Title: PSI 2016/2017 Gravitational Physics (Review) - Lecture 10

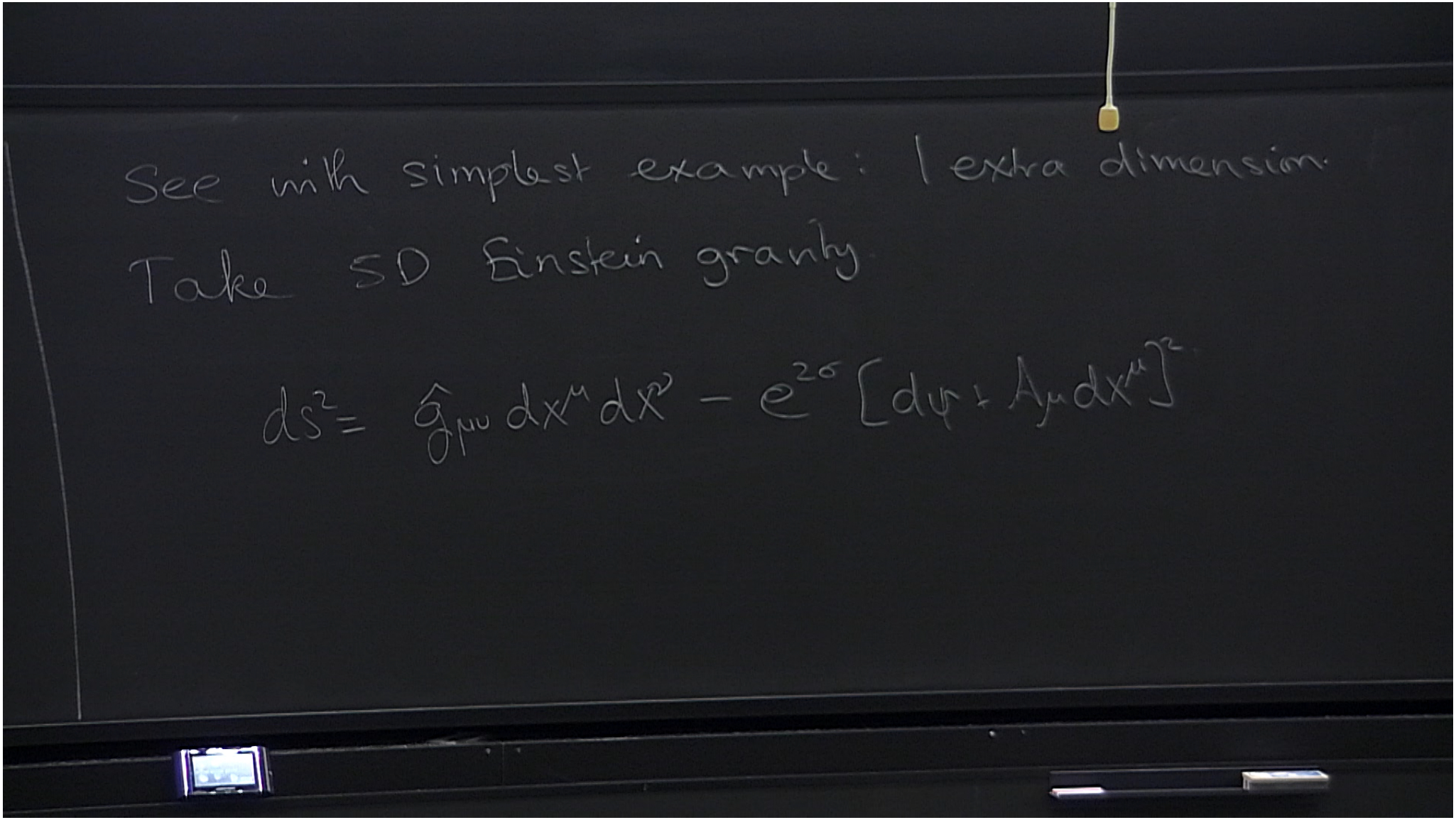
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Abstract:

LECTURE 10: Kaluza-Klein Theory

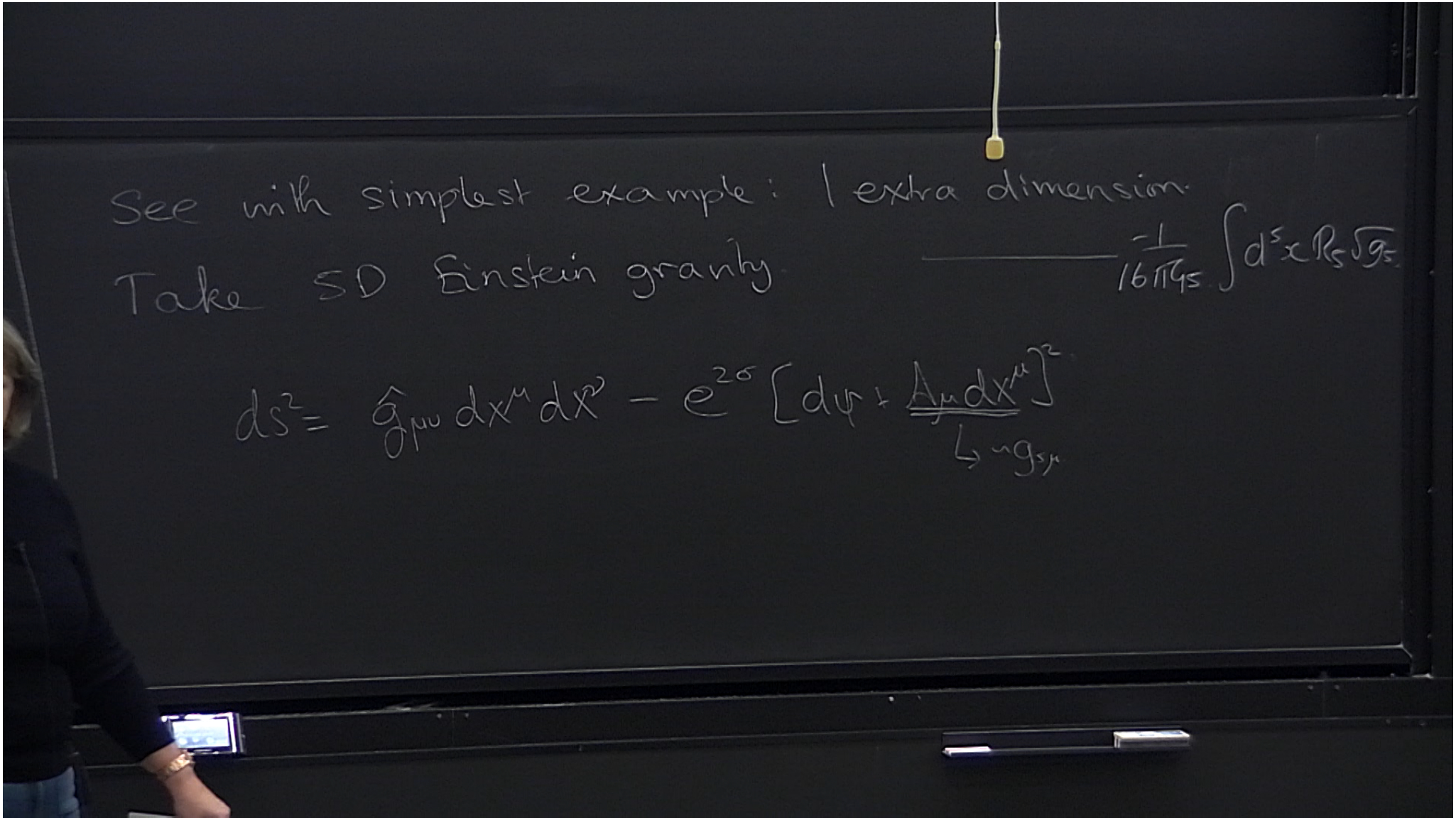
Extracts 4D gravity as a low-energy limit of a higher dimensional theory. The extra dimensions are small, & geometry does not depend on them.


