Title: PSI 2016/2017 Gravitational Physics (Review) - Lecture 6

Date: Jan 10, 2017 10:15 AM

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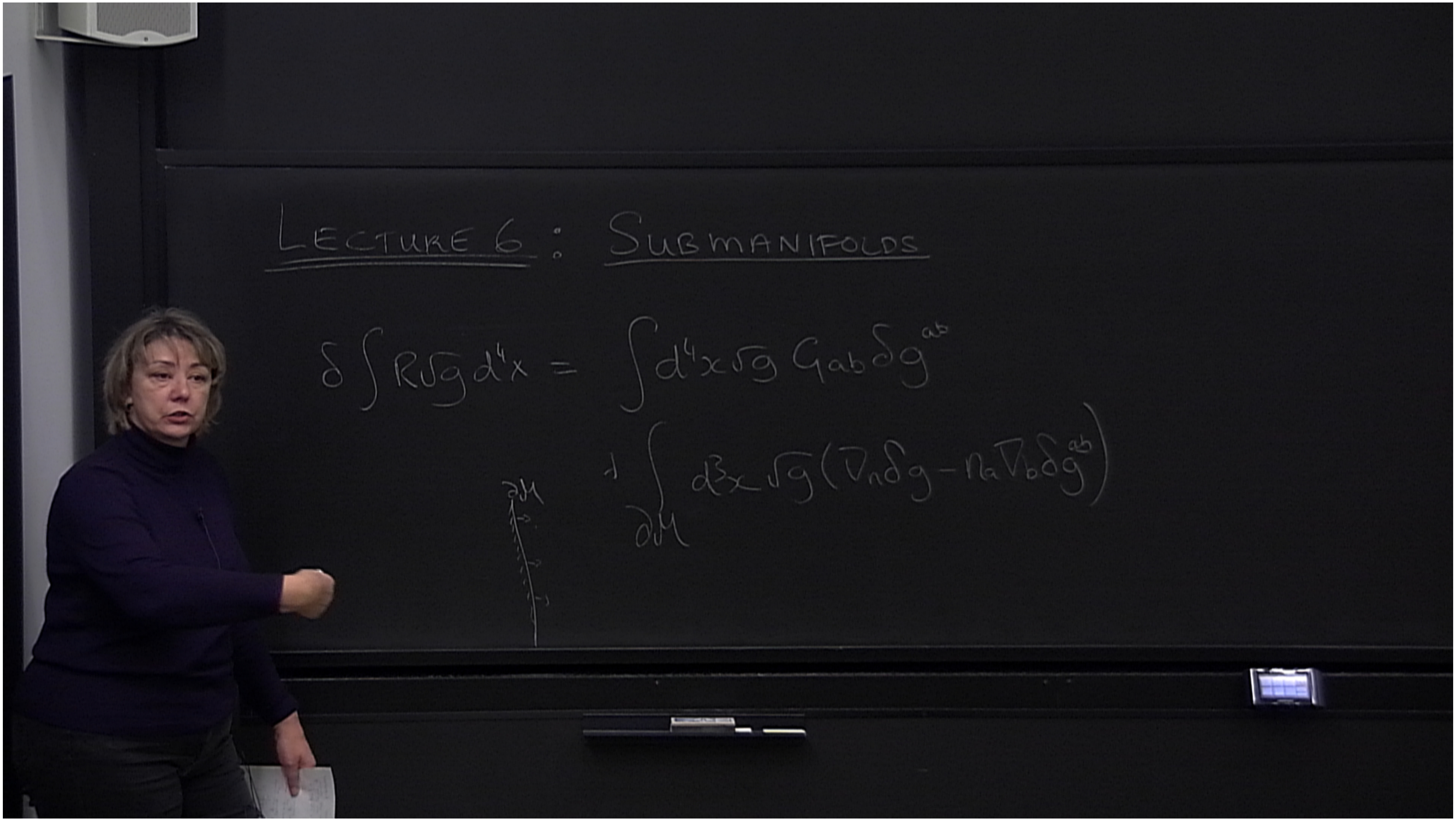
Abstract:


LECTURE 6: SUBMANIFOLDS
$\delta \int R\sqrt{g}\, d^4x = \int d^4x \sqrt{g}\, G_{ab}\, \delta g^{ab}$
$+ \int_{\partial M} d^3x \sqrt{g} \left(\nabla_n \delta g - n_a \nabla_b \delta g^{ab} \right)$
∂M

LECTURE 6: SUBMANIFOLDS

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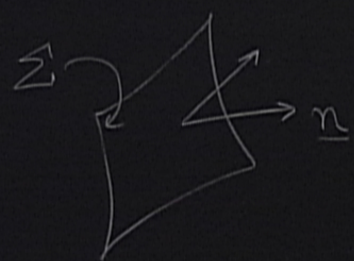
$$+ \int_{\partial M} d^3x \sqrt{g} \left(\nabla_n \delta g - n_a \nabla_b \delta g^{ab} \right)$$

$\nabla_b \delta g^{ab})$

Let $\Sigma \subset \mathcal{M}$ be a submanifold of dim n, dim $\mathcal{M} = D$.



__Defn__ The __co-dimension__ of Σ is $(D-n)$

i.e. $\exists$ $(D-n)$ linearly independent vectors in $T_p(\mathcal{M})$ $(p \in \Sigma)$ s.t.

$$\underline{n}(\sigma^A) = 0 \quad \text{for } \sigma^A \text{ coord fns on } \Sigma$$

is $h_{ab} = g_{ab} \pm \sum_{i=1}^{D-n} n_{ia} n_{ib}$ $+$ if n spacelike. $-$ if n timelike

t)]

h_{ab} is the projection of the metric parallel to Σ it is the metric Σ inherits from $\mathcal{M}$. Note $h \in T_p^*(\mathcal{M}) \otimes T_p^*(\mathcal{M})$

