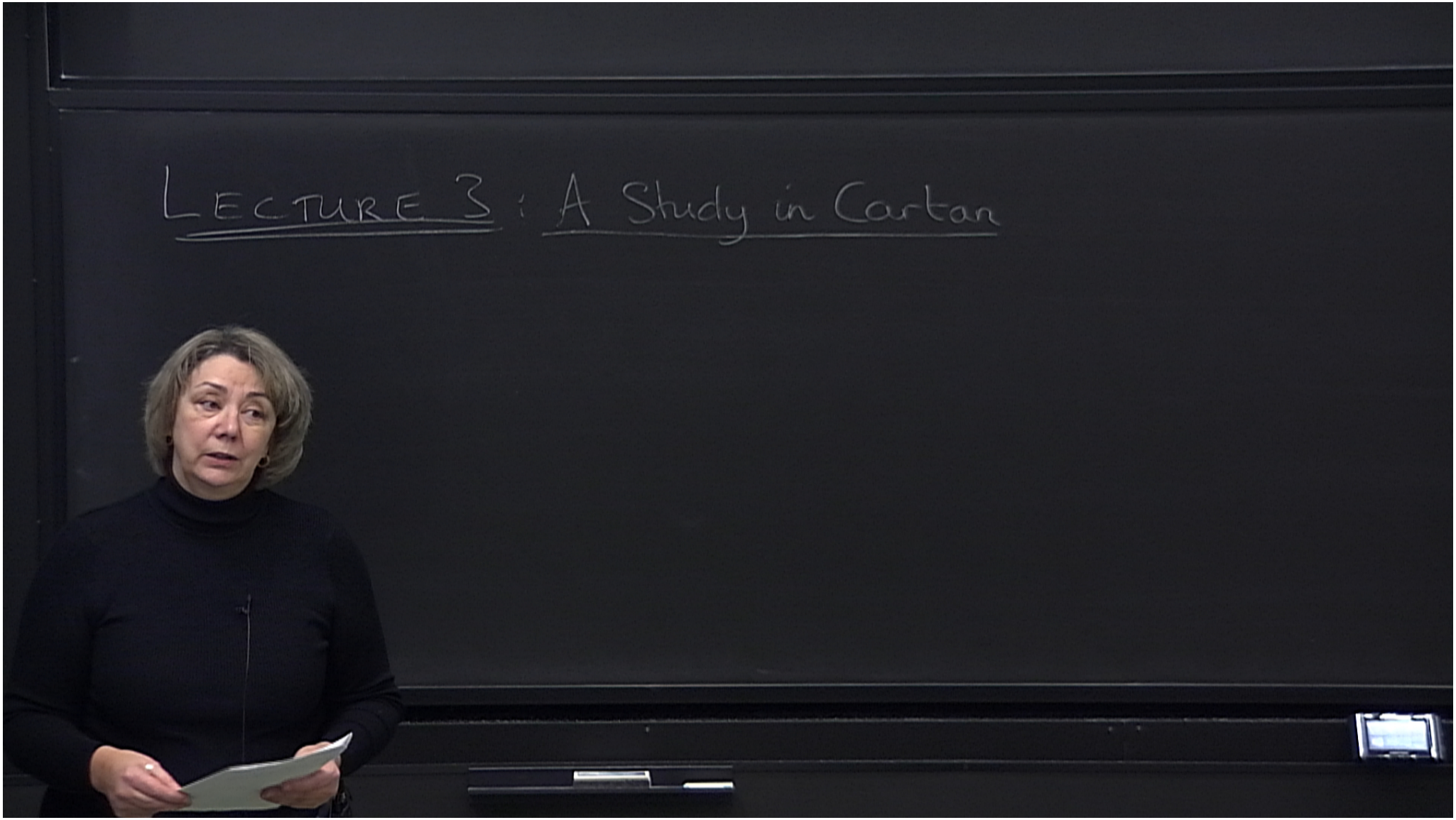
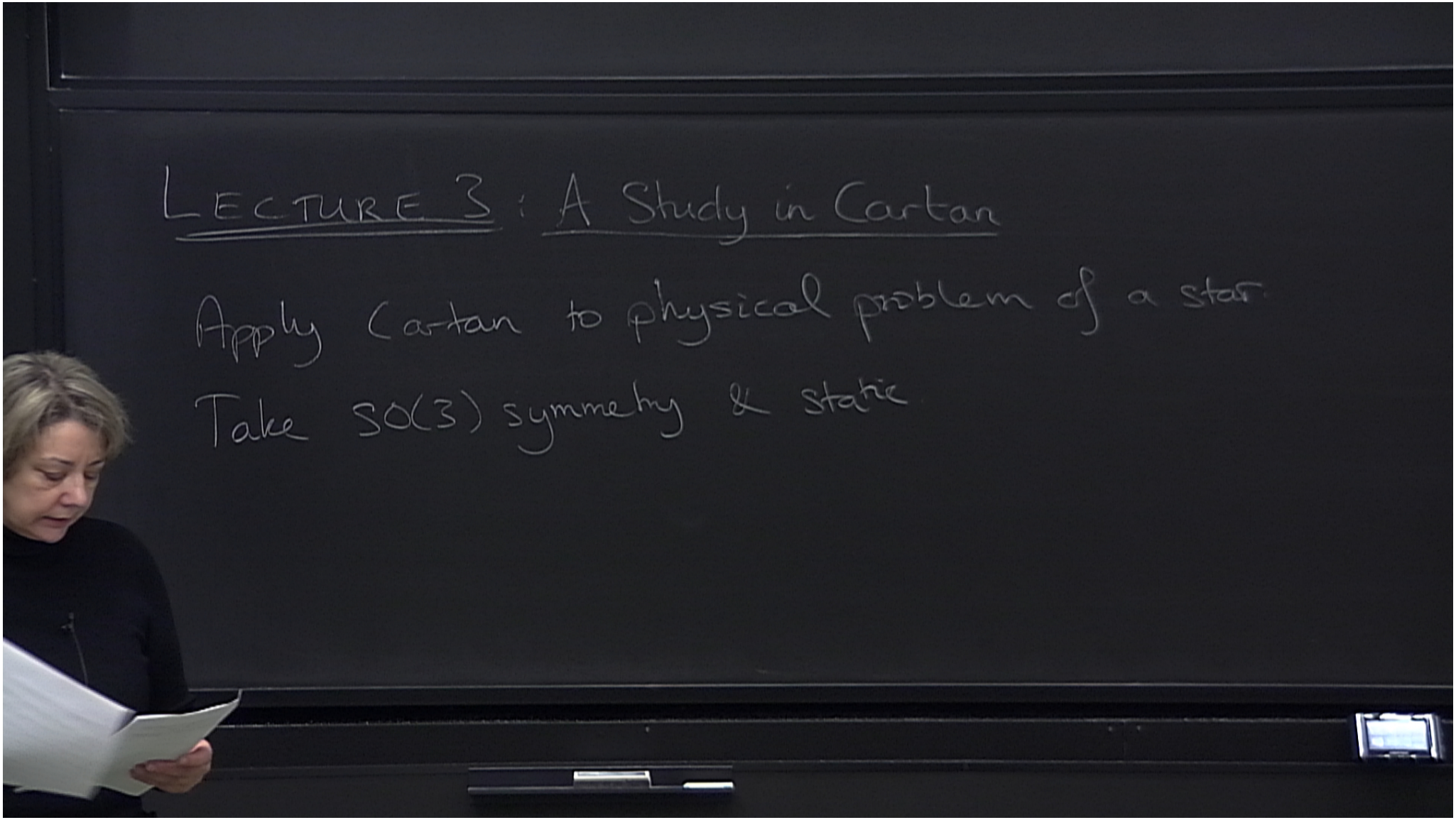
Title: PSI 2016/2017 Gravitational Physics (Review) - Lecture 3

Date: Jan 05, 2017 10:15 AM

URL: http://pirsa.org/17010021

Abstract:


LECTURE 3 : A Study in Cartan


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Apply Cartan to physical problem of a star
Take SO(3) symmetry & static

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Apply Cartan to physical problem of a star

Take SO(3) symmetry & static

$$ds^2 = A^2(r)dt^2 - B^2(r)dr^2 - C^2(r)\underbrace{[d\theta^2 + \sin^2\theta d\varphi^2]}_{d\Omega_{II}^2}$$

Use Cartan to calculate curvature

- Find o/n basis & compute $d\underline{\omega}^{\hat{a}}$.

$\underline{\omega}^{\hat{t}} = A\,dt$ $\qquad$ $\underline{\omega}^{\hat{\theta}} = C\,d\theta$

$\underline{\omega}^{\hat{r}} = B\,dr$ $\qquad$ $\underline{\omega}^{\hat{\varphi}} = C\sin\theta\,d\varphi$.

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$$\underline{d\omega^{\hat{t}}} = dA\wedge dt = A'\,dr\wedge dt = -\frac{A'}{AB}\,\underline{\omega}^{\hat{t}}\wedge\underline{\omega}^{\hat{r}}$$

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$\underline{\omega}^{\hat{t}} = A\,dt$ $\quad$ $\underline{\omega}^{\hat{\theta}} = C\,d\theta$ $\quad$ ^ emphasises o/n basis

$\underline{\omega}^{\hat{r}} = B\,dr$ $\quad$ $\underline{\omega}^{\hat{\varphi}} = C\sin\theta\,d\varphi$.

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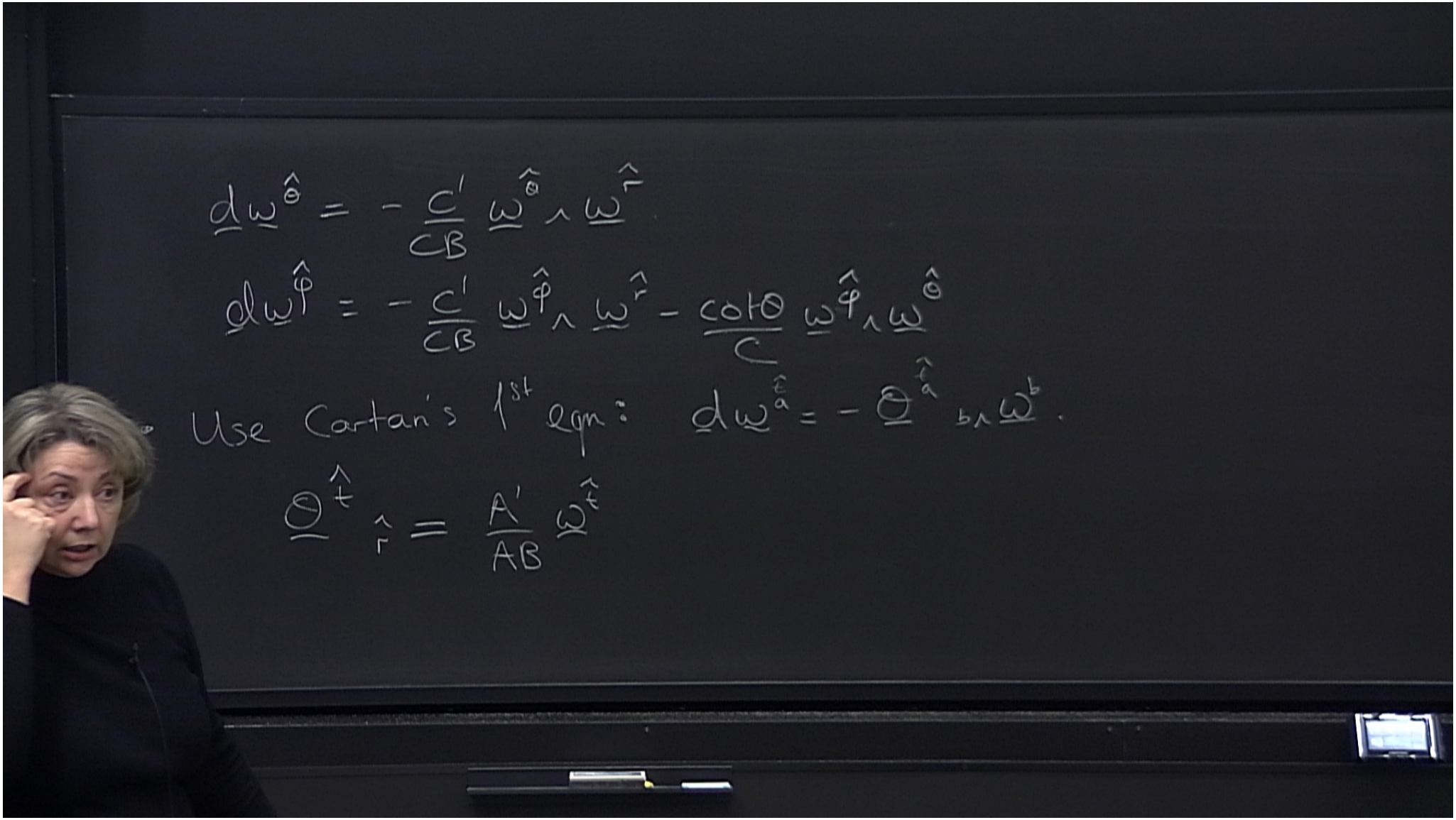
^ emphasises o/n basis

$$\underline{d\omega^{\hat{t}}} = dA\wedge dt = A'\,dr\wedge dt = -\frac{A'}{AB}\,\underline{\omega^{\hat{t}}}\wedge\underline{\omega^{\hat{r}}}$$

$$\underline{d\omega^{\hat{r}}} = 0$$

$$d\omega^{\hat{\theta}} = -\frac{c'}{cB}\,\omega^{\hat{\theta}} \wedge \omega^{\hat{r}}$$

$$d\omega^{\hat{\phi}} = -\frac{c'}{cB}\,\omega^{\hat{\phi}} \wedge \omega^{\hat{r}} - \frac{\cot\theta}{c}\,\omega^{\hat{\phi}} \wedge \omega^{\hat{\theta}}$$


$d\omega^{\hat{\theta}} = -\frac{C'}{CB}\omega^{\hat{\theta}} \wedge \omega^{\hat{r}}$
$d\omega^{\hat{\varphi}} = -\frac{C'}{CB}\omega^{\hat{\varphi}} \wedge \omega^{\hat{r}} - \frac{\cot\theta}{C}\omega^{\hat{\varphi}} \wedge \omega^{\hat{\theta}}$
Use Cartan's 1st eqn: $d\omega^{\hat{a}} = -\Theta^{\hat{a}}{}_{\hat{b}} \wedge \omega^{\hat{b}}$.
$\Theta^{\hat{t}}{}_{\hat{r}} = \frac{A'}{AB}\omega^{\hat{t}}$