See with simplest example: 1 extra dimension.
Take 5D Einstein gravity
$ds^2 = \hat{g}_{\mu\nu} dx^\mu dx^\nu - e^{2\sigma} [d\psi + A_\mu dx^\mu]^2$

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$ds^2 = \hat{g}_{\mu\nu}dx^\mu dx^\nu - e^{2\sigma}[d\psi + A_\mu dx^\mu]^2$
$\hookrightarrow g_{5\mu}$

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$$\det g_5 = \det \hat{g}_4 \cdot e^{\sigma}$$

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$$\det g_5 = \det \hat{g}_4 \cdot e^{\sigma}$$

$\frac{\partial}{\partial\psi}$ is killing vector.

Let $\underline{\omega}^a = e^a{}_\mu \underline{dx}^\mu$ be an o/n basis for $\hat{g}$.

& $\underline{\omega}^\psi = e^\sigma [d\psi + A_\mu dx^\mu]$ & use Cartan.

$$\underline{d\omega}^a = -\underline{\Theta}^a{}_{\sigma\, b} \wedge \underline{\omega}^b$$

$$\underline{d\omega}^\psi = \sigma_{,c}\, \underline{\omega}^c \wedge \underline{\omega}^\psi + e^\sigma A_{\mu,\nu} dx^\nu \wedge dx^\mu$$

$$= -\sigma_{,c} \underline{\omega}^\psi \wedge \underline{\omega}^c + e^\sigma \underline{F}$$

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$$\underline{d\omega}^a = -\underline{\Theta}^a{}_{0\,b} \wedge \underline{\omega}^b = -\underline{\Theta}^a{}_{b} \wedge \underline{\omega}^b - \underline{\Theta}^a{}_{\psi} \wedge \underline{\omega}^\psi$$

$$\underline{d\omega}^\psi = \sigma_{,c}\, \underline{\omega}^c \wedge \underline{\omega}^\psi + e^\sigma A_{\mu,\nu} dx^\nu \wedge dx^\mu$$

$$= -\sigma_{,c} \underline{\omega}^\psi \wedge \underline{\omega}^c + e^\sigma \underline{F}$$

$$\underline{F} = \underline{d}\underline{A} = \frac{1}{2} F_{ab}\, \underline{\omega}^a \wedge \underline{\omega}^b = \frac{1}{2}\left(A_{\mu\nu} - A_{\nu,\mu}\right) dx^\nu \wedge dx^\mu$$

$$\underline{\Omega}^\gamma{}_a = \sigma_{,a}\, \underline{\omega}^\gamma + \frac{1}{2} e^\sigma F_{ab}\, \underline{\omega}^b$$

$$\underline{\Omega}^a{}_b = \underline{\Omega}_0{}^a{}_b + \frac{1}{2} e^\sigma F^a{}_b\, \underline{\omega}^\gamma$$

$$\underline{F} = \underline{d}\underline{A} = \frac{1}{2} F_{ab}\, \underline{\omega}^a \wedge \underline{\omega}^b = \frac{1}{2}(A_{\mu\nu} - A_{\nu,\mu})\, dx^\nu \wedge dx^\mu$$

$$\underline{\Theta}^\gamma{}_a = \sigma_{,a}\, \underline{\omega}^\gamma + \frac{1}{2} e^\sigma F_{ab}\, \underline{\omega}^b$$

$$\underline{\Theta}^a{}_b = \underline{\Theta}_0{}^a{}_b + \frac{1}{2} e^\sigma F^a{}_b\, \underline{\omega}^\gamma$$

Use $\underline{R}^a{}_b = \underline{d}\underline{\Theta}^a{}_b + \underline{\Theta}^a{}_c \wedge \underline{\Theta}^c{}_b + \underline{\Theta}^a{}_\gamma \wedge \underline{\Theta}^\gamma{}_b$

$$\underline{d}\underline{\omega}^{\psi} = \sigma_{,c}\,\underline{\omega}^{c}\wedge\underline{\omega}^{\psi} + e^{\sigma}A_{\mu,\nu}dx^{\nu}\wedge dx^{\mu}$$

$$= -\sigma_{,c}\,\underline{\omega}^{\psi}\wedge\underline{\omega}^{c} + e^{\sigma}\underline{F}$$

$$= \underline{d}\underline{\theta}_{o}{}^{a}{}_{b} + \frac{1}{2}e^{\sigma}F^{a}{}_{b}\,\sigma_{,c}\,\underline{\omega}^{c}\wedge\underline{\omega}^{\psi} + \frac{1}{2}e^{\sigma}F^{a}{}_{b,c}\,\underline{\omega}^{c}\wedge\underline{\omega}^{\psi}$$

