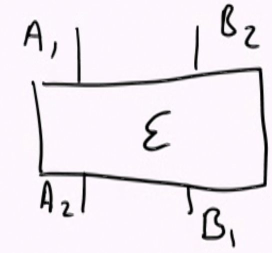
- Now comes the fun part:

$$\mathcal{L}(\mathcal{H}_A) = \mathcal{H}_{A_2} \otimes \mathcal{H}_{A_1}^{\dagger} \qquad \mathcal{L}(\mathcal{H}_B) = \mathcal{H}_{B_2} \otimes \mathcal{H}_{B_1}^{\dagger}$$

These are both $\mathcal{H}_A$ but it helps to keep track of which is the input and which is the output

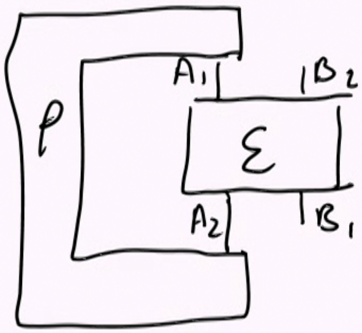
$$\therefore \mathcal{L}(\mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)) = \mathcal{L}(\mathcal{H}_{A_2} \otimes \mathcal{H}_{A_1}^{\dagger} \to \mathcal{H}_{B_2} \otimes \mathcal{H}_{B_1}^{\dagger})$$
$$= \mathcal{H}_{B_2} \otimes \mathcal{H}_{B_1}^{\dagger} \otimes \mathcal{H}_{A_2}^{\dagger} \otimes \mathcal{H}_{A_1}$$

$$\therefore \mathcal{E}_{B|A} = \sum_{jklm} \mathcal{E}^{jm}_{kl} \, |j\rangle_{A_1} \otimes {}_{A_2}\langle k| \otimes {}_{B_1}\langle l| \otimes |m\rangle_{B_2} \qquad \mathcal{E}^{j_{A_1} m_{B_2}}_{k_{A_2} l_{B_1}}$$



The View from the Smaller Church

- The action of $\mathcal{E}_{B|A}$ on a density operator ρ_A is going to be



$$\mathcal{E}_{B|A}(\rho_A) = \sum_{jklm} \mathcal{E}^{jm}_{kl} \; {}_{A_2}\langle k|\rho_A|j\rangle_{A_1} \otimes |m\rangle_{B_2}\,{}_{B_1}\langle l|$$

$$= \mathcal{E}^{j_{A_1} m_{B_2}}_{k_{A_2} l_{B_1}} \rho^{k_{A_2}}_{j_{A_1}}$$

- The space $\mathcal{L}(\mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)) = \mathcal{H}_{A_1} \otimes \mathcal{H}^{\dagger}_{A_2} \otimes \mathcal{H}^{\dagger}_{B_1} \otimes \mathcal{H}_{B_2}$ can be decomposed in a different way, which will end up giving us the operator-sum decomposition.

$$\left(\mathcal{H}_{A_1} \otimes \mathcal{H}^{\dagger}_{B_1}\right) \otimes \left(\mathcal{H}_{B_2} \otimes \mathcal{H}^{\dagger}_{A_2}\right) = \mathcal{L}(\mathcal{H}_{B_1} \to \mathcal{H}_{A_1}) \otimes \mathcal{L}(\mathcal{H}_{A_2} \to \mathcal{H}_{B_2})$$

$$= \mathcal{L}(\mathcal{H}_{A_1} \to \mathcal{H}_{B_1})^{\dagger} \otimes \mathcal{L}(\mathcal{H}_{A_2} \to \mathcal{H}_{B_2})$$

$$= \mathcal{L}\left(\mathcal{L}(\mathcal{H}_A \to \mathcal{H}_B)\right)$$

The View from the Smaller Church

- On this decomposition, we would write the action of $\mathcal{E}_{B|A}$ as

$$\mathcal{E}_{B|A}(\rho_A) = \sum_{jklm} \mathcal{E}^{jm}_{kl} \, |m\rangle_{B_2} \, {}_{A_2}\langle k| \rho_A |j\rangle_{A_1} \, {}_{B_1}\langle l|$$

- If we view $\mathcal{L}(\mathcal{H}_A \to \mathcal{H}_B)$ as a space of kets $|jk\rangle\!\rangle_{AB} = |j\rangle_B \, {}_A\langle k|$
  and $\mathcal{L}(\mathcal{H}_B \to \mathcal{H}_A)$ as a space of bras ${}_{AB}\langle\!\langle jk| = |k\rangle_A \, {}_B\langle j|$

  Then $\mathcal{E}_{B|A}$ has the form

$$\mathcal{E}_{B|A} = \sum_{jklm} \mathcal{E}^{jm}_{kl} \, |mk\rangle\!\rangle_{AB} \langle\!\langle lj|$$

