Title: PSI 2016/2017 Gravitational Physics (Review) - Lecture 1

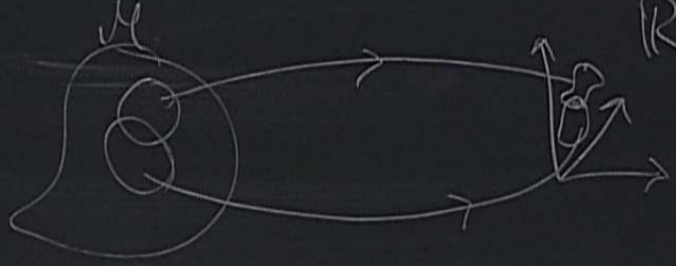
Date: Jan 03, 2017 10:15 AM

URL: http://pirsa.org/17010019

Abstract:

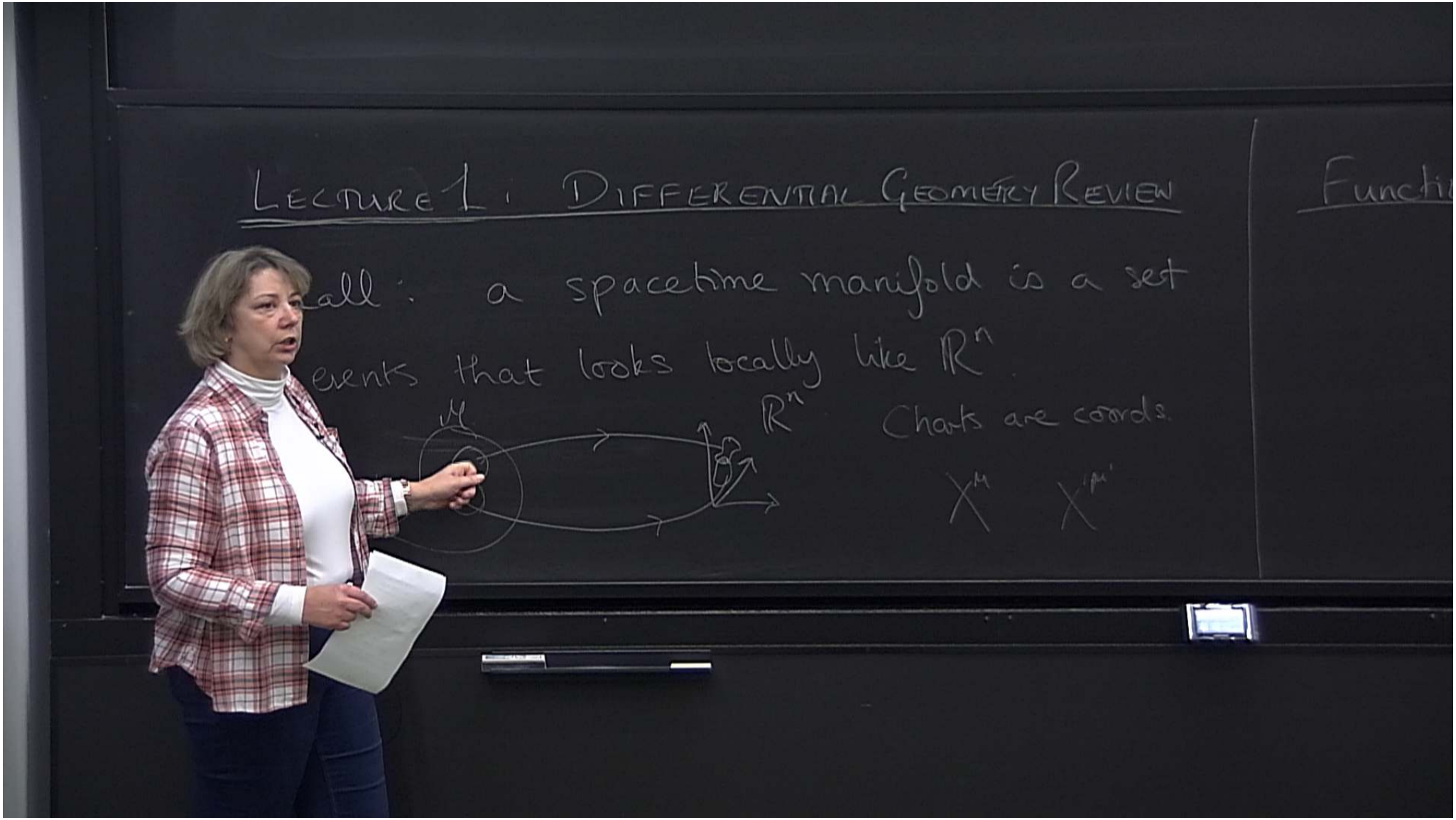
Lecture 1: Differential Geometry Review

Recall: a spacetime manifold is a set of events that looks locally like $\mathbb{R}^n$



Charts are coords.