$$+ \frac{1}{2}e^{\sigma}F^{a}{}_{b}\left(\sigma_{,c}\,\underline{\omega}^{c}\wedge\underline{\omega}^{\psi} + e^{\sigma}\underline{F}\right) + \underline{\Omega}_{o}{}^{a}{}_{c}\wedge\underline{\theta}_{o}{}^{c}{}_{b}$$

$$+ \frac{1}{2}e^{\sigma}F^{a}{}_{c}\,\underline{\omega}^{\psi}\wedge\underline{\theta}_{o}{}^{c}{}_{b} + \frac{1}{2}e^{\sigma}F^{c}{}_{b}\,\underline{\theta}_{o}{}^{a}{}_{c}\wedge\underline{\omega}^{\psi}$$

$$\underline{d}\underline{\omega}^{\psi} = \sigma_{,c}\,\underline{\omega}^{c}\wedge\underline{\omega}^{\psi} + e^{\sigma}A_{\mu,\nu}dx^{\nu}\wedge dx^{\mu}$$

$$= -\sigma_{,c}\,\underline{\omega}^{\psi}\wedge\underline{\omega}^{c} + e^{\sigma}\underline{F}$$

$$= \underline{d}\underline{\theta}_{o}{}^{a}{}_{b} + \frac{1}{2}e^{\sigma}F^{a}{}_{b}\,\sigma_{,c}\,\underline{\omega}^{c}\wedge\underline{\omega}^{\psi} + \frac{1}{2}e^{\sigma}F^{a}{}_{b,c}\,\underline{\omega}^{c}\wedge\underline{\omega}^{\psi}$$

$$+ \frac{1}{2}e^{\sigma}F^{a}{}_{b}\left(\sigma_{,c}\,\underline{\omega}^{c}\wedge\underline{\omega}^{\psi} + e^{\sigma}\underline{F}\right) + \underline{\Omega}_{o}{}^{a}{}_{c}\wedge\underline{\theta}_{o}{}^{c}{}_{b}$$

$$+ \frac{1}{2}e^{\sigma}F^{a}{}_{c}\,\underline{\omega}^{\psi}\wedge\underline{\Omega}_{o}{}^{c}{}_{b} + \frac{1}{2}e^{\sigma}F^{c}{}_{b}\,\underline{\theta}_{o}{}^{a}{}_{c}\wedge\underline{\omega}^{\psi}$$

$$+ \left(\sigma^{,a}\,\underline{\omega}^{\psi} + \frac{1}{2}e^{\sigma}F^{a}{}_{c}\,\underline{\omega}^{c}\right)\wedge\left(\sigma_{,b}\,\underline{\omega}^{\psi} + \frac{1}{2}e^{\sigma}F_{bd}\,\underline{\omega}^{d}\right)$$

$$= -\sigma_{,c}\,\underline{\omega}^{\psi} \wedge \underline{\omega}^{c} + e^{\sigma}\,\underline{F}$$

$$= \underline{d\,\Theta_{o}{}^{a}{}_{b}} + \frac{1}{2} e^{\sigma} F^{a}{}_{b}\,\sigma_{,c}\,\underline{\omega}^{c} \wedge \underline{\omega}^{\psi} + \frac{1}{2} e^{\sigma} F^{a}{}_{b,c}\,\underline{\omega}^{c} \wedge \underline{\omega}^{\psi}$$

$$+ \frac{1}{2} e^{\sigma} F^{a}{}_{b} \left(\sigma_{,c}\,\underline{\omega}^{c} \wedge \underline{\omega}^{\psi} + e^{\sigma}\,\underline{F} \right) + \underline{\Omega_{o}{}^{a}{}_{c} \wedge \Theta_{o}{}^{c}{}_{b}}$$

$$+ \frac{1}{2} e^{\sigma} F^{a}{}_{c}\,\underline{\omega}^{\psi} \wedge \underline{\Omega_{o}{}^{c}{}_{b}} + \frac{1}{2} e^{\sigma} F^{c}{}_{b}\,\underline{\Omega_{o}{}^{a}{}_{c}} \wedge \underline{\omega}^{\psi}$$

$$+ \left(\sigma^{,a}\,\underline{\omega}^{\psi} + \frac{1}{2} e^{\sigma} F^{a}{}_{c}\,\underline{\omega}^{c} \right) \wedge \left(\sigma_{,b}\,\underline{\omega}^{\psi} + \frac{1}{2} e^{\sigma} F_{bd}\,\underline{\omega}^{d} \right)$$

$$= \underline{R}_o{}^a{}_b + \tfrac{1}{2} e^\sigma \underline{\omega}^c \wedge \underline{\omega}^r \Big(F^a{}_{b\,\sigma,c} + F^a{}_{b,c} - F^a{}_d \Gamma_{bc}{}^d{}_b + F^d{}_b \Gamma_{bc}{}^a{}_d + F^a{}_c \sigma_{,b} - \sigma^a F_{bc} \Big)$$

$$+ \tfrac{1}{4} e^{2\sigma} F^a{}_c F_{bd} \underline{\omega}^c \wedge \underline{\omega}^d$$

