Title: PSI 2016/2017 Gravitational Physics (Review) - Lecture 1

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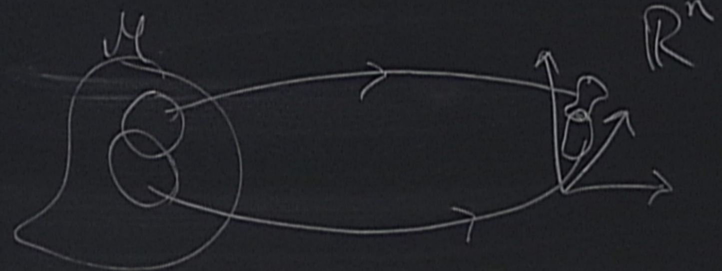
Abstract:

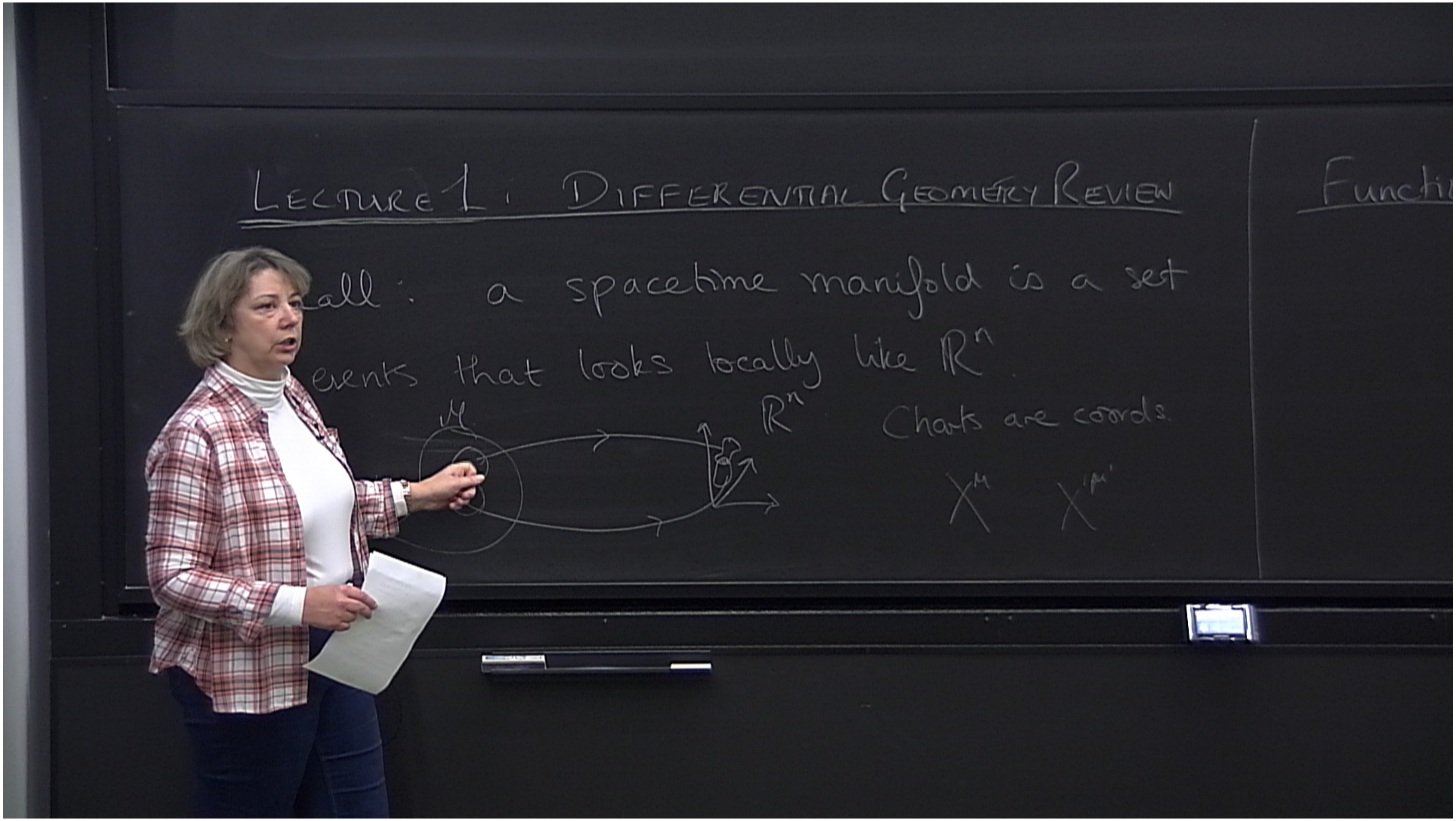
LECTURE 1: DIFFERENTIAL GEOMETRY REVIEW

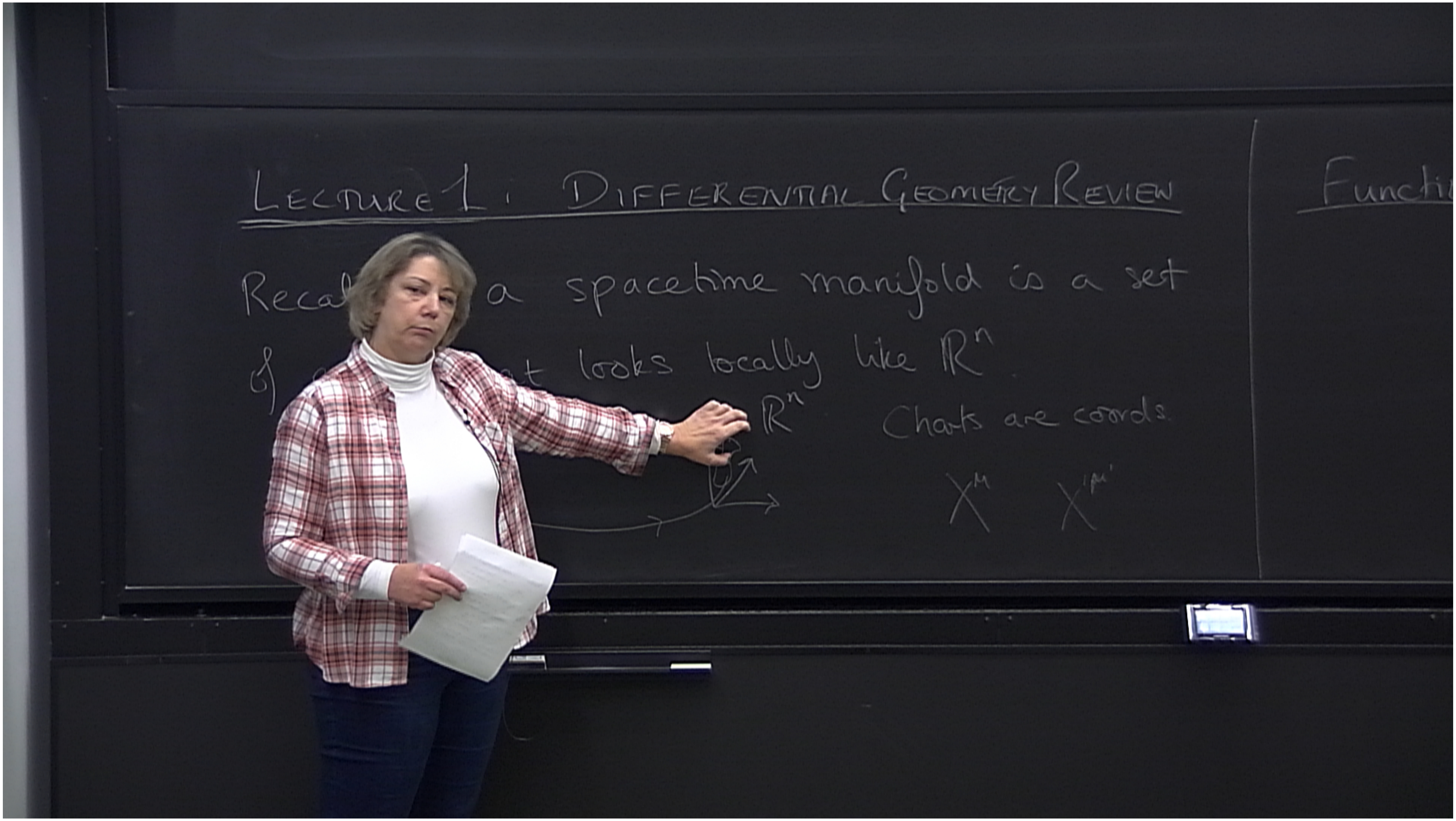
Recall: a spacetime manifold is a set of events that looks locally like $\mathbb{R}^n$.



Charts are coords.

$x^\mu \quad x^{\prime\mu'}$


LECTURE 1: DIFFERENTIAL GEOMETRY REVIEW
all: a spacetime manifold is a set
events that looks locally like $\mathbb{R}^n$.
$\mathcal{M}$
$\mathbb{R}^n$. Charts are coords.
x^μ $x'^{\mu'}$


