Title: Convex Polytopes for the Central Degeneration of the Affine Grassmannian

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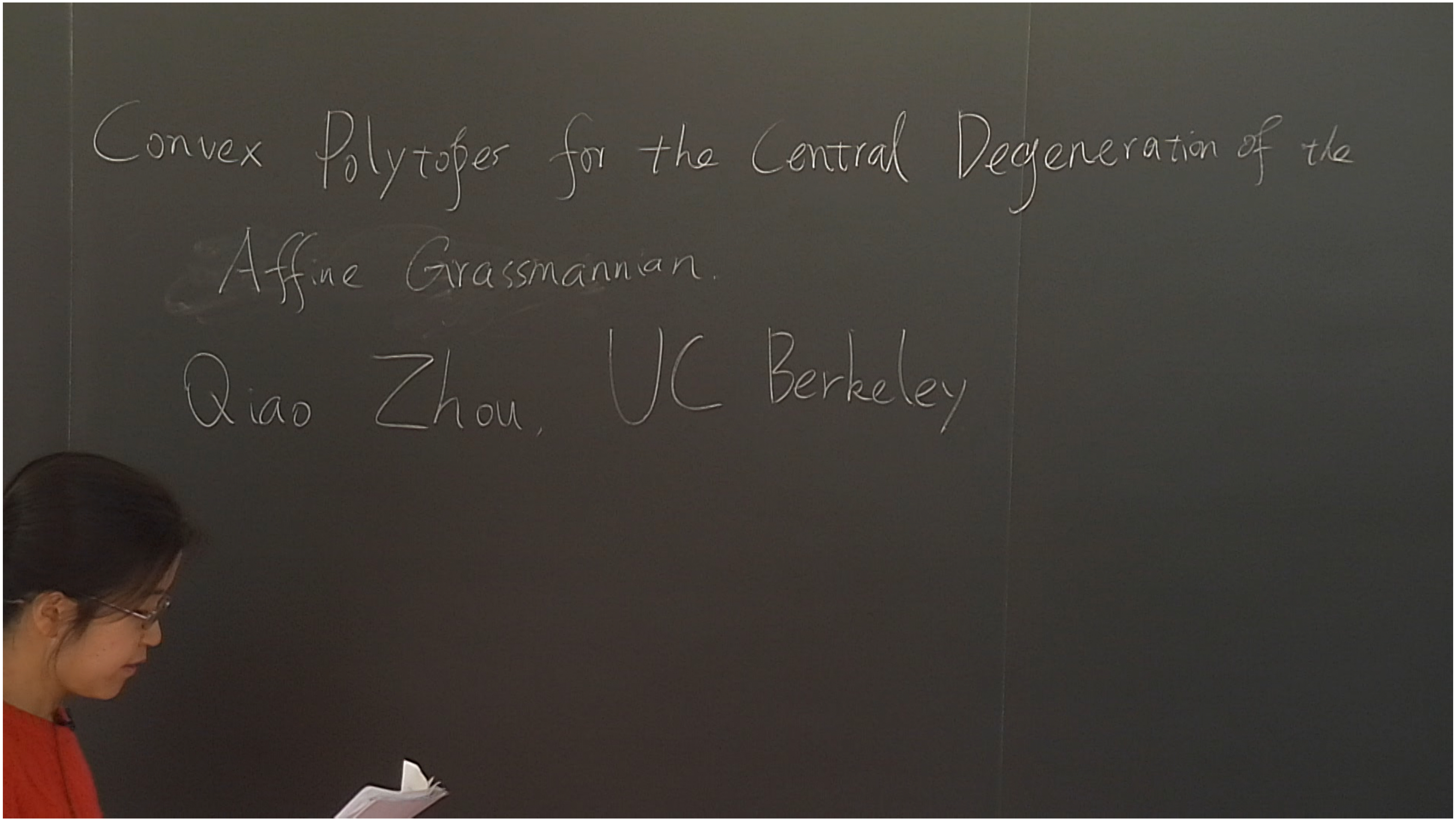
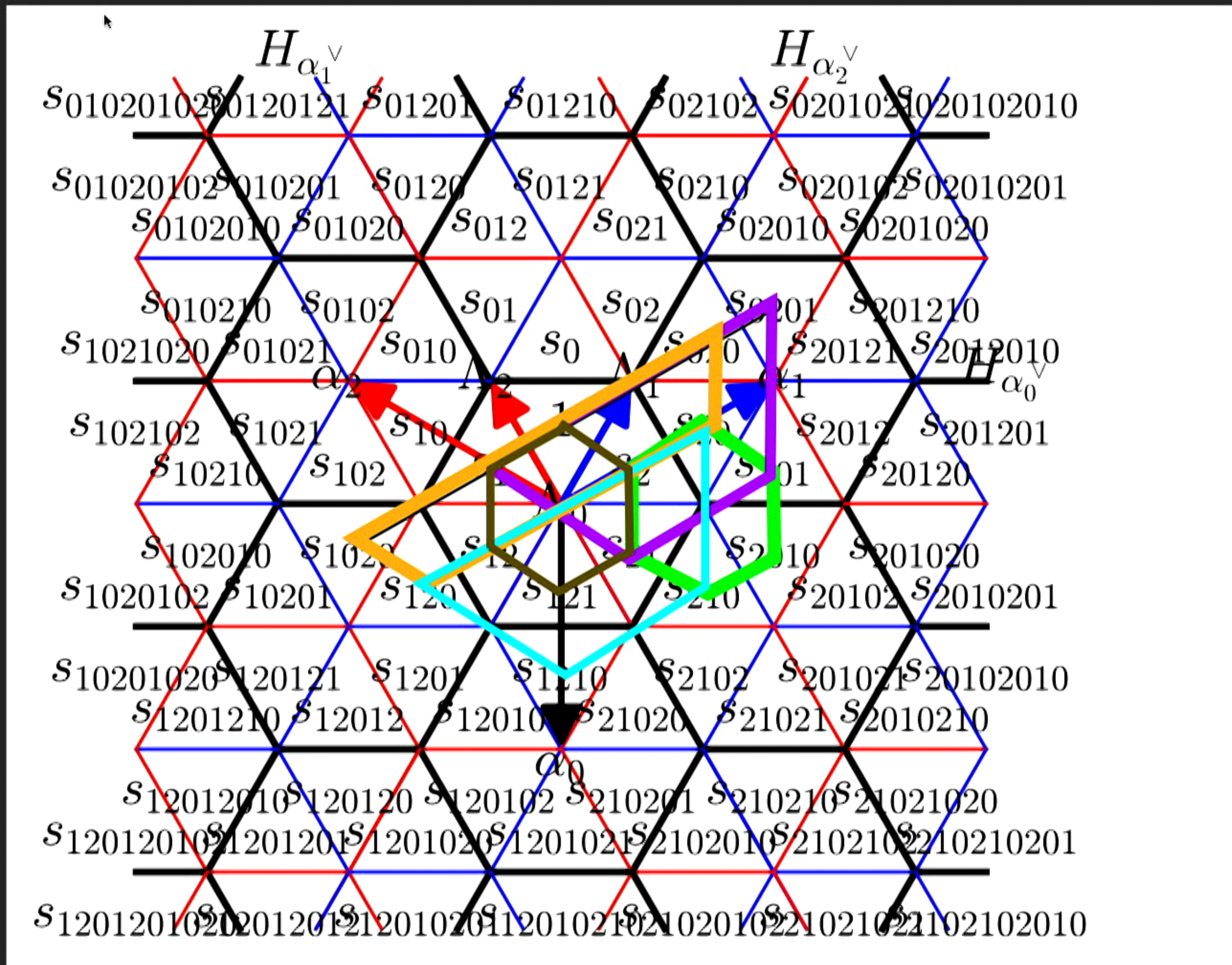
Abstract: <p>The affine Grassmannian is the analog of the Grassmannian for the loop group. They are very important objects in mathematical physics and the Geometric Langlands program. In this talk, I will explain my recent work on the central degeneration of semi-infinite orbits, Iwahori orbits and Mirkovic-Vilonen cycles in the affine Grassmannian. I will also use lots of convex polytopes to illustrate my results. In addition, I will explain the connections between my work and other parts of geometric representation theory and combinatorial algebraic geometry. </p>