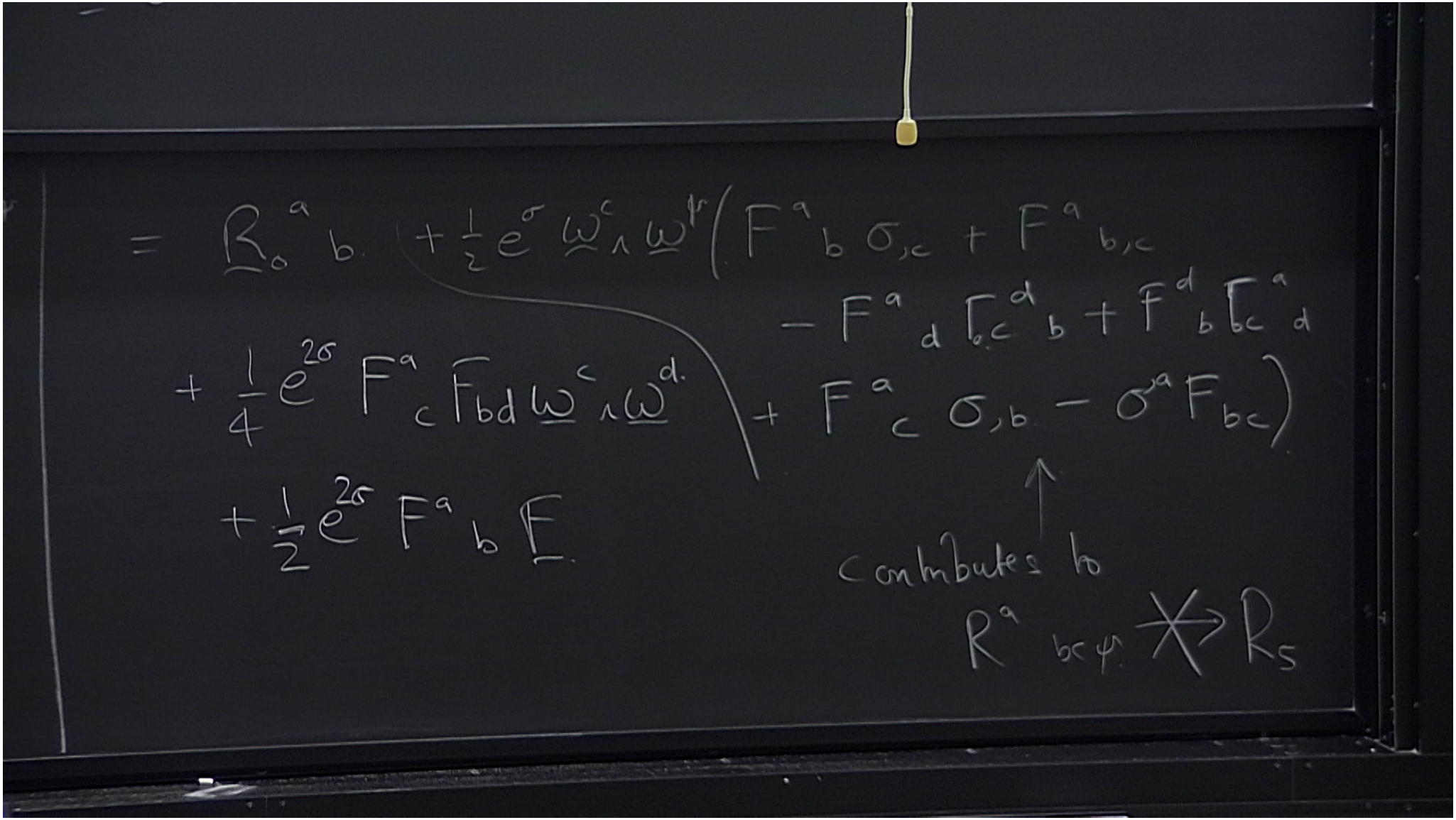
$$+ \tfrac{1}{2} e^{2\sigma} F^a{}_b \underline{F}$$

$$= \underline{R}_o{}^a{}_b + \tfrac{1}{2} e^\sigma \underline{\omega}^c \wedge \underline{\omega}^\nu \Big(F^a{}_{b\sigma,c} + F^a{}_{b,c} - F^a{}_d \Gamma_{bc}{}^d{}_b + F^d{}_b \Gamma_{bc}{}^a{}_d + F^a{}_c \sigma_{,b} - \sigma^a F_{bc} \Big)$$

$$+ \frac{1}{4} e^{2\sigma} F^a{}_c F_{bd} \underline{\omega}^c \wedge \underline{\omega}^d$$

$$+ \frac{1}{2} e^{2\sigma} F^a{}_b \underline{F}$$

contributes to $R^a{}_{bc\nu}$


$= \underline{R}^a{}_b + \frac{1}{2} e^\sigma \underline{\omega}^c \wedge \underline{\omega}^d \left(F^a{}_{b\sigma,c} + F^a{}_{b,c} \right.$
$- F^a{}_d \Gamma^d_{bc} + F^d{}_b \Gamma^a_{bc}$
$+ \frac{1}{4} e^{2\sigma} F^a{}_c F_{bd} \underline{\omega}^c \wedge \underline{\omega}^d \Big| \left. + F^a{}_c \sigma_{,b} - \sigma^a F_{bc} \right)$
$+ \frac{1}{2} e^{2\sigma} F^a{}_b \underline{F}$
contributes to
$R^a{}_{bc\psi} \not\to R_5$

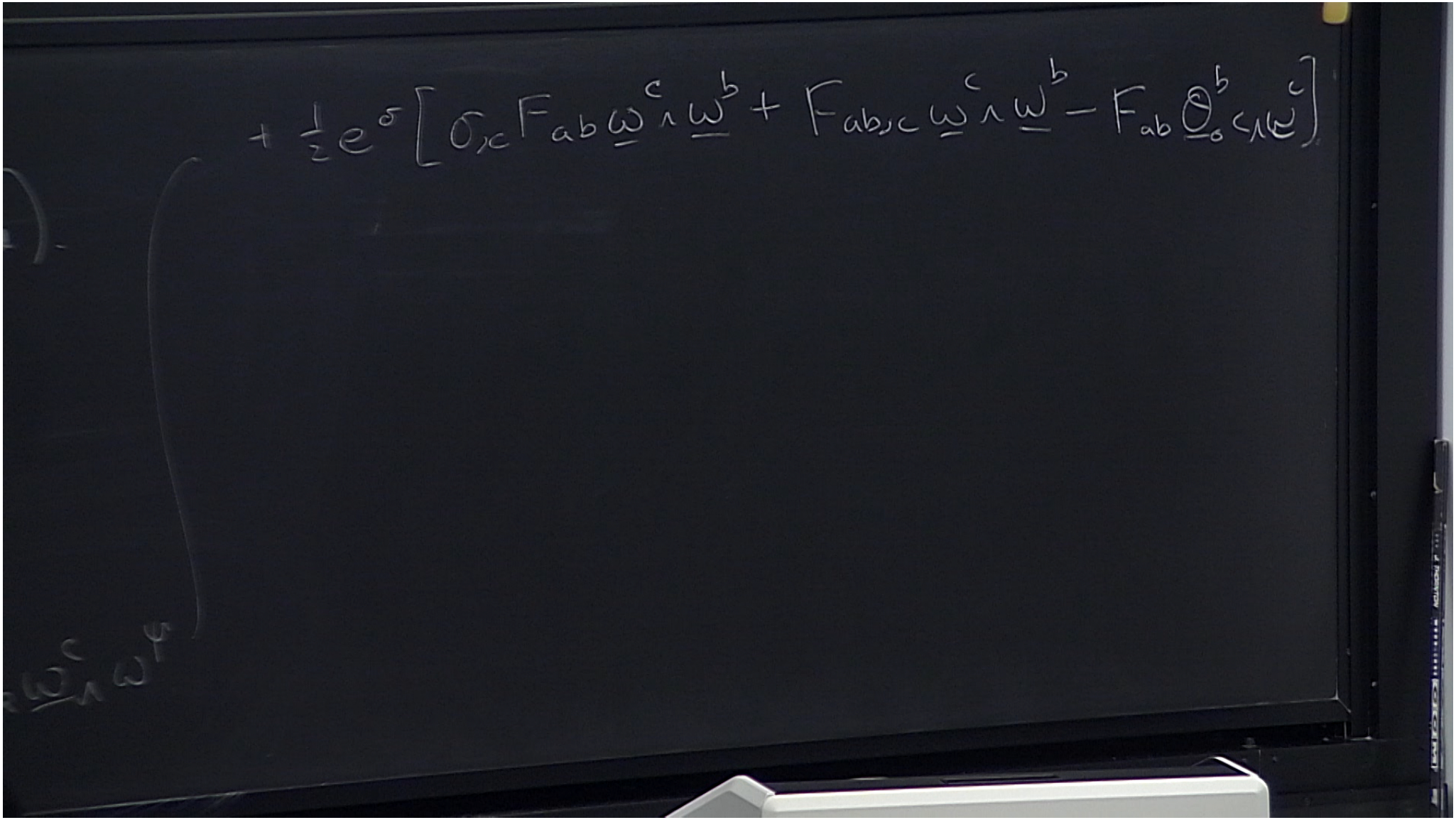
Extract:

$$\hat{R}^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}} = R_0{}^{a}{}_{bcd} + \frac{e^{2\sigma}}{4}\left(F^{a}{}_{c}F_{bd} - F^{a}{}_{d}F_{bc} + 2F^{a}{}_{b}F_{cd}\right).$$

$$\underline{R}^{\psi}{}_{a} = \underline{d}\underline{\Theta}^{\psi}{}_{a} + \underline{\Theta}^{\psi}{}_{b} \wedge \underline{\Theta}^{b}{}_{a}$$

$$= \sigma_{,ab}\,\underline{\omega}^{b} \wedge \underline{\omega}^{\psi} + \sigma_{,a}\sigma_{b}\,\underline{\omega}^{b} \wedge \underline{\omega}^{\psi} + \sigma_{,a}e^{\sigma}\underline{F}$$

$$+ \sigma_{,b}\,\underline{\omega}^{\psi} \wedge \underline{\Theta}_{0}{}^{b}{}_{a} + \frac{1}{2}e^{\sigma}F_{bc}\,\underline{\omega}^{c} \wedge \underline{\Theta}_{0}{}^{b}{}_{a} + \frac{1}{4}e^{2\sigma}F_{bc}F^{b}{}_{a}\underline{\omega}^{c} \wedge \underline{\omega}^{\psi}$$

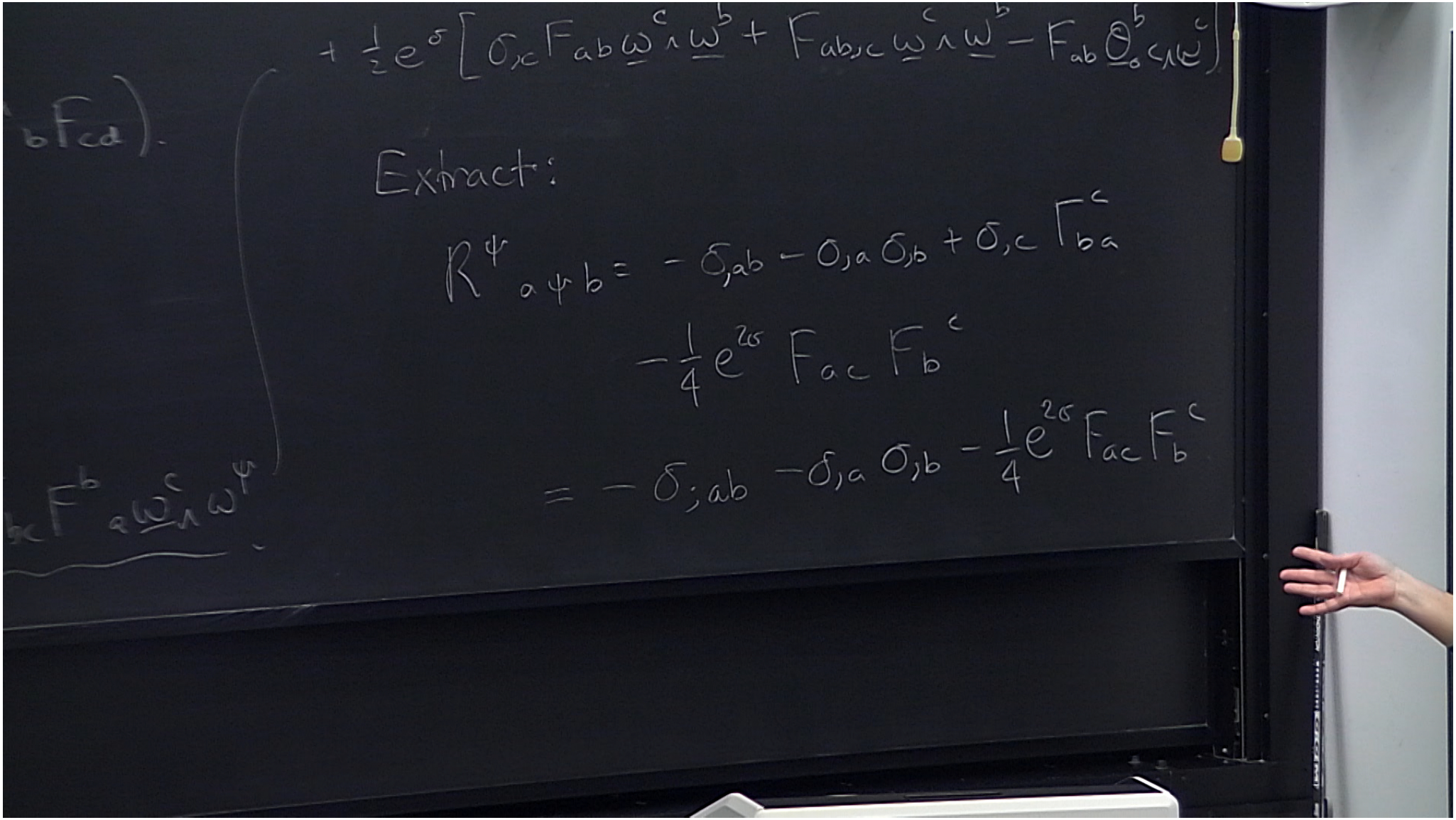

$+\frac{1}{2}e^{a}\left[\partial_{[c}F_{ab]}\underline{\omega}^{c}\wedge\underline{\omega}^{b}+F_{ab[c}\underline{\omega}^{c}\wedge\underline{\omega}^{b}-F_{ab}\underline{\theta}^{b}{}_{[c}\wedge\underline{\omega}^{c}\right]$