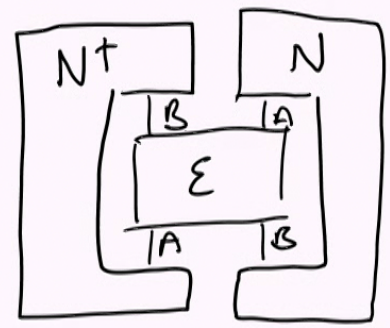
The View from the Smaller Church

- If I can show that $\mathcal{E}_{B|A}$ is a positive operator on this space

  i.e. $\forall \; N \in \mathcal{L}(\mathcal{H}_A \to \mathcal{H}_B)$

$$(N|\mathcal{E}_{B|A}|N) = \sum_{jklm} \mathcal{E}^{jm}_{kl} (N|mk)(lj|N) \geq 0$$

which means $\quad N^{\dagger\, k_A}_{\;\; m_B} \mathcal{E}^{j_A m_B}_{k_A l_B} N^{l_B}_{\;\; j_A} \geq 0$



≥ 0

Then we will have an eigenoperator decomposition with positive eigenvalues

$$\mathcal{E}_{B|A} = \sum_j \lambda_j |R^{(j)})(R^{(j)}| = \sum_j |M^{(j)})(M^{(j)}| \quad \text{where } M^{(j)} = \sqrt{\lambda_j}\, R^{(j)}$$

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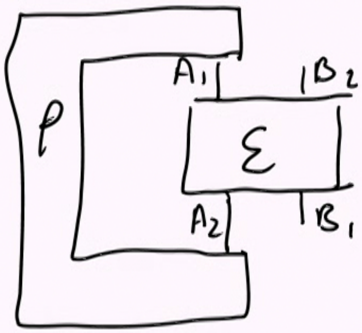

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$$\begin{aligned}
(\mathcal{H}_{A_1} \otimes \mathcal{H}^\dagger_{B_1}) \otimes (\mathcal{H}_{B_2} \otimes \mathcal{H}^\dagger_{A_2}) &= \mathcal{L}(\mathcal{H}_{B_1} \to \mathcal{H}_{A_1}) \otimes \mathcal{L}(\mathcal{H}_{A_2} \to \mathcal{H}_{B_2}) \\
&= \mathcal{L}(\mathcal{H}_{A_1} \to \mathcal{H}_{B_1})^\dagger \otimes \mathcal{L}(\mathcal{H}_{A_2} \to \mathcal{H}_{B_2}) \\
&= \mathcal{L}(\mathcal{L}(\mathcal{H}_A \to \mathcal{H}_B))
\end{aligned}$$

The View from the Smaller Church

- If I can show that $\mathcal{E}_{B|A}$ is a positive operator on this space

  i.e. $\forall \; N \in \mathcal{L}(\mathcal{H}_A \to \mathcal{H}_B)$

$$(N|\mathcal{E}_{B|A}|N) = \sum_{jklm} \mathcal{E}^{jm}_{kl} \, (N|mk)(lj|N) \geq 0$$

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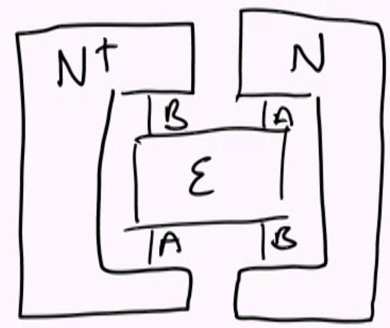

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The View from the Smaller Church

- Now if $\mathcal{E}_{B|A} = \sum_j |M^{(j)}\rangle\langle M^{(j)}|$

  then $\mathcal{E}_{B|A}(\rho_A) = \sum_j M^{(j)} \rho_A M^{(j)\dagger}$ so we'll have an operator sum decomposition

- We'll prove that the required positivity holds soon.

- However, if the operator sum decomposition holds, we can prove that

$$\sum_j M^{(j)\dagger} M^{(j)} = I$$

- If $\mathcal{E}_{B|A}(\rho_A)$ is a density operator then $\mathrm{Tr}(\mathcal{E}_{B|A}(\rho_A)) = \mathrm{Tr}(\rho_A) = 1$ for any input density operator ρ_A, i.e. $\mathcal{E}_{B|A}$ is trace preserving

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Trace Preservation

- If $\sum_j M^{(j)\dagger} M^{(j)} = I$ then

$$\mathrm{Tr}\left(\mathcal{E}_{B|A}(\rho_A)\right) = \mathrm{Tr}\left[\sum_j M^{(j)} \rho_A M^{(j)\dagger}\right] = \mathrm{Tr}\left[\left(\sum_j M^{(j)\dagger} M^{(j)}\right)\rho_A\right]$$

$$= \mathrm{Tr}\left[I \rho_A\right] = \mathrm{Tr}(\rho_A) = 1$$

- Conversely, suppose $\mathcal{E}_{B|A}$ is trace preserving and let $\rho_A = |\psi\rangle_A\langle\psi|$

Then $\mathrm{Tr}\left[\sum_j M^{(j)} |\psi\rangle_A\langle\psi| M^{(j)\dagger}\right] = 1$

$${}_A\langle\psi| \left(\sum_j M^{(j)\dagger} M^{(j)}\right) |\psi\rangle_A = 1$$

$\Rightarrow$ ${}_A\langle\psi|N|\psi\rangle_A = 1$ with $N = \sum_j M^{(j)\dagger} M^{(j)}$