$-n.$

$\Sigma.$

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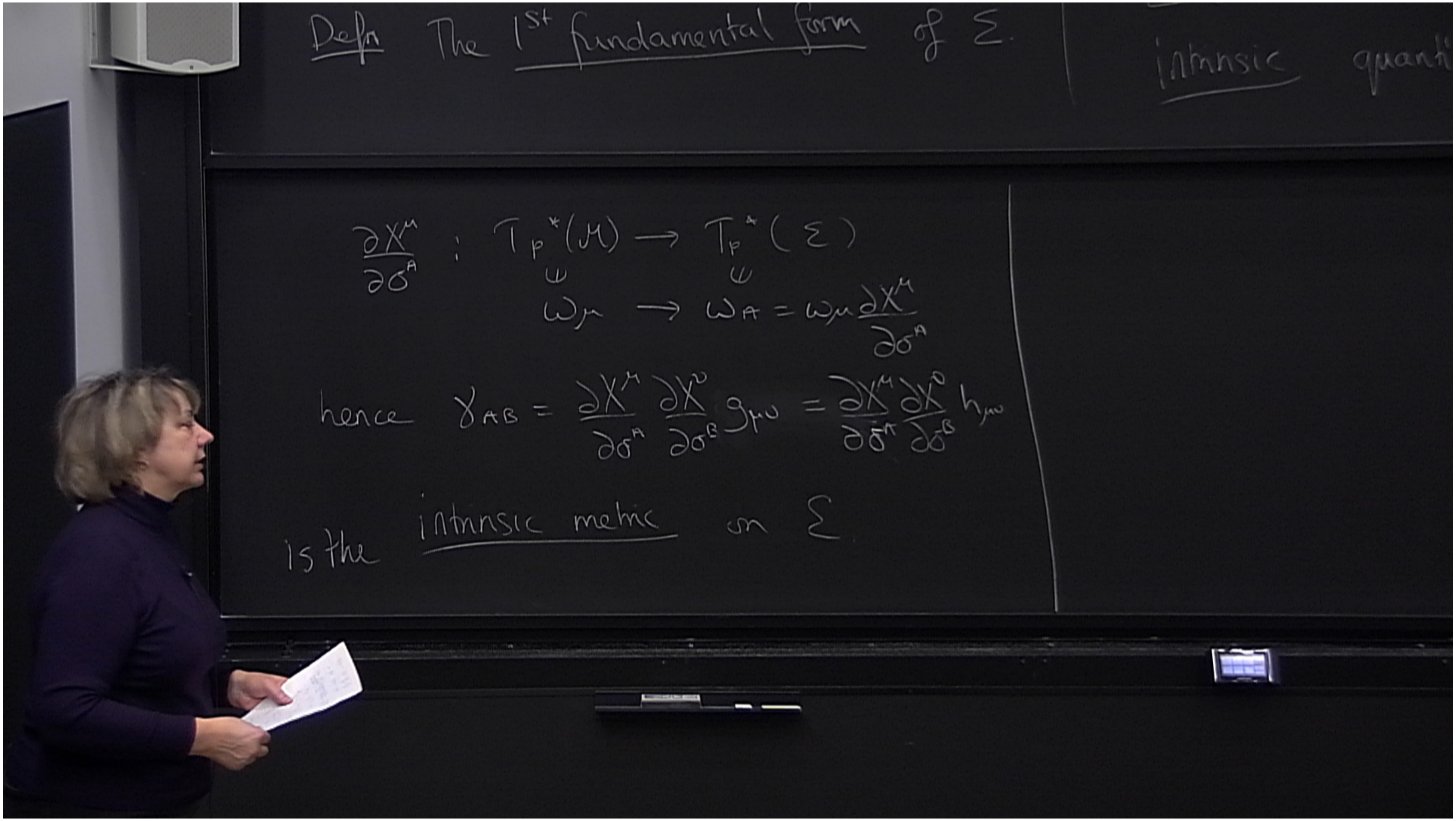
Σ is also a manifold in its own right, & can consider

<u>intrinsic</u> quantities in $T(\Sigma)$ ...


Defn The 1st fundamental form of Σ.
intrinsic
$\frac{\partial X^\mu}{\partial \sigma^A} : T_p^*(M) \longrightarrow T_p^*(\Sigma)$
$\omega_\mu \longrightarrow \omega_A = \omega_\mu \frac{\partial X^\mu}{\partial \sigma^A}$
hence $\gamma_{AB} = \frac{\partial X^\mu}{\partial \sigma^A} \frac{\partial X^\nu}{\partial \sigma^B} g_{\mu\nu} = \frac{\partial X^\mu}{\partial \sigma^A} \frac{\partial X^\nu}{\partial \sigma^B} h_{\mu\nu}$
is the intrinsic metric on Σ

... form of Σ.

Σ is also a manifold in its own right, & can consider <u>intrinsic</u> quantities in $T(\Sigma)$ ...

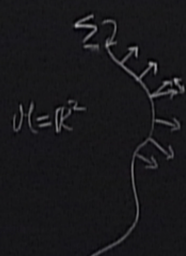
(Σ)

$= \omega_\mu \frac{\partial X^\mu}{\partial \sigma^A}$

$\mu\nu = \frac{\partial X^\mu}{\partial \sigma^A} \frac{\partial X^\nu}{\partial \sigma^B} h_{\mu\nu}$

n Σ

The 2^{nd} fundamental form or extrinsic curvature measures how Σ curves in $\mathcal{M}$



