Title: PSI 2016/2017 Foundations of Quantum Mechanics (Review) - Lecture 3

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Abstract:

# Quantum Foundations: Lecture 3

2) Basic Phenomenology

3) The Generalized Formalism

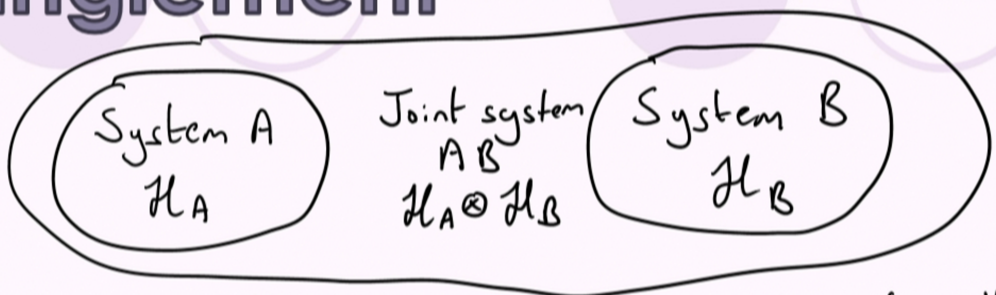
# Today's Lecture

2. Basic Phenomenology
   3. Entanglement and EPR
   4. The No-Cloning Theorem
3. The Generalized Formalism
   1. Vectors, Dual Vectors, and Inner Products
   2. Tensor Products and Partial Inner Products
   3. Some Hilbert Spaces

# 2.3) Entanglement and EPR

- In 1935, Einstein, Podolsky and Rosen pointed out a conflict between orthodox quantum mechanics and locality. – A. Einstein, B. Podolsky, N. Rosen, "Can Quantum-Mechanical Description of Physical Reality be Considered Complete?," *Phys. Rev.*, vol. 47 pp. 777–780 (1935).
- "When two systems, of which we know the states by their respective representatives, enter into temporary physical interaction due to known forces between them, and when after a time of mutual influence the systems separate again, then they can no longer be described in the same way as before, viz. by endowing each of them with a representative of its own. I would not call that *one* but rather *the* characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought. By the interaction the two representatives [the quantum states] have become entangled." – E. Schrödinger "Discussion of Probability Relations Between Separated Systems," *Proc. Cambridge Phil. Soc.*, 31, pp. 555–563 (1935).

# Entanglement




- The Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ consists of all states of the form

$$|\psi\rangle_{AB} = \sum_{jk} \alpha_{jk} |j\rangle_A \otimes |k\rangle_B$$

- A state $|\psi\rangle_{AB}$ is a product state if it can be written as

$$|\psi\rangle_{AB} = |\phi\rangle_A \otimes |\chi\rangle_B \quad \text{for some } |\phi\rangle_A \in \mathcal{H}_A, \ |\chi\rangle_B \in \mathcal{H}_B$$

- Otherwise it is an entangled state
- According to the orthodox interpretation A and B have no individual properties when AB is entangled.

# Entanglement

- For 2-qubits it is straightforward to prove that

$$|\psi\rangle_{AB} = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

is entangled iff

$$\alpha_{00}\alpha_{11} = \alpha_{01}\alpha_{10}$$

- So, in particular,

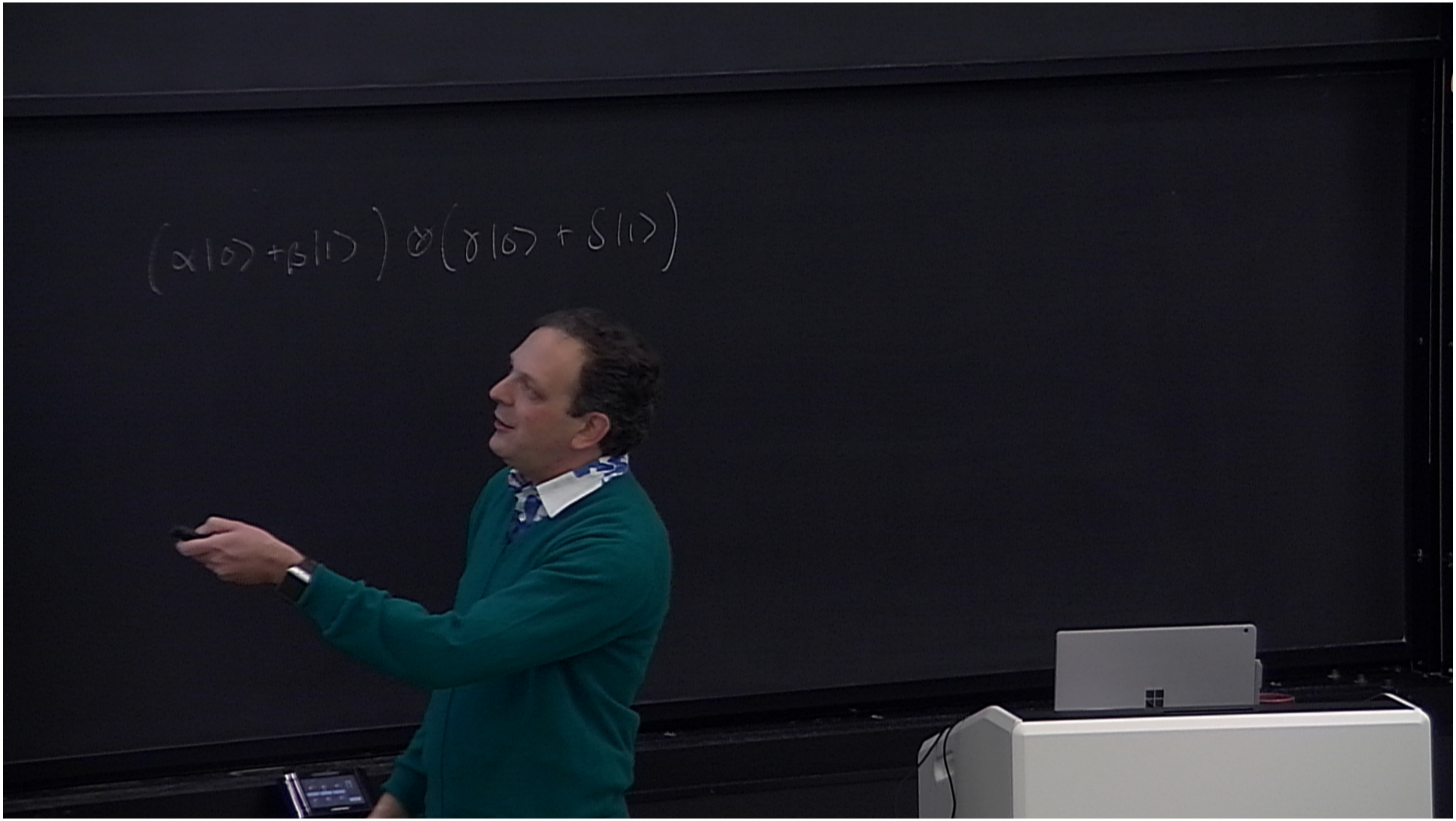
$$|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB}\right)$$