Trace Preservation

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$\Rightarrow$ ${}_A\langle\psi|N|\psi\rangle_A = 1$ with $N = \sum_j M^{(j)\dagger} M^{(j)}$

Trace Preservation

- Now, if $\langle\psi|N|\psi\rangle = 1$ for all unit vectors $|\psi\rangle$ then $N = I$

Proof: Let $|\psi\rangle = |k\rangle$ (a basis vector), then $\langle k|N|k\rangle = 1$, so diagonal entries are equal to 1

Let $|\psi\rangle = \frac{1}{\sqrt{2}}(|k\rangle + |m\rangle)$

$$\Rightarrow \frac{1}{2}\left(\langle k|N|k\rangle + \langle k|N|m\rangle + \langle m|N|k\rangle + \langle m|N|m\rangle\right) = 1$$

$$\Rightarrow \langle k|N|m\rangle + \langle m|N|k\rangle = 0 \qquad (1)$$

Let $|\psi\rangle = \frac{1}{\sqrt{2}}(|k\rangle + i|m\rangle)$

$$\Rightarrow i\langle k|N|m\rangle - i\langle m|N|k\rangle = 0$$

$$\Rightarrow \langle k|N|m\rangle - \langle m|N|k\rangle = 0 \qquad (2)$$

$\frac{(1)+(2)}{2} \Rightarrow \langle k|N|m\rangle = 0$, so off-diagonal entries are 0.

Positivity vs. Complete Positivity

- It would be nice if we could show that $\langle N|\mathcal{E}_{B|A}|N\rangle \geq 0$ follows from the requirement that $\mathcal{E}_{B|A}(\rho_A)$ is a positive operator.
- Such a superoperator is called a positive superoperator.
- What happens instead is something of an embarassment for the Church of the Smaller Hilbert space.
- If system A is correlated with its environment E, then acting with $\mathcal{E}_{B|A}$ on A alone should keep the state of AE positive

$$\mathcal{E}_{B|A}(\rho_{AE}) \text{ should be a positive operator}$$

- A superoperator with this property is called completely positive.

Positivity vs. Complete Positivity

- An example of a superoperator that is positive but not completely positive is the transpose map

$$\mathcal{E}_{B|A}\left(|j\rangle_A\langle k| \right) = |k\rangle_B\langle j| \qquad (\text{here } \mathcal{H}_A \text{ and } \mathcal{H}_B \text{ have same dimension})$$

- This maps $\rho^{k_A}_{j_A}$ to $\rho^{j_B}_{k_B}$, which preserves its eigenvalues and hence positivity.

- However, let A, B, E be qubits and consider the initial state

$$\rho_{AE} = |\Phi^+\rangle_{AE}\langle \Phi^+| \quad \text{with} \quad |\Phi^+\rangle_{AE} = \frac{1}{\sqrt{2}}\left(|00\rangle_{AE} + |11\rangle_{AE} \right)$$

$$= \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11| \right)$$

$$\mathcal{E}_{B|A}(\rho_{AE}) = \frac{1}{2}\left(|00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11| \right)$$

$\frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle \right)$ is an eigenvector of this with eigenvalue $-\frac{1}{2}$

$\Rightarrow$ not a positive operator.

Complete Positivity

- For complete positivity, we require

$$\mathcal{E}_{B|A}(\rho_{AE}) \text{ is a positive operator for any positive } \rho_{AE}$$

$$\Longleftrightarrow \; {}_{BE}\langle\psi|\mathcal{E}_{B|A}(\rho_{AE})|\psi\rangle_{BE} \geq 0 \text{ for any } |\psi\rangle_{AE} \in \mathcal{H}_A \otimes \mathcal{H}_E \text{ and any positive } \rho_{AE}$$

- In particular

$${}_{A'B}\langle\psi|\mathcal{E}_{B|A}(|\delta\rangle_{AA'}\langle\delta|)|\psi\rangle_{A'B} \geq 0 \quad \text{for} \quad |\delta\rangle_{AA'} = \sum_j |j\rangle_A \otimes |j\rangle_{A'}$$

but this turns out to be equivalent to

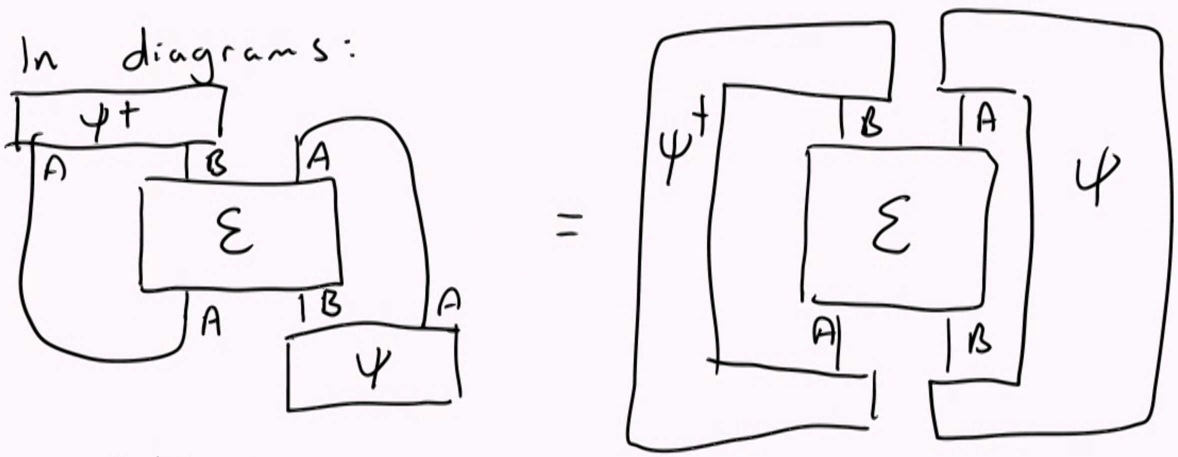
$$(\psi|\mathcal{E}_{B|A}|\psi) \geq 0 \quad \text{for any} \quad \psi \in \mathcal{L}(\mathcal{H}_A \to \mathcal{H}_B)$$