$$K_{i\mu\nu} = h_\mu^{\ \lambda} h_\nu^{\ \sigma} \nabla_\sigma n_{i\lambda}$$

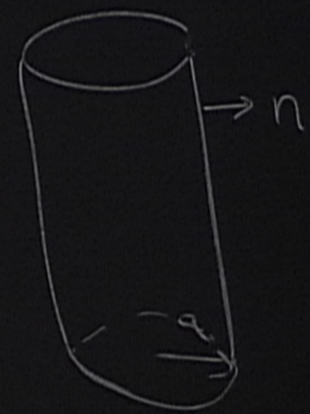
or $$K_{iAB} = \frac{\partial X^\mu}{\partial \sigma^A} \frac{\partial X^\nu}{\partial \sigma^B} \nabla_\mu n_{i\nu} = -n_{i\mu} \nabla_A^{(m)} \left(\frac{\partial X^\mu}{\partial \sigma^B} \right)$$

where $\nabla^{(n)}$ is covariant derivative inherited by Σ:

$$\nabla^{(n)}_\mu V_\nu = h_\mu{}^\lambda h_\nu{}^\sigma \nabla_\lambda V_\sigma .$$

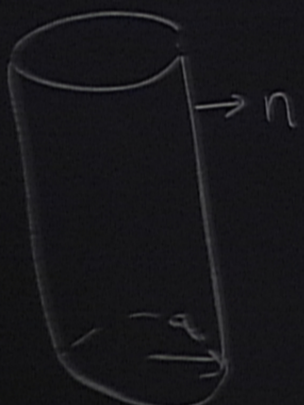
$V \in T_p^*(M)$

e.g. Cylinder $\{x^2+y^2=a^2\}$



derivative

$V \in T_p^*(M)$



$$h_{ab} = \delta_{ab} - n_a n_b \qquad \text{(cartesians)}$$

$$= \begin{pmatrix} \sin^2\theta & -\sin\theta\cos\theta & 0 \\ -\sin\theta\cos\theta & \cos^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Instead, in polars.

$$h_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$h_{ab} = \delta_{ab} - n_a n_b \quad \text{(cartesians)}$$

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Instead, in polars.

$$h_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

ie $\gamma_{AB} = \begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \sigma^A = \{\theta, z\}$

Extrinsic curvature (in polars)

$$K_{ab} = -\Gamma^r_{ab}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$K_{AB} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

If we have more than 1 normal, these can also vary normals normally.

$$\beta_{\mu ij} = n_{i\nu} \nabla_\mu n_j^{\ \nu}$$

- the <u>normal fundamental forms</u>

(Choose $\nabla_\mu n = 0$)

Gauss - eqn $\quad {}^{(n)}R^a{}_{bcd} = {}^{(D)}R^{a'}{}_{b'c'd'}\, h^a{}_{a'}\, h^{b'}{}_b\, h^{c'}{}_c\, h^{d'}{}_d \pm \sum_{i=1}^{D-n} \left[K_i{}^a{}_c K_{ibd} - K_i{}^a{}_d K_{ibc} \right]$

Demonstrate for codim 1:

$$\left[{}^{(n)}\nabla_c {}^{(n)}\nabla_d - {}^{(n)}\nabla_d {}^{(n)}\nabla_c \right] V^a = {}^{(n)}R^a{}_{bcd} V^b \qquad V \text{ tgt to } \Sigma.$$

$$h_{ab} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$K_{AB} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

Gauss - eqn

$$^{(n)}R^{a}{}_{bcd} = {}^{(D)}R^{a'}{}_{b'c'd'}\, h^{a}{}_{a'}\, h^{b'}{}_{b}\, h^{c'}{}_{c}\, h^{d'}{}_{d} \pm \sum_{i=1}^{D-n} \left[K_i{}^{a}{}_{c} K_{ibd} - K_i{}^{a}{}_{d} K_{ibc} \right]$$