is an entangled state.

- Note: If $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\vec{n}\cdot\vec{\sigma} = n_1\sigma_1 + n_3\sigma_3$ with $\vec{n} = \begin{pmatrix} n_1 \\ n_3 \end{pmatrix}$ a unit vector in the $x$-$z$ plane then

$$|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|\vec{n}_+\rangle_A|\vec{n}_+\rangle_B + |\vec{n}_-\rangle_A|\vec{n}_-\rangle_B\right)$$

with $\vec{n}\cdot\vec{\sigma}\,|\vec{n}_\pm\rangle = \pm|\vec{n}_\pm\rangle$


$(\alpha|0\rangle+\beta|1\rangle)\otimes(\gamma|0\rangle+\delta|1\rangle)$

# Partial Measurement

- If we measure one of the subsystems of a joint system in a complete orthonormal basis, then after the measurement the state gets updated to a product state.
- Joint system starts in state

$$|\psi\rangle_{AB} = \sum_{jk} \alpha_{jk} |j\rangle_A \otimes |k\rangle_B$$

- A is measured in basis $\{|\phi_m\rangle\}$, outcome $|\phi_n\rangle$ is obtained.
- B gets updated to

$$\frac{{}_A\langle\phi_n|\psi\rangle_{AB}}{\|{}_A\langle\phi_n|\psi\rangle_{AB}\|^2} = \frac{\sum_{jk} \alpha_{jk} \langle\phi_n|j\rangle \, |k\rangle_B}{\sum_k |\alpha_{jk}\langle\phi_n|j\rangle|^2}$$

# Partial Measurement

- In particular, if $|\Phi^+\rangle_{AB}$ is measured in the basis $\{|0\rangle_A, |1\rangle_A\}$ then system B ends up in the state

  $|0\rangle_B$ if $|0\rangle_A$ is found

  or $|1\rangle_B$ if $|1\rangle_A$ is found

- More generally, if $|\Phi^+\rangle_{AB}$ is measured in the basis $\{|\vec{n}+\rangle_A, |\vec{n}-\rangle_A\}$ then B ends up in

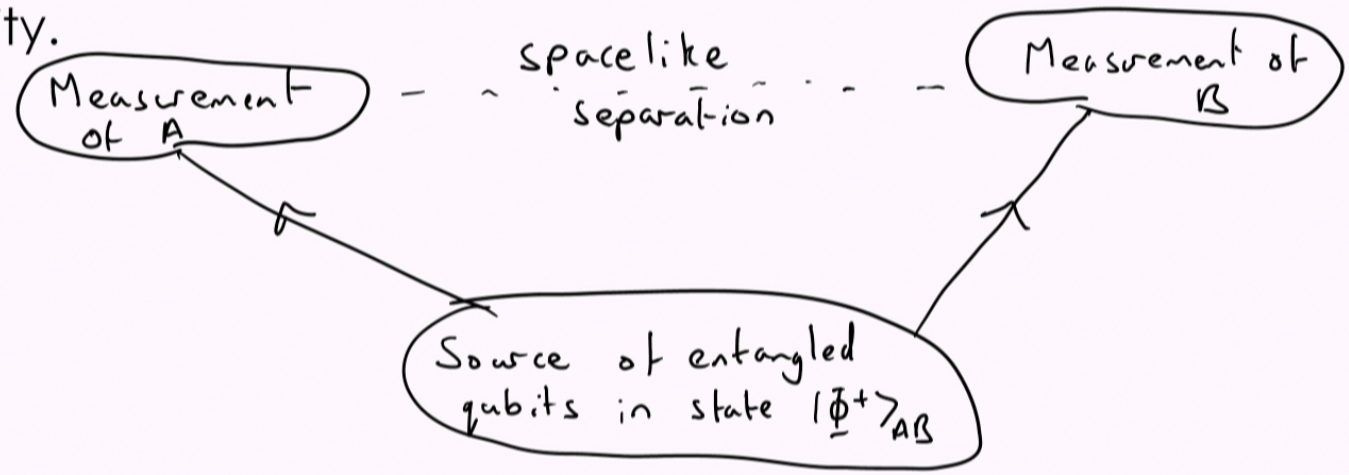
  $|\vec{n}+\rangle_B$ if $|\vec{n}+\rangle_A$ is found

  $|\vec{n}-\rangle_B$ if $|\vec{n}-\rangle_A$ is found

- We will be able to predict the result of a $\vec{n}\cdot\vec{\sigma}$ measurement on system B with certainty.

# The EPR Criterion of Reality

- "If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of reality corresponding to that quantity." – A. Einstein, B. Podolsky, N. Rosen, "Can Quantum-Mechanical Description of Physical Reality be Considered Complete?," *Phys. Rev.*, vol. 47 pp. 777–780 (1935).
- We can ensure that a measurement of $A$ "does not disturb" $B$ by locality.