Convex Polytopes for the Central Degeneration of the
Affine Grassmannian.
Qiao Zhou, UC Berkeley



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$G \quad (G = GL_n(\mathbb{C}) \text{ or } SL_n(\mathbb{C}))$

$B,\ T,\ N \subset B.\ N_w = wNw^{-1},\ w \in W$, Weyl group

$K = \mathbb{C}((t)),\ \mathcal{O} = \mathbb{C}[[t]]$

$D = \mathrm{spec}(\mathcal{O}),\ D^* = \mathrm{spec}(K)$

$G(K),\ G(\mathcal{O}) \subset G(K)$. Iwahori $I = \{a \in G(\mathcal{O}) \mid a \in ev_0^{-1}(B)\}$

Affine Grassmannian $= G(\mathcal{K})/G(\mathcal{O})$

$Gr = \{ \mathcal{P}_G, \psi \mid \mathcal{P}_G$ is a principal G-bundle on D, ψ is a trivialization of $\mathcal{P}_G$ on $D^* \}$

Affine flag variety $= G(\mathcal{K})/I$

Fl = data for Gr + a B-reduction of $\mathcal{P}_G$ at 0.

$$C = \mathbb{A}^1$$

$$\mathcal{F}l_{\mathbb{A}^1}\big|_{\varepsilon \neq 0} \cong Gr \times G/B$$

$$\mathcal{F}l_{\mathbb{A}^1}\big|_{\varepsilon = 0} \cong \mathcal{F}l$$

$$\mathcal{F}l_{\mathbb{A}^1} = \{(\varepsilon, P_G, \varphi, \phi) \mid \varepsilon \in \mathbb{A}^1,$$

P_G is a Principal G-bundle on C,

$\varphi : P_G|_{C \setminus \{\varepsilon\}} \cong$ trivial bundle

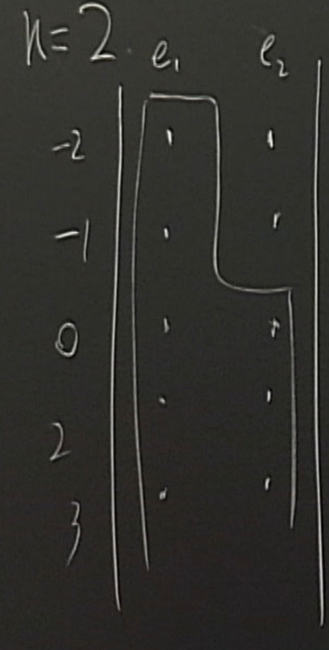
ϕ : B-reduction of P_G at $0 \in \mathbb{A}^1\}$

Type A lattice picture, $G = GL_n(\mathbb{C})$.