x^μ $x'^{\mu'}$


LECTURE 1: DIFFERENTIAL GEOMETRY REVIEW
all: a spacetime manifold is a set
events that looks locally like $\mathbb{R}^n$
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Charts are coords.
x^μ
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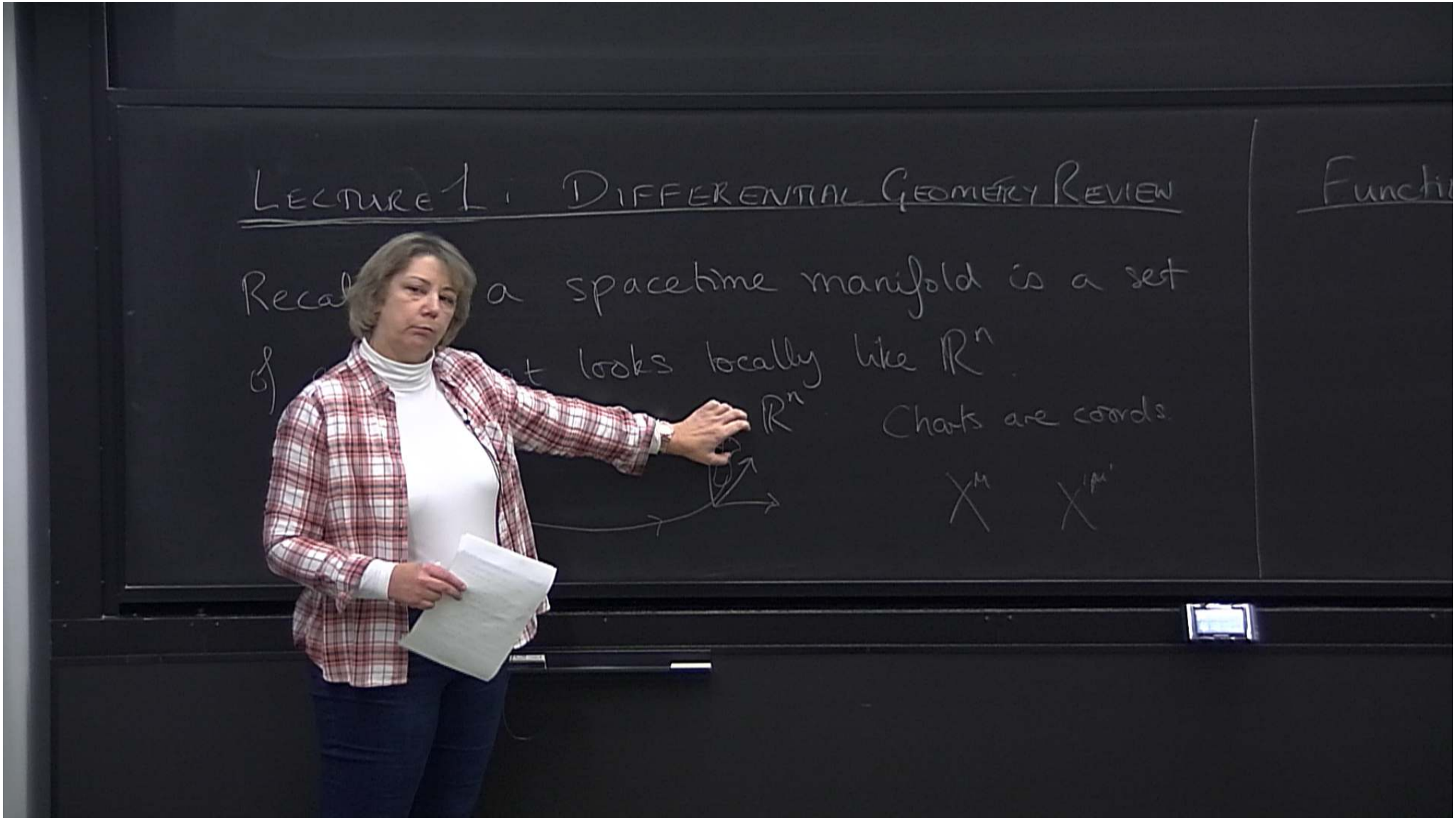
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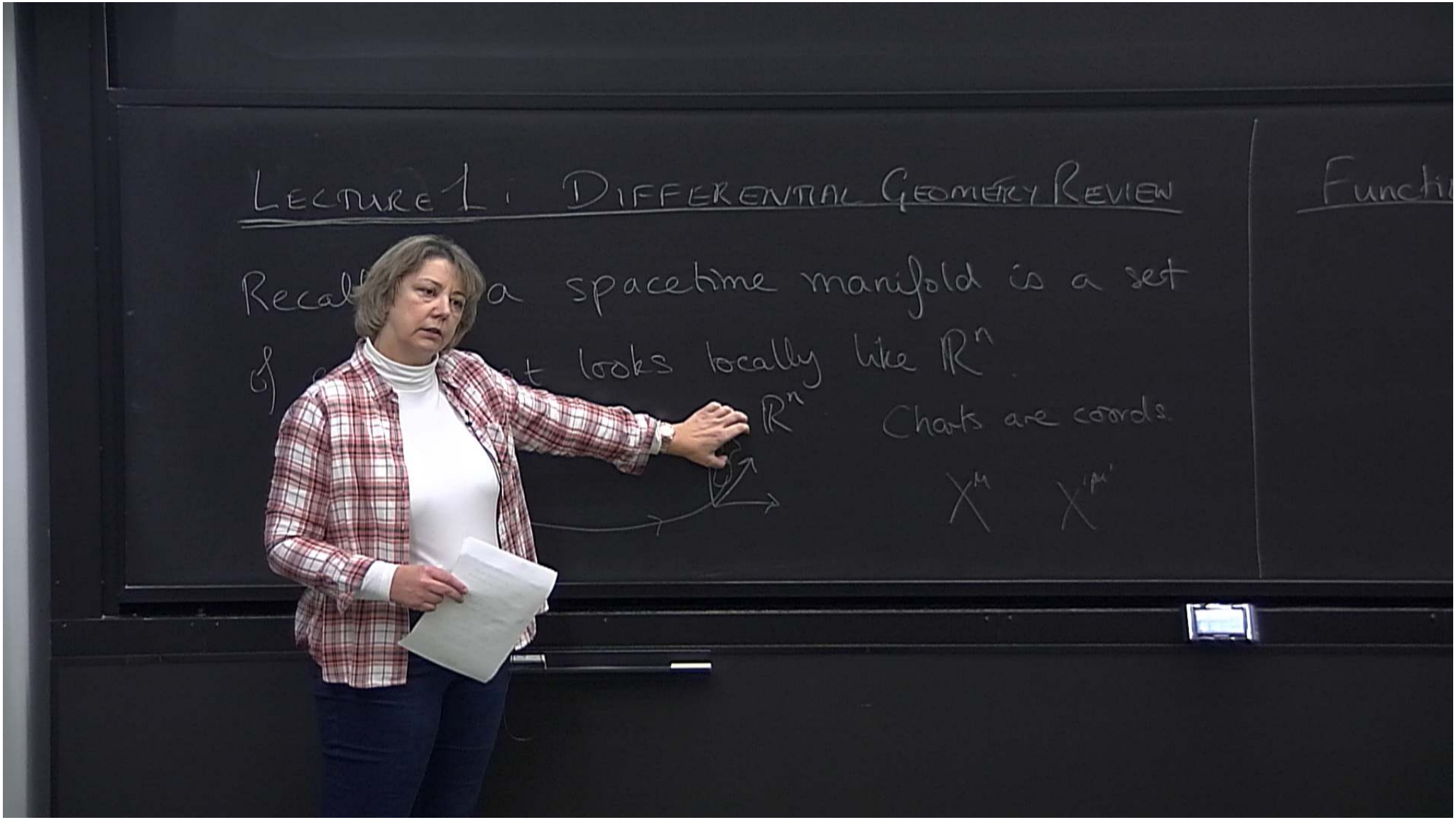
LECTURE 1: DIFFERENTIAL GEOMETRY REVIEW
Recall a spacetime manifold is a set
looks locally like $\mathbb{R}^n$
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x^μ
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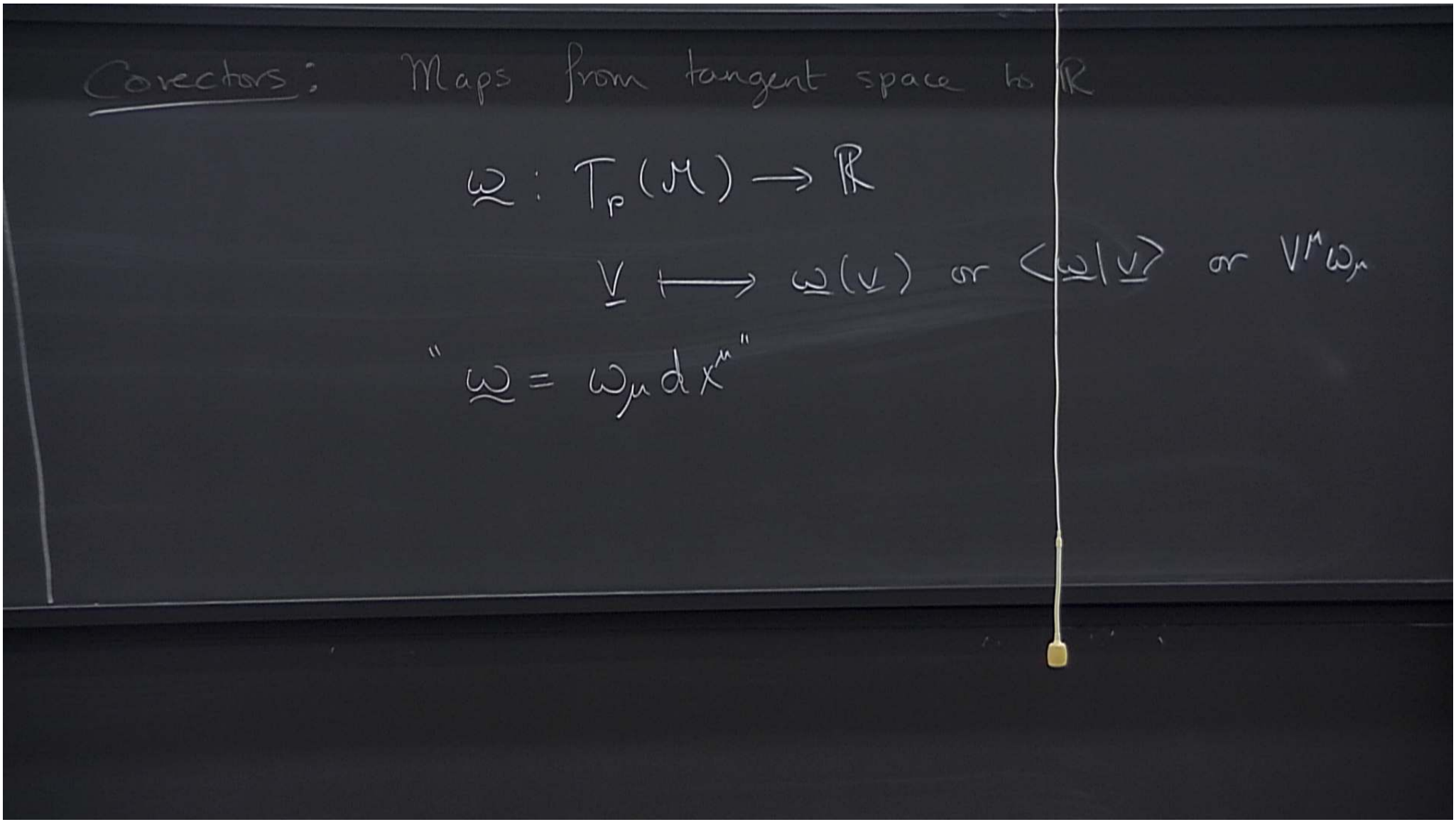
$$\underline{I} : C^\infty(M) \longrightarrow C^\infty(M)$$

