Title: PSI 2016/2017 Gravitational Physics (Review) - Lecture 2

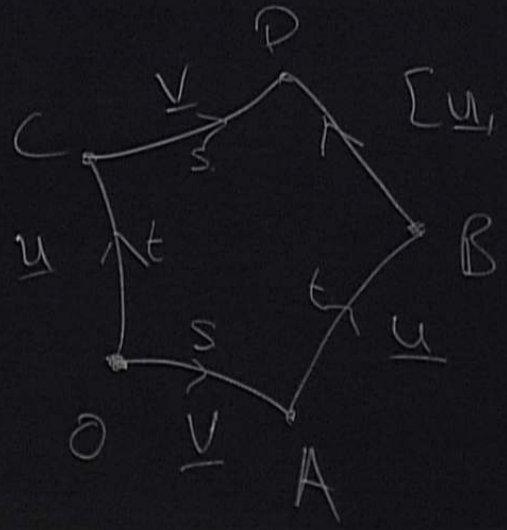
Date: Jan 04, 2017 10:15 AM

URL: http://pirsa.org/17010020

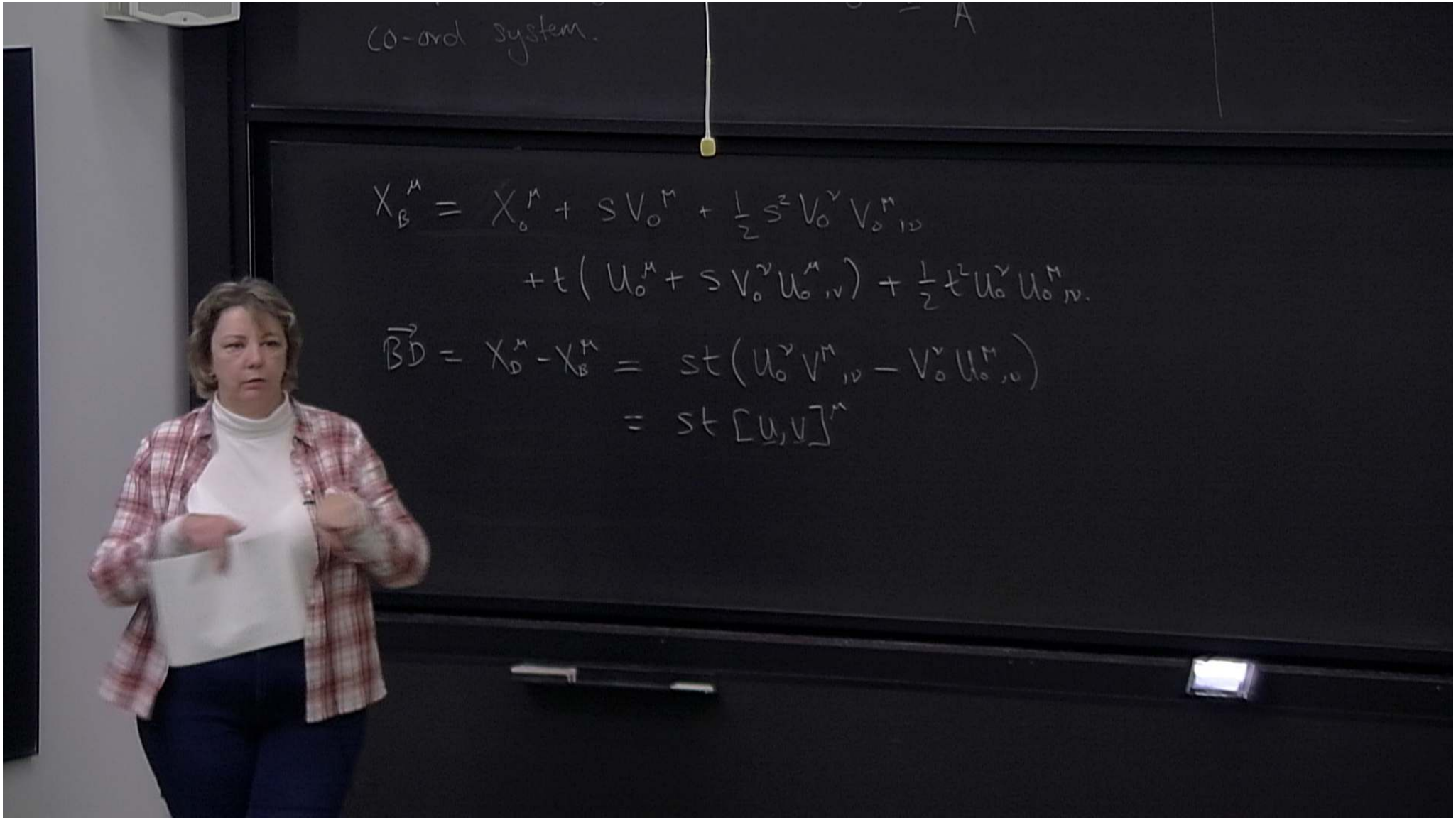
Abstract:

LECTURE 2: Connections, Curvature & Cartan

Bracket tells us how far off two vectors are from making a co-ord system.



$[u, v]$ st.


co-ord system.
$X_B^\mu = X_0^\mu + sV_0^\mu + \frac{1}{2}s^2 V_0^\nu V_{0,\nu}^\mu$
$+ t\left(U_0^\mu + sV_0^\nu U_{0,\nu}^\mu\right) + \frac{1}{2}t^2 U_0^\nu U_{0,\nu}^\mu.$
$\vec{BD} = X_D^\mu - X_B^\mu = st\left(U_0^\nu V_{,\nu}^\mu - V_0^\nu U_{0,\nu}^\mu\right)$
$= st\,[U,V]^\mu$

$$\begin{cases} X_A^\mu = X_0^\mu + s V_0^\mu + \frac{1}{2} s^2 V_0^\nu V_{0,\nu}^\mu \\ U_A^\mu = U_0^\mu + s V_0^\nu U_{0,\nu}^\mu \end{cases}$$