# The EPR Argument

- By the EPR criterion and locality, system $B$ must have an element of reality that determines the outcome of a $\{|0\rangle_B, |1\rangle_B\}$ measurement before $A$ is measured.
- The orthodox interpretation is nonlocal, because this "pops into existence" from nothing when $A$ is measured.
- But note: Any interpretation in which measurement of $\{|0\rangle_B, |1\rangle_B\}$ is undetermined before $A$ is measured would also be nonlocal by the EPR criterion.
- Note that, because of the perfect correlations in all $\{|\vec{n}+\rangle, |\vec{n}-\rangle\}$ measurements, the same is true for all possible measurement directions. Having all of these elements of reality would violate the uncertainty principle for $B$.
  - This is irrelevant to the main argument, which holds for just one measurement.
  - However, one can use this to show that a local theory must also be $\psi$-epistemic – N. Harrigan, R. Spekkens, Found. Phys. 40, 125 (2010).
- Bell's Theorem will show that no completion of quantum theory can restore locality.

# 2.4) The No-Cloning Theorem

- If $0 < |\langle\phi|\psi\rangle| < 1$ then there is no physical operation that outputs $|\psi\rangle \otimes |\psi\rangle$ when $|\psi\rangle$ is input and also $|\phi\rangle \otimes |\phi\rangle$ when $|\phi\rangle$ is input.

Proof:

- Physical operations must be unitary, so let $|\chi\rangle$ be a fixed state on the same Hilbert space as $|\psi\rangle$ and $|\phi\rangle$.
- A cloning unitary would satisfy

$$U|\psi\rangle \otimes |\chi\rangle = |\psi\rangle \otimes |\psi\rangle \qquad U|\phi\rangle \otimes |\chi\rangle = |\phi\rangle \otimes |\phi\rangle$$

- Unitaries preserve inner products, so

$$\langle\phi| \otimes \langle\chi| U^\dagger U |\psi\rangle \otimes |\chi\rangle = \left(\langle\phi| \otimes \langle\phi|\right)\left(|\psi\rangle \otimes |\psi\rangle\right)$$

$$\Rightarrow \quad \langle\phi|\psi\rangle\langle\chi|\chi\rangle = \langle\phi|\psi\rangle^2$$

$$\Rightarrow \quad \langle\phi|\psi\rangle = \langle\phi|\psi\rangle^2 \qquad \Rightarrow \quad |\langle\phi|\psi\rangle| = 0 \text{ or } \underline{1}$$

# Comments on No-Cloning

- No-cloning is related to a number of other key features of quantum theory:
  - If we could perfectly clone, we could create an arbitrarily large number of copies of the initial state. Would allow us to determine the state exactly from just one initial copy.
  - This would allow us to signal superluminally in the EPR experiment (consider what would happen if we could clone state of $B$ after measurement of $A$).
  - Could measure any observable without disturbing the state of the system (just clone first and put one copy to the side).
- So its good that no-cloning holds, but we should explain why. In particular, if the quantum state really exists then why should it be uncopiable (suggests $\psi$-epistemic interpretation).

# 3) The Generalized Formalism

# 3.1) Vectors, Dual Vectors, and Inner Products

- You are familiar with the "kets" $|\psi\rangle$ and "bras" of Dirac notation.
- $\langle\psi|$ lives in a different vector space from $|\psi\rangle$ called the dual vector space
- In quantum physics we are often quite careless with the distinction, but it will help to be explicit here.

# Dual Vector Spaces

- Given a vector space $V$ over a field $\mathbb{F}$ (always $\mathbb{C}$ or $\mathbb{R}$ here), the dual vector space $V^\dagger$ is the set of linear functionals from $V$ to $\mathbb{F}$.

$$\langle g| : V \longrightarrow \mathbb{F}$$

$$\langle g| \big( \alpha|\psi\rangle + \beta|\phi\rangle \big) = \alpha\langle g|\psi\rangle + \beta\langle g|\phi\rangle$$

- $V^\dagger$ is itself a vector space, if we define $\alpha\langle g| + \beta\langle f|$ via

$$\big( \alpha\langle g| + \beta\langle f| \big)|\psi\rangle = \alpha\langle g|\psi\rangle + \beta\langle f|\psi\rangle$$

# Inner Products

- An inner product on a vector space $V$ is a functional $V \times V \to \mathbb{F}$ satisfying
  - Conjugate symmetry $(|\phi\rangle, |\psi\rangle) = (|\psi\rangle, |\phi\rangle)^*$
  - Linearity: $(|\chi\rangle, \alpha|\psi\rangle + \beta|\phi\rangle) = \alpha(|\chi\rangle, |\psi\rangle) + \beta(|\chi\rangle, |\phi\rangle)$
  - Positive definiteness: $(|\psi\rangle, |\psi\rangle) \geq 0$

    with $(|\psi\rangle, |\psi\rangle) = 0$ iff $|\psi\rangle =$ the zero vector
- A vector space with an inner product is called a Hilbert space (we are in finite dimensions, so ignoring $\infty$-dimensional complications)
- We usually use $\mathcal{H}$ to denote a Hilbert space.

# Inner Products and Dual Vectors

- In a Hilbert space, the inner product induces an isomorphism $\mathcal{H} \equiv \mathcal{H}^\dagger$
- Given a vector $|\psi\rangle \in \mathcal{H}$, we can define a dual vector $\langle\psi| \in \mathcal{H}^\dagger$ via

$$\langle\psi|\phi\rangle = (|\psi\rangle, |\phi\rangle)$$

Linearity of inner product ensures this is a linear functional.

- Given a dual vector $\langle g| \in \mathcal{H}^\dagger$, let $\{|j\rangle\}$ be an orthonormal basis for $\mathcal{H}$ and define $g_j = \langle g|j\rangle$

Then define $|g\rangle = \sum_j g^{\dagger j}|j\rangle$ with $g^{\dagger j} = g_j^*$

Straightforward to prove that this is an isomorphism.