$$f \longmapsto \frac{df}{dt} \quad \text{at } P \text{ on } \gamma.$$

$\gamma(t)$ our curve, $\frac{d}{dt}$ is the tgt vec

"$\underline{T} = T^\mu \frac{\partial}{\partial x^\mu}$" Form a tangent Space at P, $T_p(M)$


Covectors: Maps from tangent space to ℝ
$\underset{\sim}{\omega} : T_p(\mathcal{M}) \to \mathbb{R}$
$\underset{\sim}{V} \longmapsto \underset{\sim}{\omega}(\underset{\sim}{v})$ or $\langle \underset{\sim}{\omega} | \underset{\sim}{v} \rangle$ or $V^\mu \omega_\mu$
"$\underset{\sim}{\omega} = \omega_\mu dx^\mu$"

Functions: $C^\infty(M)$ ∞^{ly} differentiable fns.

$f: M \to \mathbb{R}$

Curve: $\gamma: \mathbb{R} \to M$

Vectors: Defined as "tangents" to curve.

ctors: Maps from tangent space to $\mathbb{R}$

Abstract Index Notation

Often use T^μ to represent a vector, but strictly, T^μ are components of the vector

$$\underline{T} = T^\mu \frac{\partial}{\partial x^\mu}$$

$\underline{T}$: vector. T^μ: cpts. $\frac{\partial}{\partial x^\mu}$: basis

at p, $T_p(M)$

ector,

of

basis

Vectors transform contravariantly $V'^{\mu'} = \frac{\partial x'^{\mu'}}{\partial x^{\mu}} V^{\mu}$

& covectors covariantly $\omega'_{\mu'} = \frac{\partial x^{\mu}}{\partial x'^{\mu'}} \omega_{\mu}$ under co-ord transfm.