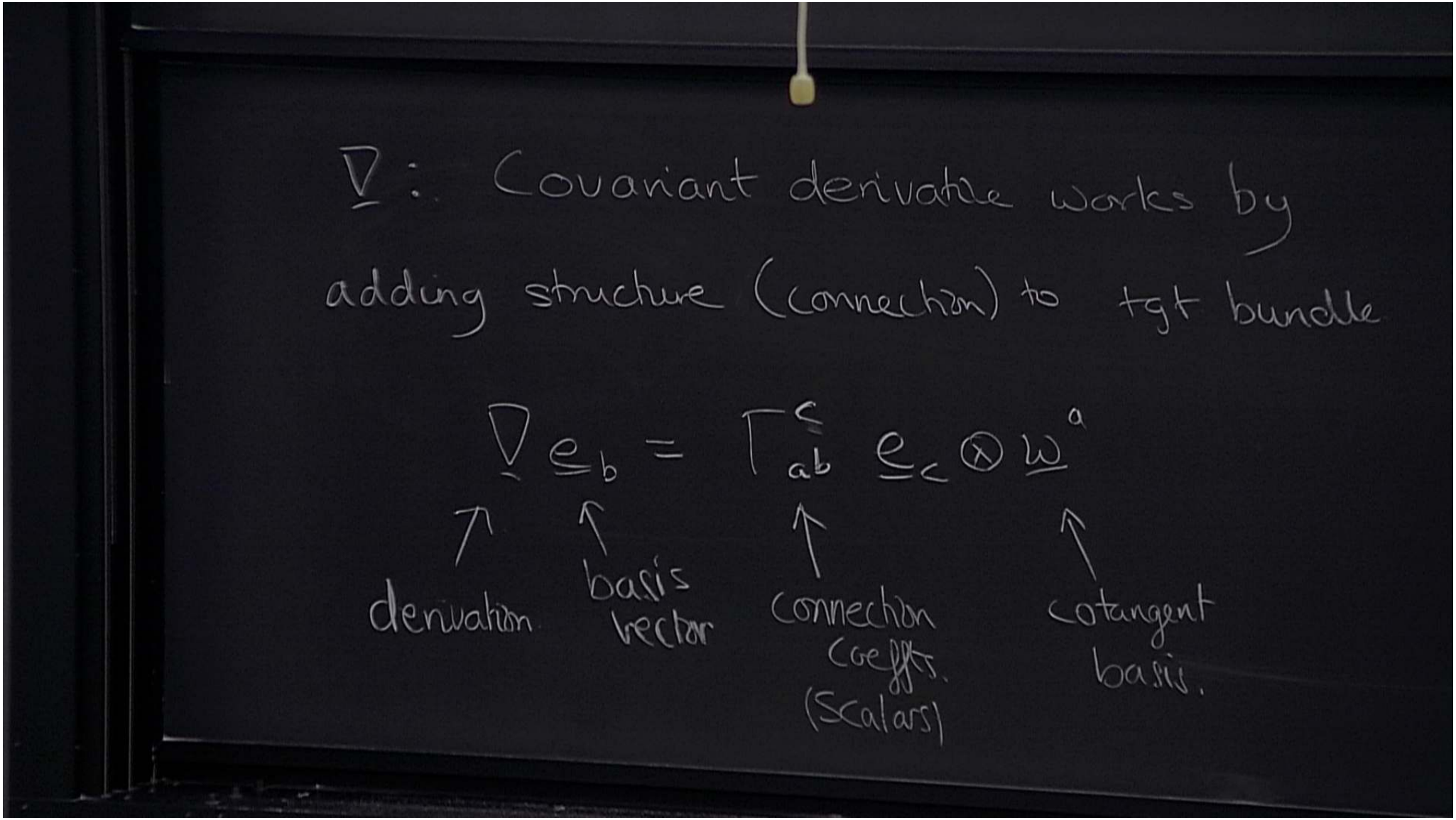
Then

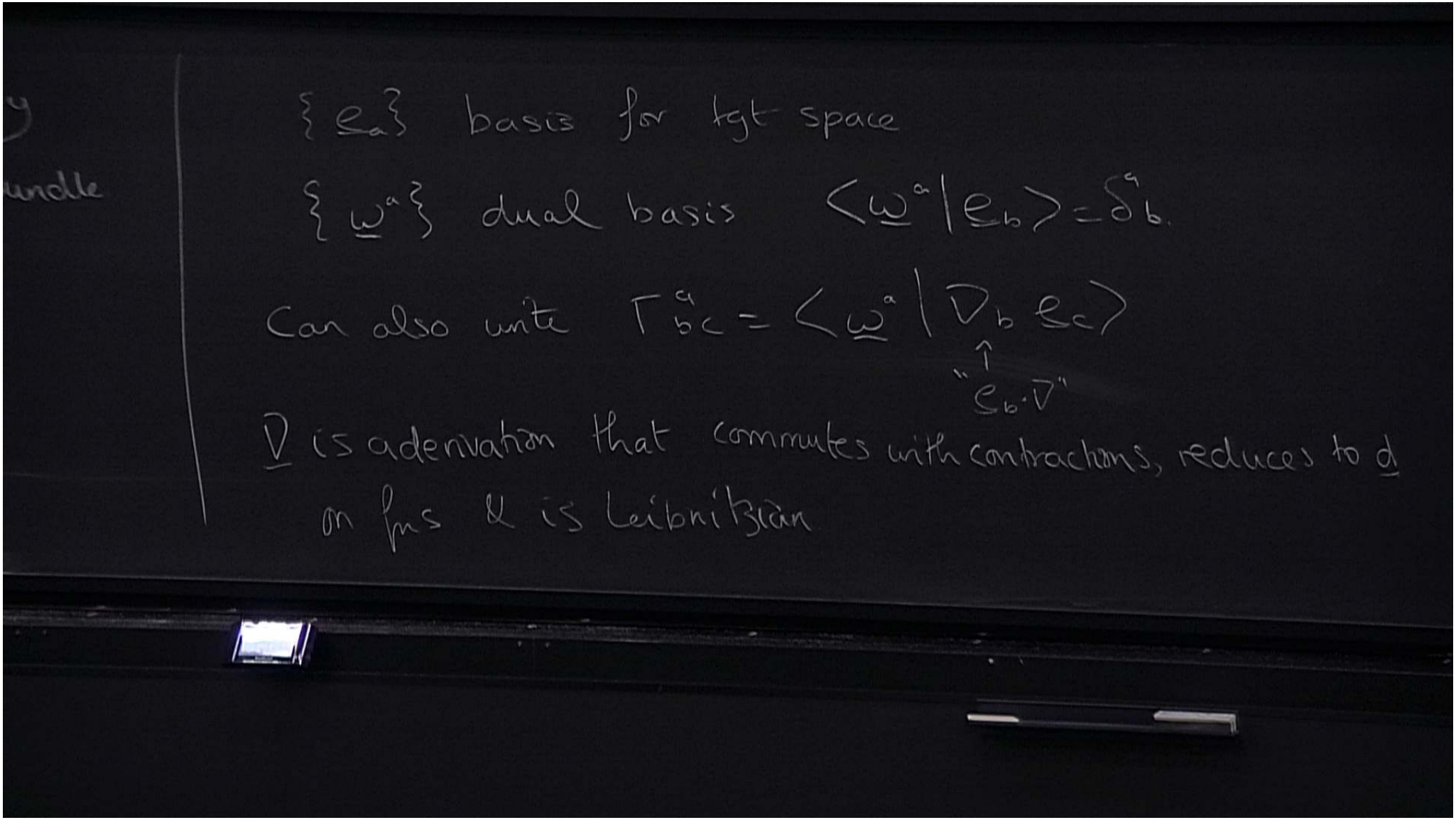
$$\begin{aligned} X_D^\mu &= X_C^\mu + s V_C^\mu + \frac{1}{2} s^2 V_C^\nu V_{C,\nu}^\mu \\ &= X_0^\mu + t U_0^\mu + \frac{1}{2} t^2 U_0^\nu U_{0,\nu}^\mu \\ &\quad + s\left(V_0^\mu + t U_0^\nu V_{0,\nu}^\mu\right) + \frac{1}{2} s^2 V_0^\nu V_{0,\nu}^\mu . \end{aligned}$$