# 3.2) Tensor Products and Partial Inner Products

- Suppose $\mathcal{H}_A$ has an orthonormal basis $|j\rangle_A$ $\quad |\psi\rangle_A = \sum_j \psi^j |j\rangle_A$

  $\mathcal{H}_B$ " " " " $|k\rangle_B$ $\quad |\phi\rangle_B = \sum_k \phi^k |k\rangle_B$

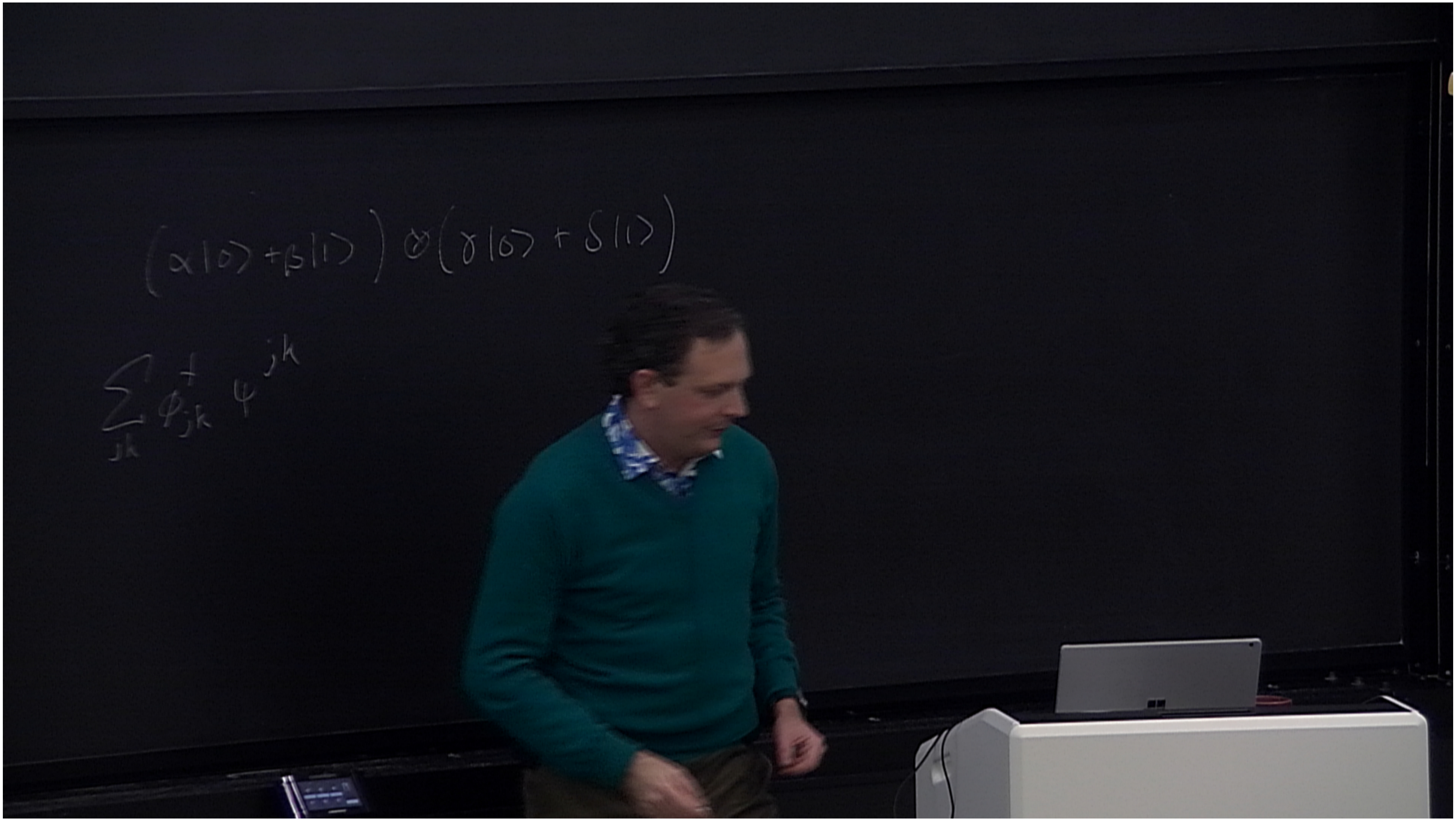
- The tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$ is the vector space spanned by $|j\rangle_A \otimes |k\rangle_B$

$$|\psi\rangle_{AB} = \sum_{jk} \psi^{jk} |j\rangle_A \otimes |k\rangle_B$$

- It is a Hilbert space, inheriting its inner product from $\mathcal{H}_A$ and $\mathcal{H}_B$

  i.e. if $|\phi\rangle_{AB} = \sum_{jk} \phi^{jk} |j\rangle_A \otimes |k\rangle_B$

$$\langle\phi|\psi\rangle_{AB} = \sum_{jklm} \phi^\dagger_{jk} \psi^{lm} \langle j|l\rangle_A \langle k|m\rangle_B = \sum_{jk} \phi^\dagger_{jk} \psi^{jk} \quad \text{where } \phi^\dagger_{jk} = \phi^{jk\,*}$$


$(\alpha|0\rangle+\beta|1\rangle)\otimes(\gamma|0\rangle+\delta|1\rangle)$
$\sum_{jk}\phi^{\dagger}_{jk}\psi^{jk}$

# Abstract Index Notation

- It is cumbersome to keep track of long strings of bras/kets

  e.g. $|j\rangle_A \otimes |k\rangle_B \otimes |l\rangle_C \otimes \ldots\ldots$

- We can develop an abstract index notation similar to that used in differential geometry and GR.

$$|\psi\rangle_A = \sum_j \psi^j |j\rangle_A \Rightarrow \psi^{j_A} \qquad {}_A\langle\phi| = \sum_j \phi_j \, {}_A\langle j| \Rightarrow \phi_{j_A}$$

$$\langle\phi|\psi\rangle_A = \phi_{j_A}\psi^{j_A}$$ ← summation convention for repeated indices

- It is necessary to include the label $A$ of the Hilbert space $\mathcal{H}_A$ in the index $j_A$ because Hilbert spaces may have different dimensions

  Only upper $A$ indices can be contracted with lower $A$ indices.