$$+\frac{1}{2}e^{\sigma}\left[\sigma_{,c}F_{ab}\underline{\omega}^{c}\wedge\underline{\omega}^{b}+F_{ab,c}\underline{\omega}^{c}\wedge\underline{\omega}^{b}-F_{ab}\underline{\theta}^{b}{}_{c}\wedge\underline{\omega}^{c}\right]$$

Extract:

$$R^{\psi}{}_{a\psi b}=-\sigma_{,ab}-\sigma_{,a}\sigma_{,b}+\sigma_{,c}\Gamma^{c}_{ba}$$

$$-\frac{1}{4}e^{2\sigma}F_{ac}F_{b}{}^{c}$$


$+\frac{1}{2}e^{\sigma}\left[\sigma_{,c}F_{ab}\underline{\omega}^{c}\wedge\underline{\omega}^{b}+F_{ab,c}\underline{\omega}^{c}\wedge\underline{\omega}^{b}-F_{ab}\underline{\theta}^{b}{}_{c}\wedge\underline{\omega}^{c}\right]$
Extract:
$R^{\psi}{}_{a\psi b}=-\sigma_{,ab}-\sigma_{,a}\sigma_{,b}+\sigma_{,c}\Gamma^{c}_{ba}$
$-\frac{1}{4}e^{2\sigma}F_{ac}F_{b}{}^{c}$
$=-\sigma_{;ab}-\sigma_{,a}\sigma_{,b}-\frac{1}{4}e^{2\sigma}F_{ac}F_{b}{}^{c}$

$$\Rightarrow R^{\psi}{}_{\psi} = -\Box\sigma - (\nabla\sigma)^2 - \frac{1}{4}e^{2\sigma}F^2$$

$$R_{ab} = R_{0ab} - \nabla_a\nabla_b\sigma - \nabla_a\sigma\nabla_b\sigma$$
$$+ \frac{1}{2}e^{2\sigma}F_{ac}F_b{}^c$$

$$\Rightarrow R^{\psi}{}_{\psi} = -\Box\sigma - (\nabla\sigma)^2 - \frac{1}{4}e^{2\sigma}F^2$$

$$R_{ab} = R_{0ab} - \nabla_a\nabla_b\sigma - \nabla_a\sigma\nabla_b\sigma + \frac{1}{2}e^{2\sigma}F_{ac}F_b{}^c$$

$$R^c{}_{acb} + R^{\psi}{}_{a\psi b}$$

$$\Rightarrow R = R_0 + \frac{1}{4}e^{2\sigma}F^2 - 2\Box\sigma - 2(\nabla\sigma)^2$$

$$\Rightarrow R^{\psi}{}_{\psi} = -\Box\sigma - (\nabla\sigma)^2 - \frac{1}{4}e^{2\sigma}F^2$$

$$R_{ab} = R_{0ab} - \nabla_a\nabla_b\sigma - \nabla_a\sigma\nabla_b\sigma + \frac{1}{2}e^{2\sigma}F_{ac}F_b{}^c$$

$$R^{c}{}_{acb} + R^{\psi}{}_{a\psi b}$$

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$$= R_0 + \frac{1}{4}e^{2\sigma}F^2 - 2e^{-\sigma}\Box e^{\sigma}$$

$$S_5 = -\frac{1}{16\pi G_5}\int d^4x\, d\psi\, \sqrt{g_5}\, R_5$$

$$L = \Delta\psi$$

$$= -\frac{L}{16\pi G_5}\int d^4x\, e^{\sigma}\, \sqrt{\hat{g}}\left[R_0 + \frac{1}{4}F^2 - 2e^{-\sigma}\Box e^{\sigma}\right]$$

$$= -\frac{L}{16\pi G_5}\int d^4x\, \sqrt{\hat{g}}\left[e^{\sigma}R(\hat{g}) + \frac{e^{\sigma}}{4}F^2\right].$$

$$= -\delta_{;ab} - \delta_{,a}\sigma_{,b} - \frac{1}{4}e\, F_{ac}F_{b}$$

$$S_5 = -\frac{1}{16\pi G_5}\int d^5x\, d\psi\, \sqrt{g_5}R_5 \qquad L = \Delta\psi$$

$$= -\frac{L}{16\pi G_5}\int d^4x\, e^{\sigma}\sqrt{\bar{g}}\left[R_0 + \frac{1}{4}F^2 - 2e^{-\sigma}\Box e^{\sigma}\right]$$

$$= -\frac{L}{16\pi G_5}\int d^4x\,\sqrt{\bar{g}}\left[e^{\sigma}R(\bar{g}) + \frac{e^{\sigma}}{4}F^2\right]$$

Although kinetic terms for σ not explicitly present - are implicitly there from integration by parts in δR_{ab}.

$= R_0 + \frac{1}{4} e^{2\phi} F^2 - 2 e^{-\phi} \Box e^{\phi}$.

there from

Note: 4D Newton's constant is G_5/L

Gravitational action ($e^{\phi} R$) has both scalar & tensor terms "scalar-tensor" theory.

Often re-define 4D metric to get Einstein-Hilbert action.

there from integration by parts in δR_{ab}.