Recall

$$\langle \underline{d}\underline{\omega} \mid \underline{u}, \underline{v} \rangle = \underline{u}(\langle \underline{\omega} \mid \underline{v} \rangle) - \underline{v}(\langle \underline{\omega} \mid \underline{u} \rangle) - \langle \underline{\omega} \mid [\underline{u}, \underline{v}] \rangle$$

reln between Lie & $\underline{d}$


$\underline{\nabla}$: Covariant derivative works by adding structure (connection) to tgt bundle
$\underline{\nabla} \underline{e}_b = \Gamma^c_{ab} \underline{e}_c \otimes \underline{\omega}^a$
derivation
basis vector
connection coeffs. (Scalars)
cotangent basis.

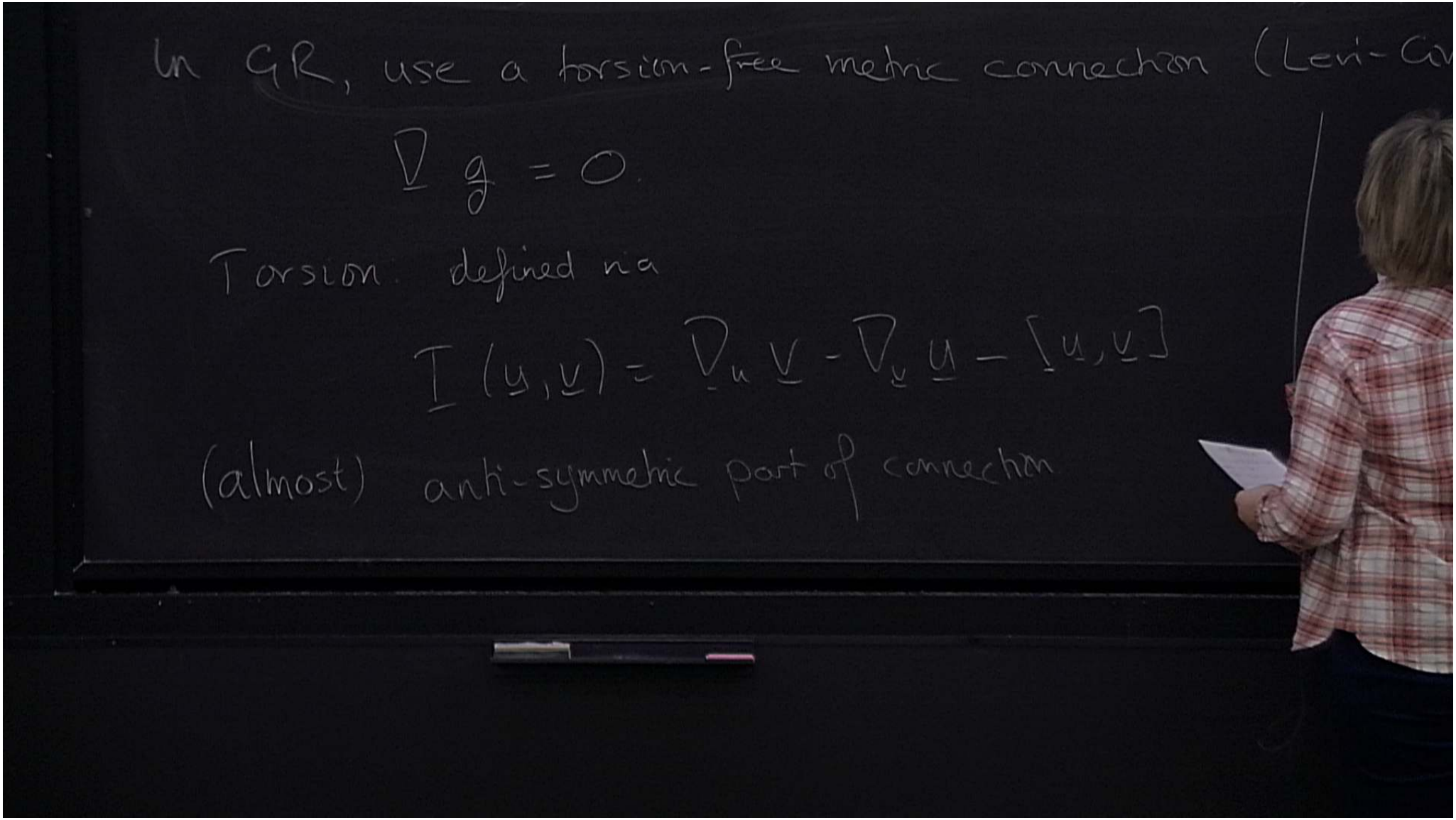

$\{e_a\}$ basis for tgt space
$\{\omega^a\}$ dual basis $\langle \omega^a | e_b \rangle = \delta^a_b$.
Can also write $\Gamma^a_{bc} = \langle \omega^a | \nabla_b e_c \rangle$
"$e_b \cdot \nabla$"
∇ is a derivation that commutes with contractions, reduces to d on fns & is Leibnitzian