LECTURE 1: DIFFERENTIAL GEOMETRY REVIEW
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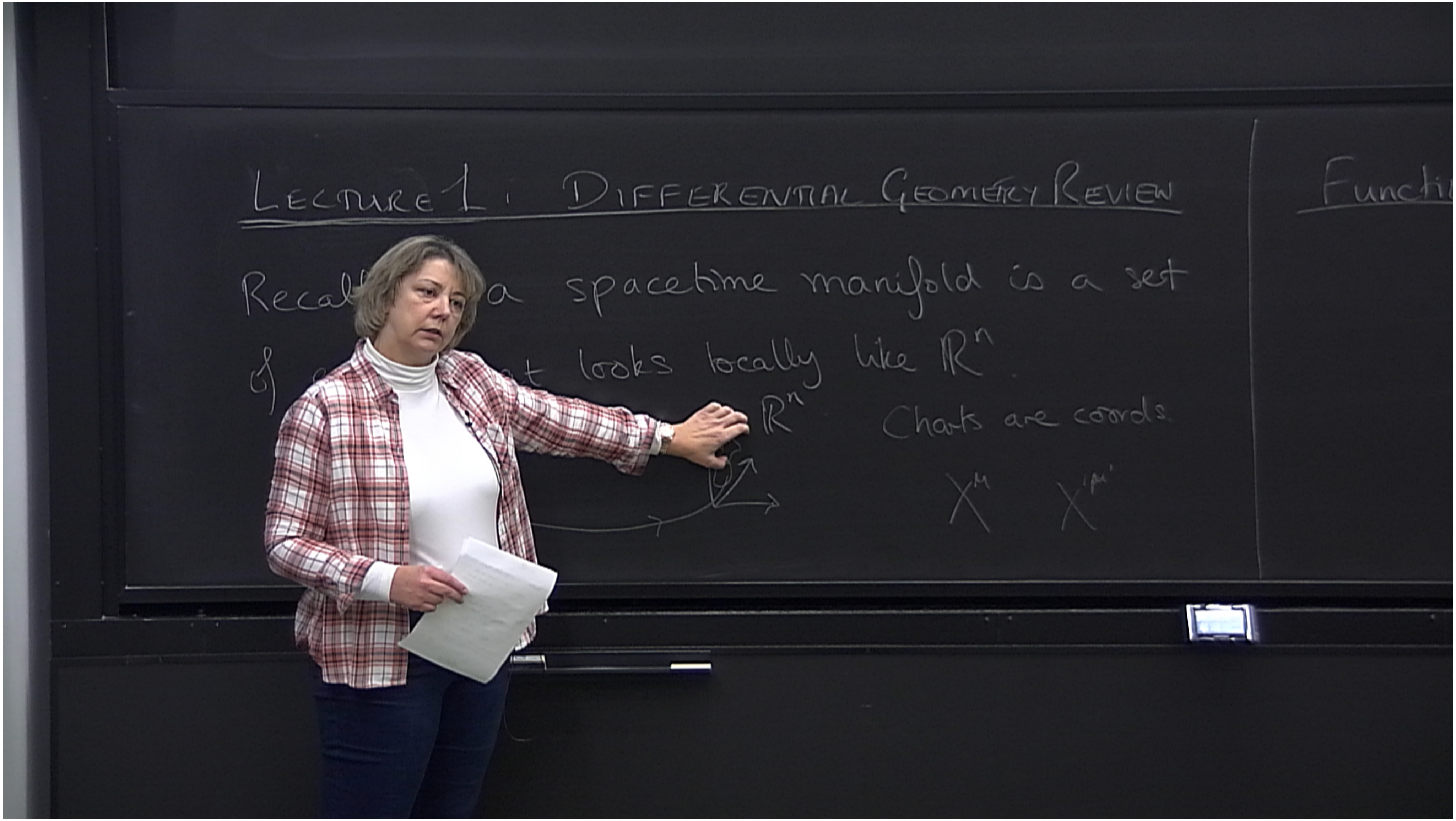
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a spacetime manifold is a set
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$$\underline{T} : C^\infty(\mathcal{M}) \longrightarrow C^\infty(\mathcal{M})$$

$$f \longmapsto \frac{df}{dt} \quad \text{at } P \text{ on } \gamma.$$

$\gamma(t)$ our curve, $\frac{d}{dt}$ is the tgt vec

" $\underline{T} = T^\mu \dfrac{\partial}{\partial x^\mu}$ "

Form a <u>tangent</u> <u>space</u> at P, $T_p(\mathcal{M})$

Covectors: Maps from tangent space to $\mathbb{R}$

$$\underset{\sim}{\omega} : T_p(\mathcal{M}) \longrightarrow \mathbb{R}$$

$$\underline{V} \longmapsto \underset{\sim}{\omega}(\underline{V}) \text{ or } \langle \underset{\sim}{\omega} | \underline{V} \rangle \text{ or } V^\mu \omega_\mu$$

" $\underset{\sim}{\omega} = \omega_\mu dx^\mu$ "

Functions: $C^\infty(\mathcal{M})$. ∞^{ly} differentiable fns.

$f: \mathcal{M} \to \mathbb{R}$

Curve: $\gamma: \mathbb{R} \to \mathcal{M}$

Vectors: Defined as "tangents" to curve.

ctors: Maps from tangent space to $\mathbb{R}$

Abstract Index Notation

Often use T^μ to represent a vector, but strictly, T^μ are components of the vector.

$$\underline{T} = T^\mu \frac{\partial}{\partial x^\mu}.$$

VECTOR. cpts. basis

at P, $T_p(M)$

ector,

of

basis

Vectors transform contravariantly $\quad V'^{\mu'} = \dfrac{\partial X'^{\mu'}}{\partial X^{\mu}} V^{\mu}$

& covectors covariantly $\quad \omega'_{\mu'} = \dfrac{\partial X^{\mu}}{\partial X'^{\mu'}} \omega_{\mu}$ under co-ord transfm.

Under differentiation:

$$\frac{\partial T^{\mu}}{\partial X^{\nu}} = \frac{\partial X'^{\nu'}}{\partial X^{\nu}} \frac{\partial}{\partial X'^{\nu'}}\left[\frac{\partial X^{\mu}}{\partial X'^{\mu'}} T'^{\mu'}\right] = \frac{\partial X'^{\nu'}}{\partial X^{\nu}}\left[\overbrace{\frac{\partial X^{\mu}}{\partial X'^{\mu'}} T'^{\mu'}}^{\text{TENSOR. ☺}} + \overbrace{\frac{\partial^2 X^{\mu}}{\partial X'^{\nu'} \partial X'^{\mu'}} T'^{\mu'}}^{\text{SOMETHING ELSE}}\right]$$

Exterior derivative & forms

Begin by noticing we can define a derivative on functions.

$$f \longmapsto \frac{\partial f}{\partial x^\mu} dx^\mu = \underline{d} f.$$

covector.