# Abstract Index Notation

- For a tensor product space $\mathcal{H}_A \otimes \mathcal{H}_B$, we would have

$$|\psi\rangle_{AB} = \sum_{jk} \psi^{jk} |j\rangle_A \otimes |k\rangle_B \Rightarrow \psi^{j_A k_B}$$

$$_{AB}\langle\phi| = \sum_{jk} \phi_{jk} \, {}_A\langle j| \otimes {}_B\langle k| \Rightarrow \phi_{j_A k_B}$$

- The inner product is

$$\langle\phi|\psi\rangle_{AB} = \phi_{j_A k_B} \psi^{j_A k_B}$$

- However $\phi_{j_A k_B} \psi^{k_A j_B}$ is not a valid contraction

# More Interesting Tensor Products

- $\mathcal{H}_A$ and $\mathcal{H}_B^\dagger$ are both Hilbert spaces, so there is no reason why we can't form the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B^\dagger$
- This would be the vector space of objects of the form

$$\sum_{jk} \psi^j_{\ k} \, |j\rangle_A \otimes {}_B\langle k| \quad \Rightarrow \quad \psi^{j_A}_{\ k_B}$$

- The dual space to $\mathcal{H}_A \otimes \mathcal{H}_B^\dagger$ is $(\mathcal{H}_A \otimes \mathcal{H}_B^\dagger)^\dagger = \mathcal{H}_A^\dagger \otimes \mathcal{H}_B$

$$\sum_{jk} \phi^{\ k}_{j} \, {}_A\langle j| \otimes |k\rangle_B \quad \Rightarrow \quad \phi^{\ k_B}_{j_A}$$

- The inner product is given by $\phi^{\ k_B}_{j_A} \psi^{j_A}_{\ k_B}$

# More Interesting Tensor Products

- An object like $\sum_{jk} \psi^{j}_{k} |j\rangle_A \otimes {}_B\langle k|$ is neither a ket nor a bra.
- However $\mathcal{H}_A \otimes \mathcal{H}_B^\dagger$ is still just a Hilbert space, like any other.
- Sometimes it will be useful to think of $\mathcal{H}_A \otimes \mathcal{H}_B^\dagger$ as a space of "kets" and its dual $\mathcal{H}_A^\dagger \otimes \mathcal{H}_B$ as a space of "bras".
- When doing so, I will use round brackets as a reminder that they are not primitive kets and bras.

$$|\psi)_{AB} = \sum_{jk} \psi^{j}_{k} |j\rangle_A \otimes {}_B\langle k|$$

$${}_{AB}(\phi| = \sum_{jk} \phi_{j}^{k} \, {}_A\langle j| \otimes |k\rangle_B$$

$$(\phi|\psi)_{AB} = \phi_{j_A}^{k_B} \psi^{j_A}_{k_B}$$

# Diagrammatic Notation

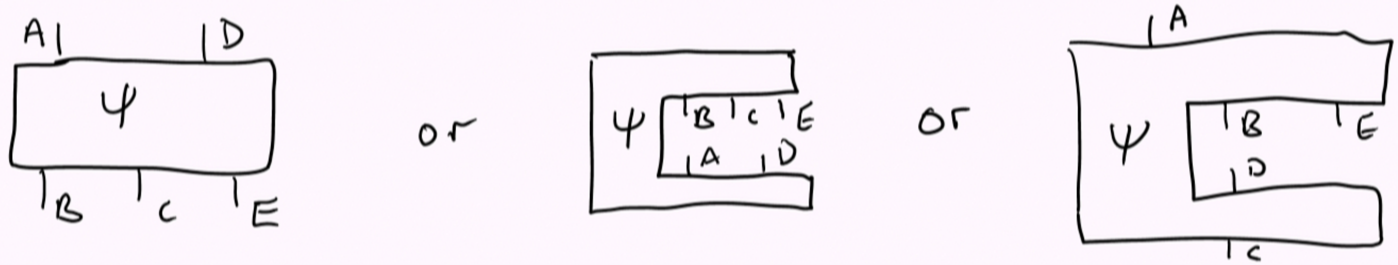
- Clearly, we can iterate this construction and consider complicated tensor products

e.g. $\mathcal{H}_1 = \mathcal{H}_A \otimes \mathcal{H}_B^\dagger \otimes \mathcal{H}_C^\dagger \otimes \mathcal{H}_D \otimes \mathcal{H}_E^\dagger$ $\quad \psi^{j_A m_D}_{\phantom{j_A m_D} k_B l_C n_E}$

$\mathcal{H}_1^\dagger = \mathcal{H}_A^\dagger \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D^\dagger \otimes \mathcal{H}_E$ $\quad \phi^{k_B l_C n_E}_{\phantom{k_B l_C n_E} j_A m_D}$

- Even abstract indices can get cumbersome after a while, so let's develop a way of representing them with diagrams
- A tensor is represented by a box of arbitrary shape
  A vector index by an upwards oriented line with Hilbert space label
  A dual index by a downwards oriented line with Hilbert space label

e.g

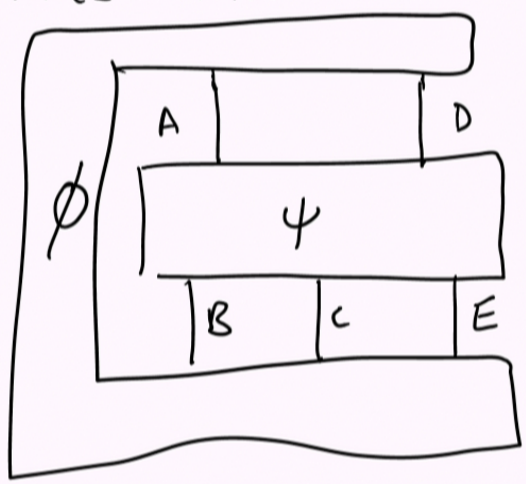


# Diagrammatic Notation

$$\mathcal{H}_1 = \mathcal{H}_A \otimes \mathcal{H}_B^\dagger \otimes \mathcal{H}_C^\dagger \otimes \mathcal{H}_D \otimes \mathcal{H}_E^\dagger$$

$$\mathcal{H}_1^\dagger = \mathcal{H}_A^\dagger \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D^\dagger \otimes \mathcal{H}_E$$

- We can then represent an inner product $\phi^{k_B l_C n_E}_{j_A m_D} \psi^{j_A m_D}_{k_B l_C n_E}$ by connecting lines like this