In GR, use a torsion-free metric connection (Levi-Civita)

$$\nabla g = 0$$

Torsion defined as

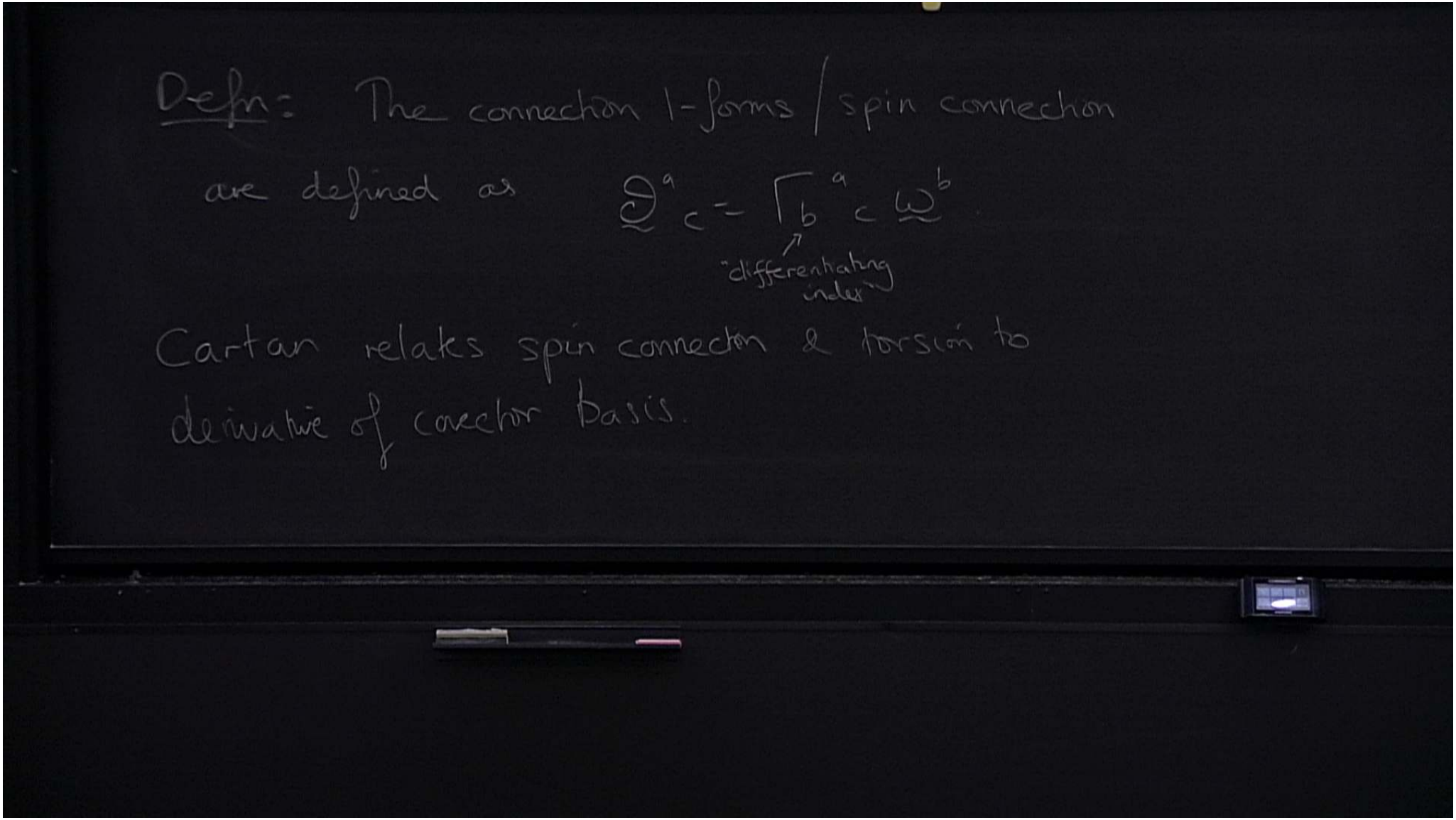
$$T(u, v) = \nabla_u v - \nabla_v u - [u, v]$$

