Title: BV formalism and derived symplectic geometry

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Abstract: <p>We will describe a formulation of the Batalin-Vilkovisky formalism using derived symplectic geometry. In this setting, the classical master equation of the BV formalism describes a space of coisotropic structures. Using this approach, we resolve a conjecture of Felder-Kazhdan regarding BRST cohomology. Time permitting, we will also describe applications of these ideas to more general quantization problems.</p>

BV formalism

$$(*) \int_X e^{iS_0(x)/\hbar} f_i(x)\,dx$$

Obs. If S_0 is a Morse function on a f.d. manifold then

$$(*) \sim \sum_{x_{crit}} (?)\, e^{iS_0(x_{crit})}$$

Idea:

1) Embed X into a larger (graded-) manifold V, and extend $S_0(x)$ to a function $S(x)$ on V (depending on $\hbar$) and express

$$(*) = \int_{V \subset T^*[-1]V} e^{iS(y)/\hbar} f(y)\,dy$$

then deform V as a Lagrangian inside $T^*[-1]V$. In order for $\int$ to not change S has to satisfy QME.

3) At $\hbar = 0$, QME becomes the classical master equation

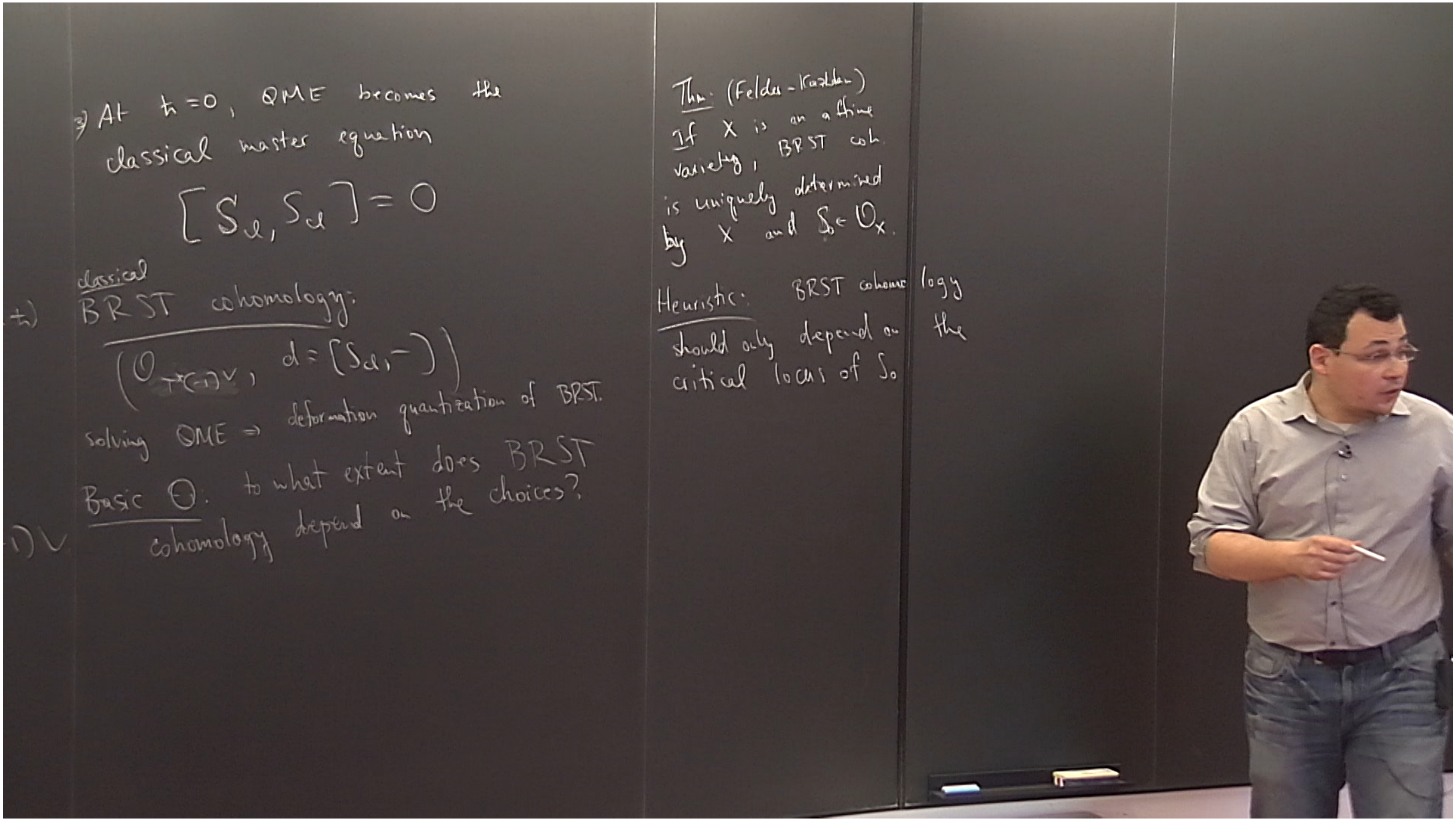
$$[S_{cl}, S_{cl}] = 0$$

classical BRST cohomology:

$\left(\mathcal{O}_{T^*(-1)V}, \ d = [S_{cl}, -]\right)$

Solving QME $\Rightarrow$ deformation quantization of BRST.

Basic Q. to what extent does BRST cohomology depend on the choices?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Thm. (Felder - Kazhdan) If X is an affine variety, BRST coh. is uniquely determined by X and $S_0 \in \mathcal{O}_X$.
Heuristic: BRST cohomology should only depend on the critical locus of S_0

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Felder-Kazhdan

X affine variety

$S_0 \in \mathcal{O}_X$

L = Lie algebra of vector fields on X annihilating S_0

$L_0 \subset L$

L_0 = sub $\mathcal{O}_X$-module gen. by

$\xi(S_0)\eta - \eta(S_0)\xi$, $\eta, \xi \in T_X$

$(\mathcal{O}_{crit(S_0)}, L/L_0)$ forms a Lie-Rinehart pair

Conj: (F-K)

$BRST(S_0) = H^*_{LR}(\mathcal{O}_{crit(S_0)}, L/L_0)$

they proved this in deg. 0 and 1

Another approach (Costell

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Another approach (Costello)

consider the derived critical locus of S_0.

solve the equation $dS_0 = 0$ "up to homotopy"

$$\begin{array}{ccc} dcrit(S_0) & \longrightarrow & X \\ \downarrow & & \downarrow dS_0 \\ X & \xrightarrow[\text{zero section}]{} & T^*X \end{array}$$

$$\mathcal{O}_{dcrit(S_0)} = \mathcal{O}_X \overset{L}{\underset{\mathcal{O}_{T^*X}}{\otimes}} \mathcal{O}_X = \left(\mathrm{Sym}_{\mathcal{O}_X} T_X[1], \iota_{dS_0}\right)$$

Def (PTVV) An n-shifted sympl. structure on X is a non-deg. closed 2-form ω on X of deg n.

Ex. A usual symplectic variety has a 0-shifted sympl. structure

$\exists$ notion of Lagrangian:

X n-shifted sympl. stack

$f : L \to X$

Lagr str. on f is $\cdot$ $f^*\omega \sim 0$ in $A^{2,cl}(L)$

Ex. X n-shifted sympl. $\Leftrightarrow$ $X \to pt_{n+1}$ Lagr. + Lagrangian condition.

Thm (PTVV) X n-shifted sympl.

$L_1, L_2 \to X$ Lagrangians

$L_1 \times_X L_2$ is $(n-1)$-shifted sympl.

Cor. $dcrit(S_0)$ has a canonical (-1)-shifted sympl. str.

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Note: BRST cohomology is a (homotopy) P_0-algebra.

Thm (R.) Let $S_0 \in \mathcal{O}_X$ be a regular function on a smooth variety.

Then:

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bracket

Thm (Costello - R.) X n-shifted sympl. stack. A coisotropic structure on $Y \longrightarrow X$ is the data of $Z(X,Y)$ n-shifted sympl. stack, + $Y \longrightarrow X$ + Lagr. str. on $Y \to X \times Z(X,Y)$

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \\ Z(X,Y) & & \end{array}$$

$Z(X,Y)$ ← coisotropic reduction

$Y \to Z(X,Y)$ is also coisotrop.

coisotropic reduction $= \hat{X}_Y$ ← formal completion of "Y in X"

Thm (Gaitsgory - R.)

Let $f: X \to Y$ be a nil isomorphism.