which is the positivity condition needed for the operator sum decomposition

Complete Positivity

$$ {}_{A'B}\langle \psi | \mathcal{E}_{B|A}\left(|\delta\rangle_{AA'}\langle \delta| \right) |\psi\rangle_{A'B} = (\psi | \mathcal{E}_{B|A} | \psi) $$

Proof: In diagrams:



In index notation:

$$ \psi^{\dagger}_{r_A m_B} \mathcal{E}^{j_A m_B}_{k_A l_B} \, \delta_{j_A n_A} \delta^{k_A r_A} \psi^{n_A l_B} = \psi^{\dagger\, k_A}_{m_B} \mathcal{E}^{j_A m_B}_{k_A l_B} \psi^{l_B}_{j_A} = (\psi | \mathcal{E}_{B|A} | \psi) $$

Complete Positivity

- We have now proved that a completely positive, trace-preserving map must have the form

$$\mathcal{E}_{B|A}(\rho_A) = \sum_j M^{(j)} \rho_A M^{(j)\dagger} \quad \text{with} \quad \sum_j M^{(j)\dagger} M^{(j)} = I_A$$

- We still need to check that $\mathcal{E}_{B|A}(\rho_{AE})$ is positive for all ρ_{AE}

  Since ρ_{AE} is positive, we can define $\rho_{AE}^{1/2}$ and then

$$M^{(j)} \rho_{AE} M^{(j)\dagger} = \left(M^{(j)} \rho_{AE}^{1/2}\right)\left(M^{(j)} \rho_{AE}^{1/2}\right)^\dagger \quad \text{is of the form } N^\dagger N$$

  for $N = \left(M^{(j)} \rho_{AE}^{1/2}\right)^\dagger$, so is positive

  A sum of positive operators is positive, so we are done.

Summary

- A physical map from density operators in $\mathcal{L}(\mathcal{H}_A)$ to density operators in $\mathcal{L}(\mathcal{H}_B)$ must be of the form

$$\mathcal{E}_{B|A}(\rho_A) = \sum_j M^{(j)} \rho_A M^{(j)\dagger} \quad \text{where } M^{(j)} \in \mathcal{L}(\mathcal{H}_A \to \mathcal{H}_B)$$
$$\text{and } \sum_j M^{(j)\dagger} M^{(j)} = I_A$$

- We can derive this from
  1) The system acts unitarily with an environment it is initially uncorrelated with.

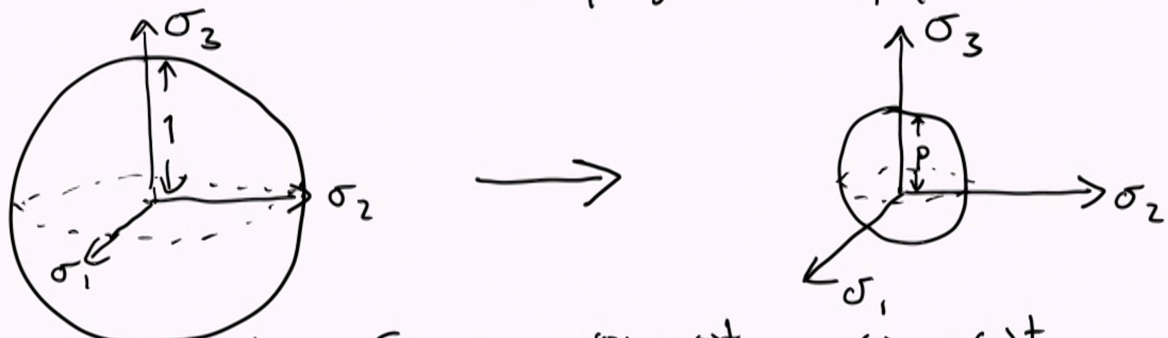
or 2) The map must be completely positive $\mathcal{E}_{B|A}(\rho_{AE})$ is positive for all positive ρ_{AE}

and trace preserving $\mathrm{Tr}(\mathcal{E}_{B|A}(\rho_A)) = \mathrm{Tr}(\rho_A)$