Under differentiation:

$$\frac{\partial T^{\mu}}{\partial x^{\nu}} = \frac{\partial x'^{\nu'}}{\partial x^{\nu}} \frac{\partial}{\partial x'^{\nu'}} \left[\frac{\partial x^{\mu}}{\partial x'^{\mu'}} T^{\mu'} \right] = \frac{\partial x'^{\nu'}}{\partial x^{\nu}} \left[\underbrace{\frac{\partial x^{\mu}}{\partial x'^{\mu'}} T^{\mu'}}_{\text{TENSOR. U}} + \underbrace{\frac{\partial^2 x^{\mu}}{\partial x'^{\nu'} \partial x'^{\mu'}} T^{\mu'}}_{\text{SOMETHING ELSE}} \right]$$

Exterior derivative & forms

Begin by noticing we can define a derivative on functions.

$$f \longmapsto \frac{\partial f}{\partial x^\mu} dx^\mu = \underline{d} f$$

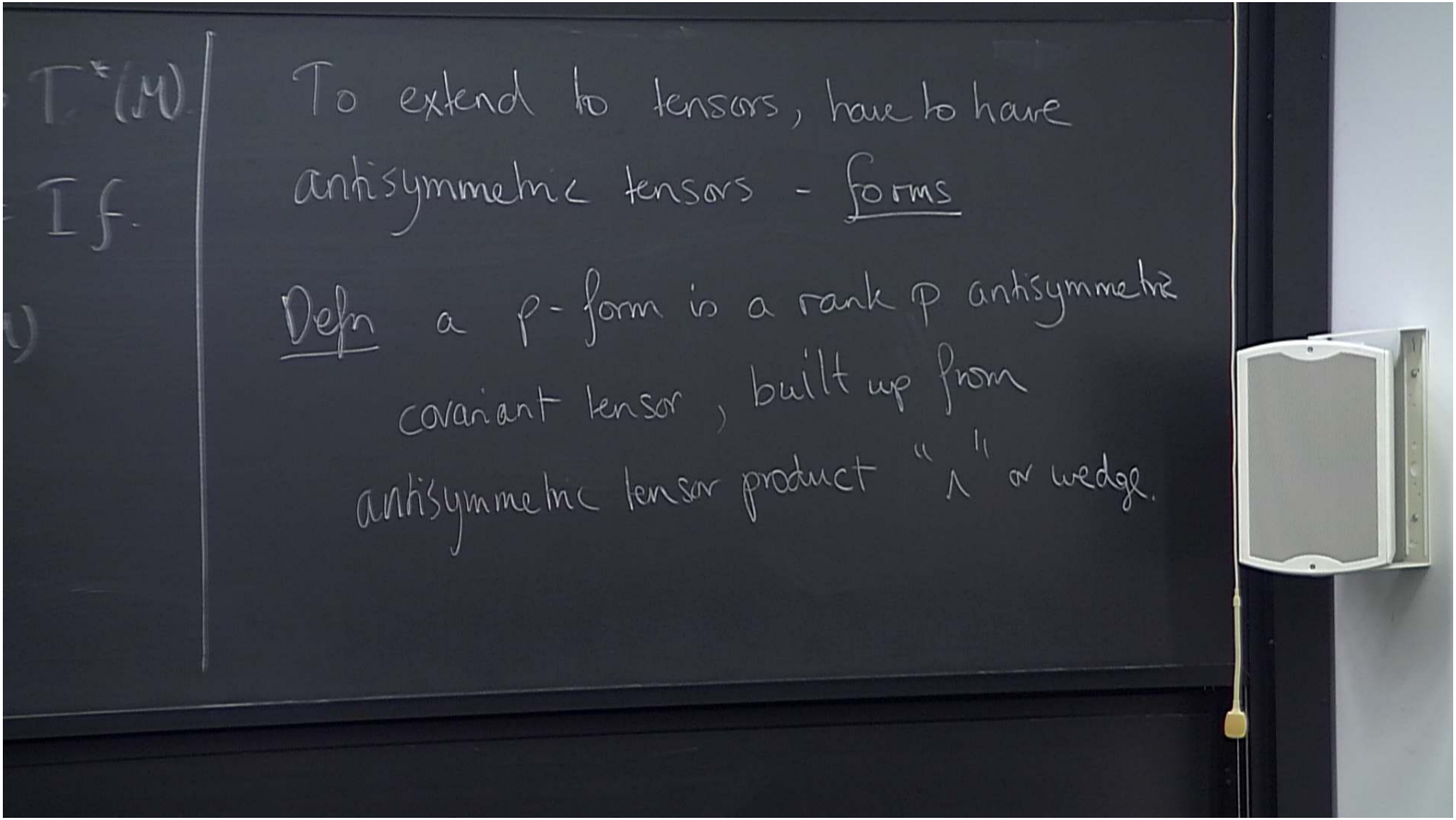
covector

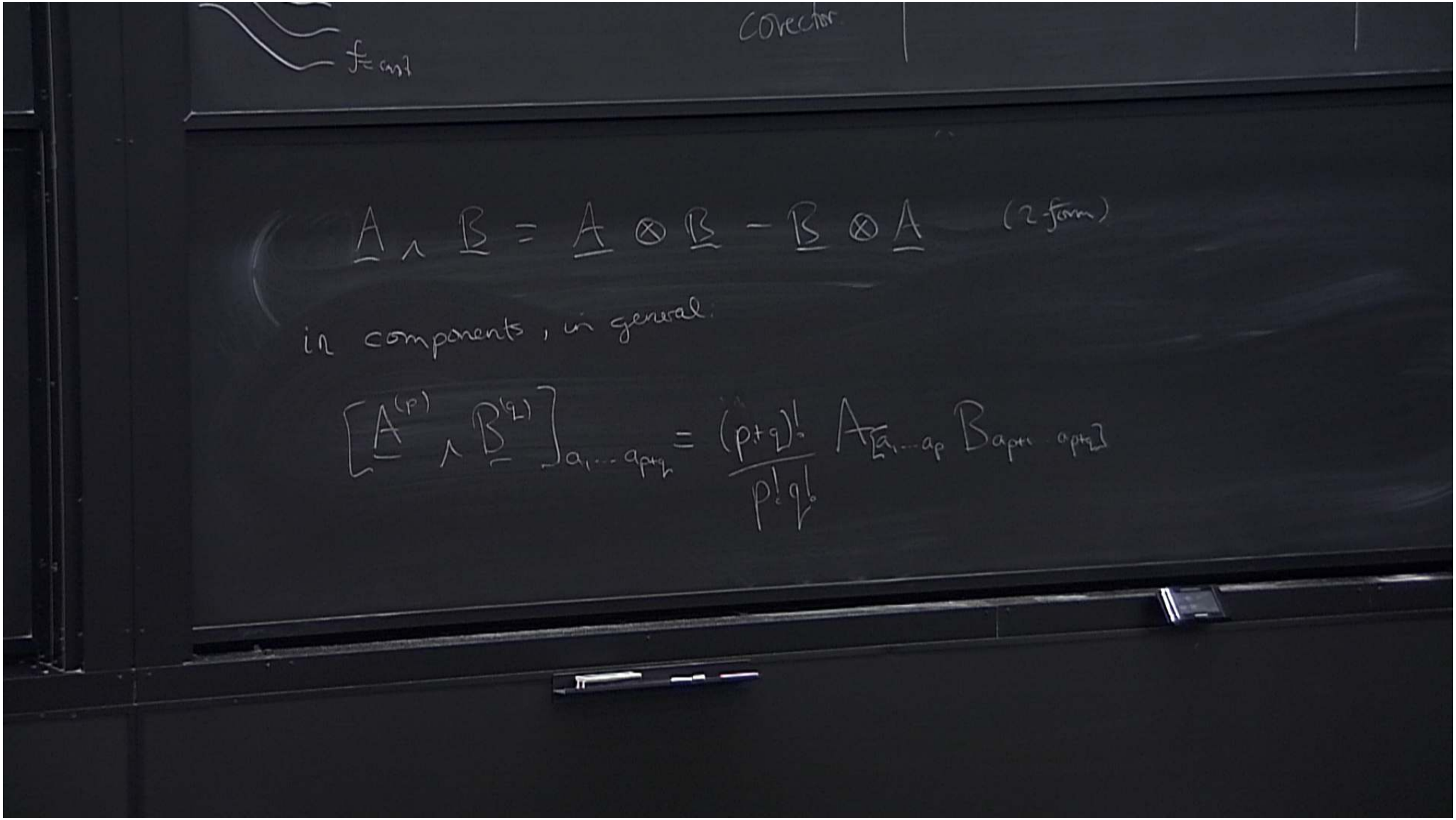
$$\underline{d} : C^\infty(M) \longrightarrow T^*(M)$$