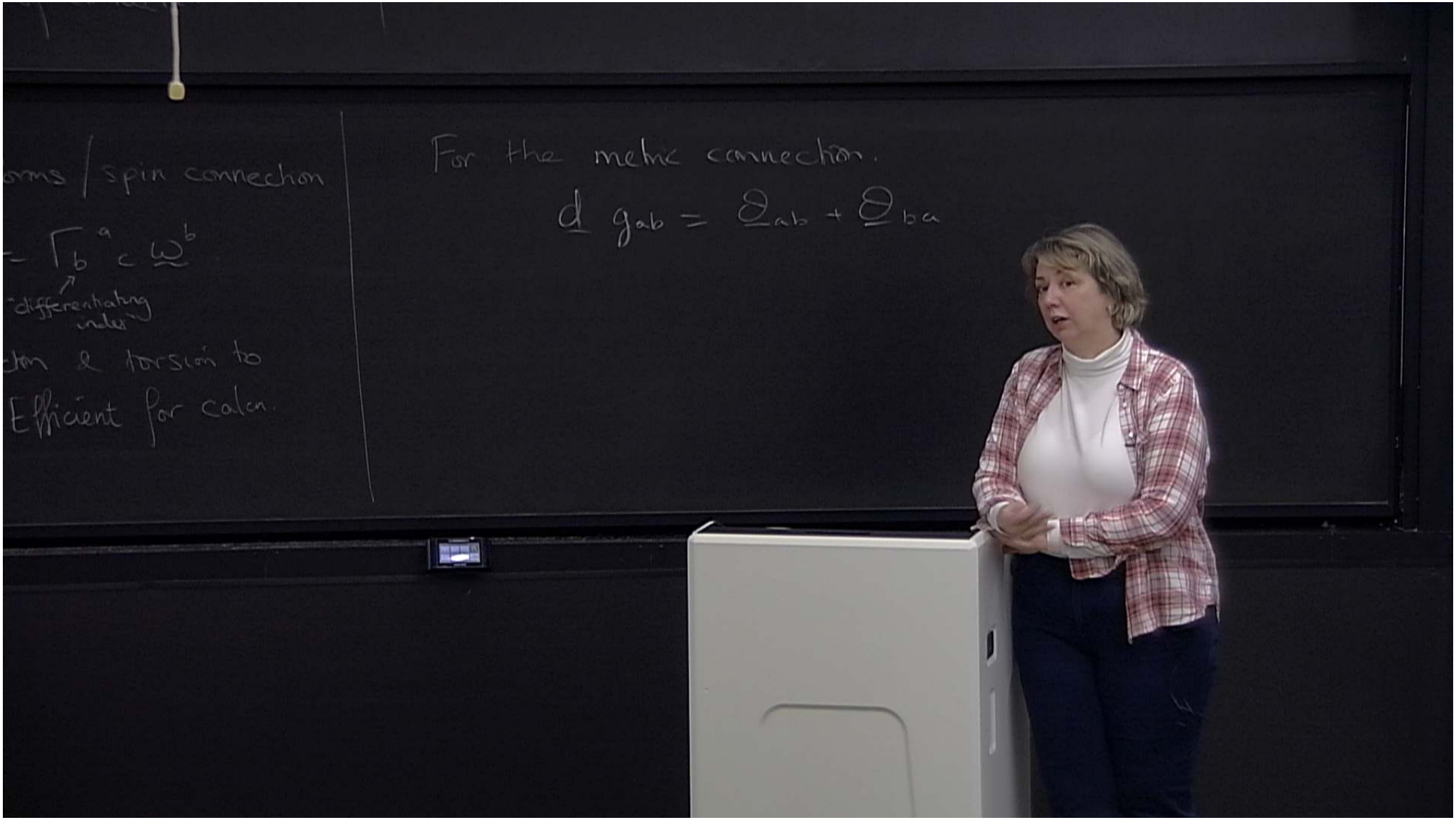

In GR, use a torsion-free metric connection (Levi-Ca
$\nabla g = 0$
Torsion defined via
$T(u,v) = \nabla_u v - \nabla_v u - [u,v]$
(almost) anti-symmetric part of connection

Levi-Civita)

$$T^{a}_{bc} = \Gamma^{a}_{bc} - \Gamma^{a}_{cb} - C^{a}_{bc}$$

where $C^{a}_{bc} = \langle \omega^{a} | [e_b, e_c] \rangle$ are structure constants of $\{e_a\}$


Defn: The connection 1-forms / spin connection
are defined as $\underset{\sim}{\omega}^a{}_c = \Gamma_b{}^a{}_c \, \underset{\sim}{\omega}^b$
"differentiating index"
Cartan relates spin connection & torsion to
derivative of covector basis.


For the metric connection.
$\underline{d}\, g_{ab} = \underline{\theta}_{ab} + \underline{\theta}_{ba}$
differentiating index
Efficient for calcn.

For the metric connection.

$$\underline{d}\, g_{ab} = \underline{\theta}_{ab} + \underline{\theta}_{ba}$$

Proof:

$$\underline{d}\, g_{ab} = \underline{d}\left(\langle \underline{g} | \underline{e}_a, \underline{e}_b \rangle\right) = \underline{\nabla}\left(\langle \underline{g} | \underline{e}_a, \underline{e}_b \rangle\right)$$

$$= \langle \underline{g} | \underline{\nabla} \underline{e}_a, \underline{e}_b \rangle + \langle \underline{g} | \underline{e}_a, \underline{\nabla} \underline{e}_b \rangle$$

$$= \Gamma^c_{da} \langle \underline{g} | \underline{e}_c, \underline{e}_b \rangle \underline{\omega}^d + \Gamma^c_{db} \langle \underline{g} | \underline{e}_c, \underline{e}_a \rangle \underline{\omega}^d$$

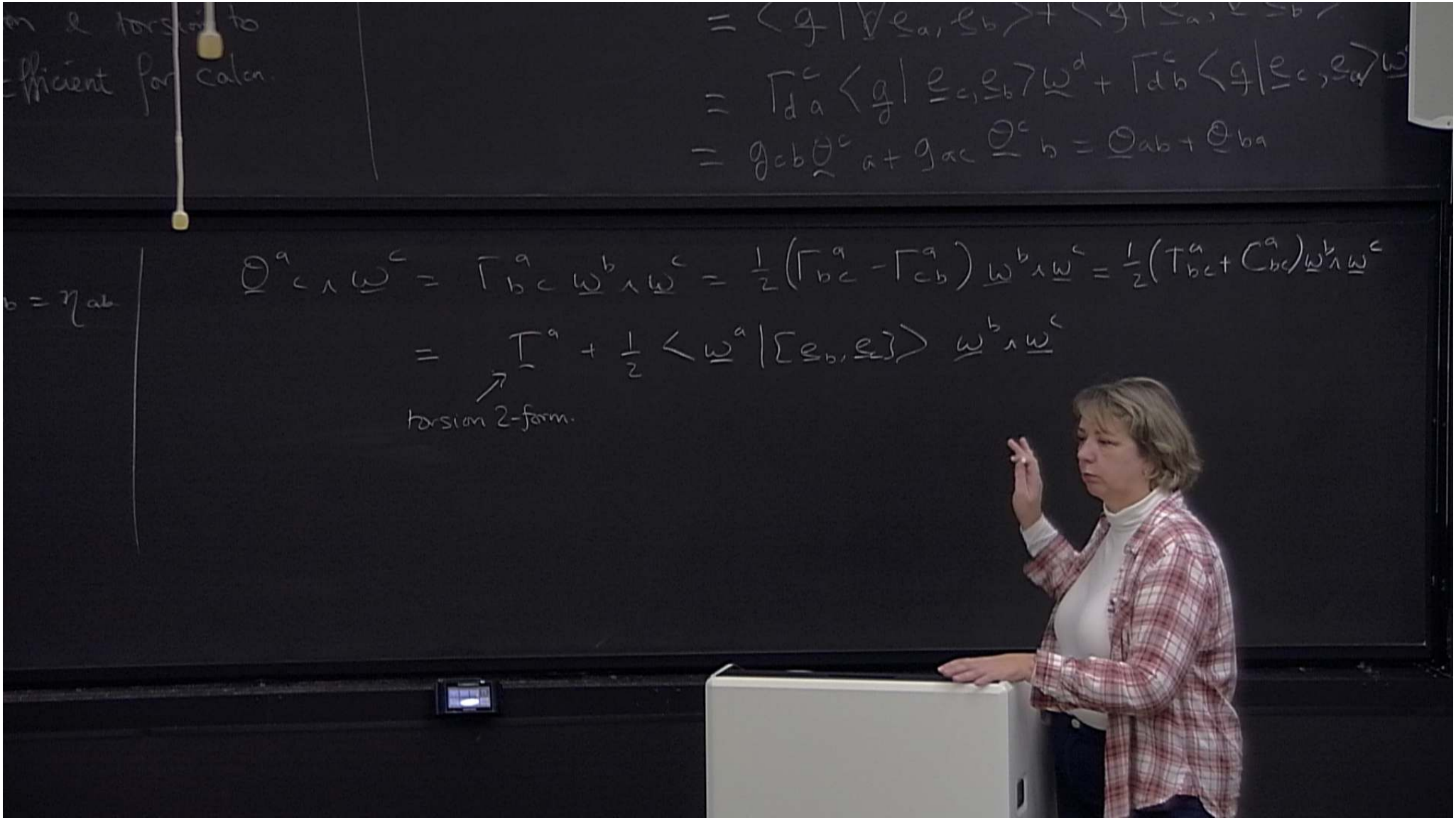
$$= g_{cb} \underline{\theta}^c{}_a + g_{ac} \underline{\theta}^c{}_b = \underline{\theta}_{ab} + \underline{\theta}_{ba}$$