Examples of Qubit CPT Maps

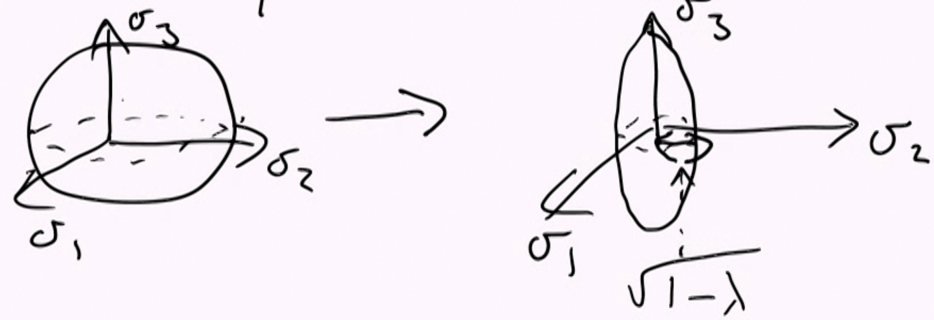
- Depolarizing channel:

$$\mathcal{E}(\rho) = p\frac{I}{2} + (1-p)\rho = \left(1 - \frac{3}{4}p\right) I \rho I + \frac{p}{4}\left(\sigma_1 \rho \sigma_1 + \sigma_2 \rho \sigma_2 + \sigma_3 \rho \sigma_3\right)$$



- Dephasing channel: $\mathcal{E}(\rho) = M^{(0)} \rho M^{(0)\dagger} + M^{(1)} \rho M^{(1)\dagger}$

$$M^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{pmatrix} \quad M^{(1)} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}$$



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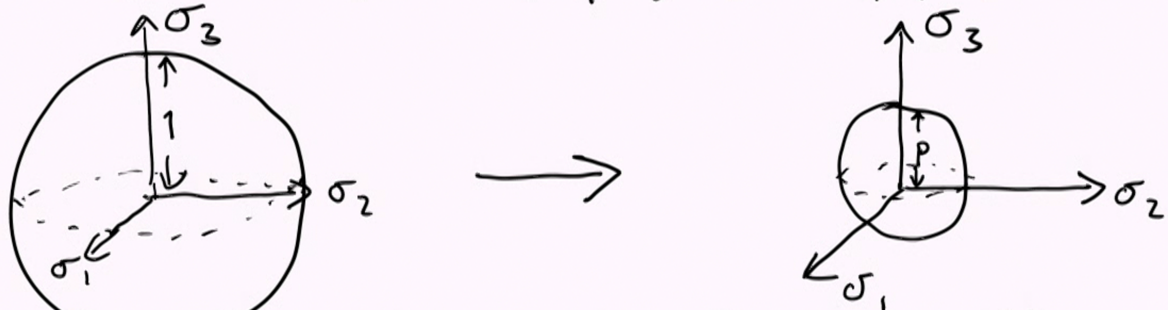

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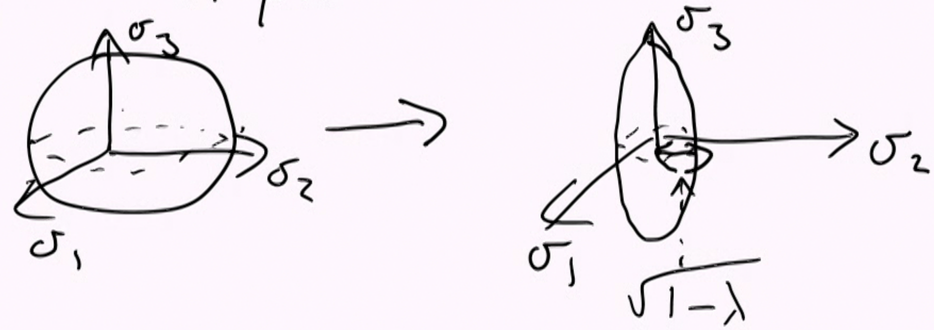
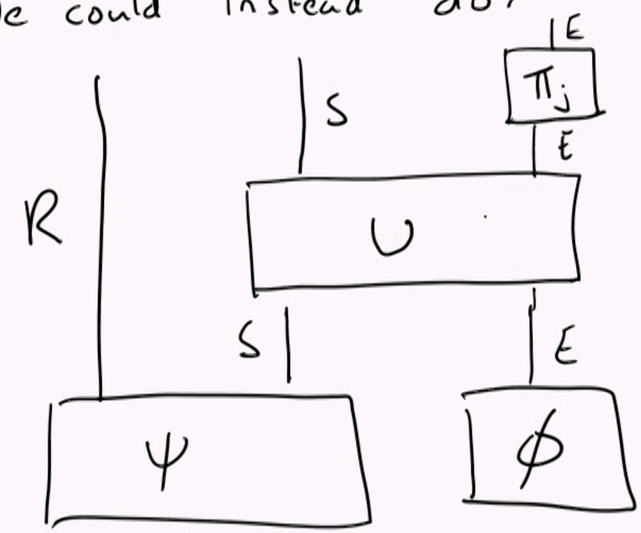


3.7) Positive Operator Valued Measures

The View from the Larger Church

- We have (temporarily) allowed projective measurements as a primitive in the larger church, but we don't need to make a measurement directly on the system.
- We could instead do:



and view the projective measurement on the environment as a measurement of the system

- Can we describe the probabilities for the measurement outcomes in terms of operators acting on $\mathcal{H}_S$ alone?