$$Fl_{/\!/\mathbb{A}^1} = \{ (\mathcal{L}, f) \mid \varepsilon \in \mathbb{A}^1,$$

$\mathcal{L}$ is a rank n $\mathcal{O}$-module in $V = K^{\oplus n}$

f is a flag in $\mathcal{L}/(t-\varepsilon)\mathcal{L} \cong \mathbb{C}^n \}$



$t^{-2} e_1 \cdot \mathcal{O}$

$\oplus\, e_2 \cdot \mathcal{O}$

is an example of $\mathcal{L}$

$Fl_{A'} = \{ (L, f) \mid \varepsilon \in A',$

L is a rank n $\mathcal{O}$-module in $V = K^{\oplus n}$

f is a flag in $L/(t-\varepsilon)L \cong \mathbb{C}^n \}$

$\bigoplus e_i \mathcal{O}$ is an example of L

Def. The Central Degeneration is the flat degeneration of some T-invariant schemes in the general fiber $Gr \times \{id\}$ to the special fiber Fl.

$A' \}$

- Previous works.

Global group scheme

[Zhu] $G(\mathcal{O})$ orbit Gr^{λ}

$\rightsquigarrow \bigcup_{\substack{\alpha \leq w\cdot\lambda \\ \text{for some } w \in W}}$ Iwahori orbits I^{α}

Gaitsgory: nearby cycles functor $\Phi: Perv_{G(\mathcal{O})}(Gr) \longrightarrow \underset{\text{center}}{Z}(Perv_{I}(Fl))$

· Previous works. Degeneration of $G(\mathcal{O})$ orbits

Global group scheme

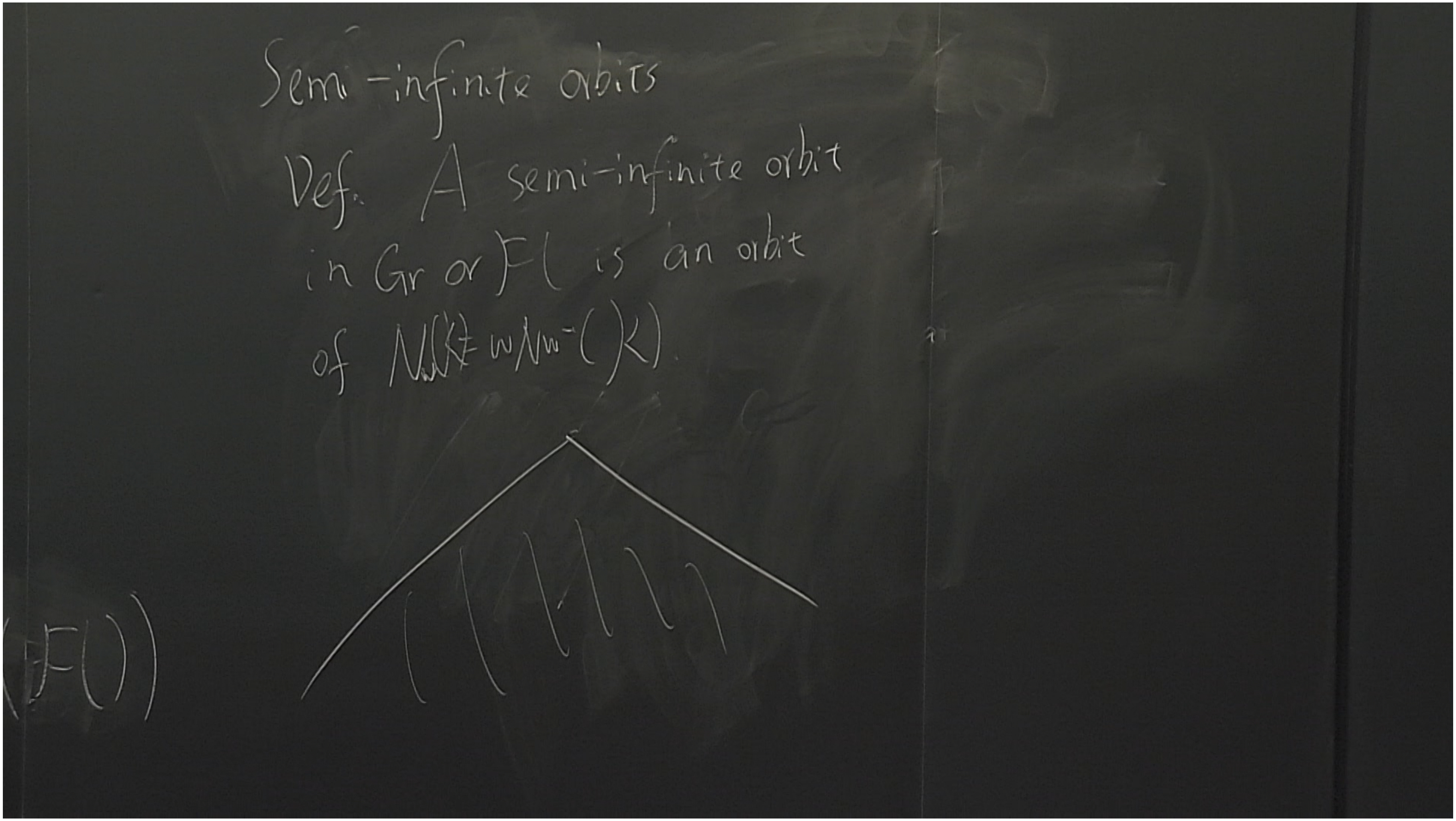
[... hu]