Demonstrate for codim 1:

$$\left[{}^{(n)}\nabla_c {}^{(n)}\nabla_d - {}^{(n)}\nabla_d {}^{(n)}\nabla_c \right] V^a = {}^{(n)}R^{a}{}_{bcd} V^b \qquad V \text{ tgt to } \Sigma.$$

$$= h^{a}{}_{e}\, h^{f}{}_{c}\, h^{g}{}_{d}\, \nabla_f \left({}^{(n)}\nabla_g V^e \right) - c \leftrightarrow d.$$

(Choose $\nabla_k n = 0$)

$$= h^a{}_e \, h^f{}_c \, h^g{}_d \, \nabla_f \left(h_g{}^p \, h^e{}_q \, \nabla_p V^q \right) - c \leftrightarrow d \qquad = h$$

$$= h^a{}_e \, h^f{}_c \, h^g{}_d \, \nabla_f \nabla_g V^e - c \leftrightarrow d$$

$$+ \, h^a{}_q \, h^f{}_c \, \cancelto{0}{h^g{}_d (\nabla_f h^p{}_g)} \nabla_p V^q \quad "$$

$$+ \, h^a{}_e \, h^f{}_c \, h^p{}_d (\nabla_f h^e{}_q) \nabla_p V^q \quad "$$

$$[{}^{(n)}\nabla_c {}^{(n)}\nabla_d - {}^{(n)}\nabla_d {}^{(n)}\nabla_c] V$$

$$= h^a{}_e h^f{}_c h^g{}_d \nabla_f \left({}^{(n)}\nabla_g V^e \right) - c \leftrightarrow d$$

$$= h^a{}_e h^f{}_c h^g{}_d R^e{}_{pfg} V^p$$

$$\pm h^a{}_e h^f{}_c h^p{}_d K^e{}_f \underbrace{n_q \nabla_p V^q}_{-V^q \nabla_p n_q} - c \leftrightarrow d$$

$$= h^a{}_{a'} h^{b'}{}_b h^{c'}{}_c h^{d'}{}_d R^{a'}{}_{b'c'd'} V^b \pm K^a{}_c K_{bd} \mp K^a{}_d K_{bc}$$

is $h_{ab} = g_{ab} \pm \sum_{i=1}^{D-n} n_{ia} n_{ib}$

$+$ if n spacelike.
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Σ is also a manifold in its own right, & can consider
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$$\delta(n^a h_{ab}) = 0 \Rightarrow h_{ab}\,\delta n^a = n_b\,\delta h^{cb} g_{ca}$$