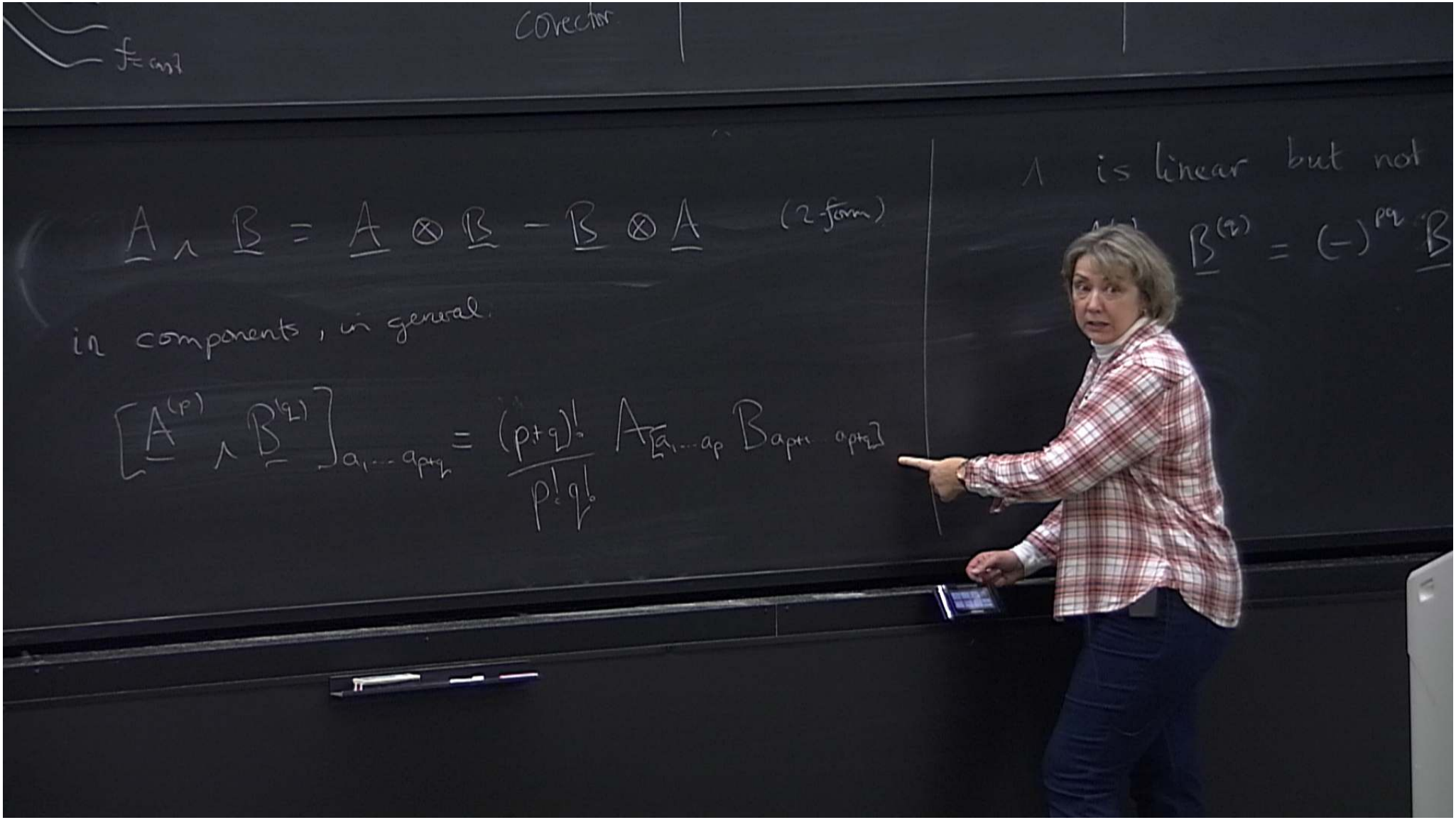
s.t. at p $\langle \underline{d}f | I \rangle = If$.

$\forall I \in T_p(M)$

$\forall p \in M$


$T^*(M)$
If.
To extend to tensors, have to have antisymmetric tensors - forms
Defn a p-form is a rank p antisymmetric covariant tensor, built up from antisymmetric tensor product "$\wedge$" or wedge.


f = const
covector
$\underline{A} \wedge \underline{B} = \underline{A} \otimes \underline{B} - \underline{B} \otimes \underline{A}$ (2-form)
in components, in general:
$[\underline{A}^{(p)} \wedge \underline{B}^{(q)}]_{a_1 \dots a_{p+q}} = \frac{(p+q)!}{p!\,q!} A_{[a_1 \dots a_p} B_{a_{p+1} \dots a_{p+q}]}$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$\wedge$ is linear but not
$\underline{B}^{(q)} = (-)^{pq} \underline{B}$

ϵ_{abcd} a tensor density

$$\underline{\underline{\epsilon}} = \frac{1}{n!}\,\epsilon_{\mu\nu\ldots\rho}\, dx^{\mu} \wedge dx^{\nu} \wedge \ldots dx^{\rho}$$

$$\frac{\partial x^{\mu}}{\partial x^{\mu'}} dx^{\mu'}$$

under coord transfm $= \frac{1}{n!} \det\left(\frac{\partial x}{\partial x'}\right) \epsilon_{\mu'\nu'\ldots\rho'}\, dx'^{\mu'} \wedge dx'^{\nu'} \wedge \ldots \wedge dx'^{\rho'}$