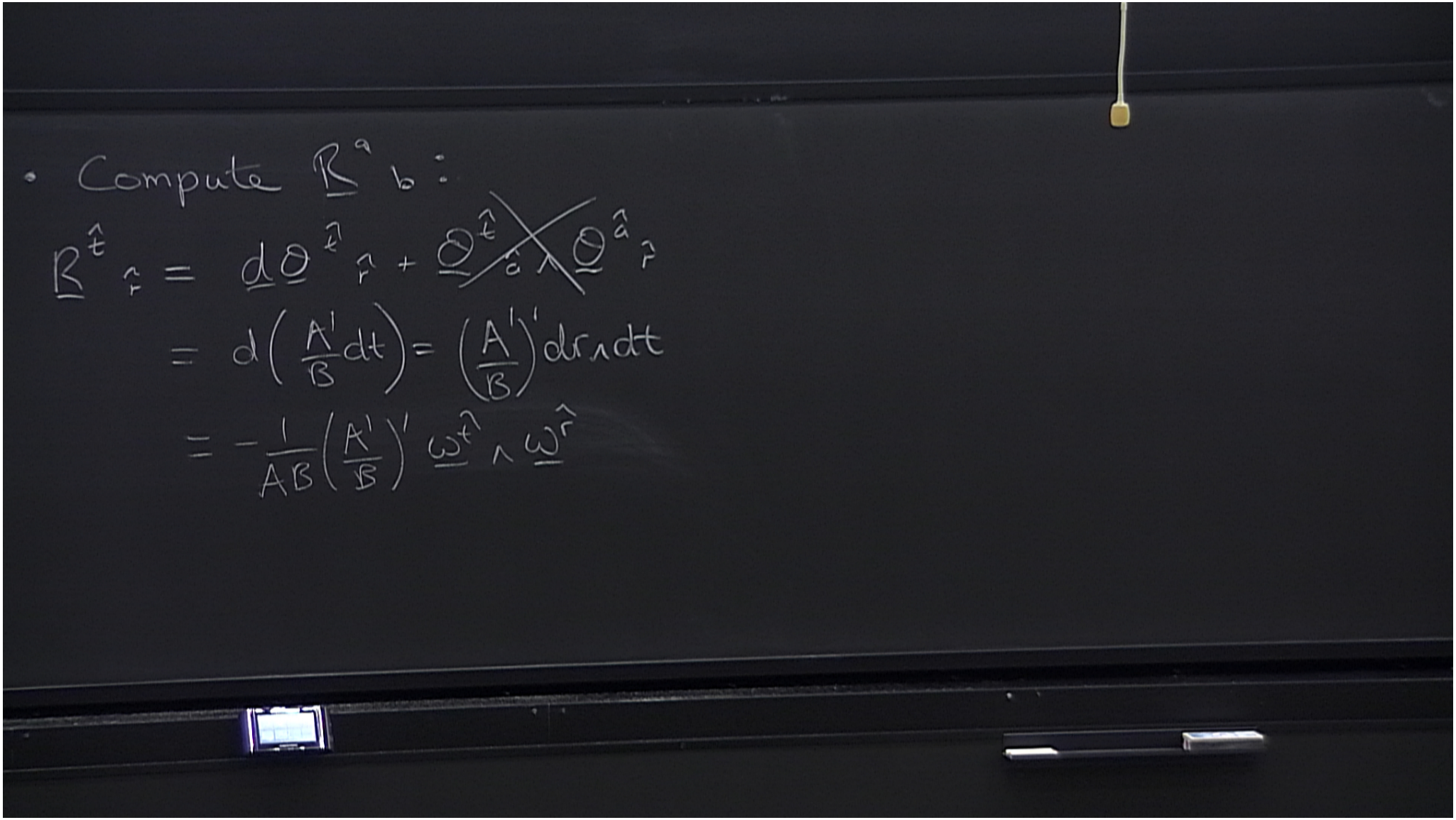
$$d\omega^{\hat{\theta}} = -\frac{C'}{CB}\,\omega^{\hat{\theta}} \wedge \omega^{\hat{r}}$$

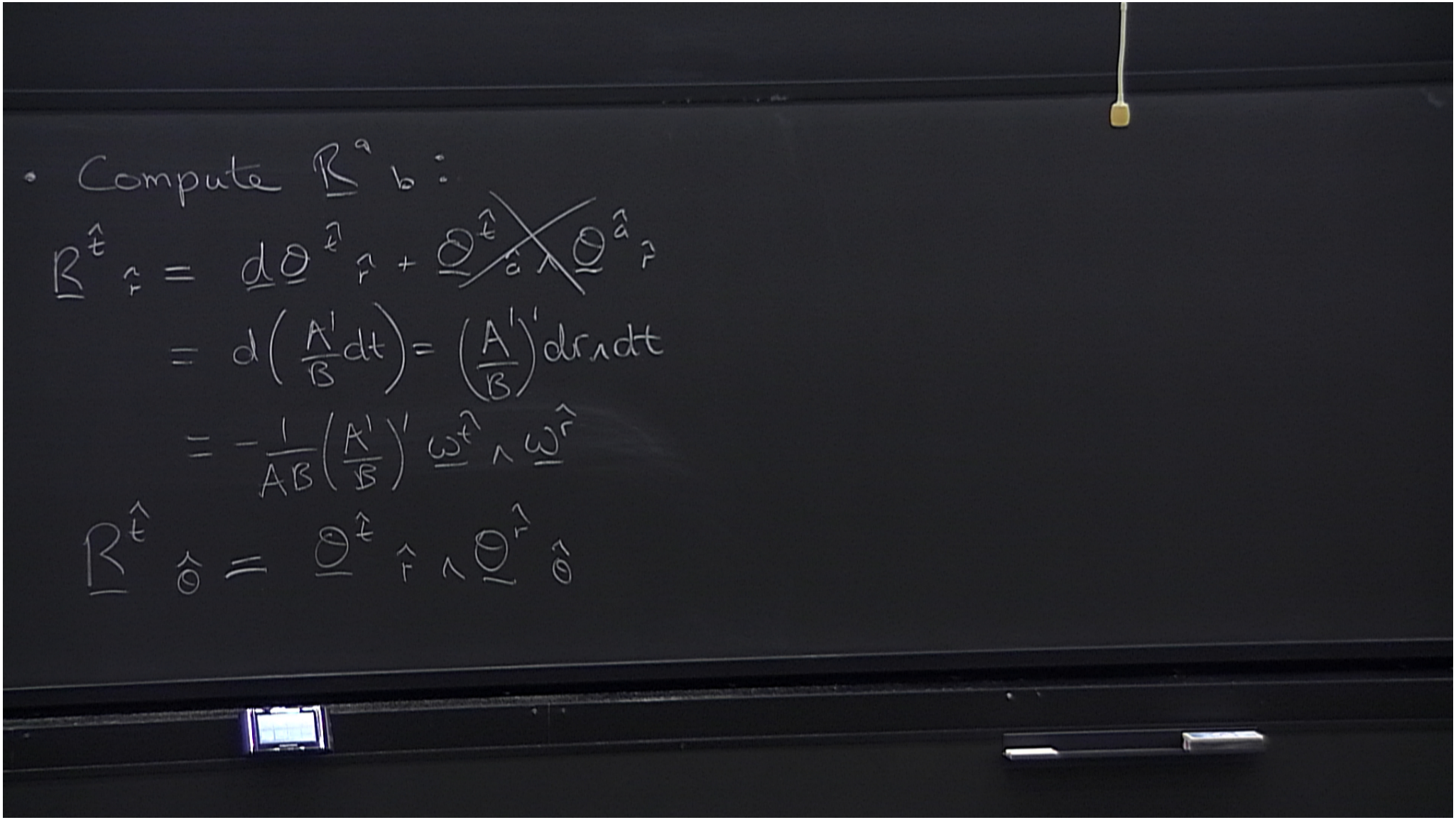
$$d\omega^{\hat{\varphi}} = -\frac{C'}{CB}\,\omega^{\hat{\varphi}} \wedge \omega^{\hat{r}} - \frac{\cot\theta}{C}\,\omega^{\hat{\varphi}} \wedge \omega^{\hat{\theta}}$$

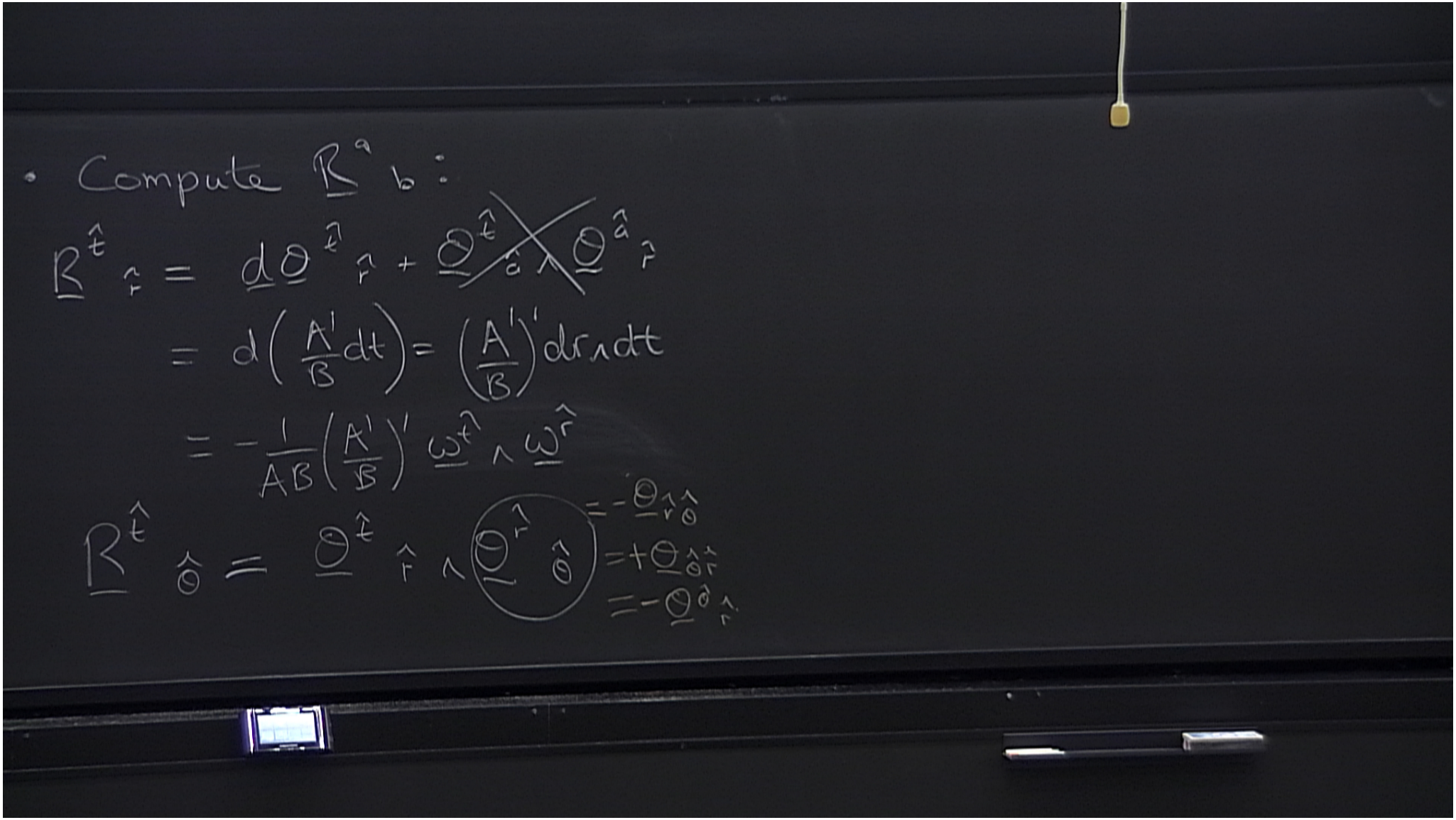
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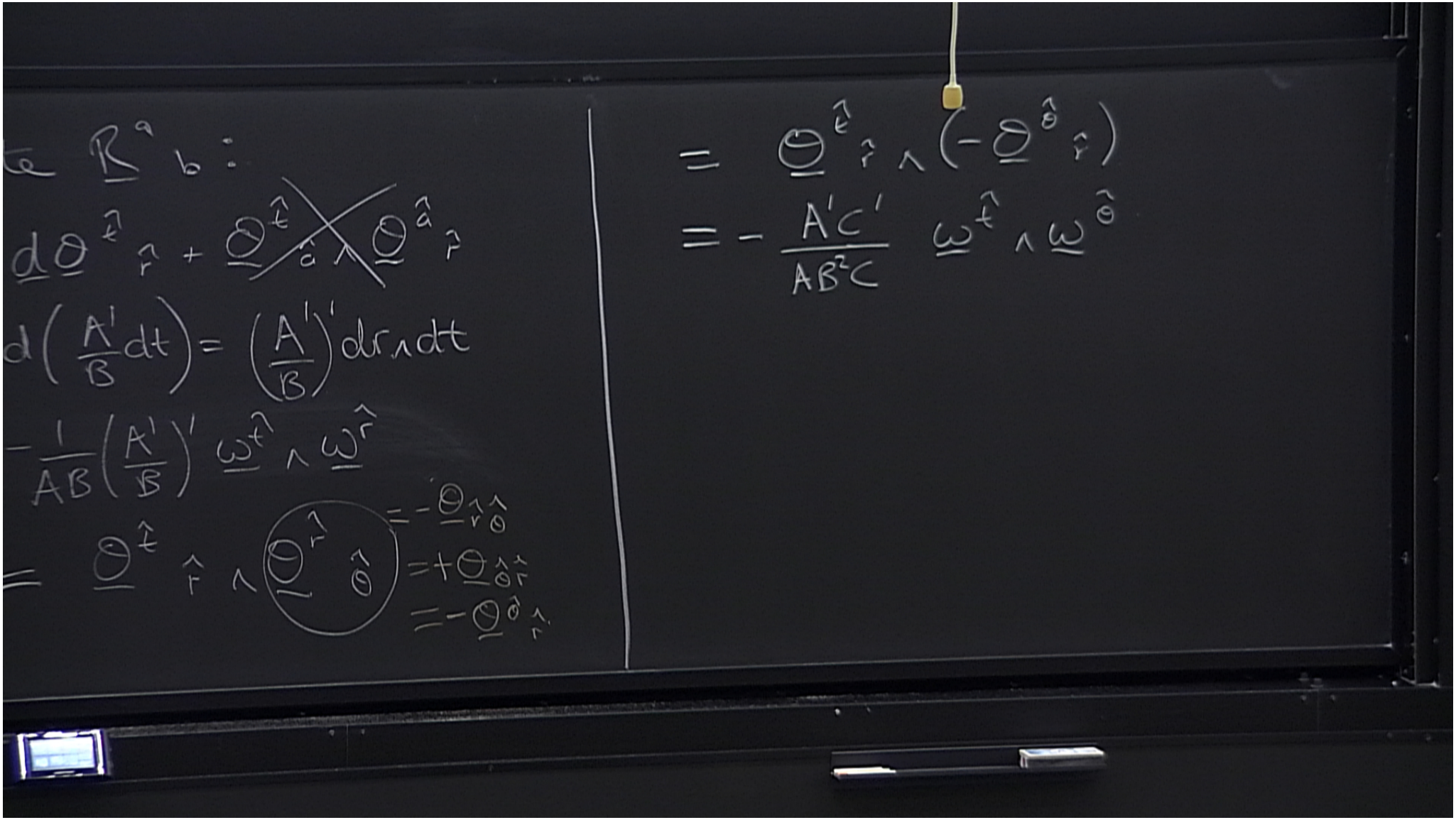
$$\Rightarrow \quad \Theta^{\hat{t}}{}_{\hat{r}} = \frac{A'}{AB}\,\omega^{\hat{t}} \qquad \Theta^{\hat{\varphi}}{}_{\hat{r}} = \frac{C'}{CB}\,\omega^{\hat{\varphi}}$$