The View from the Larger Church

$$\mathrm{Prob}(\Pi_j) = \mathrm{Tr}_{SE}\left(\Pi_j^E U_{SE}\, \rho_S \otimes |\phi\rangle_E\langle\phi|\, U_{SE}^\dagger\right)$$

$$= \mathrm{Tr}_{SE}\left(|\phi\rangle_E\langle\phi| U_{SE}^\dagger \Pi_j^E U_{SE}\, \rho_S\right)$$

$$= \mathrm{Tr}_S\left(\sum_k {}_E\langle k|\phi\rangle_E\langle\phi| U_{SE}^\dagger \Pi_j^E U_{SE} |k\rangle_E\, \rho_S\right)$$

$$= \mathrm{Tr}_S\left({}_E\langle\phi| U_{SE}^\dagger \Pi_j^E U_{SE}\left[\sum_k |k\rangle_E\langle k|\right]|\phi\rangle_E\, \rho_S\right)$$

$$= \mathrm{Tr}_S\left({}_E\langle\phi| U_{SE}^\dagger \Pi_j^E U_{SE} |\phi\rangle_E\, \rho_S\right)$$

$$= \mathrm{Tr}_S\left(E_j\, \rho_S\right)$$

where $E_j = {}_E\langle\phi| U_{SE}^\dagger \Pi_j^E U_{SE} |\phi\rangle_E \in \mathcal{L}(\mathcal{H}_S)$

The View from the Larger Church

- Properties of the E_j operators

1) E_j is a positive operator, i.e. ${}_S\langle\psi|E_j|\psi\rangle_S \geq 0 \quad \forall\ |\psi\rangle_S \in \mathcal{H}_S$.

$${}_S\langle\psi|E_j|\psi\rangle_S = \left({}_S\langle\psi| \otimes {}_E\langle\phi|\right) U_{SE}^\dagger \Pi_j^E U_{SE} \left(|\psi\rangle_S \otimes |\phi\rangle_E\right)$$

This is a special case of

${}_{SE}\langle\psi| U_{SE}^\dagger \Pi_j^E U_{SE} |\psi\rangle_{SE}$ and $U_{SE}^\dagger \Pi_j^E U_{SE}$ is of the form $N^\dagger N$ with $N = \Pi_j^E U_{SE}$ so is positive

2) $\sum_j E_j = I_S$

$$\sum_j E_j = {}_E\langle\phi| U_{SE}^\dagger \left(\sum_j \Pi_j^E\right) U_{SE} |\phi\rangle_E = {}_E\langle\phi| U_{SE}^\dagger U_{SE} |\phi\rangle = {}_E\langle\phi| I_{SE} |\phi\rangle_E$$

$$= I_S \langle\phi|\phi\rangle_E = I_S$$

It can be shown that any set of operators $\{E_j\}$ satisfying 1) and 2) can arise in this way.

The View from the Smaller Church

- According to the smaller church any consistent way of assigning probabilities to measurement outcomes should be a valid description of a measurement.
- A measurement outcome should be described by an affine functional from density operators to $\mathbb{R}$

$$f_j(p\rho_A + (1-p)\sigma_A) = p\, f_j(\rho_A) + (1-p)\, f_j(\sigma_A)$$

- We can extend this to a linear functional on $\mathcal{S}(\mathcal{H}_A)$ by defining

$$f_j(-\rho_A) := -f_j(\rho_A)$$

- $\mathcal{S}(\mathcal{H}_A)^\dagger = \mathcal{S}(\mathcal{H}_A)$ and the inner product is $\mathrm{Tr}(MN)$, so we know that

$$f_j = \mathrm{Tr}(E_j \rho_A) \quad \text{for some self adjoint operator } E_j \in \mathcal{S}(\mathcal{H}_A)$$

The View from the Smaller Church

- It is easy to see that E_j has to be positive. $\mathrm{Tr}(E_j \rho_A) \geq 0$ because it is a probability. Let $\rho_A = |\psi\rangle_A\langle\psi|$.

$$0 \leq \mathrm{Tr}\left(E_j \, |\psi\rangle_A\langle\psi|\right) = {}_A\langle\psi| E_j |\psi\rangle$$

and further, all density operators can be written as $\rho_A = \sum_j p_j |\psi_j\rangle_A\langle\psi_j|$ so this is sufficient for $\mathrm{Tr}(E_j \rho_A) \geq 0$ for all ρ_A

- Secondly, we must have

$$\sum_j \mathrm{Tr}(E_j \rho_A) = 1 \quad \text{for all } \rho_A$$

$$\Rightarrow \mathrm{Tr}\left[\left(\sum_j E_j\right)\rho_A\right] = 1$$

$$\Rightarrow {}_A\langle\psi| \left(\sum_j E_j\right) |\psi\rangle_A = 1 \quad \text{for all pure states } |\psi\rangle_A$$