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under coord transf^{n}

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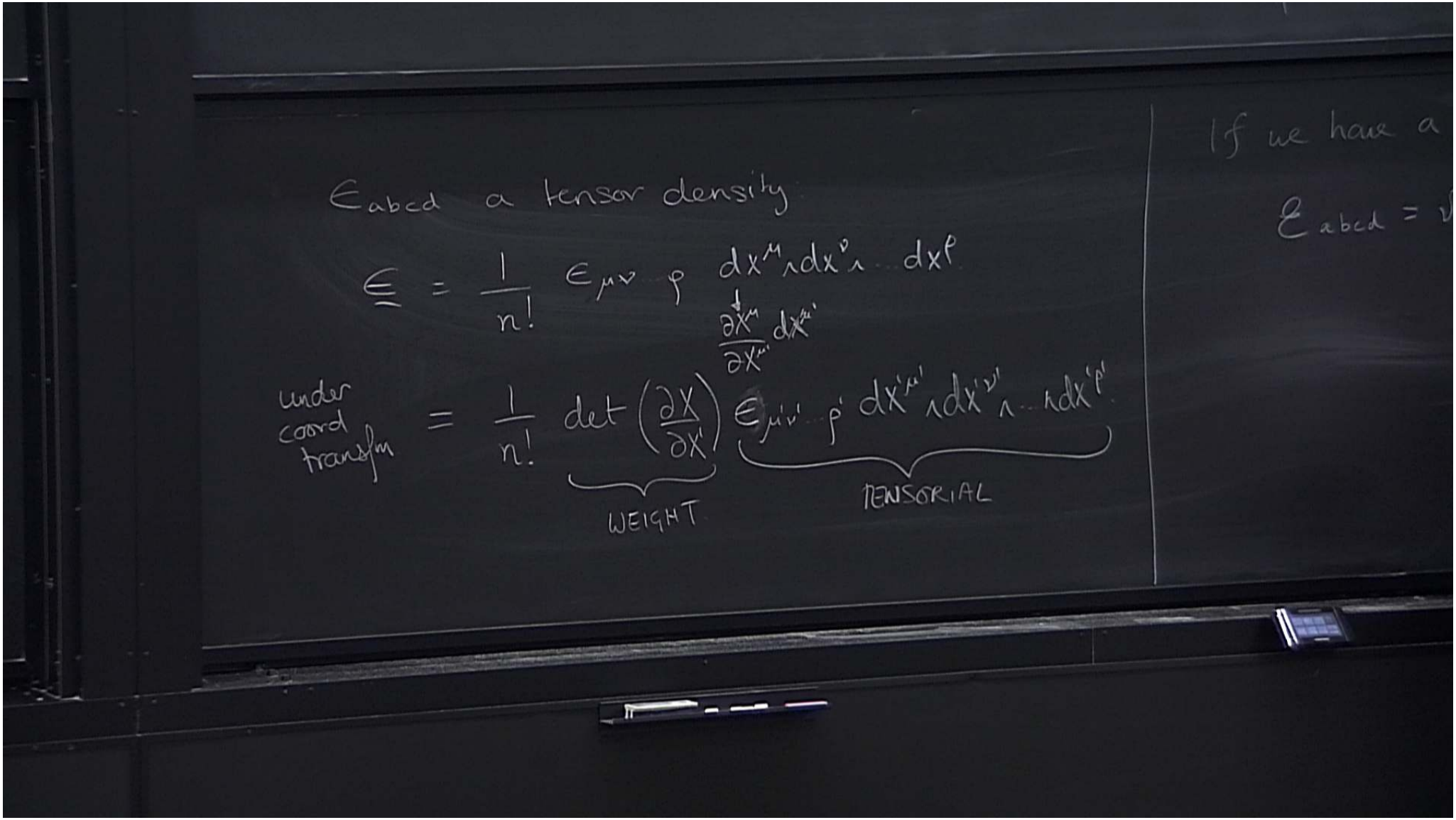
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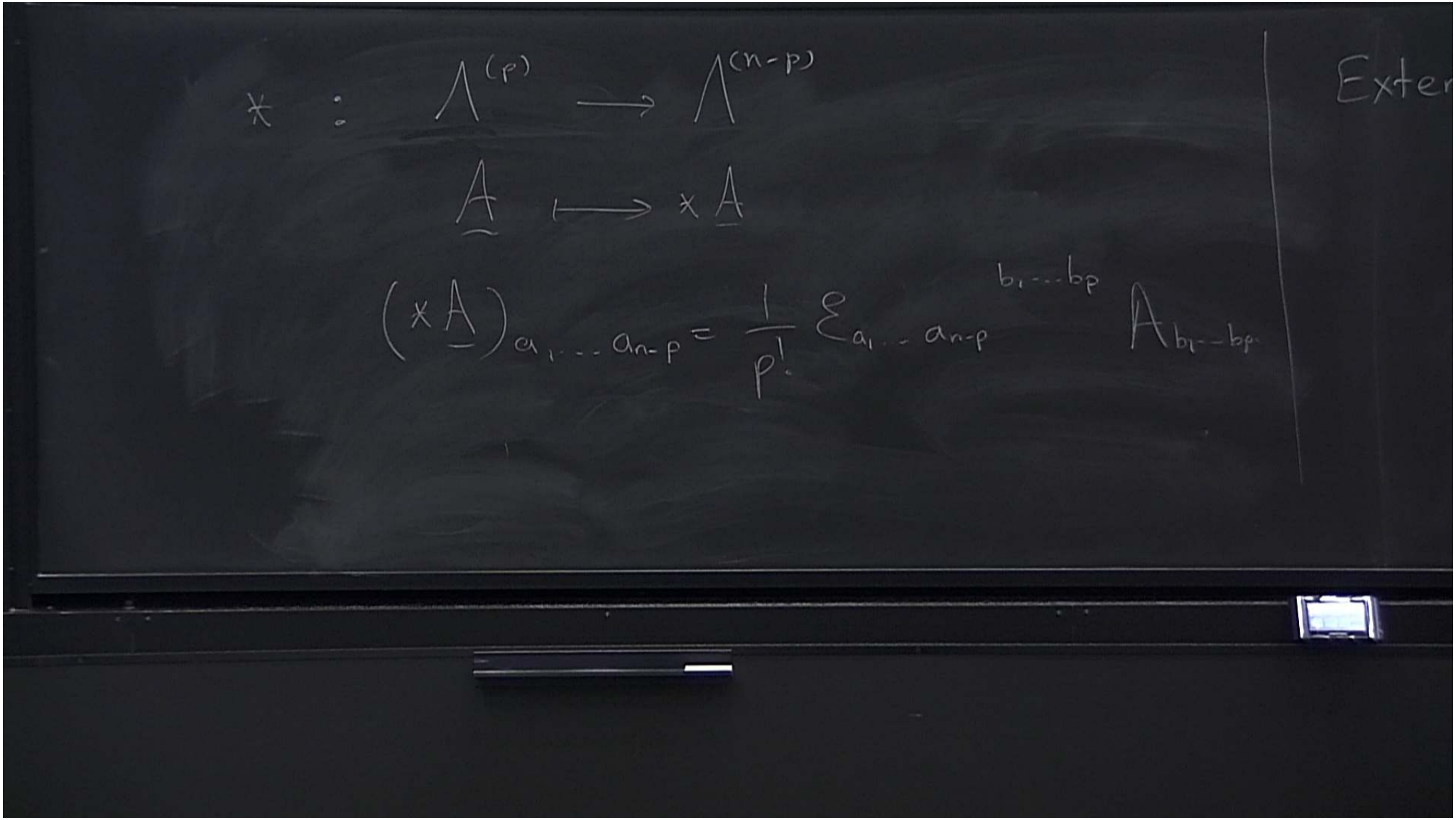
$\frac{\partial x^\mu}{\partial x^{\mu'}} dx^{\mu'}$
under coord transfm
$= \frac{1}{n!} \det\left(\frac{\partial x}{\partial x'}\right) \epsilon_{\mu'\nu'\ldots\rho'} \, dx'^{\mu'} \wedge dx'^{\nu'} \wedge \ldots \wedge dx'^{\rho'}$
WEIGHT
TENSORIAL
If we have a
$\mathcal{E}_{abcd} = \nu$