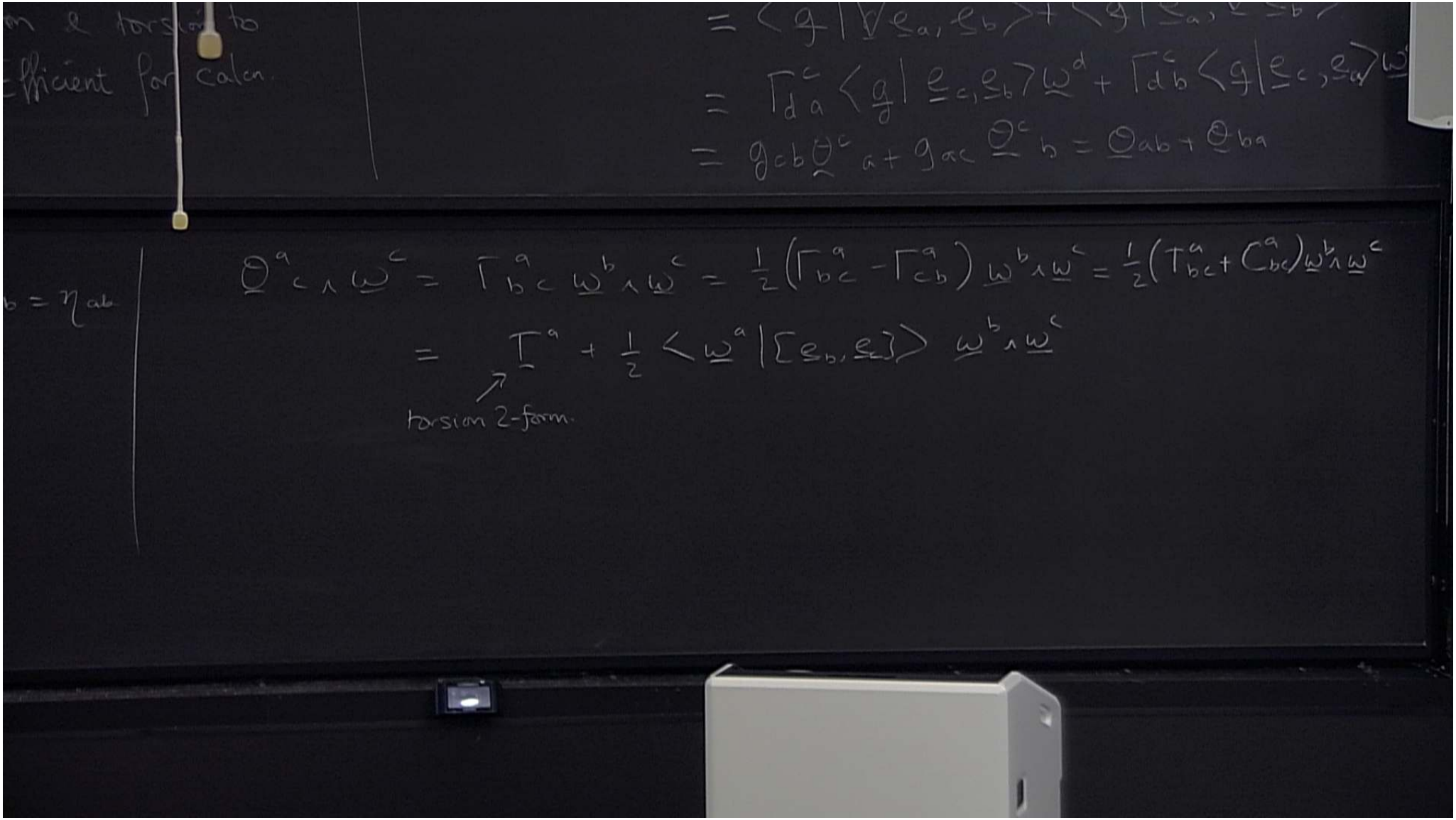
In an o/n basis, $g_{ab} = \eta_{ab}$

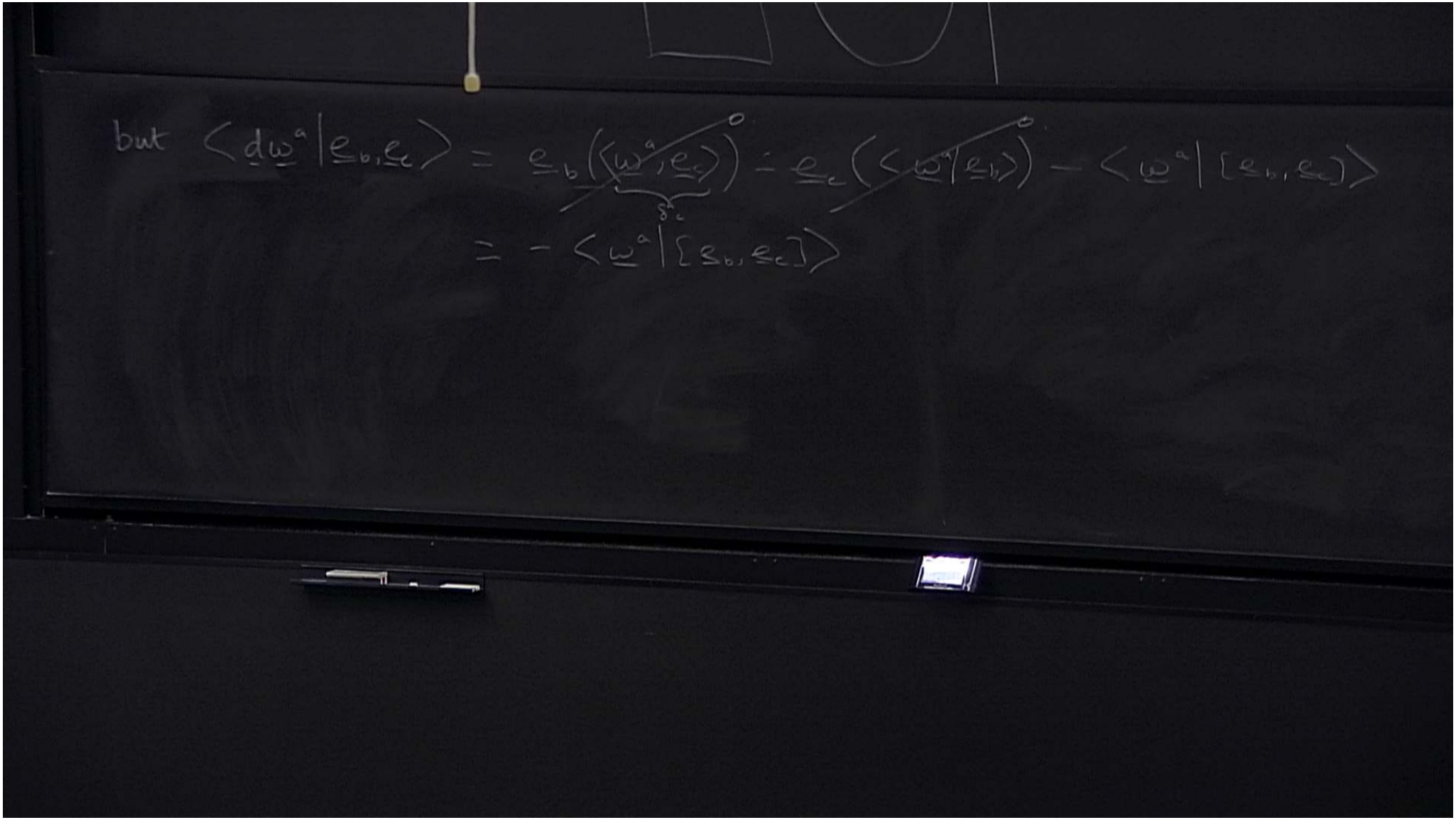
