Title: PSI 2016/2017 Gravitational Physics (Review) - Lecture 5

Date: Jan 09, 2017 10:15 AM

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Abstract:

LECTURE 5: The Einstein Action

In field theory, (almost) always work from an action. Need to integrate on a curved manifold & find Einstein action.

Consider $d^4x \leftrightarrow$ coord volume

but $d^4x' = \det\left(\frac{\partial x'}{\partial x}\right) d^4x$ $\quad [\epsilon_{abcd}\, dx^a dx^b dx^c dx^d]$

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To vary, use identity

$$\det M = \exp \operatorname{tr} \ln M.$$

$$\delta \det M = (\det M)\, \underbrace{\delta(\operatorname{tr} \ln M)}_{\operatorname{tr} M^{-1}\delta M}$$

$$\text{or } -\operatorname{tr} M \delta M^{-1}$$

Take example of massless scalar field

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$$\delta S_\phi = \int d^4x\sqrt{g}\left[\partial_\mu\phi\,\partial_\nu\delta\phi\,g^{\mu\nu}\right.$$

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$$\delta S_\phi = \int d^4x\sqrt{g}\Big[\partial_\mu\phi\,\partial_\nu\delta\phi\, g^{\mu\nu} + \frac{1}{2}\partial_\mu\phi\,\partial_\nu\phi\,\delta g^{\mu\nu} - \frac{1}{4}(\partial\phi)^2 g_{\mu\nu}\delta g^{\mu\nu}\Big]$$

$$\mathrm{tr}\, M^{-1} \delta M$$

$$\text{or} \quad -\mathrm{tr}\, M \delta M^{-1}$$

$$= \int d^4x \sqrt{g} \left[-\delta\phi \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g}\, g^{\mu\nu} \phi \right) + \frac{1}{2} \delta g^{\mu\nu} \left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} (\partial\phi)^2 g_{\mu\nu} \right) \right.$$

$$+ \text{ boundary term } \left(\int d^3x \sqrt{g}\, \nabla_n \phi\, \delta\phi \right)$$

$$\frac{\delta S_\phi}{\delta \phi}$$

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$$+ \text{ boundary term } \left(\int d^3x \sqrt{g}\, n_\mu \partial^\mu \phi \, \delta\phi \right)$$

$$-\frac{\delta S_\phi}{\delta \phi} = \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} \partial^\mu \phi \right) = \Box \phi$$

$\mathrm{tr}\, M^{-1} \delta M$

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ϕ - eqn of motion

(curved) wave eqn

$$2\frac{\delta S_\phi}{\delta g^{\mu\nu}} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}(\partial\phi)^2 g_{\mu\nu} = T_{\phi\,\mu\nu}.$$

– a source term for gravity

Need a "Kinetic term" for g.

Have a scalar, R, containing derivatives of g,

but instead of $(\partial g)^2$, contains $(\partial^2 g)$.

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$\delta R_{\mu\nu} = \delta R^{\lambda}{}_{\mu\lambda\nu}$ find using normal coordinates (NC)

NC's are a local inertial frame, at P, $\Gamma = 0$

$g_{\mu\nu,\lambda} = 0$ at P; but $g_{\mu\nu,\lambda\sigma} \neq 0$.

In NC, $\partial_\mu = \nabla_\mu$, so compute with $\Gamma = 0 \, (\partial\Gamma \neq 0)$

then take $\partial \to \nabla$, & if geometric, then we have found

an answer!

$$\delta R_{\mu\nu} = \delta\Gamma^{\lambda}_{\mu\nu,\lambda} - \delta\Gamma^{\lambda}_{\mu\lambda,\nu}$$

$$= \delta\Gamma^{\lambda}_{\mu\nu;\lambda} - \delta\Gamma^{\lambda}_{\mu\lambda;\nu}$$

$\delta\Gamma$ is a tensor, so this is a covariant expression

$$\delta R_{ab} = \nabla_c \delta\Gamma^{c}_{ab} - \nabla_a \delta\Gamma^{c}_{bc}$$ (Palatini lemma)

To get $\delta\Gamma$, use NC again (remember $d\Gamma$ appears)

$$\delta\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma}\left(\delta g_{\sigma\nu,\lambda} + \delta g_{\sigma\lambda,\nu} - \delta g_{\nu\lambda,\sigma}\right)$$

Hence

$$\delta R_{\mu\nu} = -\frac{1}{2} D_\lambda D_\mu \delta g^{-1}{}_\nu{}^\lambda - \frac{1}{2} D_\lambda D_\nu \delta g^{-1}{}_\mu{}^\lambda$$

$$+ \frac{1}{2} \Box \delta g^{-1}{}_{\mu\nu} + \frac{1}{2} D_\mu D_\nu \delta g^{-1}$$

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c.f. gravitational perturbation theory.

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Thus $\delta R = \delta g^{\mu\nu} R_{\mu\nu} + \Box \delta g - \nabla_\mu \nabla_\nu \delta g^{\mu\nu}$

(note δg now means $\delta(g^{-1})$!!)