Do this via a conformal transformation

Rewrite $\hat{g}_{ab} = \Omega^2(x)\, g_{ab}$. ← lengths rescaled, angles invt.

$$\hat{R}(\hat{g}) = \Omega^{-2}\left(R(g) - 6\Omega^{-1}\Box_g \Omega\right)$$

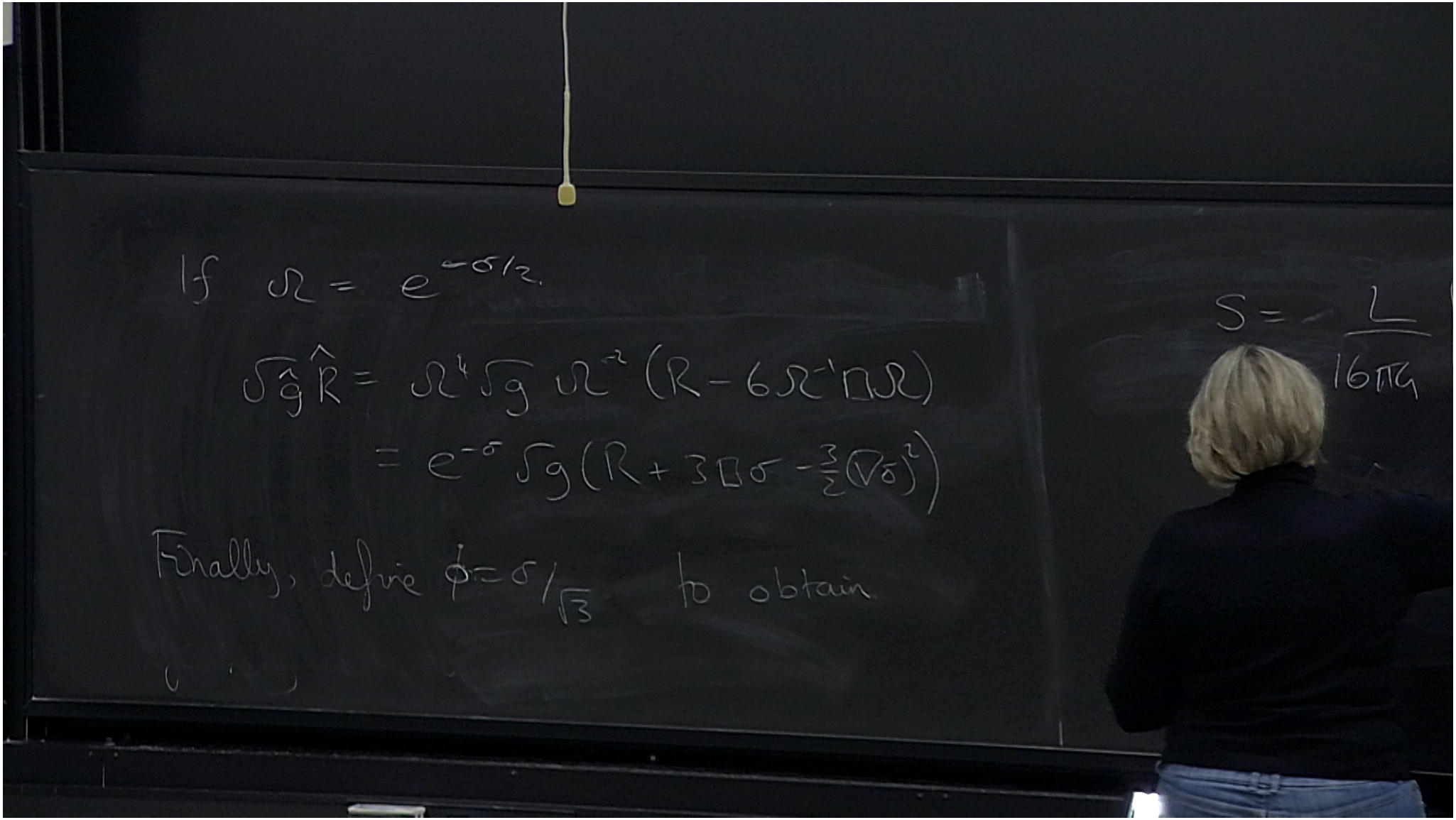
there from integration by p

Do this via a conformal transformation

Rewrite $\hat{g}_{ab} = \Omega^2(x)\, g_{ab}$. ← lengths rescaled, angles invt.