$\epsilon_{abcd} = \pm 1$ even/odd permutation

If we have a metric, define

$\mathcal{E}_{abcd} = \sqrt{|g|}\,\epsilon_{abcd}$ ← now a tensor

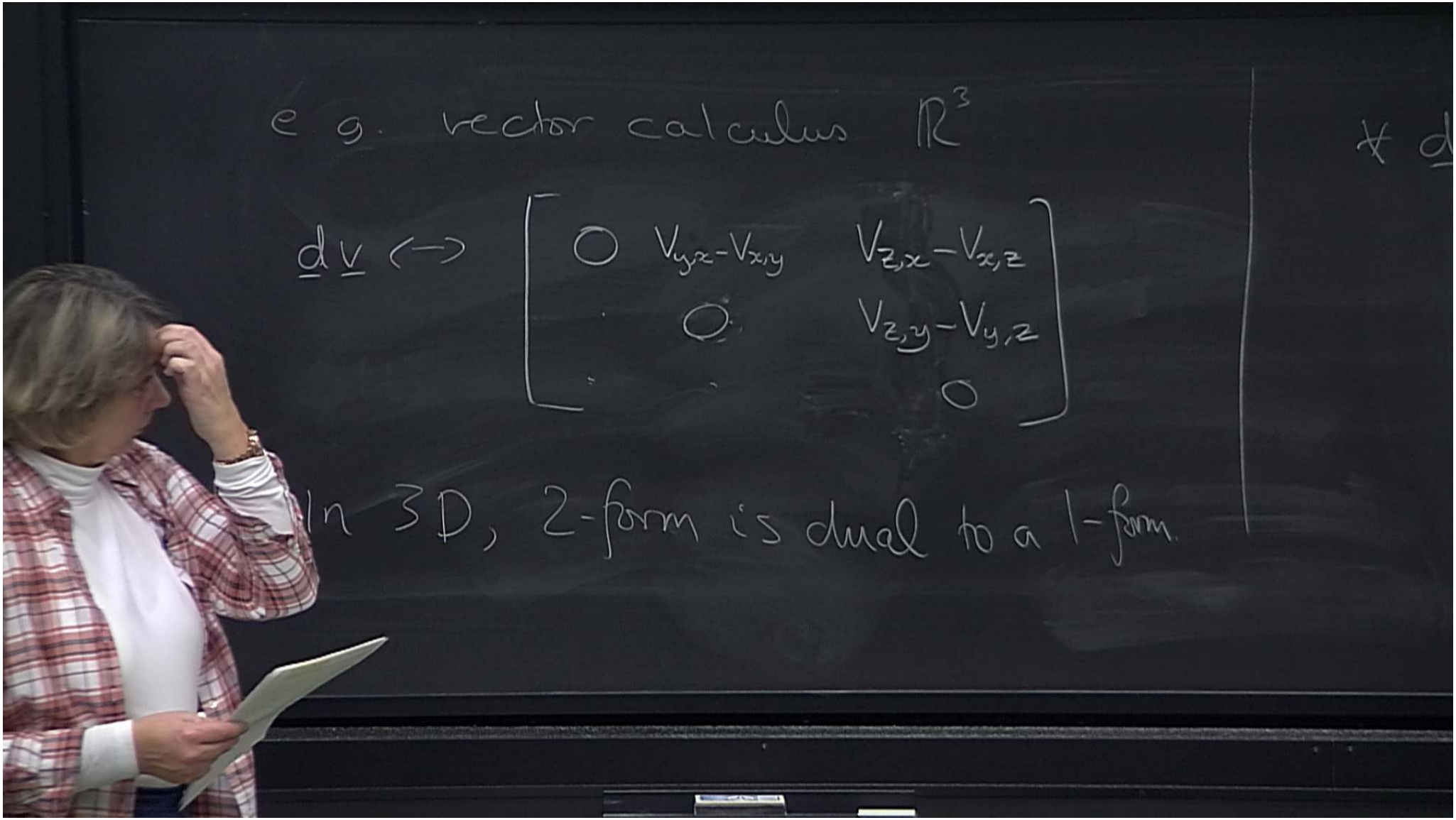
Allows us to define Hodge dual "$*$"