hence $\underline{d}\, g_{ab} = 0$

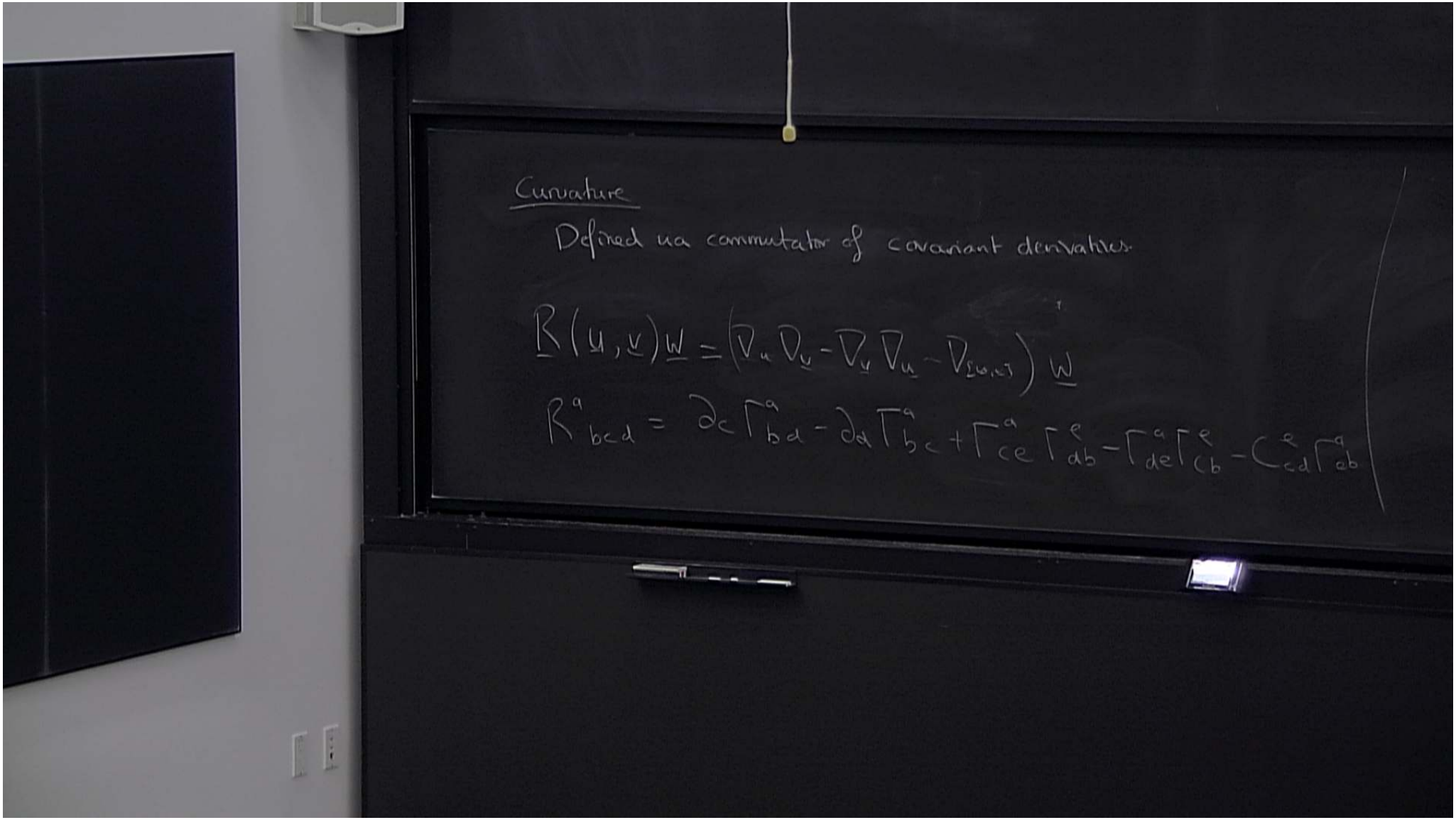
& $\underset{\sim}{\Theta}_{ab}$ anti-symmetric