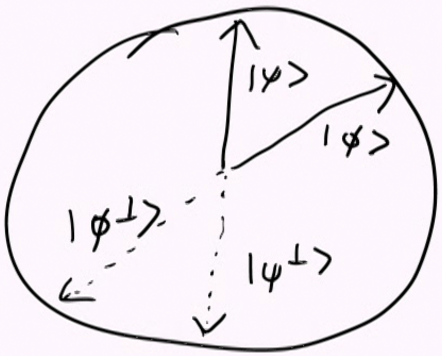
We already proved that this implies $\sum_j E_j = I_A$

Summary

- A general measurement is described by a set $\{E_j\}$ of positive operators such that $\sum_j E_j = I$.
- This is called a positive operator valued measure.
- The probability rule is $\mathrm{Prob}(E_j) = \mathrm{Tr}(E_j \rho)$
- This can be derived from:
  - A system interacts with an initially uncorrelated environment. We perform a projective measurement on the environment.
  - The requirement that measurements should assign well-defined probability distributions to all density operators.

Example

- You shouldn't always think of a POVM as a noisy version of a projective measurement. Sometimes it is the optimal measurement to do.
- E.g. Unambiguous state discrimination



$$E_\phi = a|\psi^\perp\rangle\langle\psi^\perp| \qquad E_\psi = b|\phi^\perp\rangle\langle\phi^\perp|$$

$$E_? = I - E_\phi - E_\psi$$

Maximise a and b such that $E_?$ is a positive operator

$$p_{succ} = 1 - |\langle\phi|\psi\rangle|$$

Continuous Time Dynamics

- A density operator evolves under unitary dynamics according to

$$\rho \longrightarrow U \rho U^{\dagger}$$

- If the unitary is generated by a fixed Hamiltonian $U(t) = e^{-iH(t-t_0)}$ then

$$\rho(t) = e^{-iH(t-t_0)} \rho(t_0) e^{iH(t-t_0)}$$

$$\rho(t+\Delta t) - \rho(t) = [I - iH\Delta t]\rho(t)[I + iH\Delta t] - \rho(t) \quad \text{to 1st order}$$

$$= -i\Delta t \left(H\rho(t) - \rho(t) H \right)$$

$$= -i\Delta t \, [H, \rho(t)]$$

$$\Rightarrow \boxed{\frac{d\rho}{dt} = -i[H, \rho]}$$

This is called the von-Neuman equation.

Continuous Time Dynamics

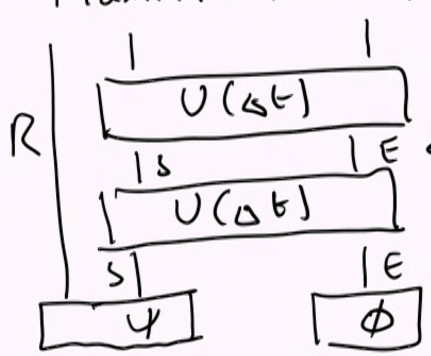
- But we know that finite time dynamics need not be unitary. We can have a completely positive, trace preserving map.

$$\rho \to \mathcal{E}(\rho) = \sum_j M^{(j)} \rho M^{(j)\dagger}$$

- What is the corresponding continuous-time dynamics?
- You might have thought that we can just parameterize $\mathcal{E}$ by t and assume that $\mathcal{E}_{t+\Delta t} = \mathcal{E}_{\Delta t} \circ \mathcal{E}_t$ i.e. $\rho(t_0+t+\Delta t) = \mathcal{E}_{\Delta t}\left(\mathcal{E}_t(\rho(t_0))\right)$
- This would give $\mathcal{E}_t$ the structure of a continuous semi-group.
- But there is a problem with this from the point of view of the larger church.

The View from the Larger Church

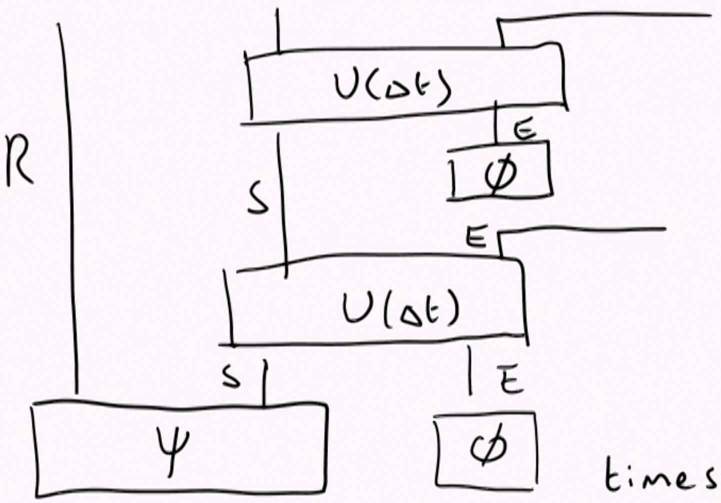
- Recall that, in order to derive CPT maps, we assumed that the system was initially uncorrelated from its environment.
- Thus, if we want $\mathcal{E}_{2\Delta t} = \mathcal{E}_{\Delta t} \circ \mathcal{E}_{\Delta t}$ with $\mathcal{E}_{\Delta t}$ CPT, we need the system to be uncorrelated with its environment after every Δt timestep.
- If the system is interacting with the environment under a fixed Hamiltonian H_{SE} then this won't be true in general



Expect a correlated state at this stage already.