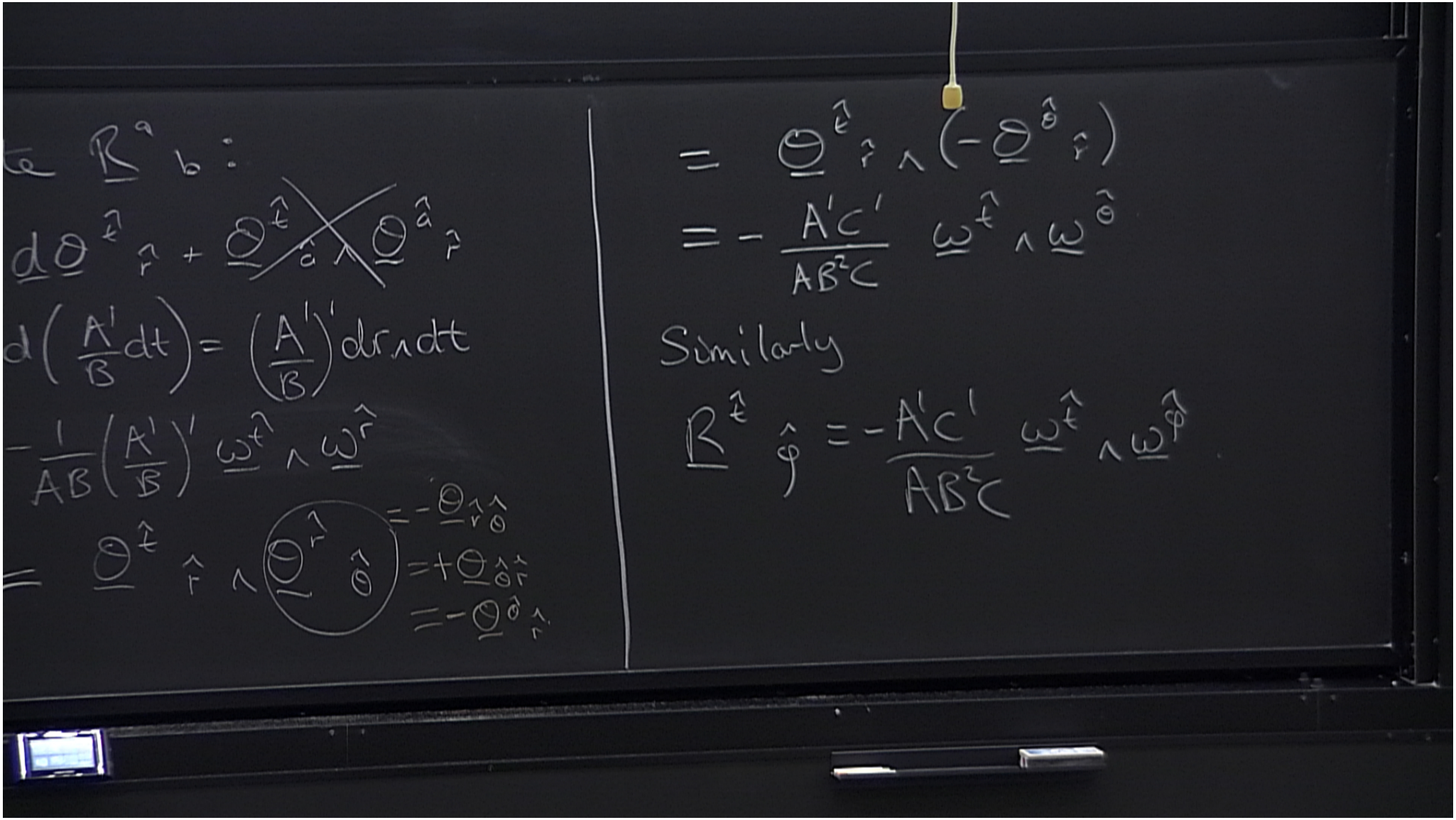
$$\Theta^{\hat{\theta}}{}_{\hat{r}} = \frac{C'}{CB}\,\omega^{\hat{\theta}} \qquad \Theta^{\hat{\varphi}}{}_{\hat{\theta}} = \frac{\cot\theta}{C}\,\omega^{\hat{\varphi}}$$

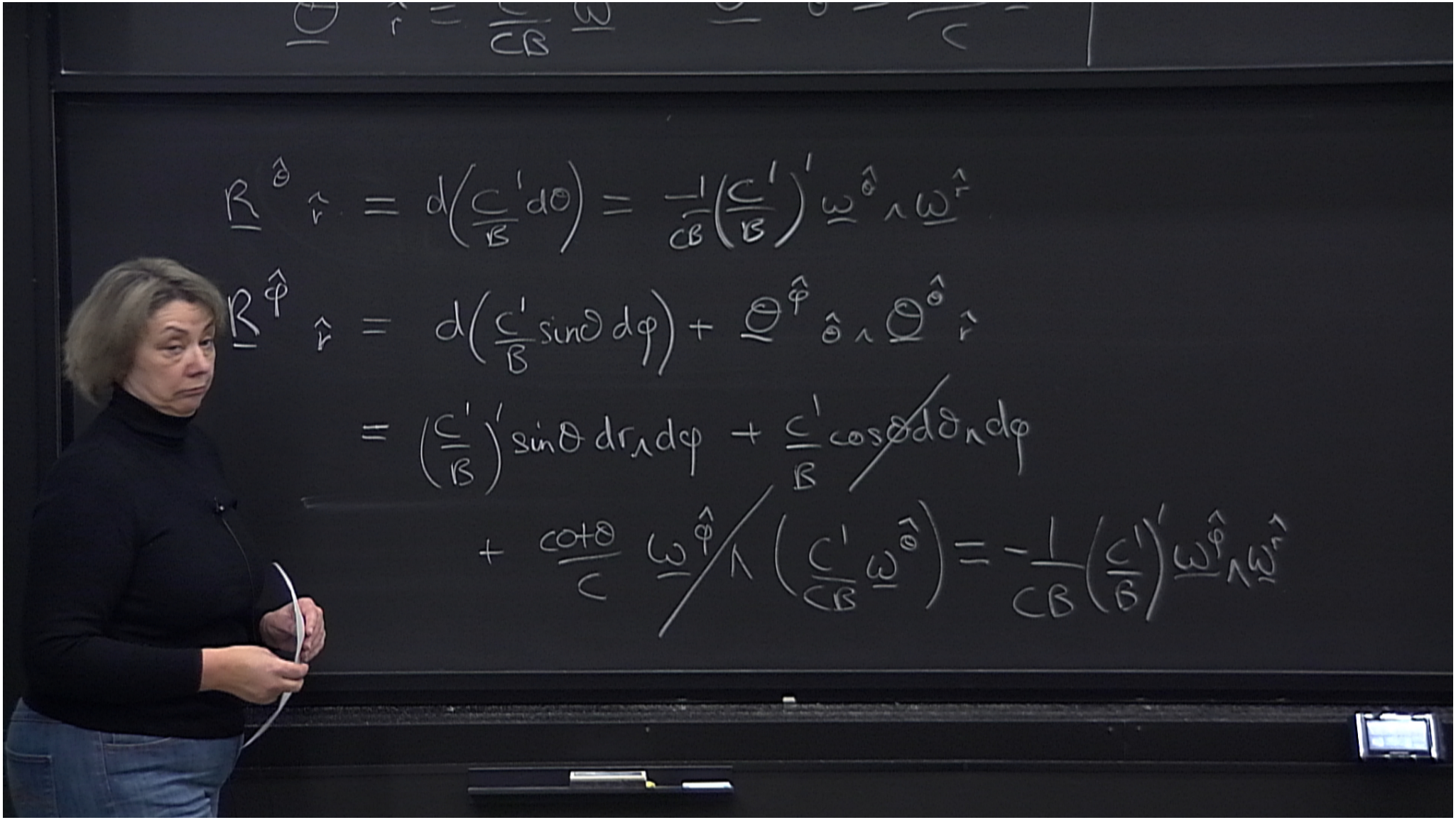

• Compute $\underline{R}^{a}{}_{b}$:
$\underline{R}^{\hat{t}}{}_{\hat{r}} = \underline{d\omega}^{\hat{t}}{}_{\hat{r}} + \underline{\omega}^{\hat{t}}{}_{\hat{a}} \wedge \underline{\omega}^{\hat{a}}{}_{\hat{r}}$
$= d\left(\frac{A'}{B}dt\right) = \left(\frac{A'}{B}\right)' dr \wedge dt$
$= -\frac{1}{AB}\left(\frac{A'}{B}\right)' \underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{r}}$

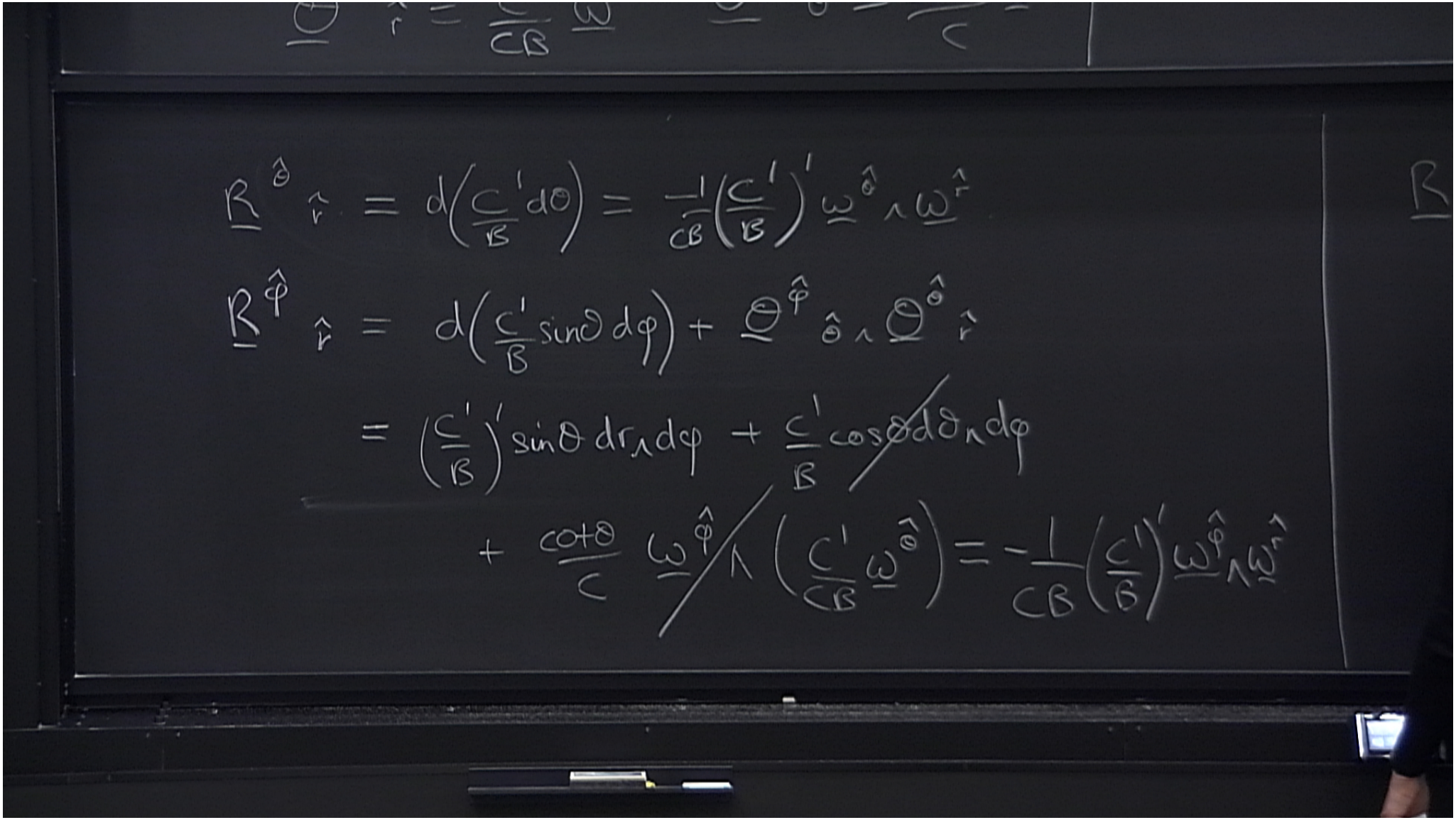

• Compute $\underline{R}^{a}{}_{b}$:
$\underline{R}^{\hat{t}}{}_{\hat{r}} = \underline{d\omega}^{\hat{t}}{}_{\hat{r}} + \underline{\omega}^{\hat{t}}{}_{a} \wedge \underline{\omega}^{a}{}_{\hat{r}}$
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$= -\frac{1}{AB}\left(\frac{A'}{B}\right)' \omega^{\hat{t}} \wedge \omega^{\hat{r}}$
$\underline{R}^{\hat{t}}{}_{\hat{\theta}} = \underline{\omega}^{\hat{t}}{}_{\hat{r}} \wedge \underline{\omega}^{\hat{r}}{}_{\hat{\theta}}$