$f = $ const

$\underline{d} : C^\infty(M) \longrightarrow T^*(M).$

s.t. at p $\langle \underline{d}f \,|\, I \rangle = I f.$

$\forall\, I \in T_p(M)$

$\forall\, p \in M$

$T^*(M)$

$If.$

To extend to tensors, have to have antisymmetric tensors - <u>forms</u>

Defn a p-form is a rank p antisymmetric covariant tensor, built up from antisymmetric tensor product "$\wedge$" or wedge.

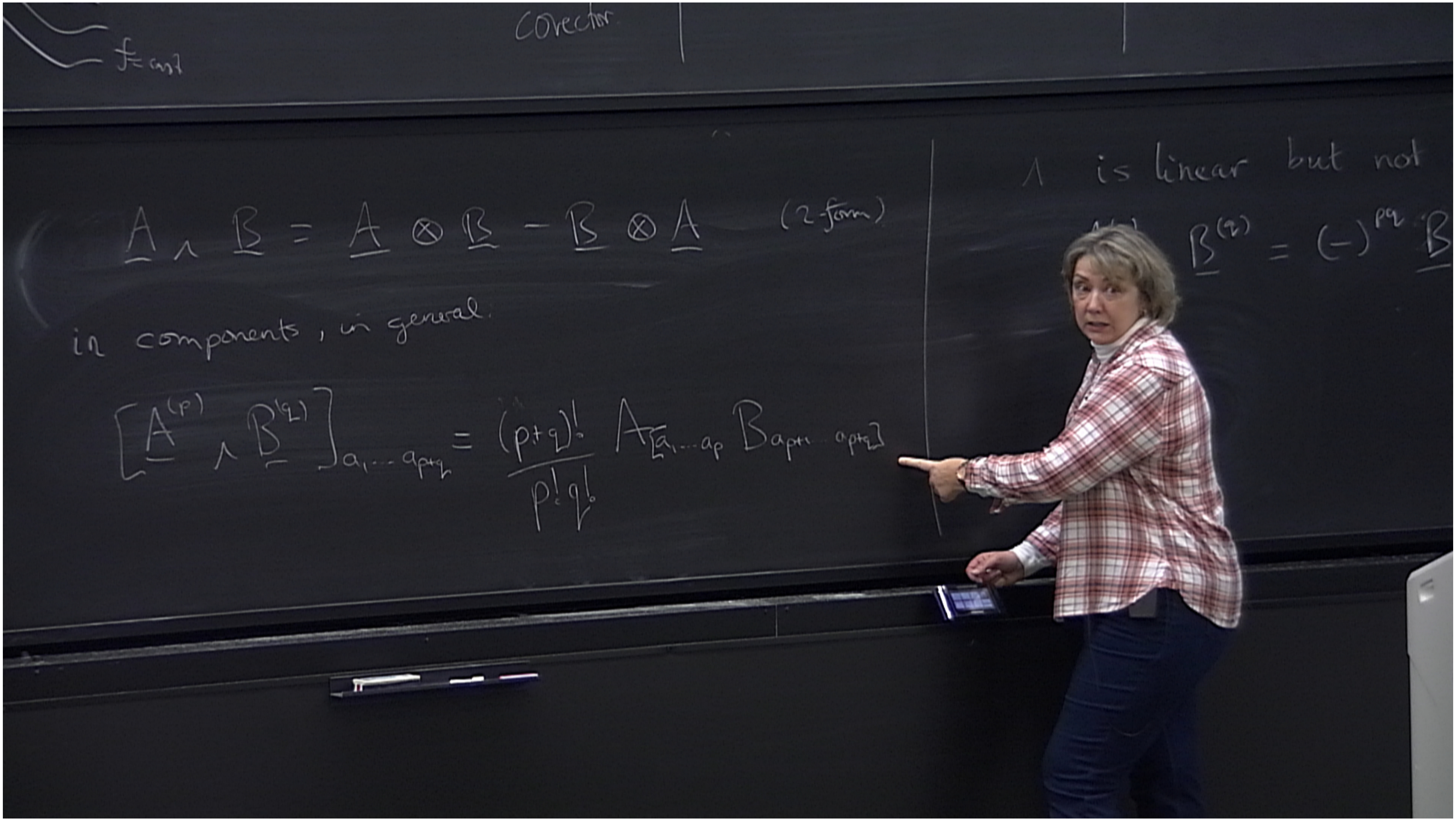
$f = \text{const}$

covector

$$\underline{A} \wedge \underline{B} = \underline{A} \otimes \underline{B} - \underline{B} \otimes \underline{A} \qquad \text{(2-form)}$$

in components, in general:

$$\left[\underline{A}^{(p)} \wedge \underline{B}^{(q)}\right]_{a_1 \ldots a_{p+q}} = \frac{(p+q)!}{p!\,q!} A_{[a_1 \ldots a_p} B_{a_{p+1} \ldots a_{p+q}]}$$