The View from the Larger Church

- So we will have to assume that the interaction with the environment is approximately like this



- The system behaves as if it is interacting with a new uncorrelated environment at every time step.
- This is called the weak coupling limit.
- E.g. suppose the environment is a thermal bath

timescale for rethermalization of the bath $\ll$ timescale on which system gets significantly correlated with environment.

Deriving the Lindblad Equation

- $\mathcal{E}_{\Delta t}$ will have the usual operator sum form

$$\rho(t+\Delta t) = \mathcal{E}_{\Delta t}(\rho(t)) = \sum_{j=0}^{N} M^{(j)} \rho(t) M^{(j)\dagger} \simeq \rho(t) + O(\Delta t)$$

- We want to expand each term up to order Δt.
- We can, without loss of generality, put all of the $O(1)$ term in a single Kraus operator

$$M^{(0)} = I + (\underbrace{L^{(0)} - iH}_{\text{general decomposition of an operator into two Hermitian operators}})\Delta t + O(\Delta t^2)$$

- In order for $M^{(j)} \rho M^{(j)\dagger}$ to contribute for $j = 1, 2, \ldots, N$ we need

$$M^{(j)} = L^{(j)} \sqrt{\Delta t} + O(\Delta t)$$

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Deriving the Lindblad Equation

- Plugging these terms into $\rho(t+\Delta t) = \mathcal{E}_{\Delta t}(\rho(t))$ gives

$$\rho(t+\Delta t) - \rho(t) = \left[\left(L^{(0)} - iH\right)\rho(t) + \rho(t)\left(L^{(0)} + iH\right) + \sum_{j=1}^{N} L^{(j)}\rho(t)L^{(j)\dagger} \right]\Delta t$$

$$= \Big[\underbrace{-i\,[H,\rho]}_{\substack{\text{unitary} \\ \text{part}}} + \underbrace{\{L^{(0)},\rho(t)\}}_{\substack{\text{anticommutator} \\ L^{(0)}\rho(t)+\rho(t)L^{(0)}}} + \sum_{j=1}^{N} L^{(j)}\rho(t)L^{(j)\dagger} \Big]\Delta t$$

$$\therefore \quad \frac{d\rho}{dt} = -i\,[H,\rho] + \{L^{(0)},\rho(t)\} + \sum_{j=1}^{N} L^{(j)}\rho(t)L^{(j)\dagger}$$

Deriving the Lindblad Equation

- We still have to impose the trace preserving condition

$$\sum_j M^{(j)\dagger} M^{(j)} = I$$

$$M^{(0)\dagger} M^{(0)} = \left[I + (L^{(0)} + iH)\Delta t \right] \left[I + (L^{(0)} - iH)\Delta t \right]$$

$$= I + (L^{(0)} + \cancel{iH} + L^{(0)} - \cancel{iH})\Delta t + O(\Delta t^2)$$

$$= I + 2L^{(0)}\Delta t + O(\Delta t^2)$$

$$\sum_{j=1}^{N} M^{(j)\dagger} M^{(j)} = \left(\sum_{j=1}^{N} L^{(j)\dagger} L^{(j)} \right)\Delta t + O(\Delta t^2)$$

$$\Rightarrow L^{(0)} = -\frac{1}{2} \sum_{j=1}^{N} L^{(j)\dagger} L^{(j)}$$

$$\boxed{\frac{d\rho}{dt} = -i[H, \rho] + \sum_{j=1}^{N} \left(L^{(j)} \rho L^{(j)\dagger} - \frac{1}{2} \left\{ L^{(j)\dagger} L^{(j)}, \rho \right\} \right)}$$

Deriving the Lindblad Equation

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$$\boxed{\frac{d\rho}{dt} = -i[H,\rho] + \sum_{j=1}^{N}\left(L^{(j)}\rho L^{(j)\dagger} - \frac{1}{2}\left\{ L^{(j)\dagger} L^{(j)}, \rho \right\} \right)}$$

Example: Decoherence

- Consider a qubit with Hamiltonian $H=0$ and a single Lindblad operator

$$L = \gamma \sigma_3$$

- Then we get $\frac{d\rho}{dt} = \gamma^2 (\sigma_3 \rho \sigma_3 - \rho)$
- In terms of components $\begin{pmatrix} \dot\rho_{00} & \dot\rho_{01} \\ \dot\rho_{10} & \dot\rho_{11} \end{pmatrix} = \begin{pmatrix} 0 & -2\gamma^2\rho_{01} \\ -2\gamma^2\rho_{10} & 0 \end{pmatrix}$

so we get the solution:

$$\rho(t) = \begin{pmatrix} \rho_{00}(0) & \rho_{01}(0)e^{-2\gamma^2 t} \\ \rho_{10}(0)e^{-2\gamma^2 t} & \rho_{11}(0) \end{pmatrix}$$

The off-diagonal elements decay exponentially

System decoheres in the $|0\rangle, |1\rangle$ basis.