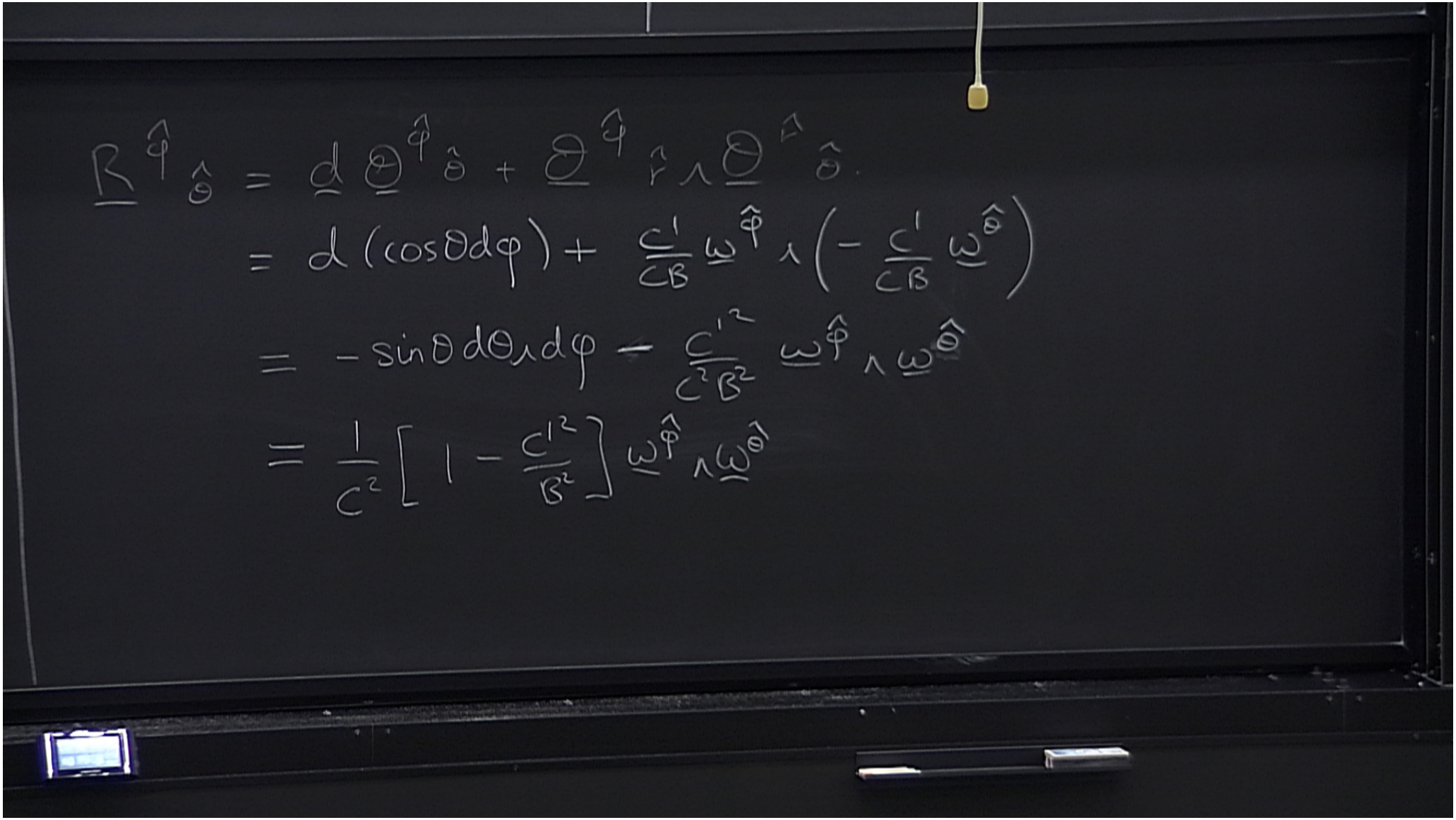

• Compute $R^a{}_b$:
$R^{\hat{t}}{}_{\hat{r}} = d\omega^{\hat{t}}{}_{\hat{r}} + \omega^{\hat{t}}{}_{\hat{a}} \wedge \omega^{\hat{a}}{}_{\hat{r}}$
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$R^{\hat{t}}{}_{\hat{\theta}} = \omega^{\hat{t}}{}_{\hat{r}} \wedge \omega^{\hat{r}}{}_{\hat{\theta}}$
$= -\omega^{\hat{r}}{}_{\hat{r}} \wedge \omega^{\hat{r}}{}_{\hat{\theta}}$
$= +\omega^{\hat{\theta}}{}_{\hat{r}}$
$= -\omega^{\hat{\theta}}{}_{\hat{r}}$


$\underline{R}^a{}_b$:
$d\underline{\theta}^{\hat{t}}{}_{\hat{r}} + \underline{\theta}^{\hat{t}}{}_{\hat{a}} \wedge \underline{\theta}^{\hat{a}}{}_{\hat{r}}$
$d\left(\frac{A'}{B}dt\right) = \left(\frac{A'}{B}\right)' dr \wedge dt$
$-\frac{1}{AB}\left(\frac{A'}{B}\right)' \underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{r}}$
$= \underline{\theta}^{\hat{t}}{}_{\hat{r}} \wedge \underline{\theta}^{\hat{r}}{}_{\hat{\theta}}$
$= -\underline{\theta}_{\hat{r}\hat{\theta}}$
$= +\underline{\theta}_{\hat{\theta}\hat{r}}$
$= -\underline{\theta}^{\hat{\theta}}{}_{\hat{r}}$
$= \underline{\theta}^{\hat{t}}{}_{\hat{r}} \wedge \left(-\underline{\theta}^{\hat{\theta}}{}_{\hat{r}}\right)$
$= -\frac{A'C'}{AB^2C} \underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{\theta}}$


$\underline{R}^{a}{}_{b}$:
$d\underline{\theta}^{\hat{t}}{}_{\hat{r}} + \underline{\theta}^{\hat{t}}{}_{\hat{a}} \wedge \underline{\theta}^{\hat{a}}{}_{\hat{r}}$
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$-\frac{1}{AB}\left(\frac{A'}{B}\right)' \underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{r}}$
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$= -\frac{A'C'}{AB^2C}\underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{\theta}}$
Similarly
$\underline{R}^{\hat{t}}{}_{\hat{\varphi}} = -\frac{A'C'}{AB^2C}\underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{\varphi}}$


$\underline{R}^{\hat{\theta}}{}_{\hat{r}} = d\left(\frac{c'}{B}d\theta\right) = \frac{-1}{cB}\left(\frac{c'}{B}\right)'\underline{\omega}^{\hat{\theta}}\wedge\underline{\omega}^{\hat{r}}$
$\underline{R}^{\hat{\varphi}}{}_{\hat{r}} = d\left(\frac{c'}{B}\sin\theta\, d\varphi\right) + \underline{\Theta}^{\hat{\varphi}}{}_{\hat{\theta}}\wedge\underline{\Theta}^{\hat{\theta}}{}_{\hat{r}}$
$= \left(\frac{c'}{B}\right)'\sin\theta\, dr\wedge d\varphi + \frac{c'}{B}\cos\theta\, d\theta\wedge d\varphi$
$+ \frac{\cot\theta}{c}\underline{\omega}^{\hat{\varphi}}\wedge\left(\frac{c'}{cB}\underline{\omega}^{\hat{\theta}}\right) = -\frac{1}{cB}\left(\frac{c'}{B}\right)'\underline{\omega}^{\hat{\varphi}}\wedge\underline{\omega}^{\hat{r}}$


$\underline{R}^{\hat{\theta}}{}_{\hat{r}} = d\left(\frac{c'}{B}d\theta\right) = \frac{-1}{cB}\left(\frac{c'}{B}\right)'\underline{\omega}^{\hat{\theta}}\wedge\underline{\omega}^{\hat{r}}$
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$= \left(\frac{c'}{B}\right)'\sin\theta\, dr\wedge d\varphi + \frac{c'}{B}\cos\theta\, d\theta\wedge d\varphi$
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$\underline{R}^{\hat{\varphi}}{}_{\hat{\theta}} = \underline{d}\,\underline{\Theta}^{\hat{\varphi}}{}_{\hat{\theta}} + \underline{\Theta}^{\hat{\varphi}}{}_{\hat{r}} \wedge \underline{\Theta}^{\hat{r}}{}_{\hat{\theta}}$
$= d(\cos\theta d\varphi) + \frac{c'}{cB}\underline{\omega}^{\hat{\varphi}} \wedge \left(-\frac{c'}{cB}\underline{\omega}^{\hat{\theta}}\right)$
$= -\sin\theta\, d\theta \wedge d\varphi - \frac{c'^2}{c^2B^2}\underline{\omega}^{\hat{\varphi}} \wedge \underline{\omega}^{\hat{\theta}}$
$= \frac{1}{c^2}\left[1 - \frac{c'^2}{B^2}\right]\underline{\omega}^{\hat{\varphi}} \wedge \underline{\omega}^{\hat{\theta}}$

• Read off Riemann: $R^a{}_b = \frac{1}{2} R^a{}_{bcd}\, \omega^c \wedge \omega^d$

$$R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} = -\frac{1}{AB}\left(\frac{A'}{B}\right)'$$

$$R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} = -\frac{A'C'}{AB^2C} = R^{\hat{t}}{}_{\hat{\rho}\hat{t}\hat{\rho}}$$

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$$R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} = -\frac{1}{AB}\left(\frac{A'}{B}\right)'$$

$$R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} = -\frac{A'C'}{AB^2C} = R^{\hat{t}}{}_{\hat{\varphi}\hat{t}\hat{\varphi}}$$

$$R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} = -\frac{1}{CB}\left(\frac{C'}{B}\right)' = R^{\hat{\varphi}}{}_{\hat{r}\hat{\varphi}\hat{r}}$$

• Read off Riemann: $R^a{}_b = \frac{1}{2} R^a{}_{bcd}\, \omega^c \wedge \omega^d$

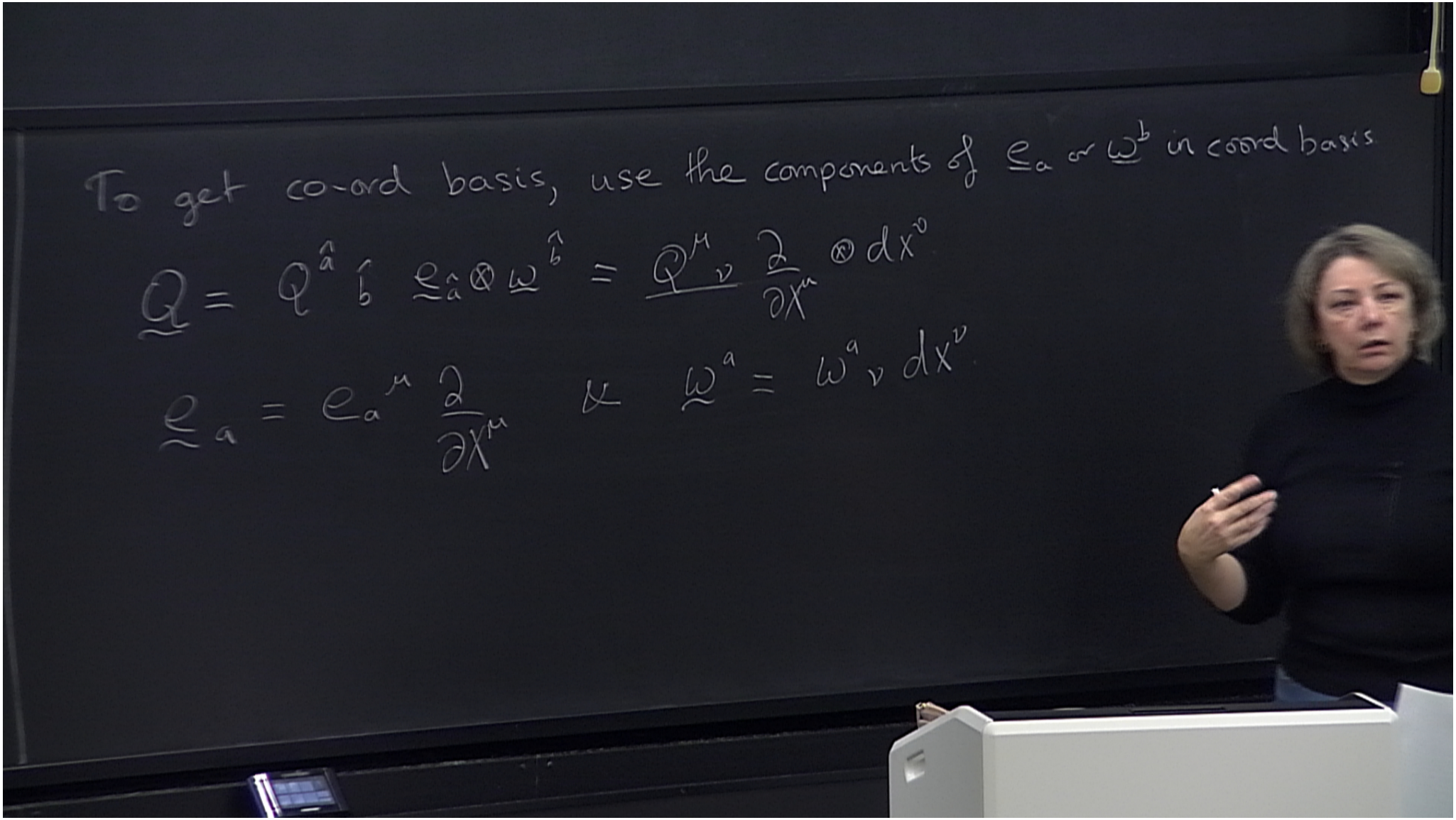
$$R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} = -\frac{1}{AB}\left(\frac{A'}{B}\right)'$$