$\hat{R}(\hat{g}) = \Omega^{-2}\left(R(g) - 6\Omega^{-1}\Box_g \Omega\right)$

$\det \hat{g} = \Omega^8 \det g$

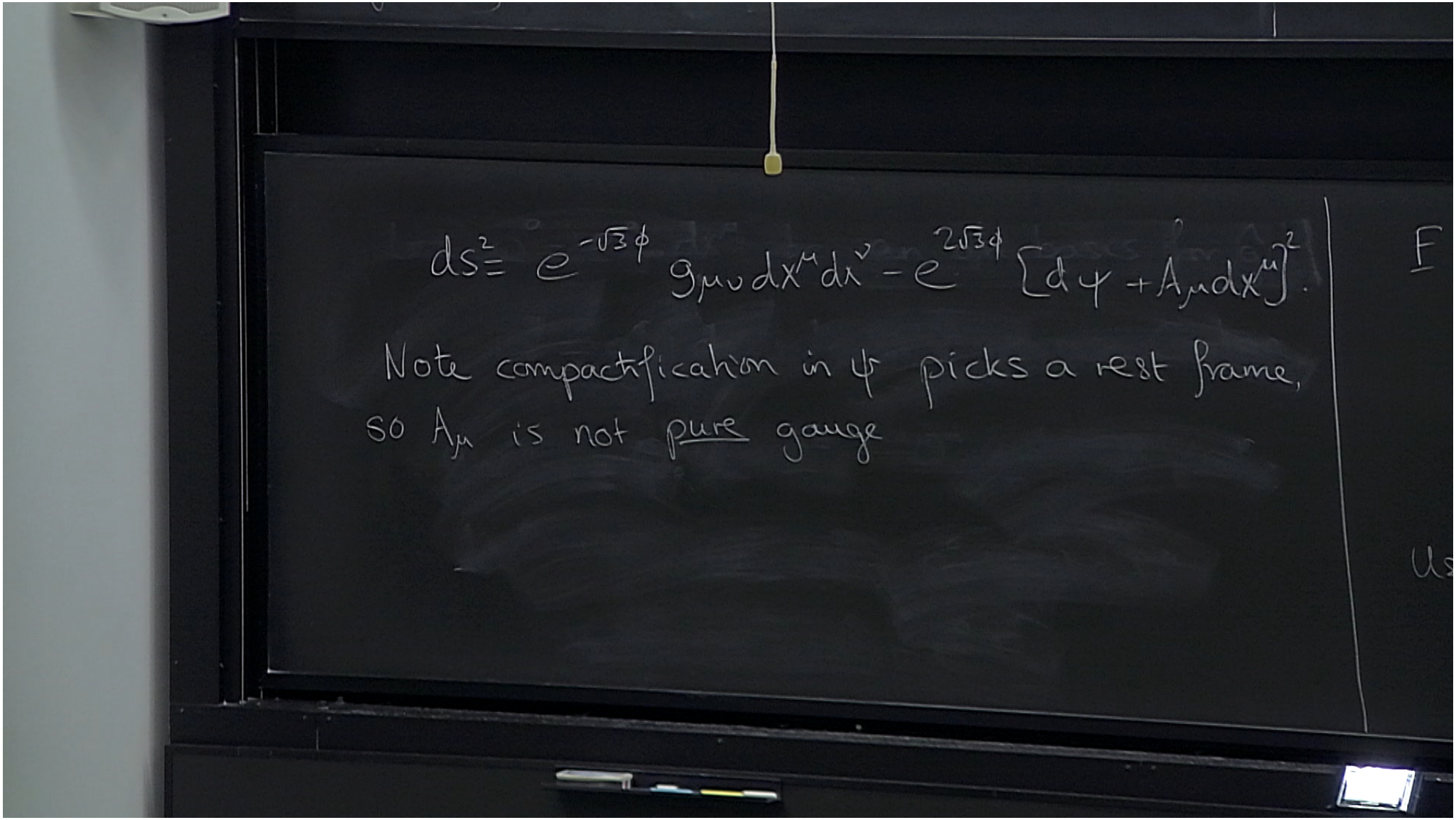

If $\Omega = e^{-\sigma/2}$.
$\sqrt{\hat{g}}\hat{R} = \Omega^4\sqrt{g}\,\Omega^{-2}(R - 6\Omega^{-1}\Box\Omega)$
$= e^{-\sigma}\sqrt{g}\left(R + 3\Box\sigma - \frac{3}{2}(\nabla\sigma)^2\right)$
Finally, define $\phi = \sigma/\sqrt{3}$ to obtain
$S = \frac{1}{16\pi G}$

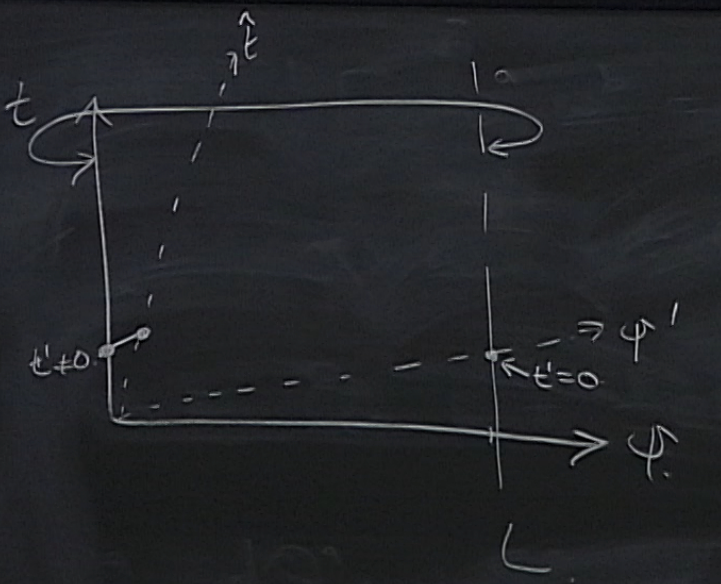
$$\underline{F} = \underline{d}\underline{A} = \frac{1}{2} F_{ab}\, \underline{\omega}^a \wedge \underline{\omega}^b = \frac{1}{2}(A_{\mu,\nu} - A_{\nu,\mu})\, dx^\nu \wedge dx^\mu$$

$$\underline{\Theta}^\psi{}_a = \sigma_{,a}\, \underline{\omega}^\psi + \frac{1}{2} e^\sigma F_{ab}\, \underline{\omega}^b$$

$$\underline{\Theta}^a{}_b = \underline{\Theta}_0{}^a{}_b + \frac{1}{2} e^\sigma F^a{}_b\, \underline{\omega}^\psi$$

Use $$\underline{R}^a{}_b = \underline{d}\underline{\Theta}^a{}_b + \underline{\Theta}^a{}_c \wedge \underline{\Theta}^c{}_b + \underline{\Theta}^a{}_\psi \wedge \underline{\Theta}^\psi{}_b$$


$ds^2 = e^{-\sqrt{3}\phi} g_{\mu\nu} dx^\mu dx^\nu - e^{2\sqrt{3}\phi} [d\psi + A_\mu dx^\mu]^2$.
Note compactification in ψ picks a rest frame,
so A_μ is not pure gauge
F



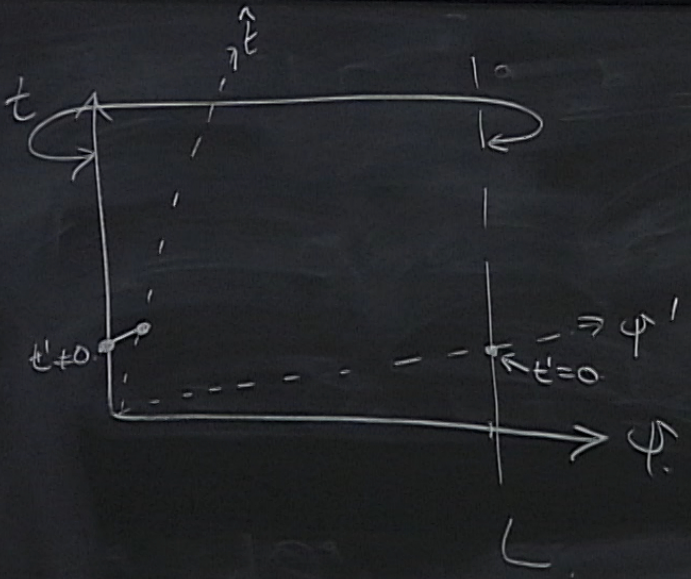
$\varphi \sim \varphi + L$ at const t.

$t' = \gamma(t - v\varphi)$

$\varphi' = \gamma(\varphi - vt)$.

$t' \sim t' - v\gamma L$.

$\varphi' \sim \varphi' + \gamma L$



$\psi \sim \psi + L$ at const t.

$t' = \gamma(t - v\psi)$

$\psi' = \gamma(\psi - vt)$.

$t' \sim t' - v\gamma L$.

$\psi' \sim \psi' + \gamma L$

'Motion' in ψ direction gives $g_{0r} \sim A_0$ - electric charge.