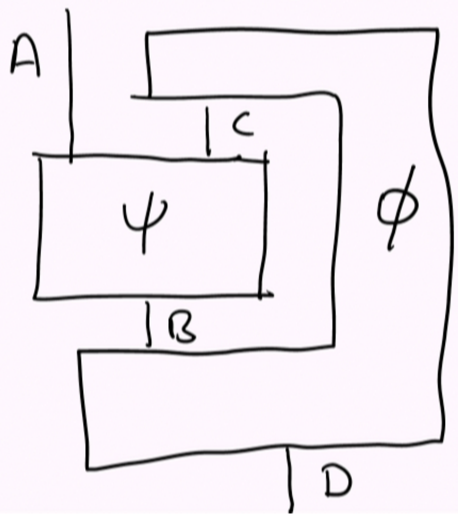


# Partial Inner Products

- We can also define partial inner products where we only contract over some of the indicies

e.g. $\psi^{j_A \, l_C}_{\;\; k_B} \in \mathcal{H}_A \otimes \mathcal{H}_B^\dagger \otimes \mathcal{H}_C \qquad \phi^{k_B}_{\;\; l_C m_D} \in \mathcal{H}_B \otimes \mathcal{H}_C^\dagger \otimes \mathcal{H}_D^\dagger$

$$\phi^{k_B}_{\;\; l_C m_D} \psi^{j_A \, l_C}_{\;\; k_B} = \chi^{j_A}_{\;\; m_D}$$

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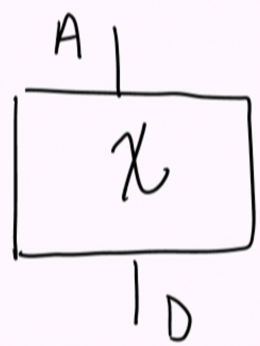

# Raising and Lowering Indices

- Consider the vector $|\delta\rangle_{AA} = \sum_j |j\rangle_A \otimes |j\rangle_A = \sum_{jk} \delta^{jk} |j\rangle_A \otimes |k\rangle_A$

  where $\delta^{jk} = \begin{cases} 1, & j=k \\ 0, & j \neq k \end{cases}$. This lives in $\mathcal{H}_A \otimes \mathcal{H}_A$

- Abstract index notation : $\delta^{j_A k_A}$

- Diagramatic notation :