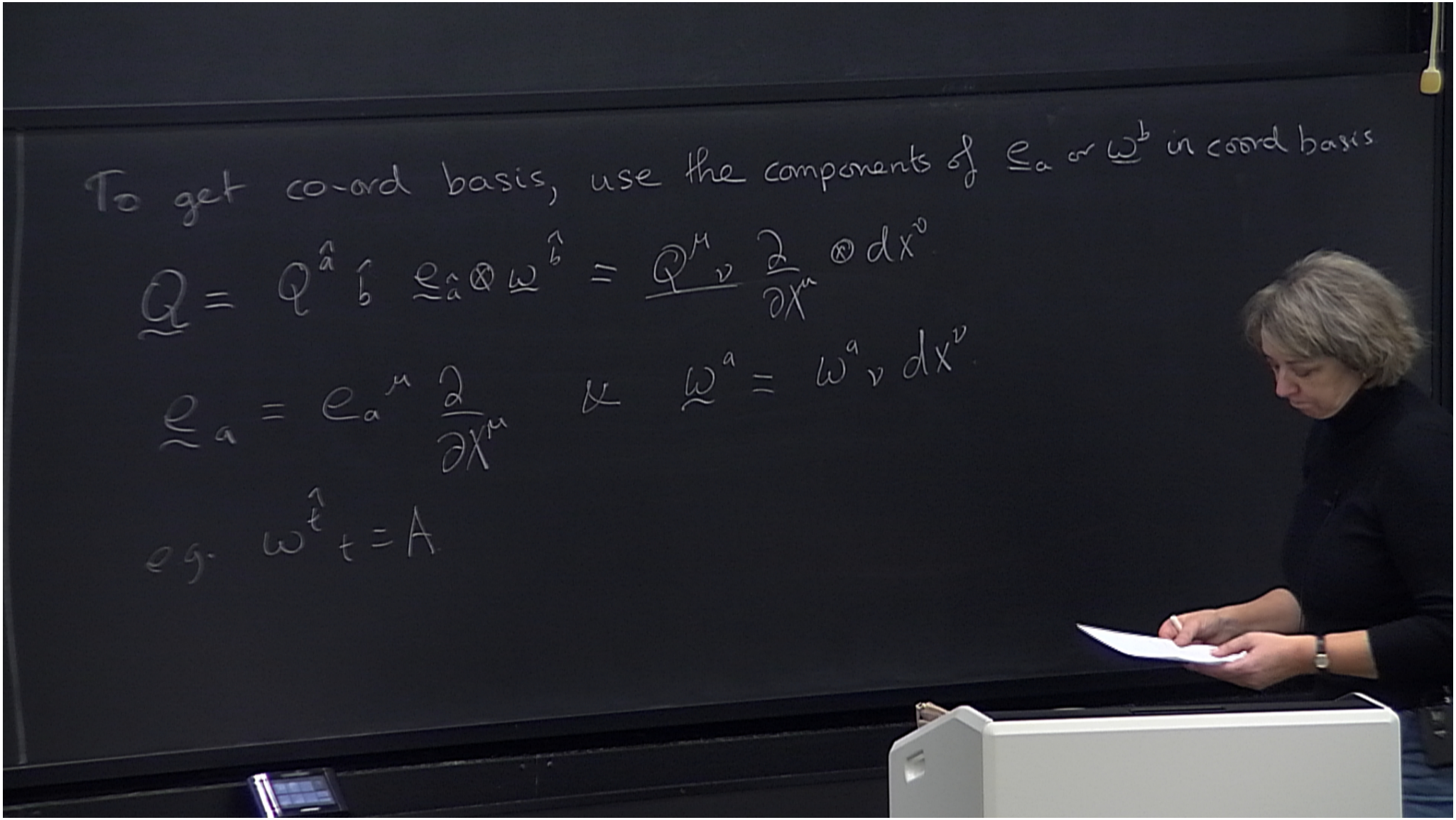
$$R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} = -\frac{A'C'}{AB^2C} = R^{\hat{t}}{}_{\hat{\varphi}\hat{t}\hat{\varphi}}$$

$$R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} = -\frac{1}{CB}\left(\frac{C'}{B}\right)' = R^{\hat{\varphi}}{}_{\hat{\theta}\hat{\varphi}\hat{r}}$$

$$R^{\hat{\varphi}}{}_{\hat{\theta}\hat{\varphi}\hat{\theta}} = \frac{1}{C^2}\left[1 - \frac{C'^2}{B^2}\right]$$

o/n basis


To get co-ord basis, use the components of $\underset{\sim}{e}_a$ or $\underset{\sim}{\omega}^b$ in coord basis
$\underset{\sim}{Q} = Q^{\hat{a}}{}_{\hat{b}}\, \underset{\sim}{e}_{\hat{a}} \otimes \underset{\sim}{\omega}^{\hat{b}} = \underline{Q^{\mu}{}_{\nu}} \frac{\partial}{\partial x^{\mu}} \otimes dx^{\nu}$
$\underset{\sim}{e}_a = e_a{}^{\mu} \frac{\partial}{\partial x^{\mu}}$ & $\underset{\sim}{\omega}^a = \omega^a{}_{\nu}\, dx^{\nu}$


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$\underset{\sim}{e}_a = e_a{}^{\mu} \frac{\partial}{\partial x^{\mu}}$ & $\underset{\sim}{\omega}^a = \omega^a{}_{\nu}\, dx^{\nu}$
eg. $\omega^{\hat{t}}{}_t = A$

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$$\underset{\sim}{Q} = Q^{\hat{a}}{}_{\hat{b}}\, \underset{\sim}{e}_{\hat{a}} \otimes \underset{\sim}{\omega}^{\hat{b}} = \underline{Q^{\mu}{}_{\nu}}\, \frac{\partial}{\partial x^{\mu}} \otimes dx^{\nu}$$

$$\underset{\sim}{e}_a = e_a{}^{\mu} \frac{\partial}{\partial x^{\mu}} \quad \& \quad \underset{\sim}{\omega}^a = \omega^a{}_{\nu}\, dx^{\nu}$$

e.g. $\omega^{\hat{t}}{}_t = A \Rightarrow R^{tr}{}_{tr} = \frac{1}{B^2}\left(\frac{A''}{A} - \frac{A'B'}{AB}\right)$. $R^{t\theta}{}_{t\theta} = \frac{A'C'}{AB^2C} = R^{t\varphi}{}_{t\varphi}$

$$R^{\theta r}{}_{\theta r} = \frac{1}{B^2}\left(\frac{C''}{C} - \frac{C'B'}{CB}\right) = R^{\varphi r}{}_{\varphi r} \quad R^{\varphi\theta}{}_{\varphi\theta} = \frac{-1}{C^2}\left(1 - \frac{C'^2}{B^2}\right)$$

Hence $R^t_{tt} = \frac{1}{B^2}\left(\frac{A''}{A} - \frac{A'B'}{AB} + 2\frac{A'C'}{AC}\right)$

$R^r_{rr} = \frac{1}{B^2}\left(\frac{A''}{A} + 2\frac{C''}{C} - \frac{B'}{B}\left(\frac{A'}{A} + 2\frac{C'}{C}\right)\right)$

$R^\theta_{\theta\theta} = \frac{1}{B^2}\left(\frac{C''}{C} - \frac{C'B'}{CB} + \frac{A'C'}{AC} + \frac{C'^2}{C^2}\right) - \frac{1}{C^2} = R^\varphi_{\varphi\varphi}$

Hence $R^t_{tt} = \frac{1}{B^2}\left(\frac{A''}{A} - \frac{A'B'}{AB} + 2\frac{A'C'}{AC}\right)$

$R^r_{rr} = \frac{1}{B^2}\left(\frac{A''}{A} + 2\frac{C''}{C} - \frac{B'}{B}\left(\frac{A'}{A} + 2\frac{C'}{C}\right)\right)$

$R^\theta_{\theta\theta} = \frac{1}{B^2}\left(\frac{C''}{C} - \frac{C'B'}{CB} + \frac{A'C'}{AC} + \frac{C'^2}{C^2}\right) - \frac{1}{C^2} = R^\varphi_\varphi$

Now look at gauge:

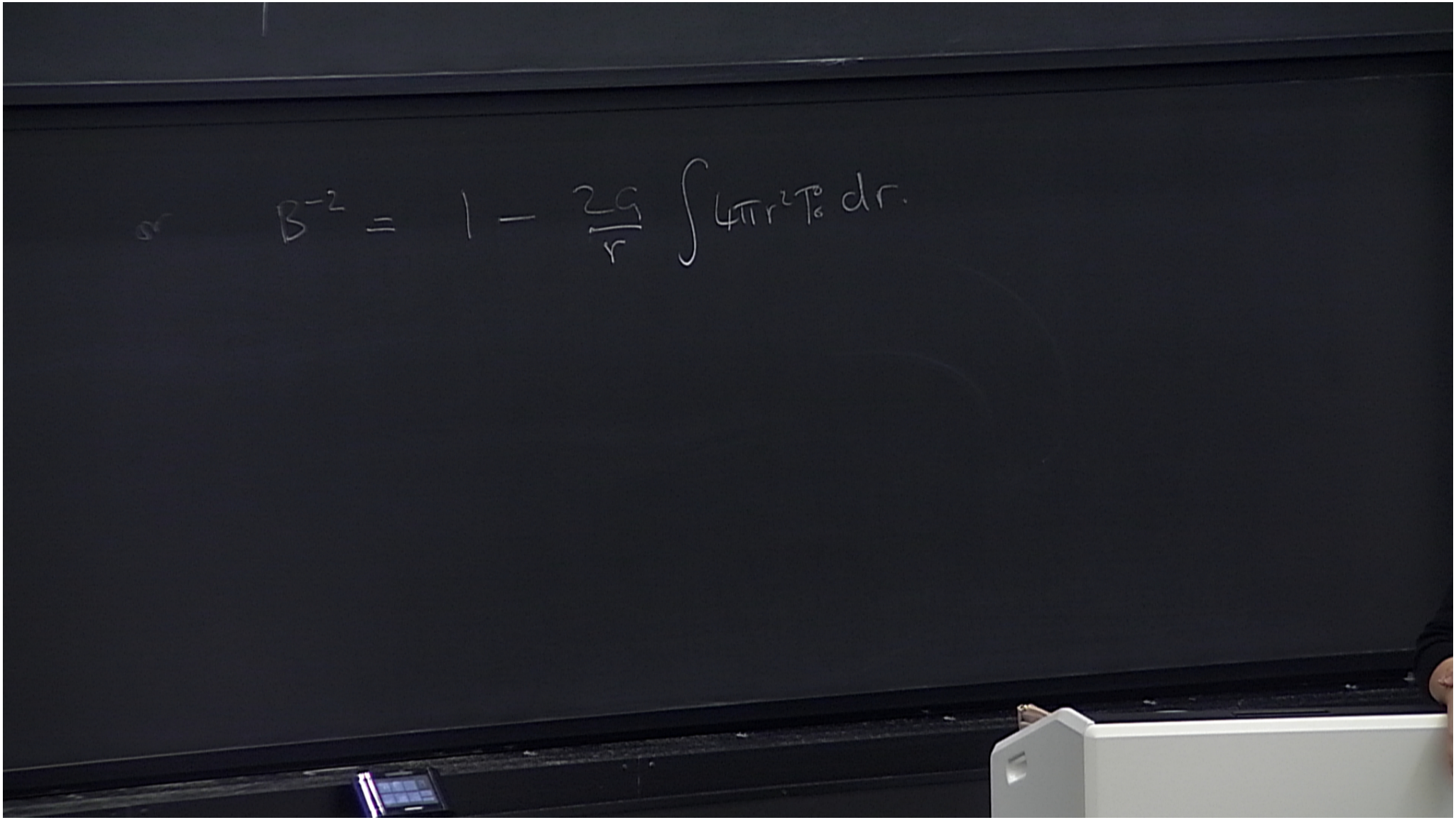
- $B = 1$ $\rightarrow$ r proper distance radially.
- $B = A^{-1}$ "SCH"
- $C = r$ "area gauge" $A(S^2)$ $4\pi r^2$.

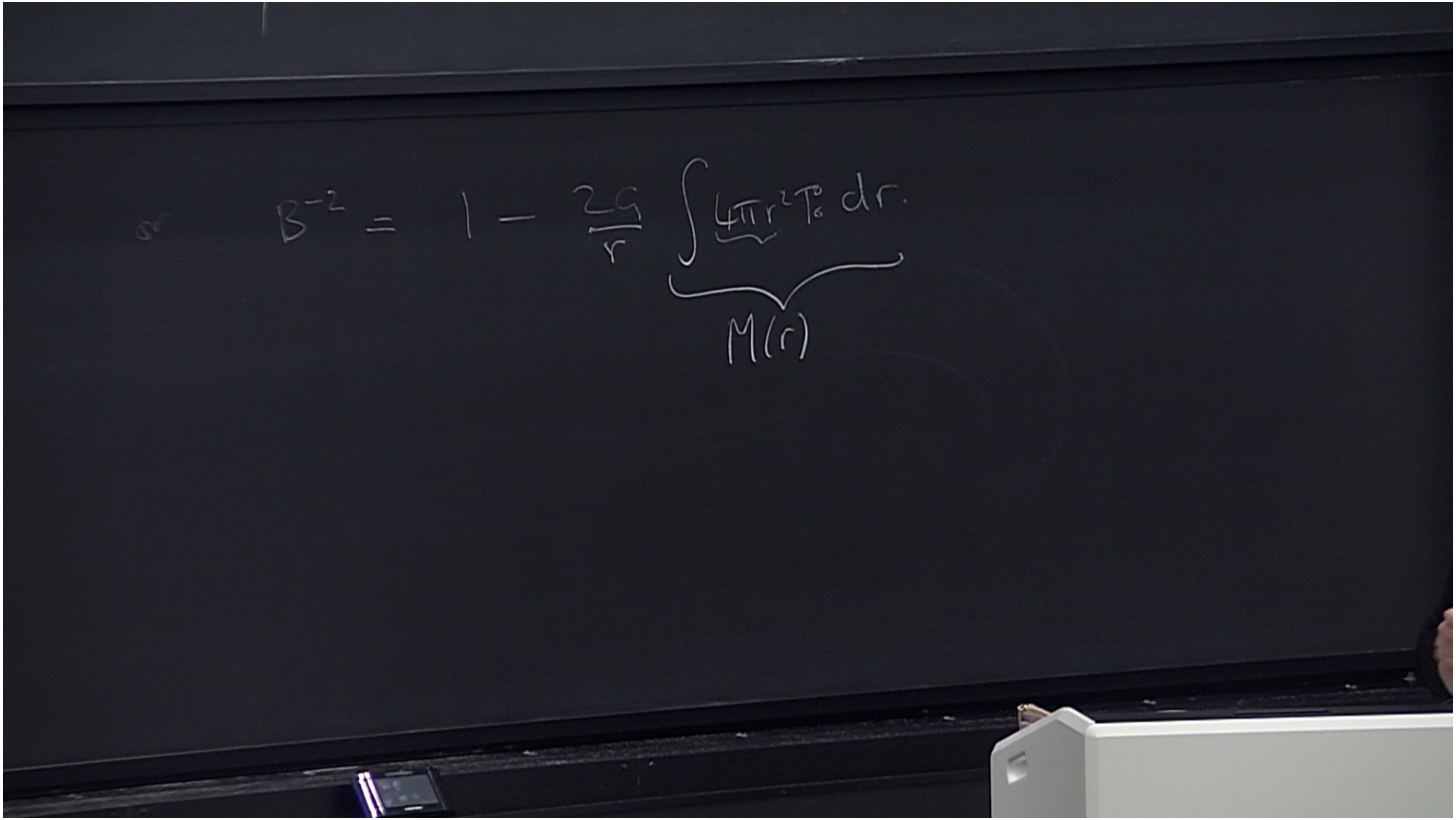
Area gauge:

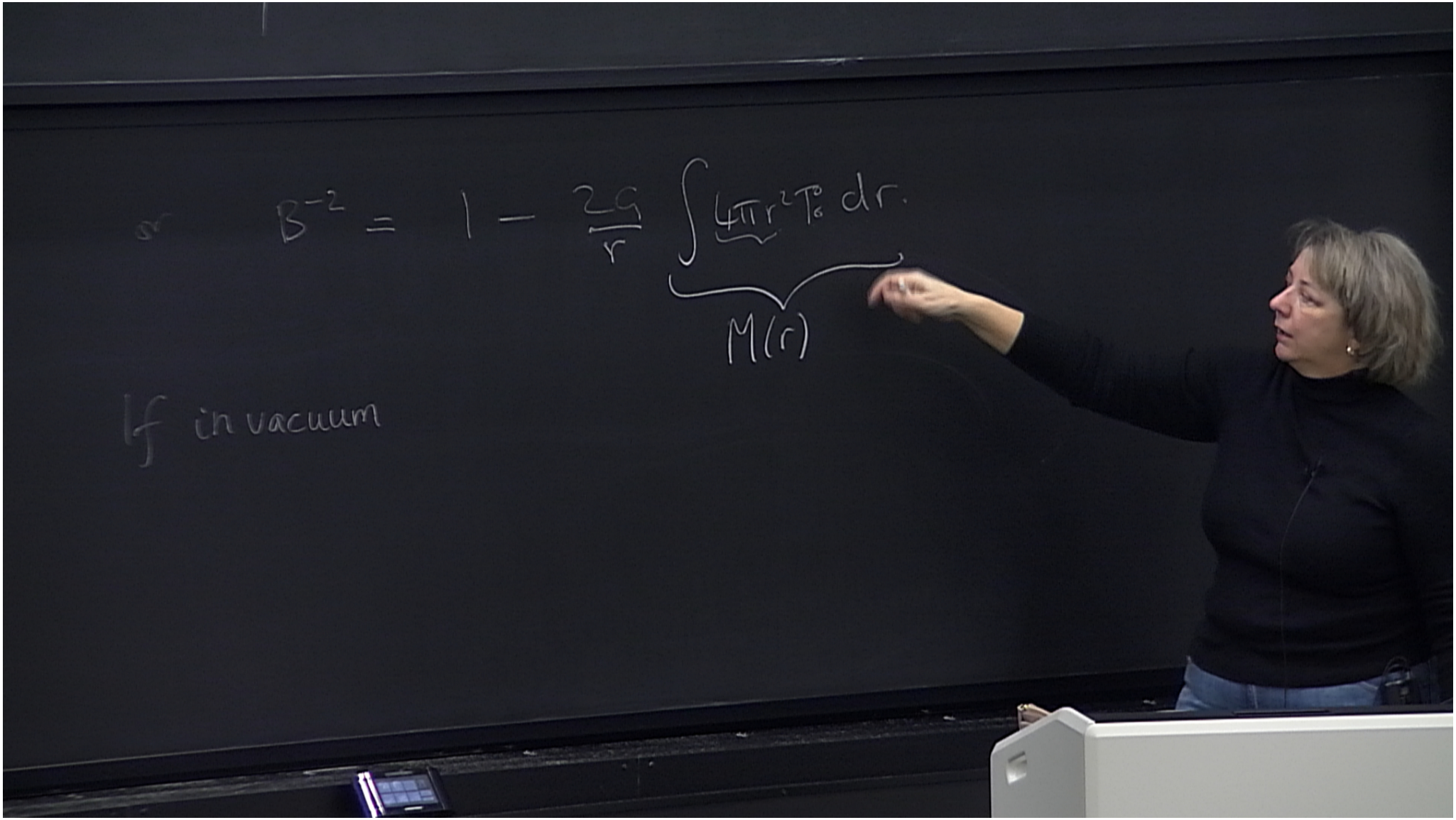
$$G^t_t = \frac{1}{C^2} - \frac{1}{B^2}\left(2\frac{C''}{C} - 2\frac{C'B'}{CB} + \frac{C'^2}{C^2}\right)$$

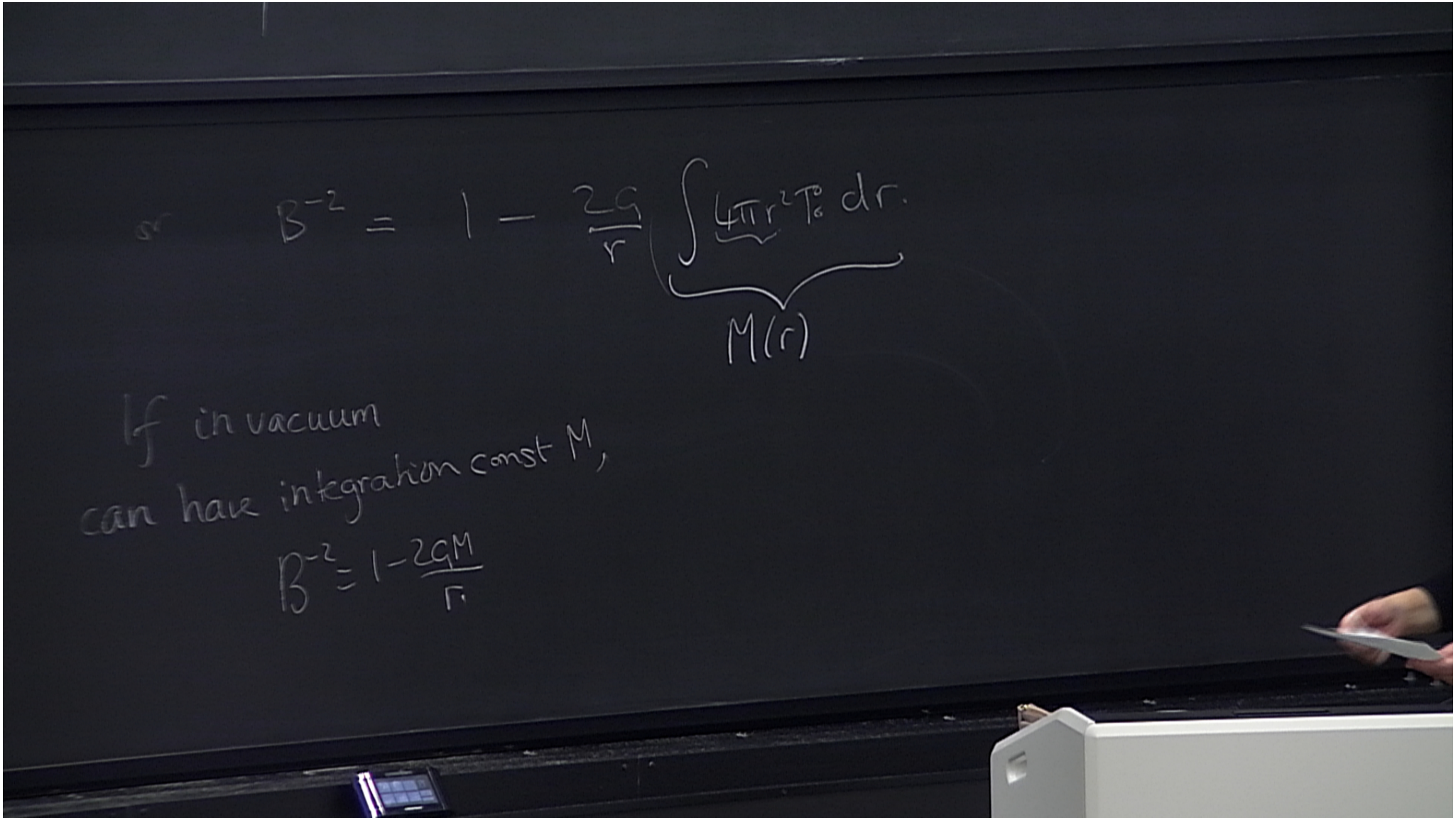
$$= \frac{1}{r^2} - \frac{1}{r^2}\left(\frac{r}{B^2}\right)' = 8\pi G T^0_0$$

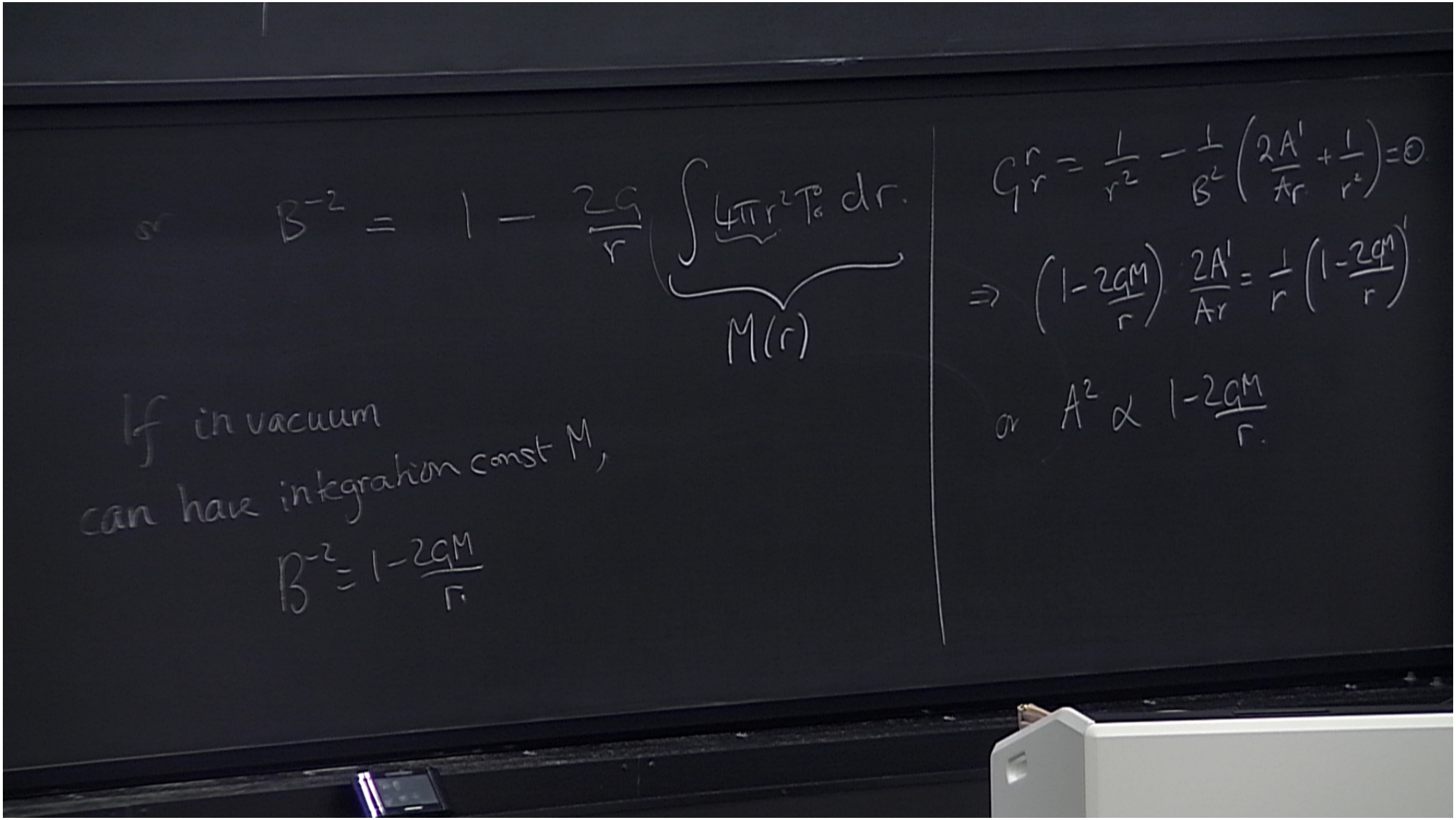
$$\Rightarrow \left(\frac{r}{B^2}\right)' = 1 - 8\pi G r^2 T^0_0.$$

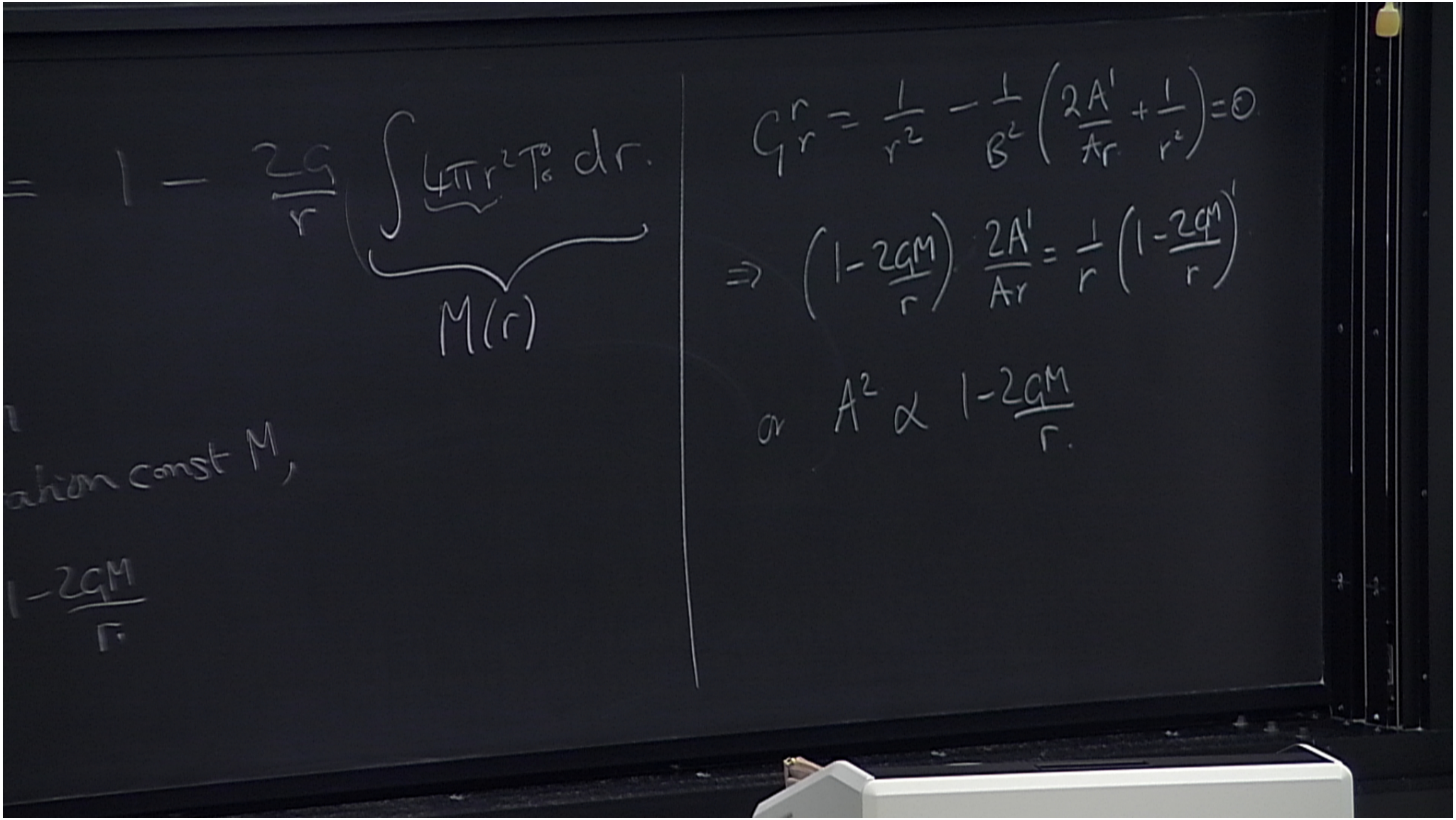

$B^{-2} = 1 - \frac{2G}{r} \int 4\pi r^2 T^0_0 \, dr.$


or $B^{-2} = 1 - \frac{2G}{r} \underbrace{\int 4\pi r^2 T_0^0 \, dr}_{M(r)}$.