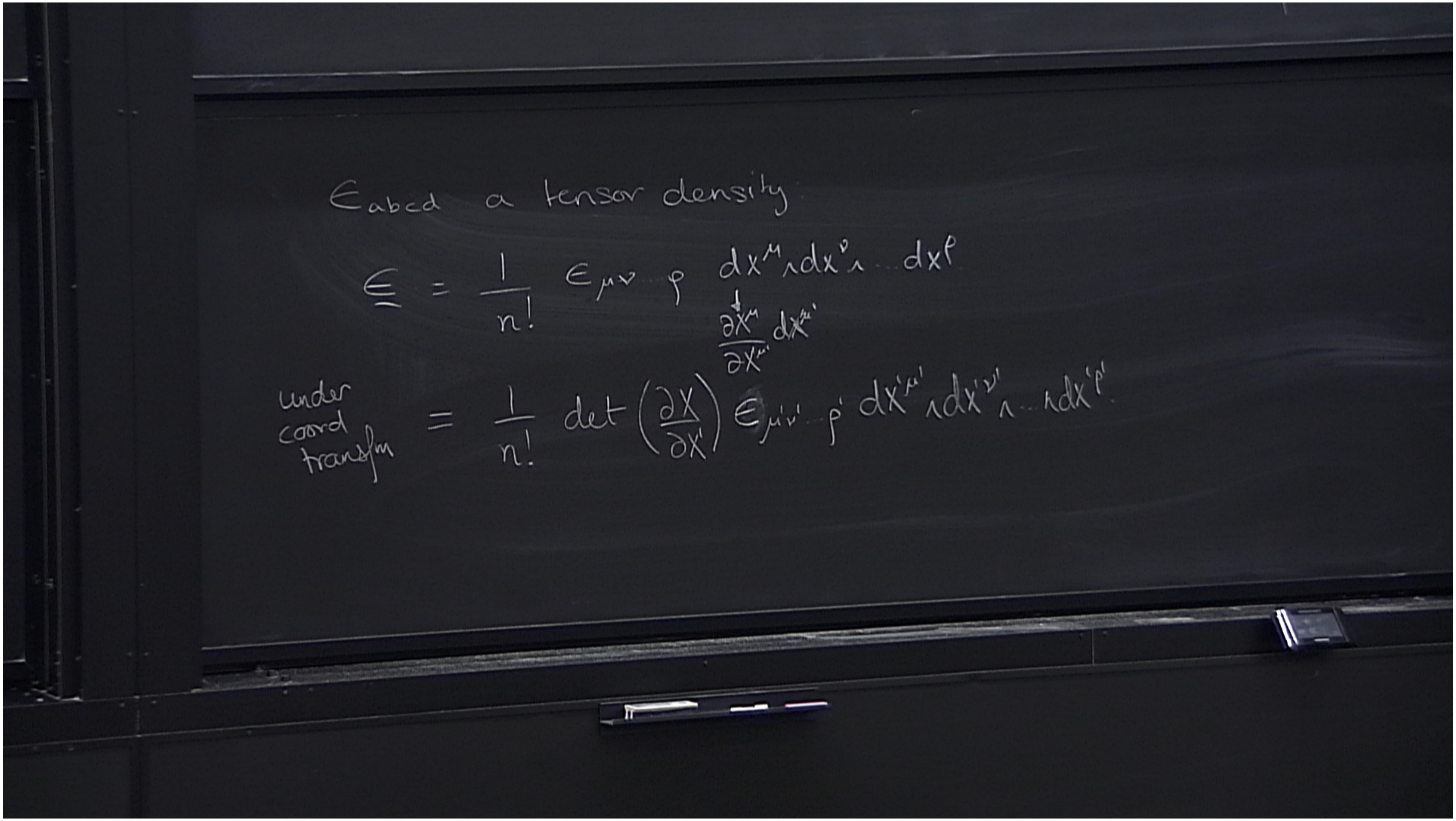

covector
$\underline{A} \wedge \underline{B} = \underline{A} \otimes \underline{B} - \underline{B} \otimes \underline{A}$ (2-form)
in components, in general
$[\underline{A}^{(p)} \wedge \underline{B}^{(q)}]_{a_1 \dots a_{p+q}} = \frac{(p+q)!}{p!\,q!} A_{[a_1 \dots a_p} B_{a_{p+1} \dots a_{p+q}]}$
$\wedge$ is linear but not
$\underline{B}^{(q)} = (-)^{pq}$

ϵ_{abcd} a tensor density

$$\underset{\sim}{\epsilon} = \frac{1}{n!}\,\epsilon_{\mu\nu\cdots\rho}\,\underbrace{dx^{\mu}}_{\frac{\partial x^{\mu}}{\partial x^{\mu'}}dx^{\mu'}}\wedge dx^{\nu}\wedge\cdots dx^{\rho}$$

under coord transfm

$$= \frac{1}{n!}\,\det\left(\frac{\partial x}{\partial x'}\right)\epsilon_{\mu'\nu'\cdots\rho'}\,dx'^{\mu'}\wedge dx'^{\nu'}\wedge\cdots\wedge dx'^{\rho'}$$


ϵ_{abcd} a tensor density
$\underset{=}{\epsilon} = \frac{1}{n!} \epsilon_{\mu\nu\dots\rho}\, dx^{\mu} \wedge dx^{\nu} \wedge \dots dx^{\rho}$
$\frac{\partial x^{\mu}}{\partial x'^{\mu'}} dx'^{\mu'}$
under coord transfm
$= \frac{1}{n!} \det\left(\frac{\partial x}{\partial x'}\right) \epsilon_{\mu'\nu'\dots\rho'}\, dx'^{\mu'} \wedge dx'^{\nu'} \wedge \dots \wedge dx'^{\rho'}$