i.e

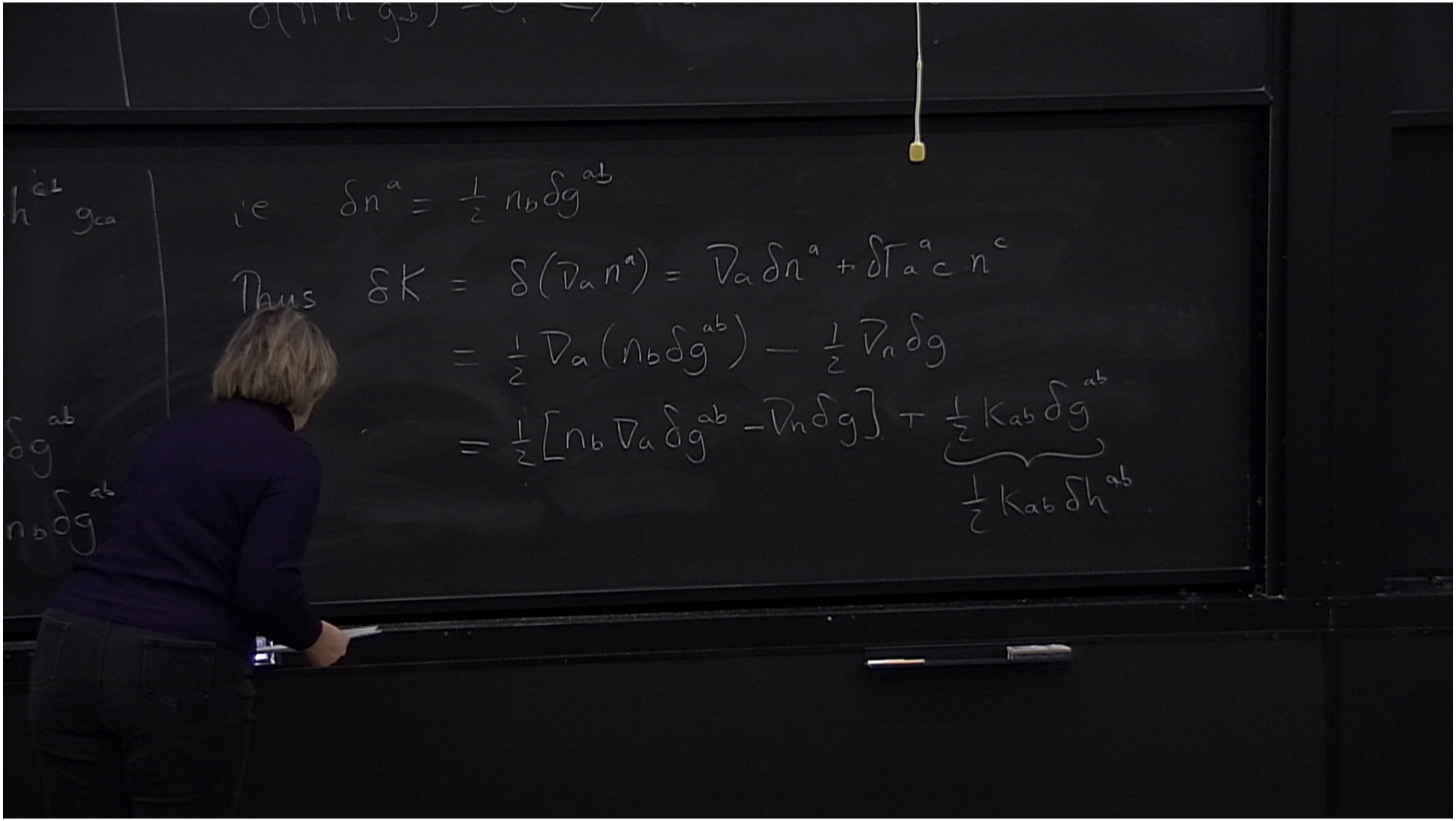
Finally,

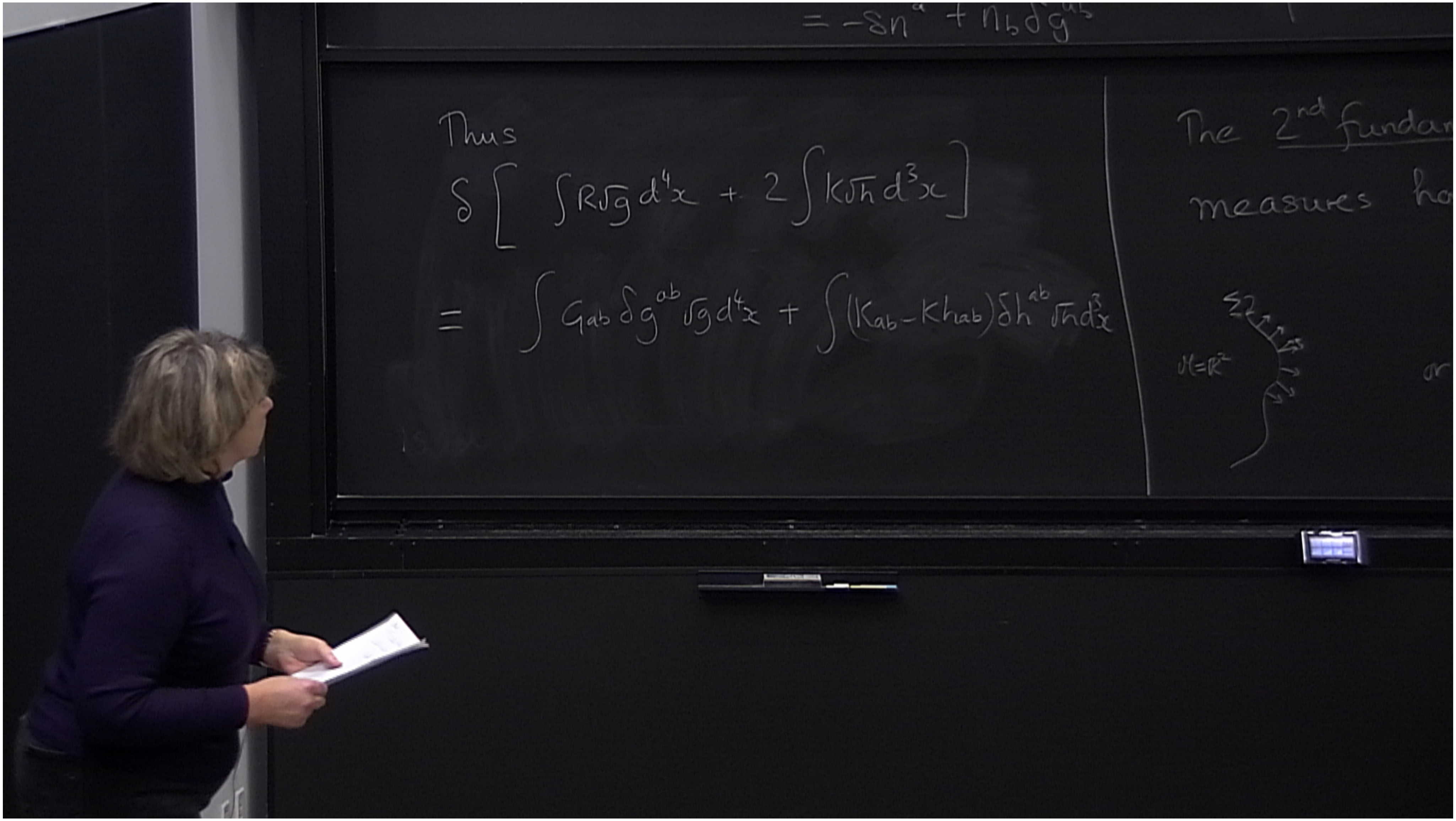
$$\delta n^a n^b + n^a \delta n^b = \delta h^{ab} - \delta g^{ab}$$

$$\Rightarrow -\delta n^a + n^a n_b \delta n^b = n_b \delta h^{ab} - n_b \delta g^{ab}$$

$$\Rightarrow \delta n^a = n^a n_b \delta n^b - h^a{}_b \delta n^b + n_b \delta g^{ab}$$

$$= -\delta n^a + n_b \delta g^{ab}$$


i.e. $\delta n^a = \frac{1}{2} n_b \delta g^{ab}$
Thus $\delta K = \delta(\nabla_a n^a) = \nabla_a \delta n^a + \delta \Gamma_a{}^a{}_c n^c$
$= \frac{1}{2} \nabla_a (n_b \delta g^{ab}) - \frac{1}{2} \nabla_n \delta g$
$= \frac{1}{2}[n_b \nabla_a \delta g^{ab} - \nabla_n \delta g] + \underbrace{\frac{1}{2} K_{ab} \delta g^{ab}}_{\frac{1}{2} K_{ab} \delta h^{ab}}$


Thus
$\delta\left[\int R\sqrt{g}\,d^4x + 2\int K\sqrt{h}\,d^3x\right]$
$= \int G_{ab}\,\delta g^{ab}\sqrt{g}\,d^4x + \int (K_{ab} - K h_{ab})\,\delta h^{ab}\sqrt{h}\,d^3x$
The 2^{nd}
measures
Σ