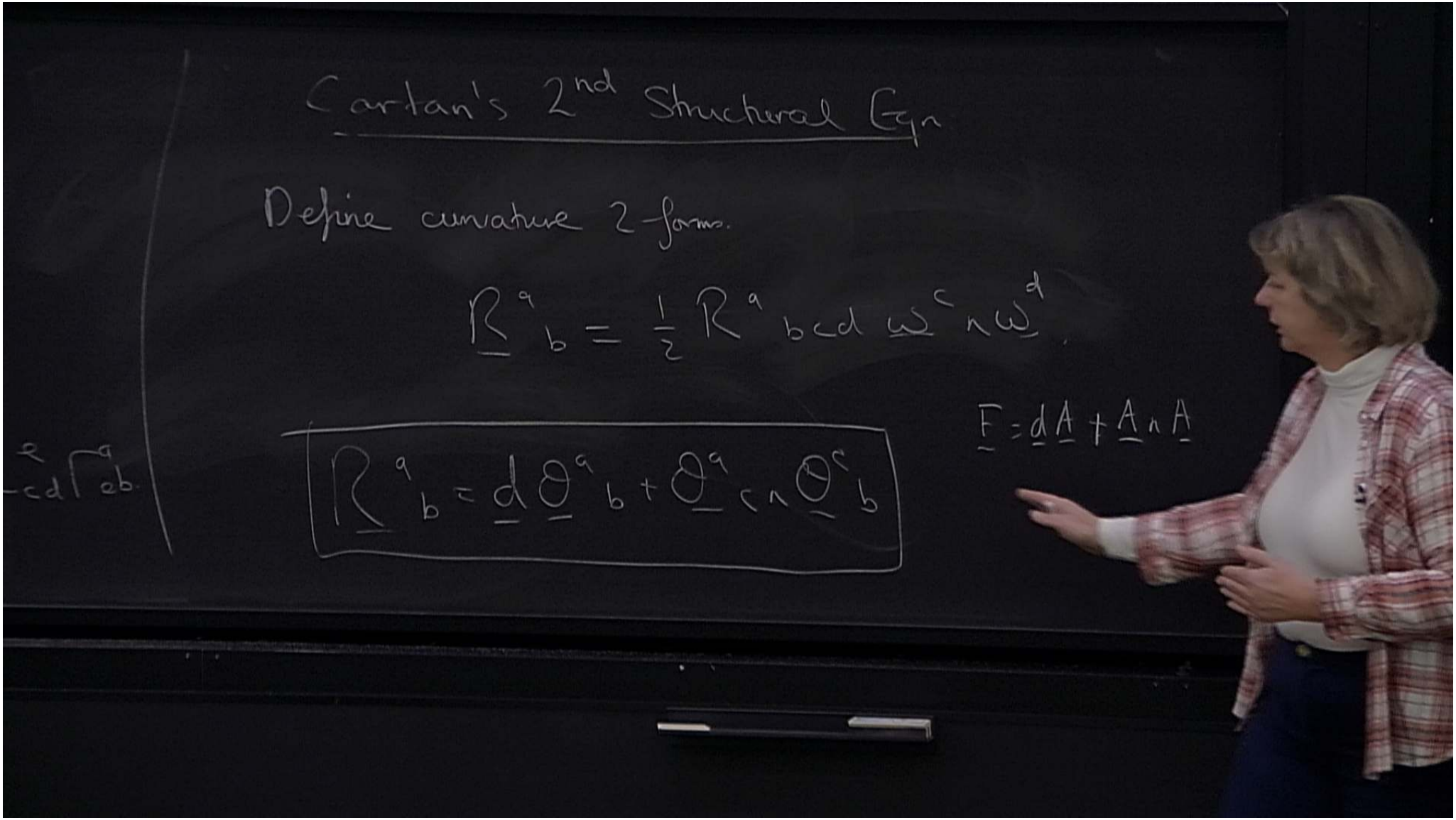
Consider $\underset{\sim}{\Theta}^{a}{}_{c} \wedge \underset{\sim}{\omega}^{c}$


$= \langle g | \nabla_{e_a} e_b \rangle + \langle g | e_a, e_b \rangle$
$= \Gamma^c_{da} \langle g | e_c, e_b \rangle \omega^d + \Gamma^c_{db} \langle g | e_c, e_a \rangle \omega$
$= g_{cb} \Theta^c{}_a + g_{ac} \Theta^c{}_b = \Theta_{ab} + \Theta_{ba}$
$\Theta^a{}_c \wedge \omega^c = \Gamma^a_{bc} \omega^b \wedge \omega^c = \frac{1}{2}(\Gamma^a_{bc} - \Gamma^a_{cb}) \omega^b \wedge \omega^c = \frac{1}{2}(T^a_{bc} + C^a_{bc}) \omega^b \wedge \omega^c$
$= \Gamma^a + \frac{1}{2} \langle \omega^a | [e_b, e_c] \rangle \omega^b \wedge \omega^c$
torsion 2-form.


$= \langle g | \nabla_{e_a} e_b \rangle + \langle g | e_a, e_b \rangle$
$= \Gamma^c_{da} \langle g | e_c, e_b \rangle \omega^d + \Gamma^c_{db} \langle g | e_c, e_a \rangle \omega$
$= g_{cb} \Theta^c{}_a + g_{ac} \Theta^c{}_b = \Theta_{ab} + \Theta_{ba}$
$\Theta^a{}_c \wedge \omega^c = \Gamma^a_{bc} \omega^b \wedge \omega^c = \frac{1}{2}(\Gamma^a_{bc} - \Gamma^a_{cb}) \omega^b \wedge \omega^c = \frac{1}{2}(T^a_{bc} + C^a_{bc}) \omega^b \wedge \omega^c$
$= T^a + \frac{1}{2} \langle \omega^a | [e_b, e_c] \rangle \omega^b \wedge \omega^c$
torsion 2-form.


but $\langle d\omega^a | e_b, e_c \rangle = e_b(\langle \omega^a, e_c \rangle)^0 - e_c(\langle \omega^a | e_b \rangle)^0 - \langle \omega^a | [e_b, e_c] \rangle$
$= - \langle \omega^a | [e_b, e_c] \rangle$


Curvature
Defined via commutator of covariant derivatives.
$\underline{R}(\underline{u},\underline{v})\underline{w} = \left(\nabla_{\underline{u}}\nabla_{\underline{v}} - \nabla_{\underline{v}}\nabla_{\underline{u}} - \nabla_{[\underline{u},\underline{v}]}\right)\underline{w}$
$R^{a}{}_{bcd} = \partial_c \Gamma^{a}_{bd} - \partial_d \Gamma^{a}_{bc} + \Gamma^{a}_{ce}\Gamma^{e}_{db} - \Gamma^{a}_{de}\Gamma^{e}_{cb} - C^{e}_{cd}\Gamma^{a}_{eb}$


Cartan's 2nd Structural Eqn
Define curvature 2-forms.
$\underline{R}^a{}_b = \frac{1}{2} R^a{}_{bcd}\, \underline{\omega}^c \wedge \underline{\omega}^d$
$\underline{R}^a{}_b = \underline{d}\,\underline{\theta}^a{}_b + \underline{\theta}^a{}_c \wedge \underline{\theta}^c{}_b$
$\underline{F} = \underline{d}\,\underline{A} + \underline{A} \wedge \underline{A}$