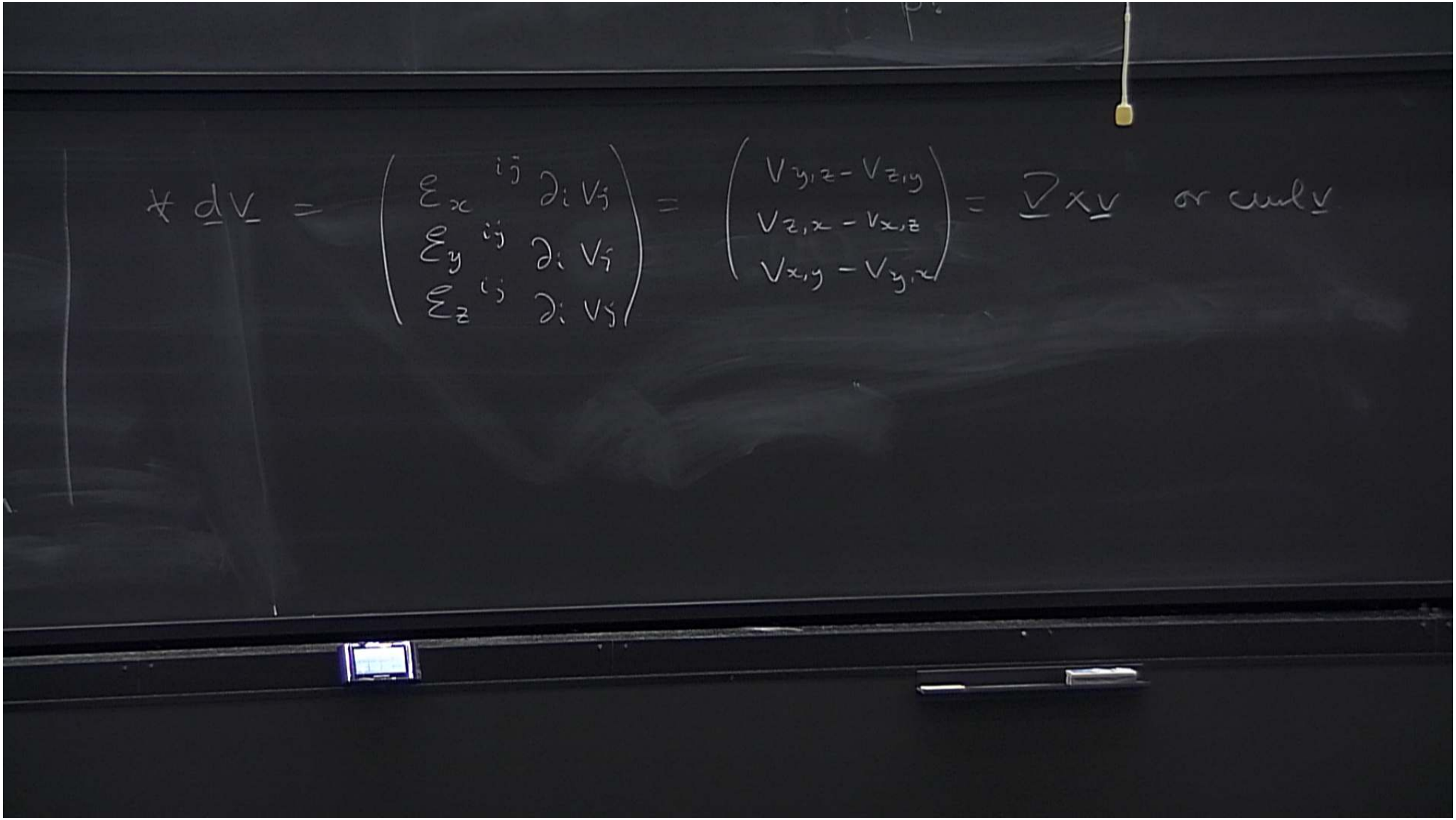

$* : \Lambda^{(p)} \longrightarrow \Lambda^{(n-p)}$
$\underset{\sim}{A} \longmapsto * \underset{\sim}{A}$
$(* \underset{\sim}{A})_{a_1 \ldots a_{n-p}} = \frac{1}{p!} \varepsilon_{a_1 \ldots a_{n-p}}{}^{b_1 \ldots b_p} A_{b_1 \ldots b_p}$
Exter

Exterior derivative: $\underline{d}$ is a map $\Lambda^p \to \Lambda^{p+1}$

- Reduces to $\underline{d}f$ on fns
- Pseudo Leibnitzian $\underline{d}(\underline{A}^{(p)} \wedge \underline{B}^{(q)}) = \underline{d}\underline{A} \wedge \underline{B} + (-1)^p \underline{A} \wedge \underline{d}\underline{B}$
- $\underline{d}^2 = 0$

$$\left(\underline{d}\,\underset{\sim}{A}^{(p)}\right)_{a_1 \ldots a_{p+1}} = \frac{(p+1)!}{p!}\, \partial_{[a_1} A_{a_2 \ldots a_{p+1}]}$$


e.g. vector calculus $\mathbb{R}^3$
$\underline{d}\underline{v} \leftrightarrow \begin{bmatrix} 0 & V_{y,x}-V_{x,y} & V_{z,x}-V_{x,z} \\ & 0 & V_{z,y}-V_{y,z} \\ & & 0 \end{bmatrix}$
In 3D, 2-form is dual to a 1-form.
$\star \underline{d}$


$* d\underline{v} = \begin{pmatrix} \varepsilon_x{}^{ij}\,\partial_i v_j \\ \varepsilon_y{}^{ij}\,\partial_i v_j \\ \varepsilon_z{}^{ij}\,\partial_i v_j \end{pmatrix} = \begin{pmatrix} v_{y,z} - v_{z,y} \\ v_{z,x} - v_{x,z} \\ v_{x,y} - v_{y,x} \end{pmatrix} = \underline{\nabla} \times \underline{v}$ or curl $\underline{v}$

using $*$, can define $\delta = *d*$: $\Lambda \to \Lambda$

a vector,

ments of

basis

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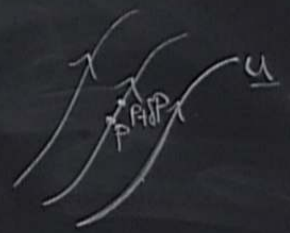
Lie Derivative:

Recall, Lie derivative uses a vector field to link tangent spaces, providing a means of comparing vectors at nearby points



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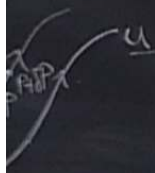

$$(\mathcal{L}_u V)^\mu = u^\nu V^\mu{}_{,\nu} - V^\nu u^\mu{}_{,\nu} = [\underline{u}, \underline{v}]$$

$$(\mathcal{L}_u \omega)_\mu = u^\nu \omega_{\mu,\nu} + u^\nu{}_{,\mu}\, \omega_\nu$$

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Lie bracket defines an algebra on vectors

$$[\underline{u}, \underline{v}] = \underline{u}(\underline{v} f) - \underline{v}(\underline{u} f)$$

under coord transfo