- A partial inner product with this turns a bra into a ket

$$\delta^{j_A k_A} \psi_{k_A} = \psi^{k_A}$$

# Raising and Lowering Indices

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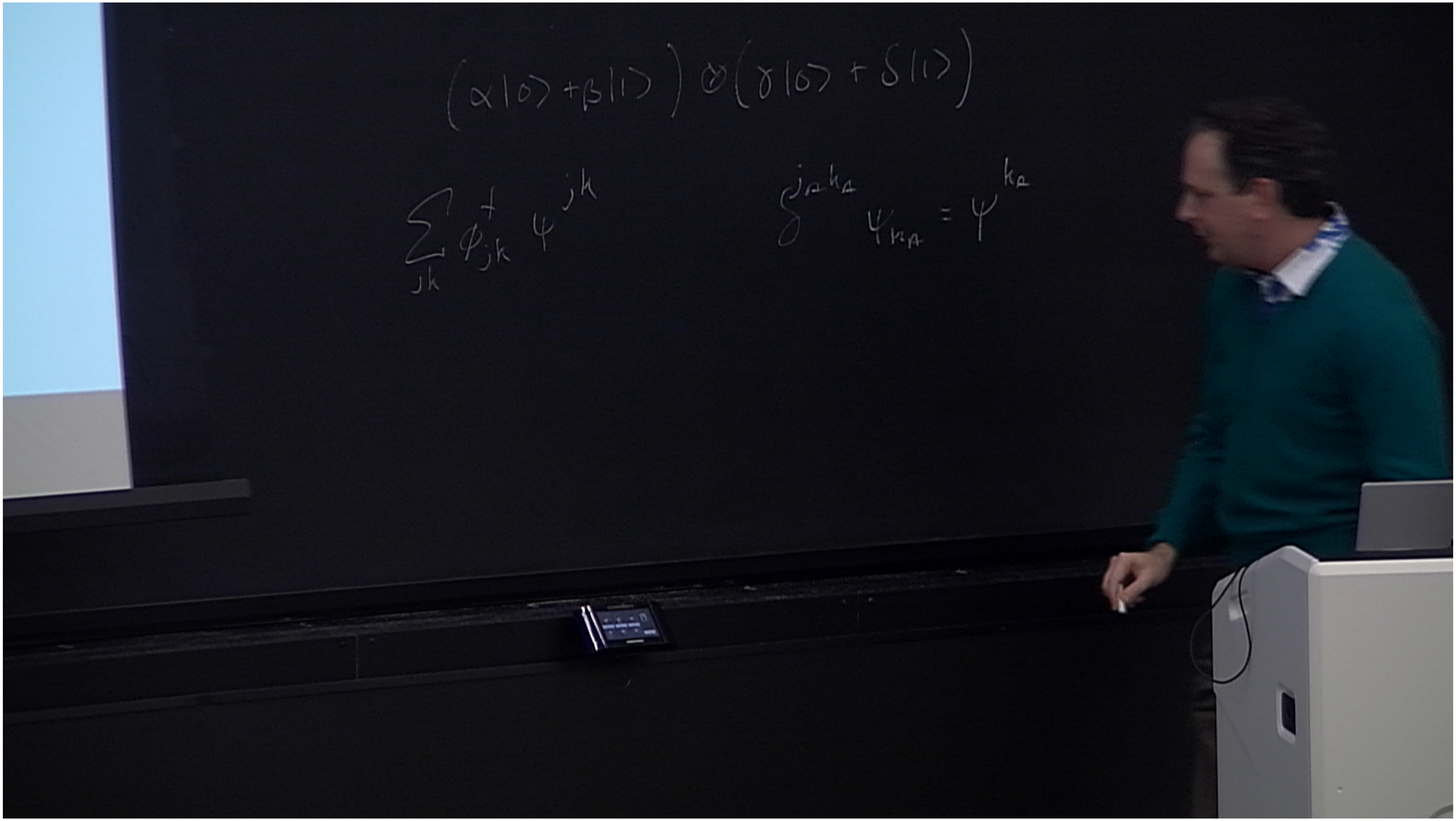
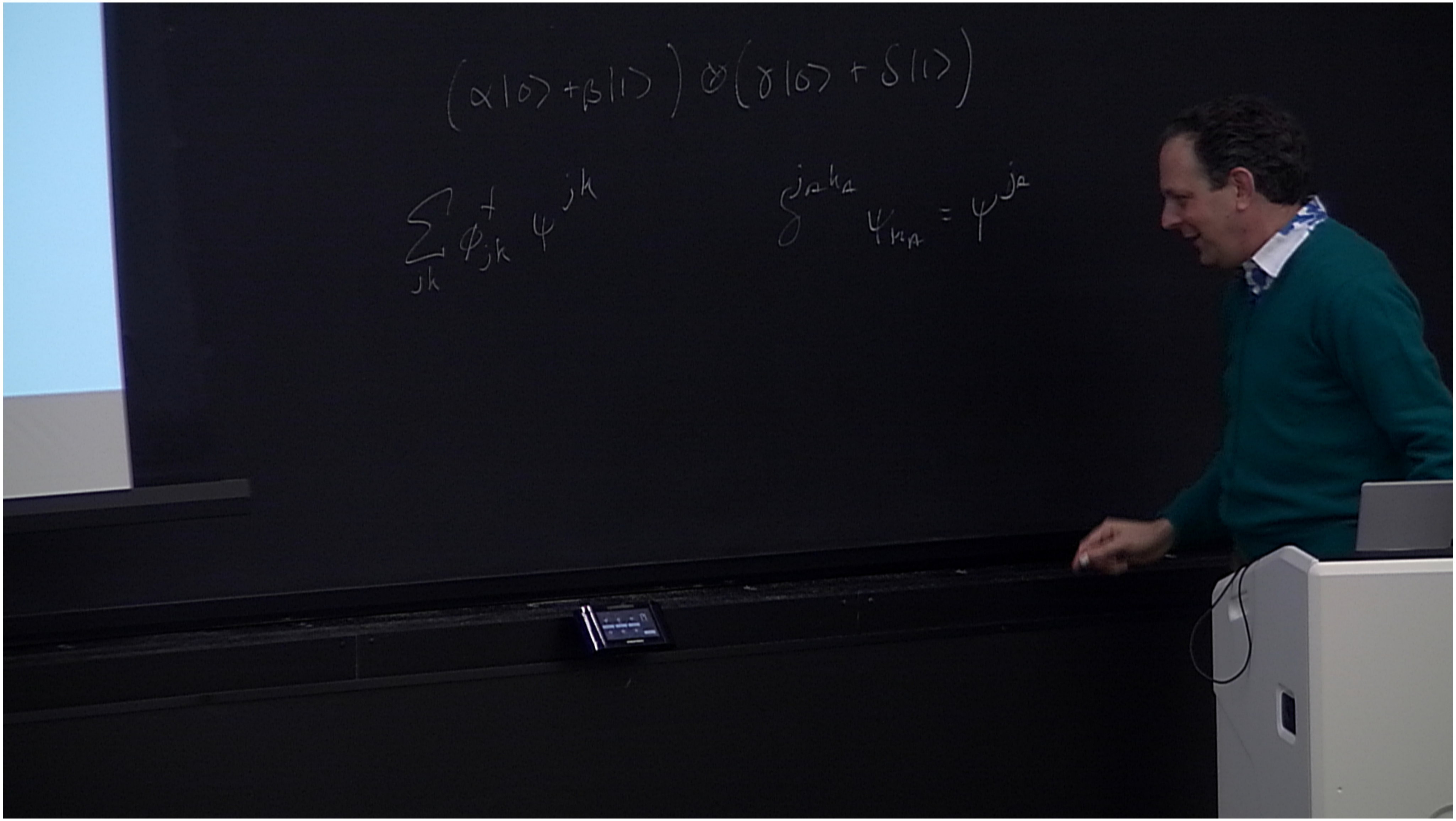

$(\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$
$\sum_{jk} \phi^{\dagger}_{jk} \psi^{jk}$
$g^{j_A k_A} \psi_{k_A} = \psi^{k_A}$


$(\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$
$\sum_{jk} \phi^{\dagger}_{jk} \psi^{jk}$
$g^{j_A k_A} \psi_{k_A} = \psi^{j_A}$

# Raising and Lowering Indices

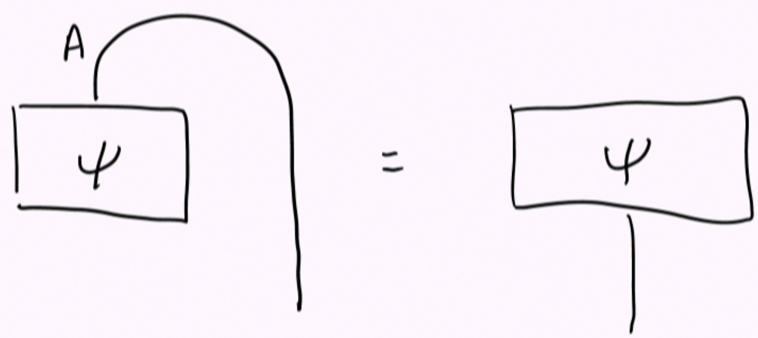
- Similarly the dual vector ${}_{AA}\langle\delta| = \sum_j {}_A\langle j| \otimes {}_A\langle j| = \sum_{jk} \delta_{jk}\, {}_A\langle j| \otimes {}_A\langle k|$

  turns kets into bras

- Abstract notation $\delta_{j_A k_A}$

- Diagramatic notation


$$\delta_{j_A k_A} \psi^{k_A} = \psi_{k_A}$$

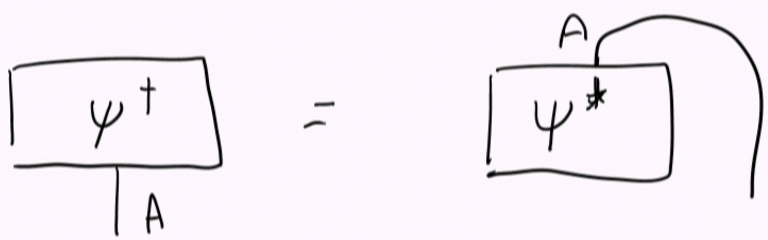


# Taking Duals

- Note that $\psi^{j_A} \to \psi_{j_A}$ is not the same as $|\psi\rangle_A \longrightarrow {}_A\langle\psi|$ if we are in a complex Hilbert space
- To take a dual, you have to change bras $\Leftrightarrow$ kets <u>AND</u> take the complex conjugate of coefficients

$$|\psi\rangle_A = \sum_j \psi^j |j\rangle_A \longrightarrow {}_A\langle\psi| = \sum_j \psi^\dagger{}_j \; {}_A\langle j|$$