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If in vacuum
can have integration const M,
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or $B^{-2} = 1 - \frac{2G}{r}\underbrace{\int 4\pi r^2 T^0_0 \, dr}_{M(r)}$
If in vacuum
can have integration const M,
$B^{-2} = 1 - \frac{2GM}{r}$
$G^r_r = \frac{1}{r^2} - \frac{1}{B^2}\left(\frac{2A'}{Ar} + \frac{1}{r^2}\right) = 0$
$\Rightarrow \left(1 - \frac{2GM}{r}\right)\frac{2A'}{Ar} = \frac{1}{r}\left(1 - \frac{2GM}{r}\right)'$
or $A^2 \propto 1 - \frac{2GM}{r}$


$= 1 - \frac{2G}{r}\underbrace{\int 4\pi r^2 T^0_0\, dr}_{M(r)}$
ation const M,
$1 - \frac{2GM}{r}$
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$\Theta^{\hat{r}} = \frac{c'}{c}\,\omega^{\hat{\theta}}$ $\quad$ $\Theta^{\hat{\theta}} = \cot\theta\,\omega^{\hat{\varphi}}$

What if not in vacuum?

$$-8\pi G p_r = \frac{1}{r^2} - \left(1 - \frac{2GM(r)}{r}\right)\left(\frac{2A'}{Ar} + \frac{1}{r^2}\right)$$

$\partial_{\hat{r}} = \partial_r \quad \omega^{\hat{r}} = dr \qquad \partial_{\hat{\theta}} = \frac{1}{r}\partial_\theta \quad \omega^{\hat{\theta}} = r\,d\theta \qquad \partial_{\hat{\varphi}} = \frac{1}{r\sin\theta}\partial_\varphi \quad \omega^{\hat{\varphi}} = r\sin\theta\,d\varphi$

What if not in vacuum?

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or A^2.

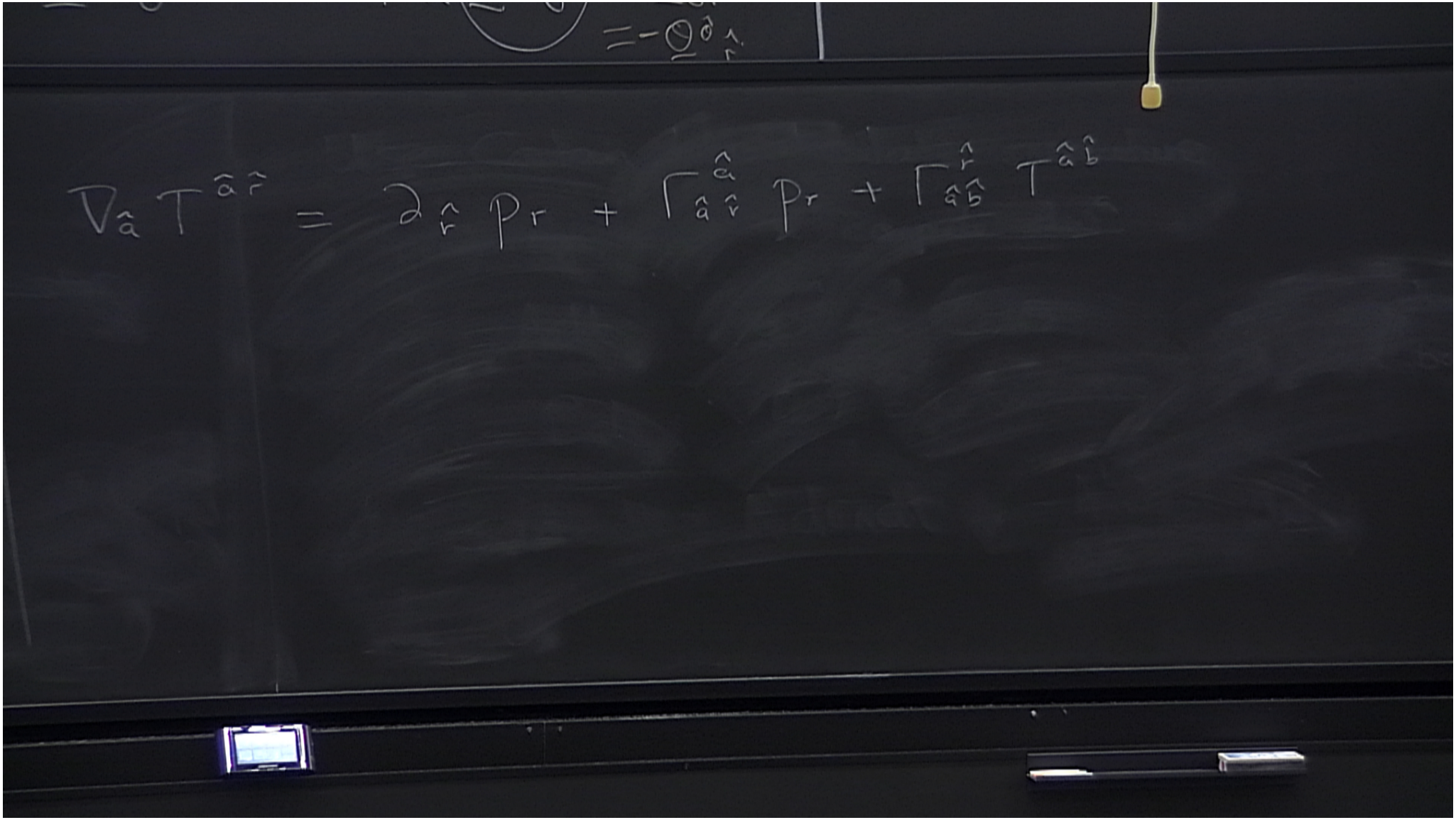
$\Theta^{\hat{r}} = \frac{c'}{c}\,\omega^{\hat{\theta}}$ $\quad \Theta^{\hat{\theta}} = \cot\theta\,\omega^{\hat{\phi}}$

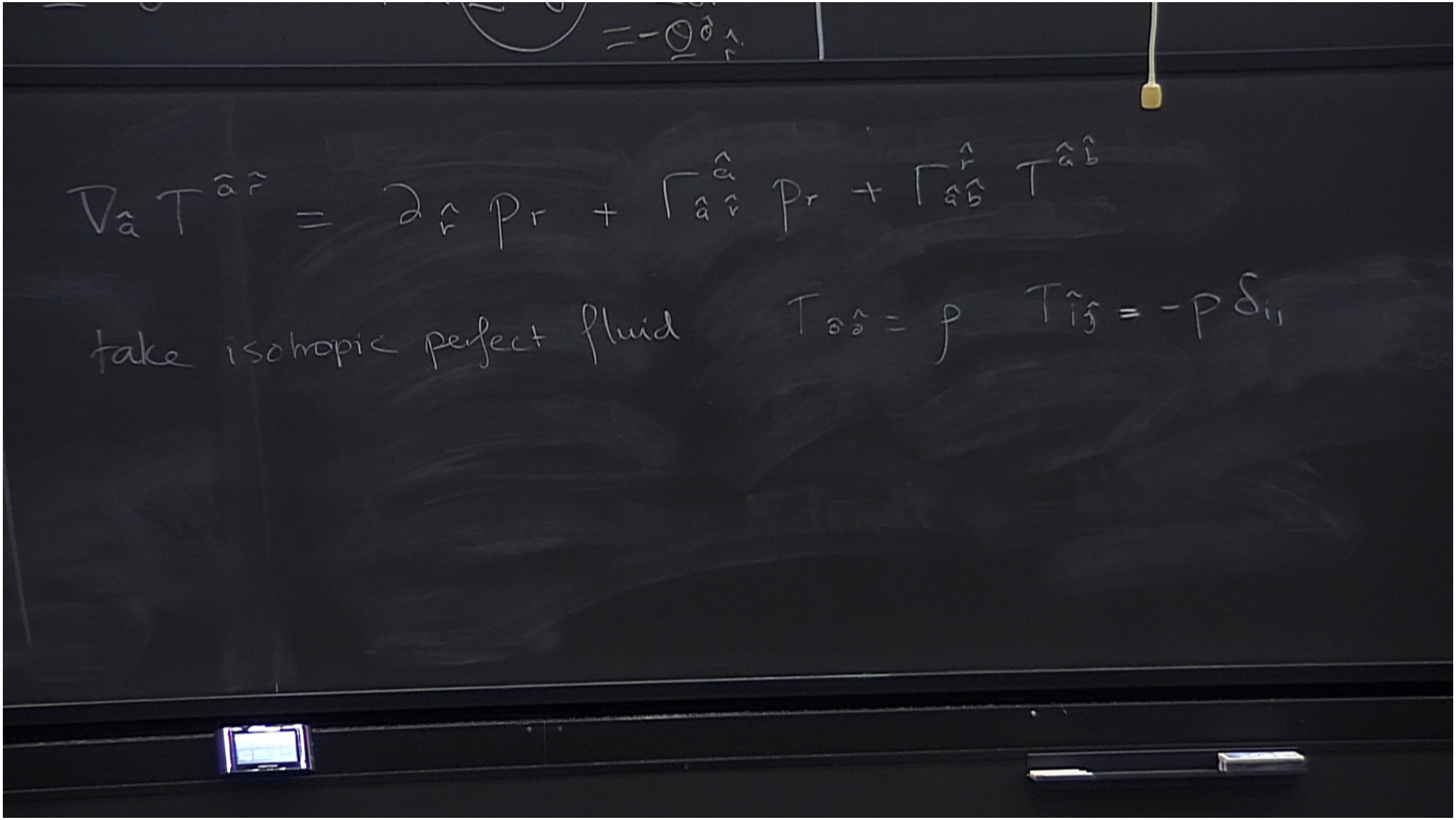
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$$\text{or } \frac{(A^2)'}{A^2} = \frac{2GM(r) + 8\pi G p_r r^3}{r(r - 2GM(r))}$$

$\nabla_{\hat{a}} T^{\hat{a}\hat{r}}$


$\nabla_{\hat{a}} T^{\hat{a}\hat{r}} = \partial_{\hat{r}} p_r + \Gamma^{\hat{a}}_{\hat{a}\hat{r}} p_r + \Gamma^{\hat{r}}_{\hat{a}\hat{b}} T^{\hat{a}\hat{b}}$


$\nabla_{\hat{a}} T^{\hat{a}\hat{r}} = \partial_{\hat{r}} p_r + \Gamma^{\hat{a}}_{\hat{a}\hat{r}} p_r + \Gamma^{\hat{r}}_{\hat{a}\hat{b}} T^{\hat{a}\hat{b}}$
take isotropic perfect fluid $T_{\hat{0}\hat{0}} = \rho$ $T_{\hat{i}\hat{j}} = -p\delta_{ij}$

$$= -\Omega^{\hat{0}}{}_{\hat{r}}$$

$$\nabla_{\hat{a}} T^{\hat{a}\hat{r}} = \partial_{\hat{r}} p_r + \Gamma^{\hat{a}}_{\hat{a}\hat{r}} p_r + \Gamma^{\hat{r}}_{\hat{a}\hat{b}} T^{\hat{a}\hat{b}}$$