ϵ_{abcd} a tensor density

$$\underset{=}{\epsilon} = \frac{1}{n!}\, \epsilon_{\mu\nu\cdots\rho}\, dx^{\mu} \wedge dx^{\nu} \wedge \cdots dx^{\rho}$$

$$dx^{\mu} \to \frac{\partial x^{\mu}}{\partial x'^{\mu'}}\, dx'^{\mu'}$$

under coord transfm

$$= \frac{1}{n!}\, \underbrace{\det\left(\frac{\partial x}{\partial x'}\right)}_{\text{WEIGHT}}\, \underbrace{\epsilon_{\mu'\nu'\cdots\rho'}\, dx'^{\mu'} \wedge dx'^{\nu'} \wedge \cdots \wedge dx'^{\rho'}}_{\text{TENSORIAL}}$$

If we have a

$\mathcal{E}_{abcd} = \sqrt{}$

$\epsilon_{abcd} = \pm 1$ even/odd

If we have a metric, define

$\varepsilon_{abcd} = \sqrt{|g|}\, \epsilon_{abcd}$ ← now a tensor

Allows us to define Hodge dual "$*$"

$$* : \Lambda^{(p)} \longrightarrow \Lambda^{(n-p)}$$

$$\underline{A} \longmapsto * \underline{A}$$

$$(* \underline{A})_{a_1 \ldots a_{n-p}} = \frac{1}{p!} \varepsilon_{a_1 \ldots a_{n-p}}{}^{b_1 \ldots b_p} A_{b_1 \ldots b_p}$$

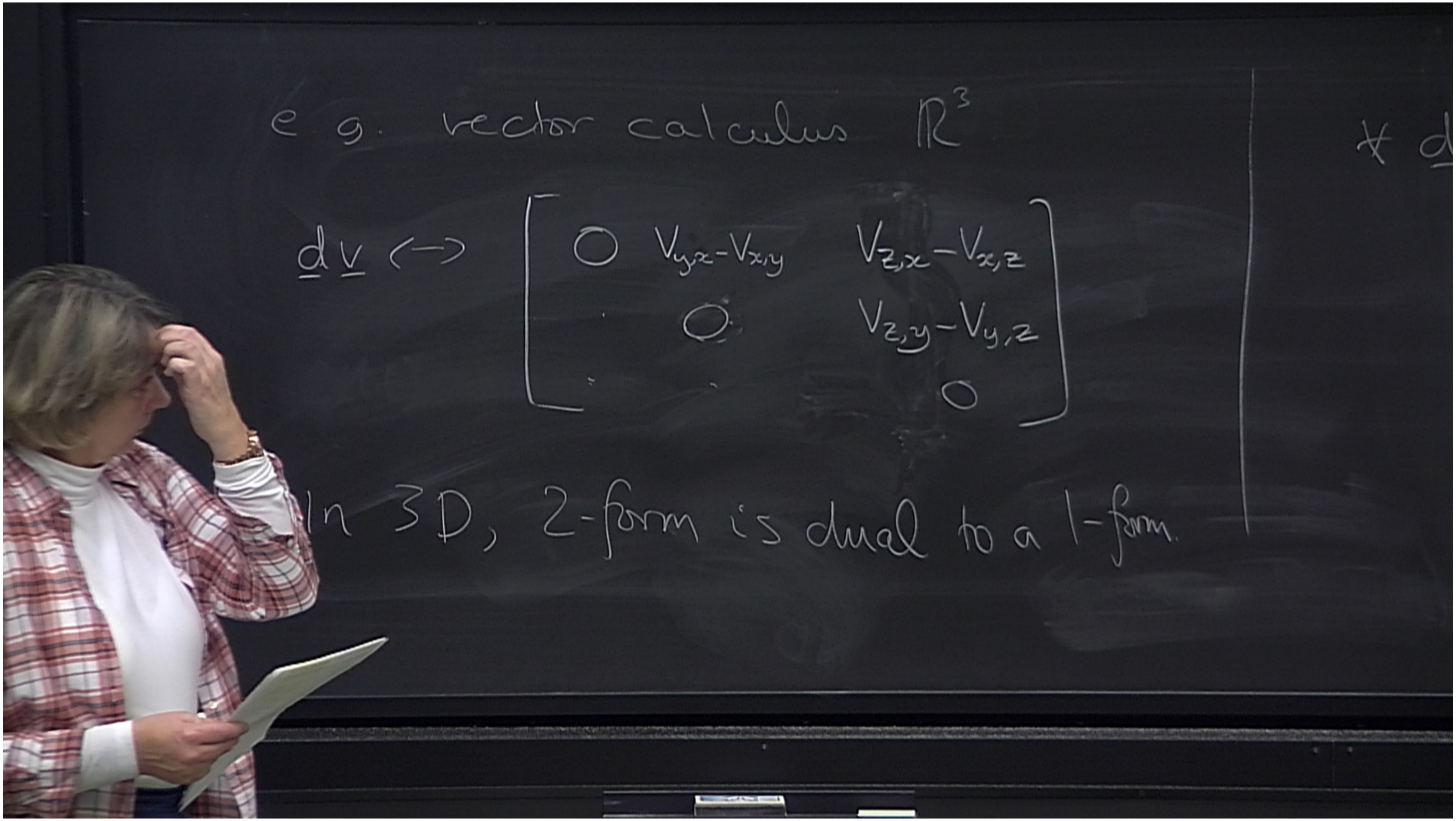
Exter

$A_{b_1 \dots b_p}$

Exterior derivative: $\underset{\sim}{d}$ is a map $\Lambda^p \to \Lambda^{p+1}$

- Reduces to $\underset{\sim}{d} f$ on fns.
- Pseudo Leibnitzian: $\underset{\sim}{d}(\underset{\sim}{A}^{(p)} \wedge \underset{\sim}{B}^{(q)}) = \underset{\sim}{d}\underset{\sim}{A} \wedge \underset{\sim}{B} + (-)^p \underset{\sim}{A} \wedge \underset{\sim}{d}\underset{\sim}{B}$
- $\underset{\sim}{d}^2 = 0$