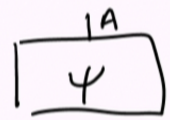
$$\text{where} \quad \psi^\dagger{}_{j_A} = \psi^{*j_A} \delta_{j_A k_A}$$



# 3.3) Some Useful Hilbert Spaces

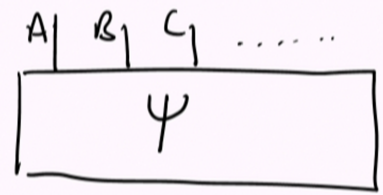
- The space of (pure) quantum states is just a boring old ket space

$|\psi\rangle_A \in \mathcal{H}_A$ $\quad\quad$ $\psi^{j_A}$




- If we have multiple subsystems $A, B, C, \ldots$ then there is a tensor product factor for each subsystem

$|\psi\rangle_{ABC\ldots} \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \ldots$ $\quad\quad$ $\psi^{j_A k_B l_C \ldots}$




- But it is still just kets so relatively boring.

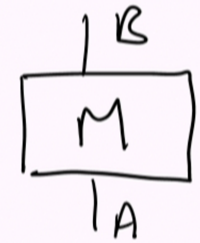
# The Space of Linear Operators

- Now let's consider the space of linear operators from $\mathcal{H}_A$ to $\mathcal{H}_B$ denoted $\mathcal{L}(\mathcal{H}_A \to \mathcal{H}_B)$ [just $\mathcal{L}(\mathcal{H}_A)$ if $\mathcal{H}_A = \mathcal{H}_B$]
- In Dirac notation, we know that an operator can be written in terms of its matrix elements

$$M = \sum_{jk} M^j_k \, |j\rangle_B \, {}_A\langle k| \qquad \text{where} \qquad M^j_k = {}_B\langle j | M | k \rangle_A$$

- But this looks just like an object in $\mathcal{H}_B \otimes \mathcal{H}_A^\dagger$

$$\sum_{jk} M^j_k \, |j\rangle_B \otimes {}_A\langle k| \Longleftrightarrow M^{j_B}_{k_A} \Longleftrightarrow$$

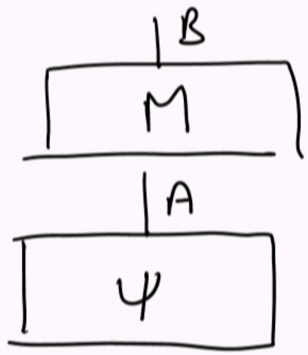


# The Space of Linear Operators

- If we treat $M$ as an element of $\mathcal{H}_B \otimes \mathcal{H}_A^\dagger$ then the action of $M$ on a vector $|\psi\rangle_A \in \mathcal{H}_A$ is just partial inner product

$$M = \sum_{jk} M^j_k \, |j\rangle_B \otimes {}_A\langle k| \qquad |\psi\rangle_A = \sum_l \psi^l \, |l\rangle_A$$

$$M|\psi\rangle_A = \sum_{jk} M^j_k \psi^k \, |j\rangle_B \qquad \text{or} \qquad M^{j_A}_{k_B} \psi^{k_B}$$



# The Space of Linear Operators

- In general, the space of linear operators from $\mathcal{H}_A$ to $\mathcal{H}_B$ is (isomorphic to) $\mathcal{H}_B \otimes \mathcal{H}_A^\dagger \equiv \mathcal{L}(\mathcal{H}_A \to \mathcal{H}_B)$

  output space. → $\mathcal{H}_B$ ; $\mathcal{H}_A^\dagger$ ← dual of input space

- Everything can be done with tensor products and partial inner products!
- Since $\mathcal{H}_B \otimes \mathcal{H}_A^\dagger \equiv \mathcal{L}(\mathcal{H}_A \to \mathcal{H}_B)$ is a Hilbert space, it must have an inner product.
- General prescription for inner products:

  Take 2 vectors $|\psi\rangle$ $|\phi\rangle$

  Dualize one of them $|\phi\rangle \to \langle\phi|$

  Act with $\langle\phi|$ on $|\psi\rangle$ $\langle\phi|\psi\rangle$

# Duperators and Inner products of Operators

- The dual of an operator is a linear functional from operators to scalars (a duperator)
- $(\mathcal{H}_B \otimes \mathcal{H}_A^\dagger)^\dagger = \mathcal{H}_A \otimes \mathcal{H}_B^\dagger$ so $\mathcal{L}(\mathcal{H}_A \to \mathcal{H}_B)^\dagger = \mathcal{L}(\mathcal{H}_B \to \mathcal{H}_A)$

  i.e. The duperators from A to B are the operators from B to A
- Let $|M) = M^{j_B}_{\ k_A}$ $\quad |N) = N^{j_B}_{\ k_A} \to (N| = N^{\dagger\, k_A}_{\ \ j_B} = \delta_{j_B l_B}\delta^{k_A m_a} N^{*\, l_B}_{\ m_a}$

$$(N|M) = N^{\dagger\, k_A}_{\ \ j_B} M^{j_B}_{\ k_A} = \mathrm{Tr}(N^\dagger M)$$

- This is called the Hilbert-Schmidt inner product.
- It is just the standard inner product on $\mathcal{H}_B \otimes \mathcal{H}_A^\dagger$