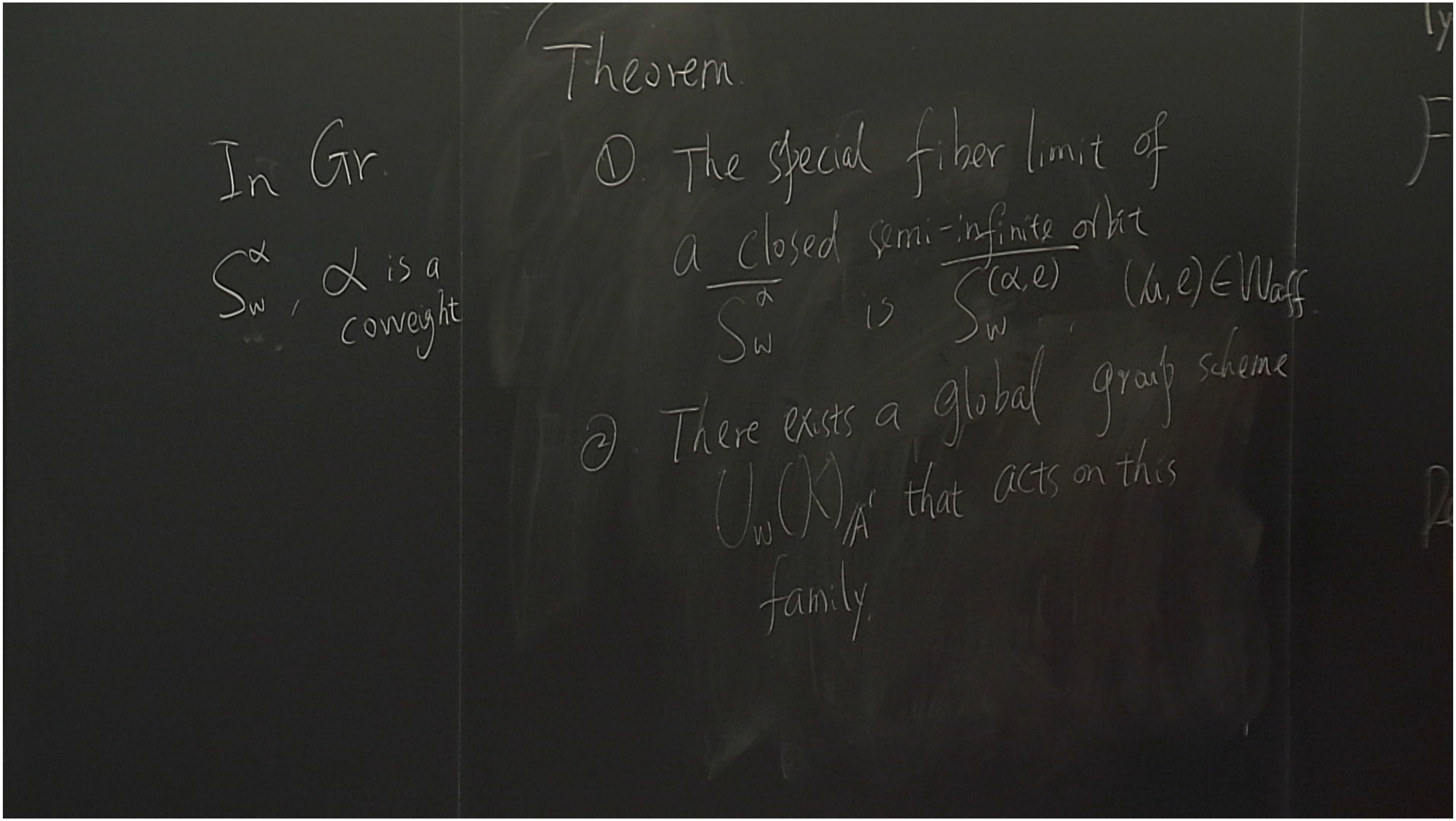
$G(\mathcal{O})$ orbit Gr^{λ}

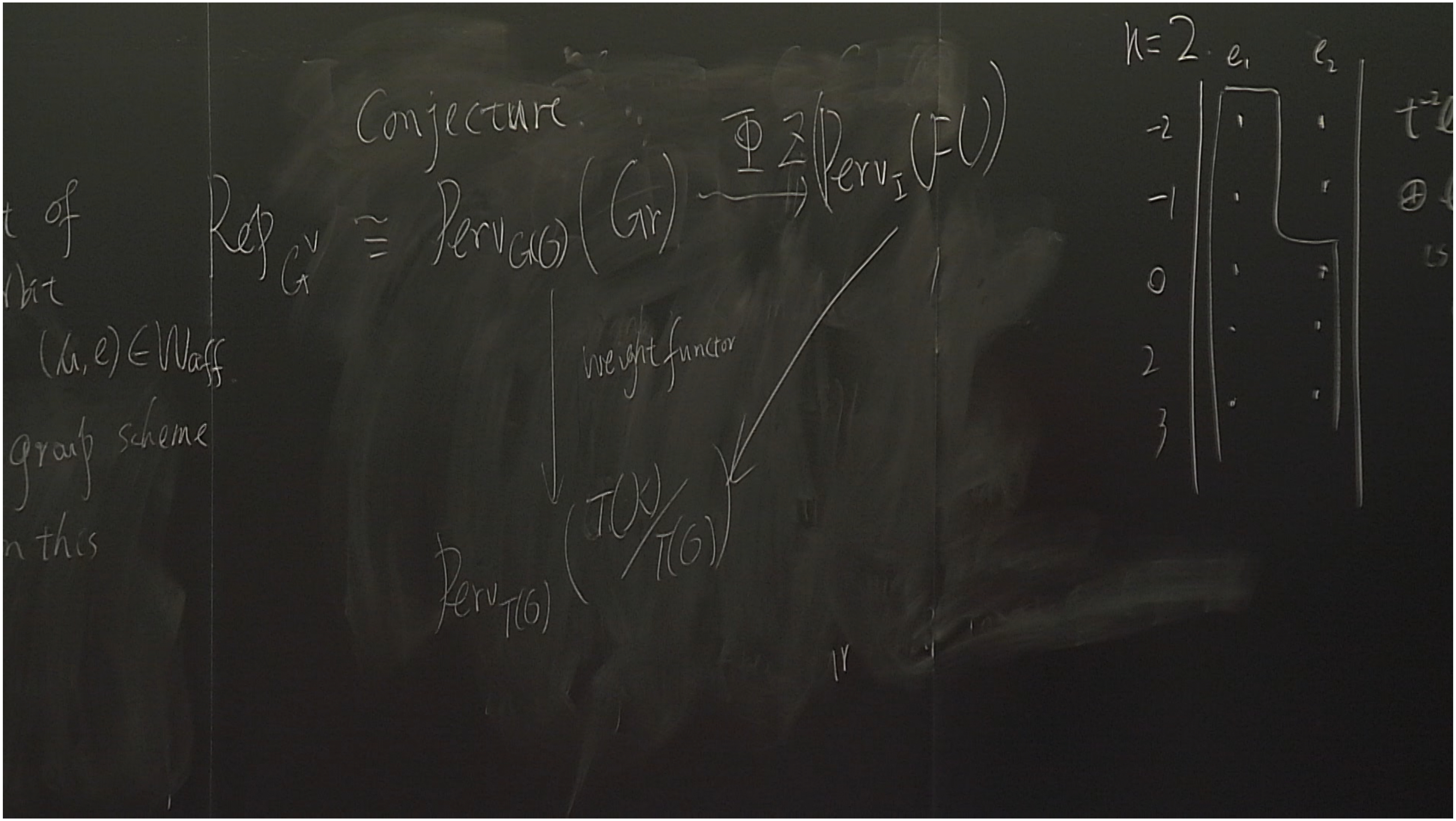
$\rightsquigarrow \bigcup_{\substack{\alpha \leq w\cdot\lambda \\ \text{for some } w\in W}}$ Iwahori orbits I^{α}