$$\left(\underset{\sim}{d}\,\underset{\sim}{A}^{(p)}\right)_{a_1 \dots a_{p+1}} = \frac{(p+1)!}{p!}\, \partial_{[a_1} A_{a_2 \dots a_{p+1}]}$$


e.g. vector calculus $\mathbb{R}^3$
$\underline{d}\,\underline{V} \longleftrightarrow \begin{bmatrix} 0 & V_{y,x}-V_{x,y} & V_{z,x}-V_{x,z} \\ \cdot & 0 & V_{z,y}-V_{y,z} \\ \cdot & \cdot & 0 \end{bmatrix}$
In 3D, 2-form is dual to a 1-form.

$$\ast\, \underline{d}\,\underline{V} = \begin{pmatrix} \varepsilon_x{}^{ij}\, \partial_i V_j \\ \varepsilon_y{}^{ij}\, \partial_i V_j \\ \varepsilon_z{}^{ij}\, \partial_i V_j \end{pmatrix} = \begin{pmatrix} V_{y,z} - V_{z,y} \\ V_{z,x} - V_{x,z} \\ V_{x,y} - V_{y,x} \end{pmatrix} = \underline{\nabla} \times \underline{V} \quad \text{or } \operatorname{curl} \underline{V}$$

1-form

Using $*$, can define $\delta = *d* : \Lambda \to \Lambda$

a vector,

nents of

basis

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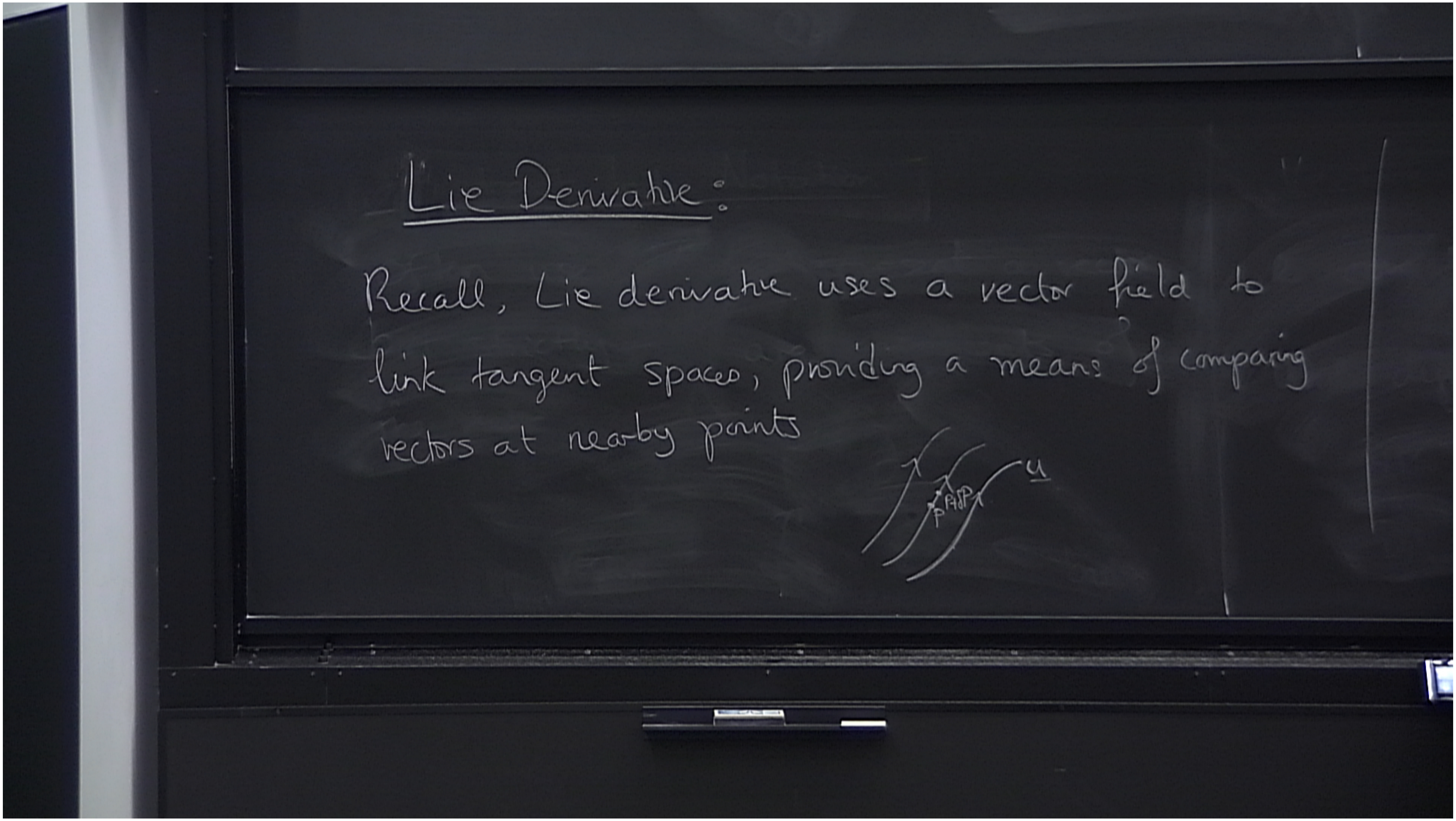