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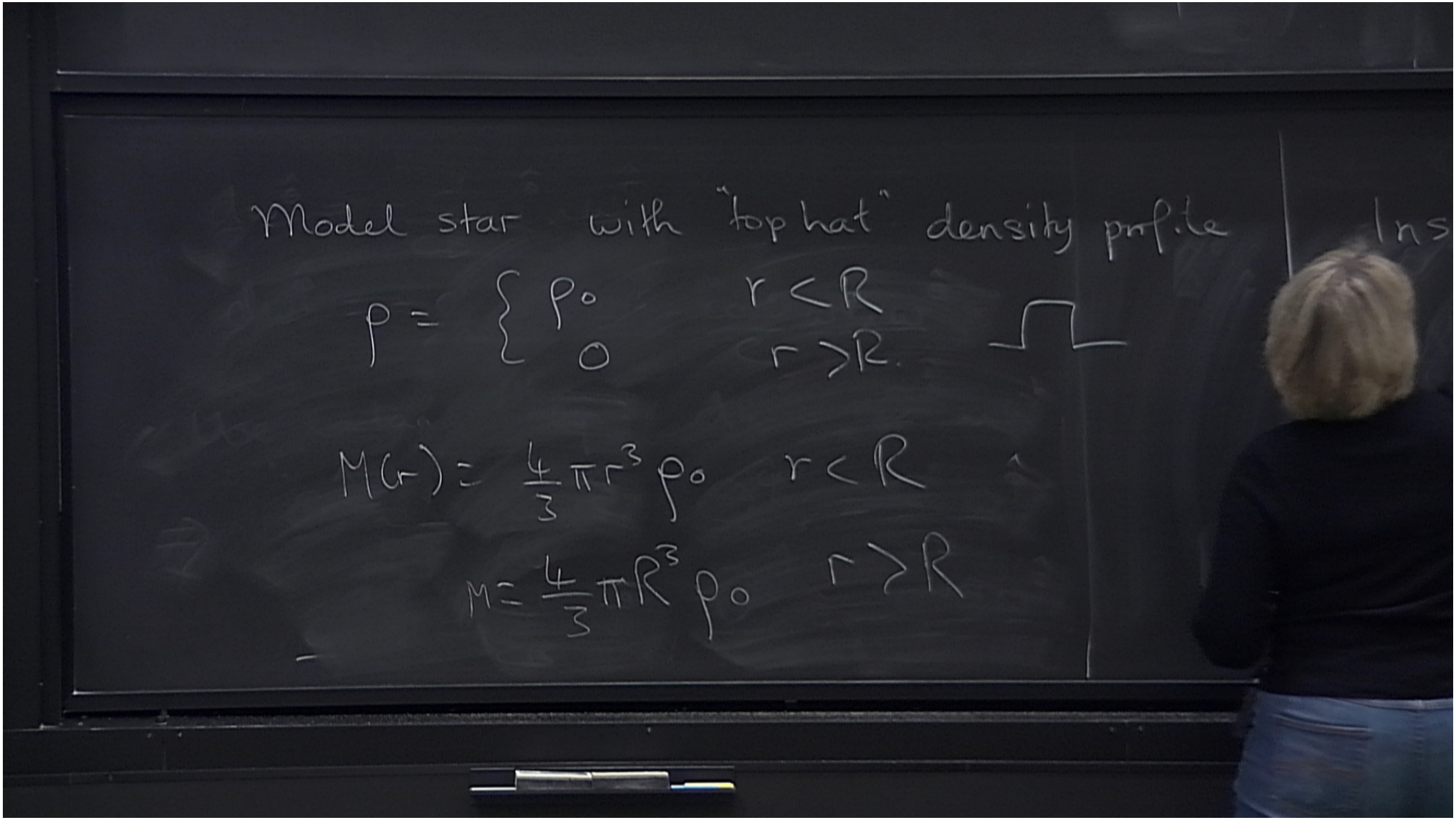
$$= \frac{p'}{B} + \left(\frac{A'}{AB} + \frac{2}{rB}\right) p + \rho \frac{A'}{AB} - \frac{2p}{rB}$$

$$\nabla_{\hat{a}} T^{\hat{a}\hat{r}} = \partial_{\hat{r}} p_r + \Gamma^{\hat{a}}_{\hat{a}\hat{r}} p_r + \Gamma^{\hat{r}}_{\hat{a}\hat{b}} T^{\hat{a}\hat{b}}$$

take isotropic perfect fluid $T_{\hat{0}\hat{0}} = \rho \quad T_{\hat{i}\hat{j}} = -p\,\delta_{ij}$

$$= \frac{\rho'}{B} + \left(\frac{A'}{AB} + \frac{2}{rB}\right)p + \rho\frac{A'}{AB} - \frac{2p}{rB}$$

$$= -\frac{1}{B}\left(p' + \frac{A'}{A}(p+\rho)\right) = 0.$$


Model star with "top hat" density profile
$\rho = \begin{cases} \rho_0 & r < R \\ 0 & r > R \end{cases}$
$M(r) = \frac{4}{3}\pi r^3 \rho_0 \quad r < R$
$M = \frac{4}{3}\pi R^3 \rho_0 \quad r > R$

Inside star

$$P' = -(\rho + p_0)\frac{[GM(r) + 4\pi G p r^3]}{r(r - 2GM(r))}$$

Inside star.

$$p' = -(p+\rho_0)\frac{[GM(r)+4\pi G p r^3]}{r(r-2GM(r))}$$

$$= -(p+\rho_0)\cdot\frac{4\pi G r^3}{3}\,\frac{(3p+\rho_0)}{r\left(r-\frac{8\pi G}{3}\rho_0 r^3\right)}$$

Inside star.

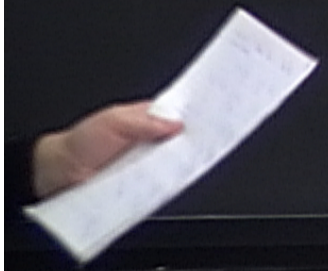
$$p' = -(p+\rho_0)\frac{[GM(r) + 4\pi G p r^3]}{r(r-2GM(r))}$$

$$= -(p+\rho_0)\cdot\frac{4\pi G r^3}{3}\,\frac{(3p+\rho_0)}{r\left(r-\frac{8\pi G}{3}\rho_0 r^3\right)}$$

$$= -(p+\rho_0)\frac{4\pi G r(3p+\rho_0)}{(3-8\pi G r^2\rho_0)}$$

Solved by:

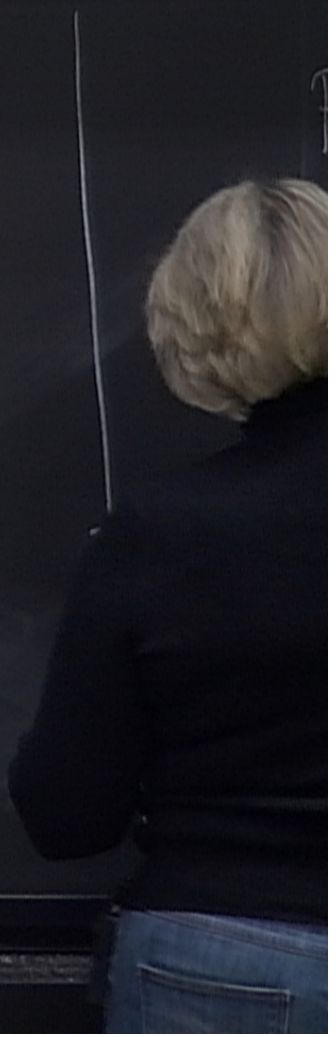
$$P = \rho_0 \left[\frac{\sqrt{1-\frac{2GM}{R}} - \sqrt{1-\frac{2GMr^2}{R^3}}}{\sqrt{1-\frac{2GMr^2}{R^3}} - 3\sqrt{1-\frac{2GM}{R}}} \right]$$

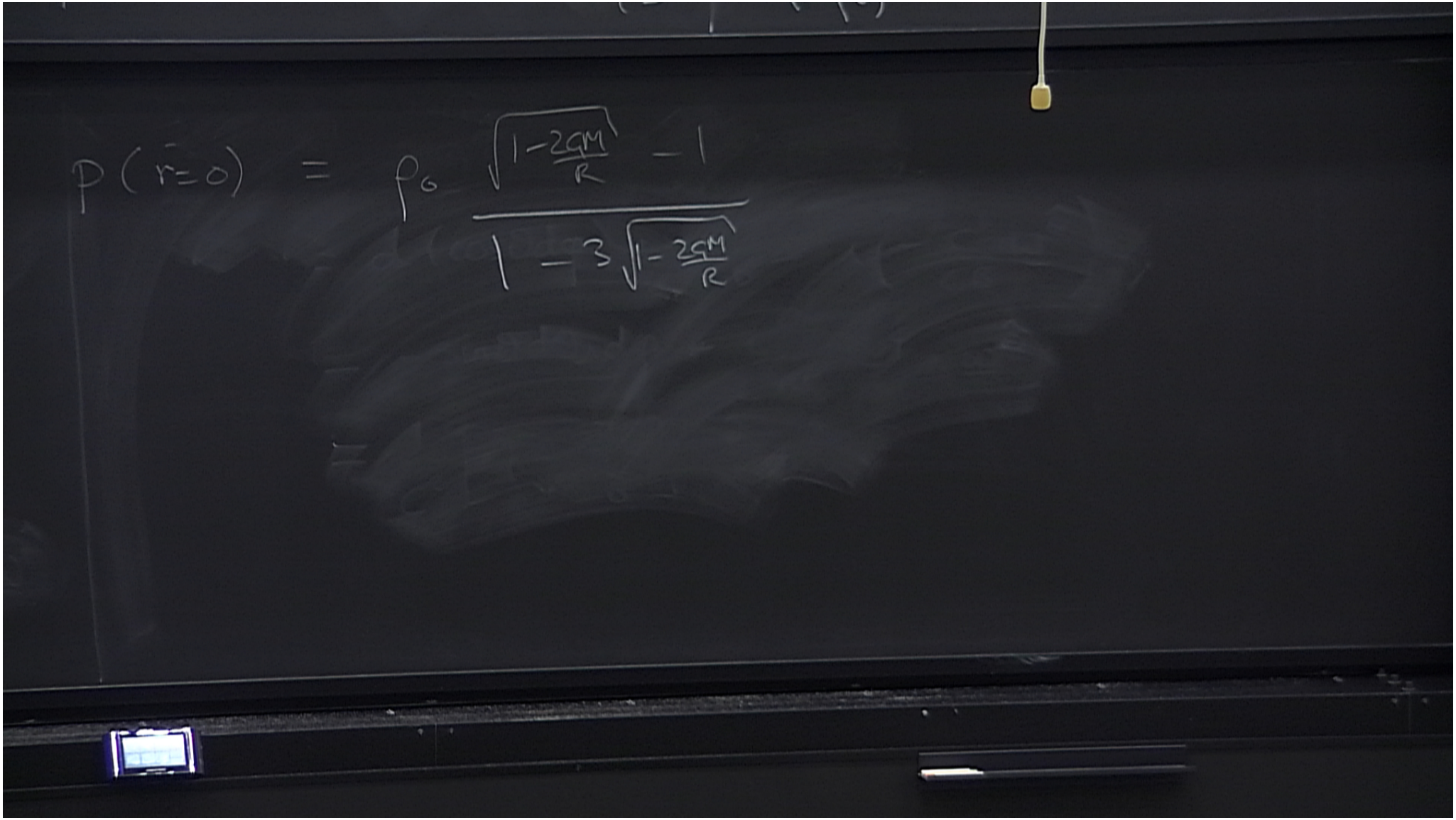


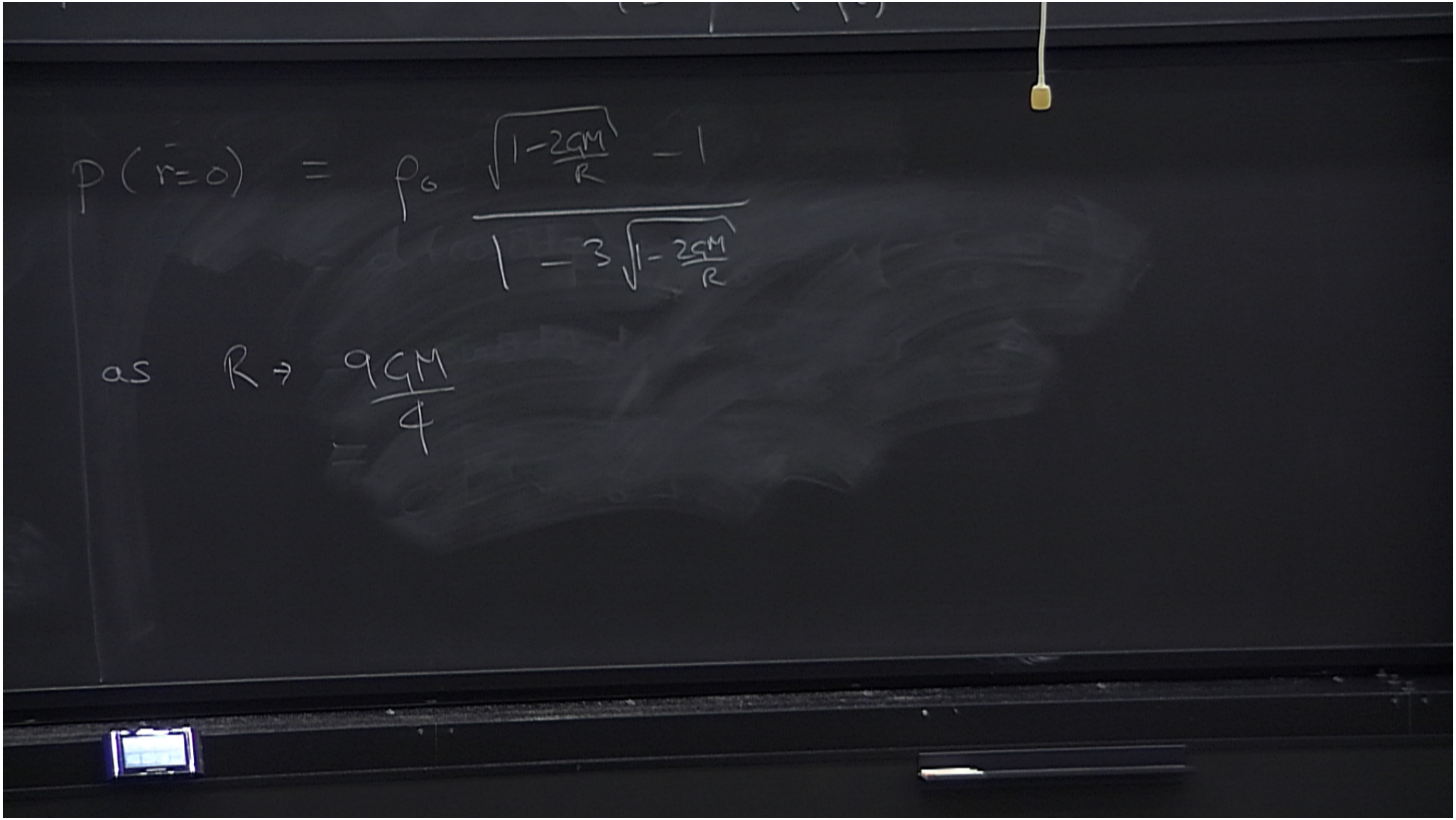
Solved by:

$$p = \rho_0 \left[\frac{\sqrt{1-\frac{2GM}{R}} - \sqrt{1-\frac{2GMr^2}{R^3}}}{\sqrt{1-\frac{2GMr^2}{R^3}} - 3\sqrt{1-\frac{2GM}{R}}} \right]$$

$p \to 0$ as $r \to R$.




$p(r=0) = \rho_0 \frac{\sqrt{1-\frac{2GM}{R}} - 1}{1 - 3\sqrt{1-\frac{2GM}{R}}}$


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as $R \to \frac{9GM}{4}$

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$$p(r=0) = \rho_0 \frac{\sqrt{1-\frac{2GM}{R}} - 1}{1 - 3\sqrt{1-\frac{2GM}{R}}}$$

as $R \to \frac{9GM}{4}$ $\quad p(0) \to \infty$

Indicates a limit as to size of star. $R^2 \gtrsim O(\frac{1}{\rho_0})$