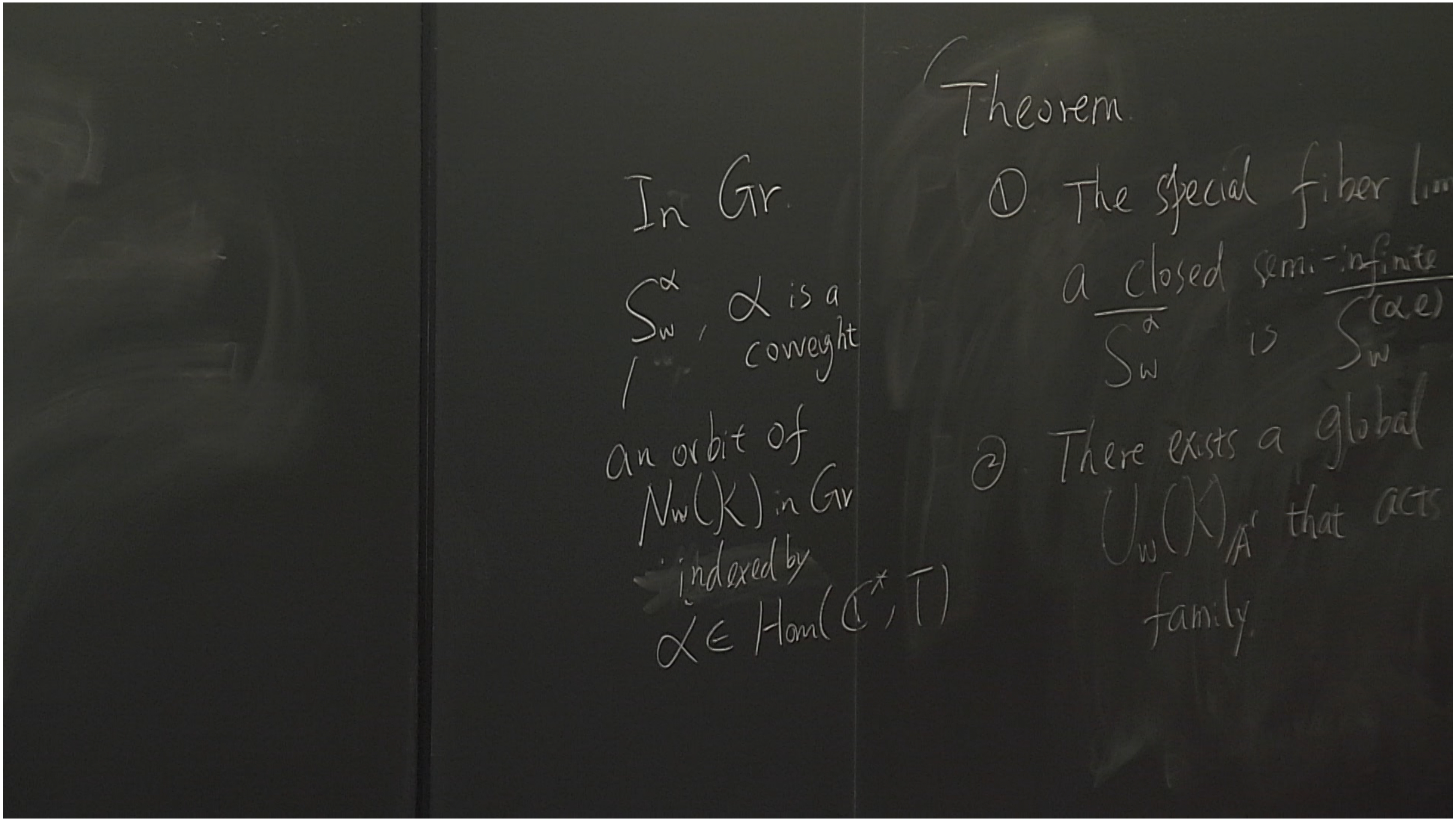
center

Gaitsgory: nearby cycles functor $\Phi : Perv_{G(\mathcal{O})}(Gr) \to Z(Perv_I(\mathcal{F}\ell))$


Semi-infinite orbits
Def. A semi-infinite orbit
in Gr or Fl is an orbit


In Gr.
S_w^α, α is a coweight
Theorem.
① The special fiber limit of
a closed semi-infinite orbit
$\overline{S_w^\alpha}$ is $S_w^{(\alpha,e)}$, $(\alpha,e)\in W_{aff}$.
② There exists a global group scheme
$U_w(K)_{/\mathbb{A}^1}$ that acts on this
family.


Conjecture
Rep G^∨ ≅ Perv_{G(O)}(Gr) —Φ→ Z(Perv_I(Fl))
weight functor
Perv_{T(O)}(T(K)/T(O))
(w, e) ∈ W_aff
group scheme
n = 2
e_1 e_2
-2
-1
0
2
3


Theorem.
In Gr.
S_w^α, α is a coweight
an orbit of $N_w(K)$ in Gr
indexed by
$\alpha \in Hom(\mathbb{C}^*, T)$
① The special fiber lim
a closed semi-infinite
$\overline{S_w^\alpha}$ is $S_w^{(\alpha e)}$
② There exists a global
$U_w(K)_{\mathbb{A}^1}$ that acts
family.

Def. An Mirković–Vilonen (MV) cycle S is an irreducible component in the intersection of a $G(\mathcal{O})$ orbit Gr_λ $(\lambda \in X_*^+(T))$ and an orbit of $N^-(K)$ S^ν, $\nu \in X_*(T)$

Their T-equivariant moment polytopes are called MV polytope

[Kamnitzer] each MV cycle $S = \bigcap_{w \in W} S_w^{\mu_w}$ for $\mu_w \in X_*(T)$

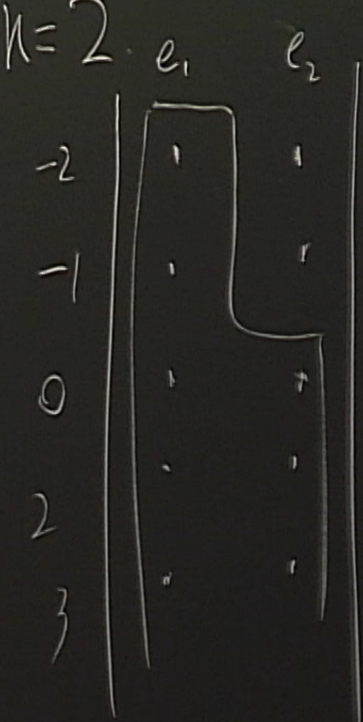
Def. A generalized MV cycle in the affine flag variety is an irreducible component of an Iwahori orbit and an orbit of $N^-(K)$

Theorem. The special fiber limit of an MV cycle is contained in (and agree with in some special cases) a union of certain generalized MV cycles.