Lie Derivative:
Recall, Lie derivative uses a vector field to
link tangent spaces, providing a means of comparing
vectors at nearby points
X
u
p

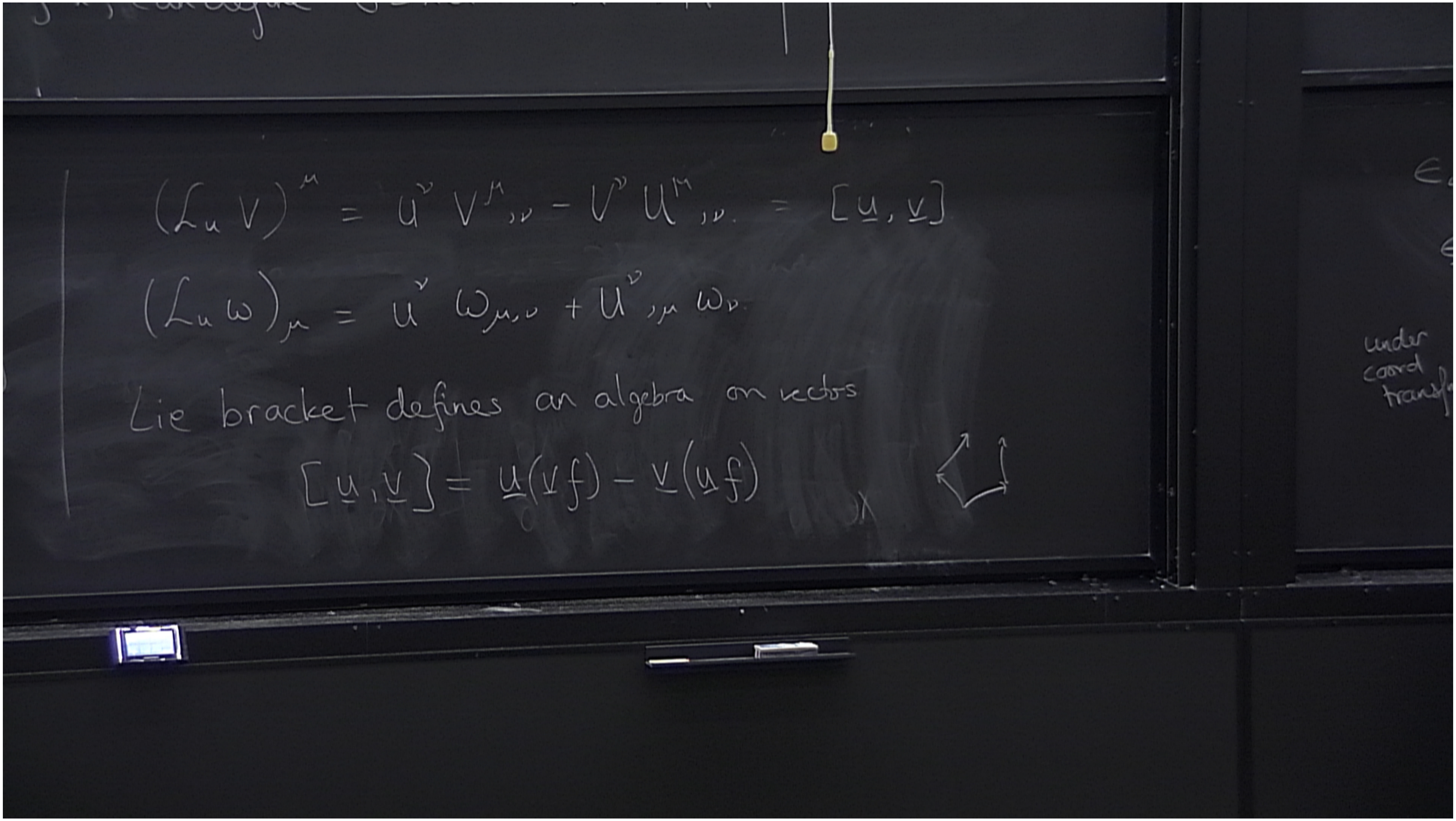
vector field to

a means of comparing

$\underline{u}$

$$(\mathcal{L}_u V)^\mu = u^\nu V^\mu{}_{,\nu} - V^\nu U^\mu{}_{,\nu} = [\underline{u}, \underline{v}]$$

$$(\mathcal{L}_u \omega)_\mu = u^\nu \omega_{\mu,\nu} + U^\nu{}_{,\mu} \omega_\nu$$


$(\mathcal{L}_u V)^\mu = u^\nu V^\mu{}_{,\nu} - V^\nu u^\mu{}_{,\nu} = [\underline{u}, \underline{v}]$
$(\mathcal{L}_u \omega)_\mu = u^\nu \omega_{\mu,\nu} + u^\nu{}_{,\mu} \omega_\nu$
Lie bracket defines an algebra on vectors
$[\underline{u}, \underline{v}] = \underline{u}(\underline{v} f) - \underline{v}(\underline{u} f)$
under
coord