Extra terms are a total derivative.

$\nabla_\nu \delta g^{\mu\nu}$

$$\delta \int \sqrt{g} R d^4x = \int \sqrt{g} d^4x \left[\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \delta g^{\mu\nu} + \Box \delta g - \nabla_\mu \nabla_\nu \delta g^{\mu\nu} \right]$$

$$= \int \sqrt{g} d^4x \, G_{\mu\nu} \delta g^{\mu\nu} + \int_{\partial M} d^3x \sqrt{g} \left[\nabla_n \delta g - n_\mu \nabla_\nu \delta g^{\mu\nu} \right]$$

EINSTEIN TENSOR + BOUNDARY TERM.

$\mathrm{tr}\, M \delta M^{-1}$

Einstein-Hilbert action, $S_E = \frac{-1}{16\pi G} \int d^4x \sqrt{g}\, R$

so that $\frac{\delta S_\phi}{\delta g^{\mu\nu}} + \frac{\delta S_{EH}}{\delta g^{\mu\nu}} = \frac{1}{2}\left[T_{\phi\mu\nu} - \frac{1}{8\pi G} G_{\mu\nu} \right]$

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$\sim - \operatorname{tr} M \delta M^{-1}$

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SCHWARZSCHILD:

$$ds^2 = \left(1 - \frac{2GM}{r}\right) d\tau^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 d\Omega_{II}^2$$

Metric now not defined for $r < 2GM$

For $\lambda = 8_4 M$, ρ is proper distance from $r = 2_4 M$.

$$ds^2_{\tau,r} = d\rho^2 + \frac{\rho^2}{8_4 M} \cdot \frac{d\tau^2}{2_4 M}$$

$$= d\rho^2 + \rho^2 d\left(\frac{\tau}{4_4 M}\right)^2$$

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Origin of polar coords, provided τ is an angular coord with periodicity

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coords,

angular

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Periodic Euclidean time a signal of thermal equilibrium at temp.

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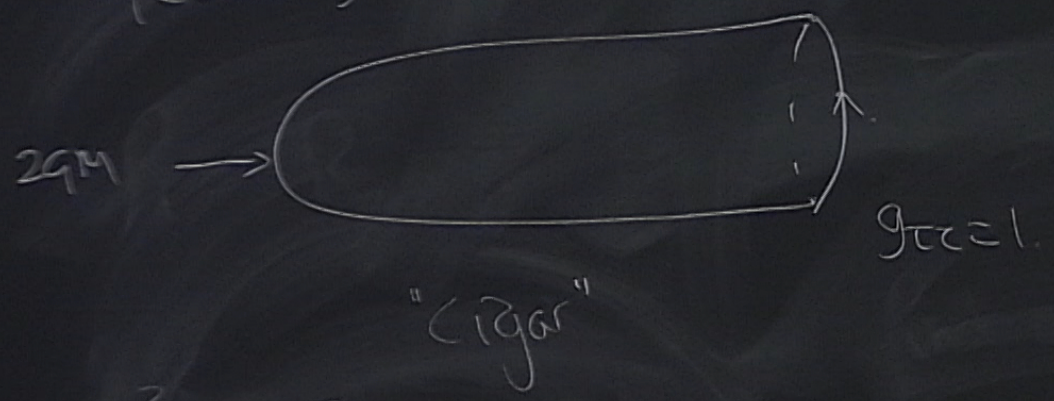
1

– convenient way of calculating Hawking temperature.

Geometry.



$\mathbb{R}^2 \times S^2$

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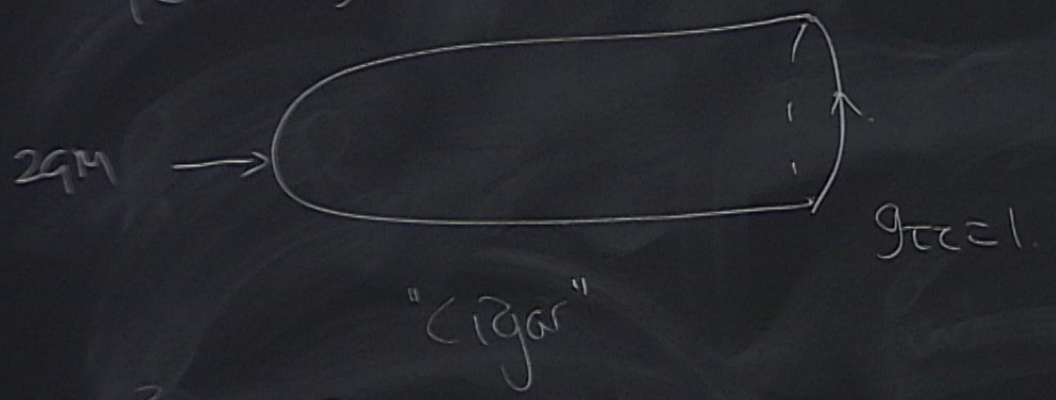

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