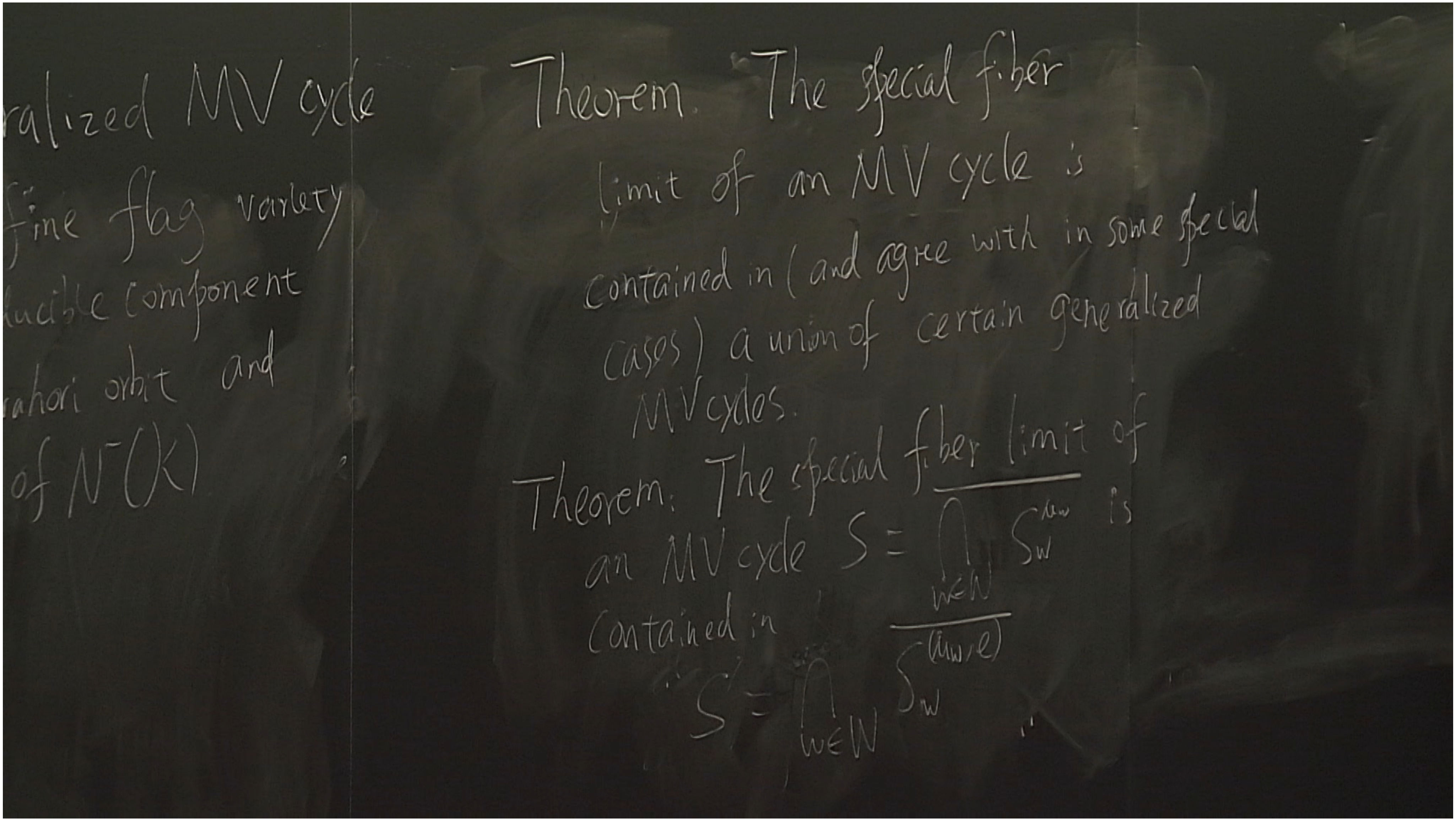
Theorem: The special fiber limit of an MV cycle $S = \overline{\bigcap_{w \in W} S_w^{\lambda_w}}$ is contained in

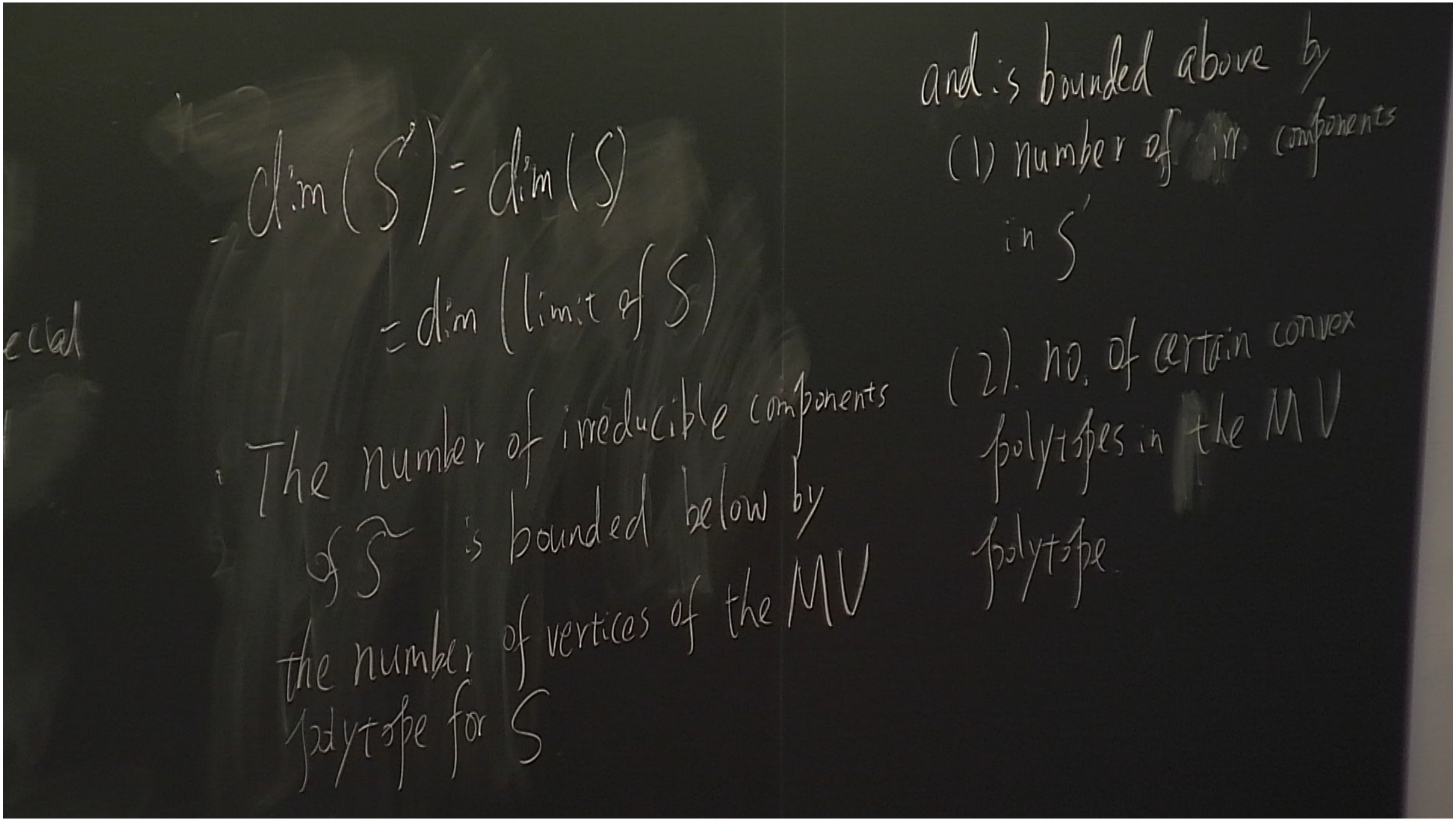
$$S' = \bigcap_{w \in W} \overline{S_w^{(\lambda_w, e)}}$$