Then $T_{X/Y}$ is a Lie algebroid on X and $Y \simeq X / T_{X/Y}$; in particular

$$\mathcal{O}_Y = \left(\mathrm{Sym}_{\mathcal{O}_X}(T^\vee_{X/Y}[-1]), d\right)$$

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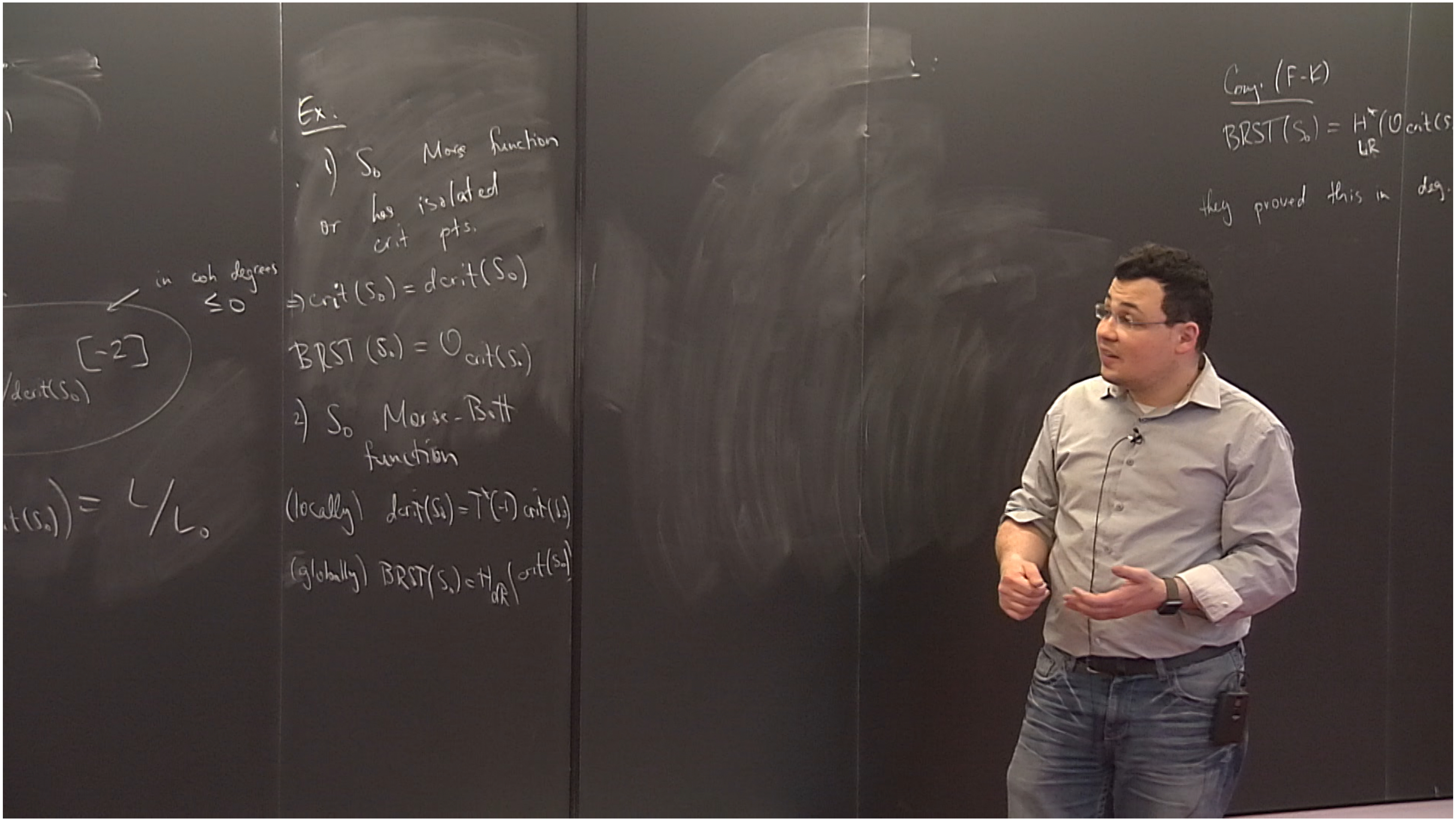
$Z \leftarrow$ coisotropic reduction

$$T_{\mathrm{crit}(S_0)/Z} \simeq \mathbb{L}_{\mathrm{crit}(S_0)/\mathrm{dcrit}(S_0)}[-2]$$

$\leftarrow$ in coh degrees ≤ 0

Obs: $H^0(\mathbb{L}_{\mathrm{crit}(S_0)/\mathrm{dcrit}(S_0)}) = L/L_0$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[-2]
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Ex.
1) S0 Morse function
or has isolated crit pts.
⇒ crit(S0) = dcrit(S0)
BRST(S0) = O crit(S0)
2) S0 Morse-Bott function
(locally) dcrit(S0) = T*(-1) crit(S0)
(globally) BRST(S0) = H dR(crit(S0))
Conj. (F-K)
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they proved this in deg.