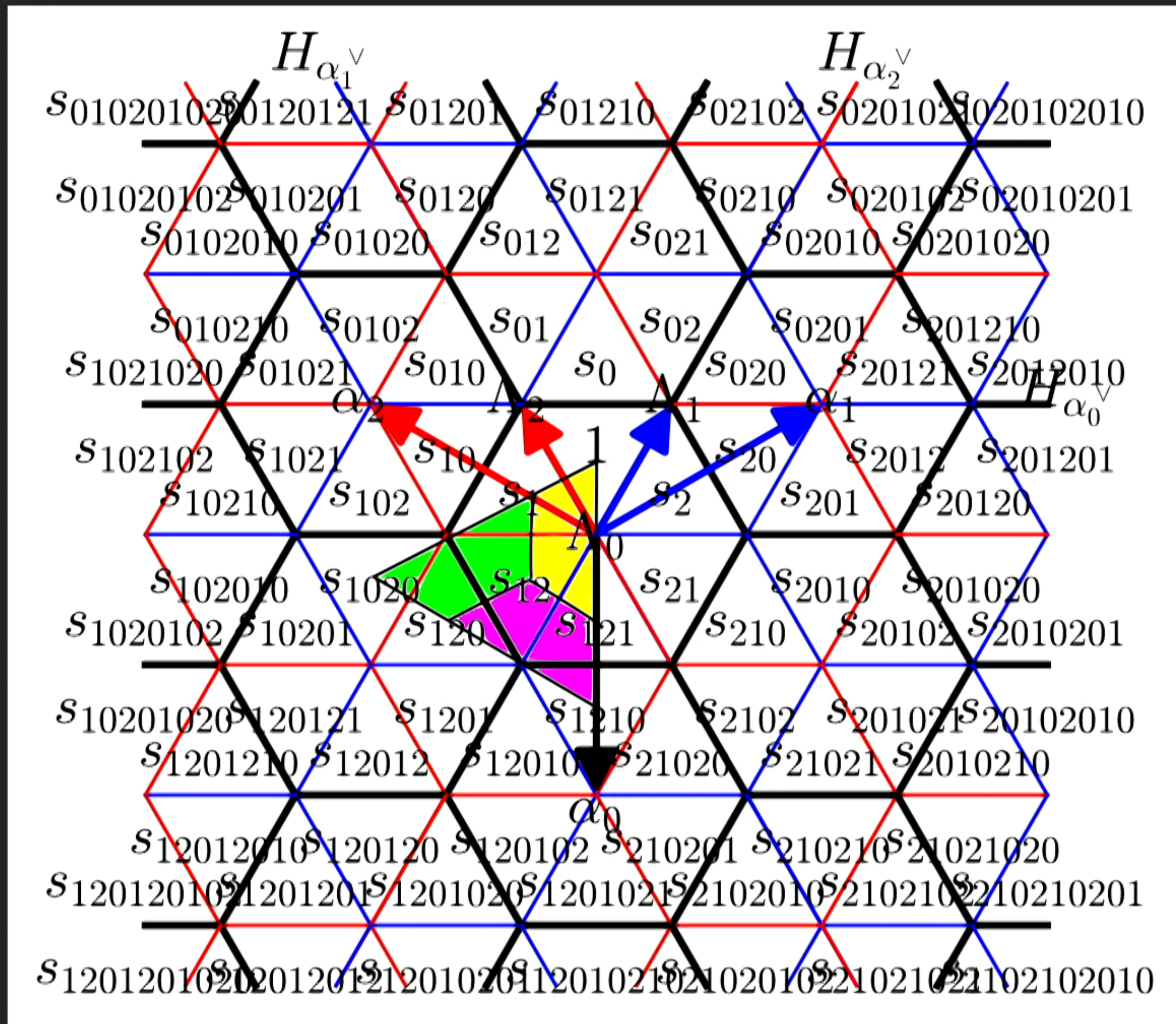
$n = 2$.



$t^{-2} e_1 \cdot \mathcal{O} \oplus e_2 \cdot \mathcal{O}$ is an exa[mple] of L


ralized MV cycle
fine flag variety
lucible component
ahori orbit and
of N⁻(K)
Theorem. The special fiber
limit of an MV cycle is
contained in (and agree with in some special
cases) a union of certain generalized
MV cycles.
Theorem: The special fiber limit of
an MV cycle S = ∩_{w∈W} S_w^{λ_w} is
contained in
S' = ∩_{w∈W} S_w^{(λ_w, e)}


dim(S') = dim(S)
= dim (limit of S)
The number of irreducible components of $\tilde{S}$ is bounded below by
the number of vertices of the MV polytope for S
and is bounded above by
(1) number of irr. components in S'
(2). no. of certain convex polytopes in the MV polytope.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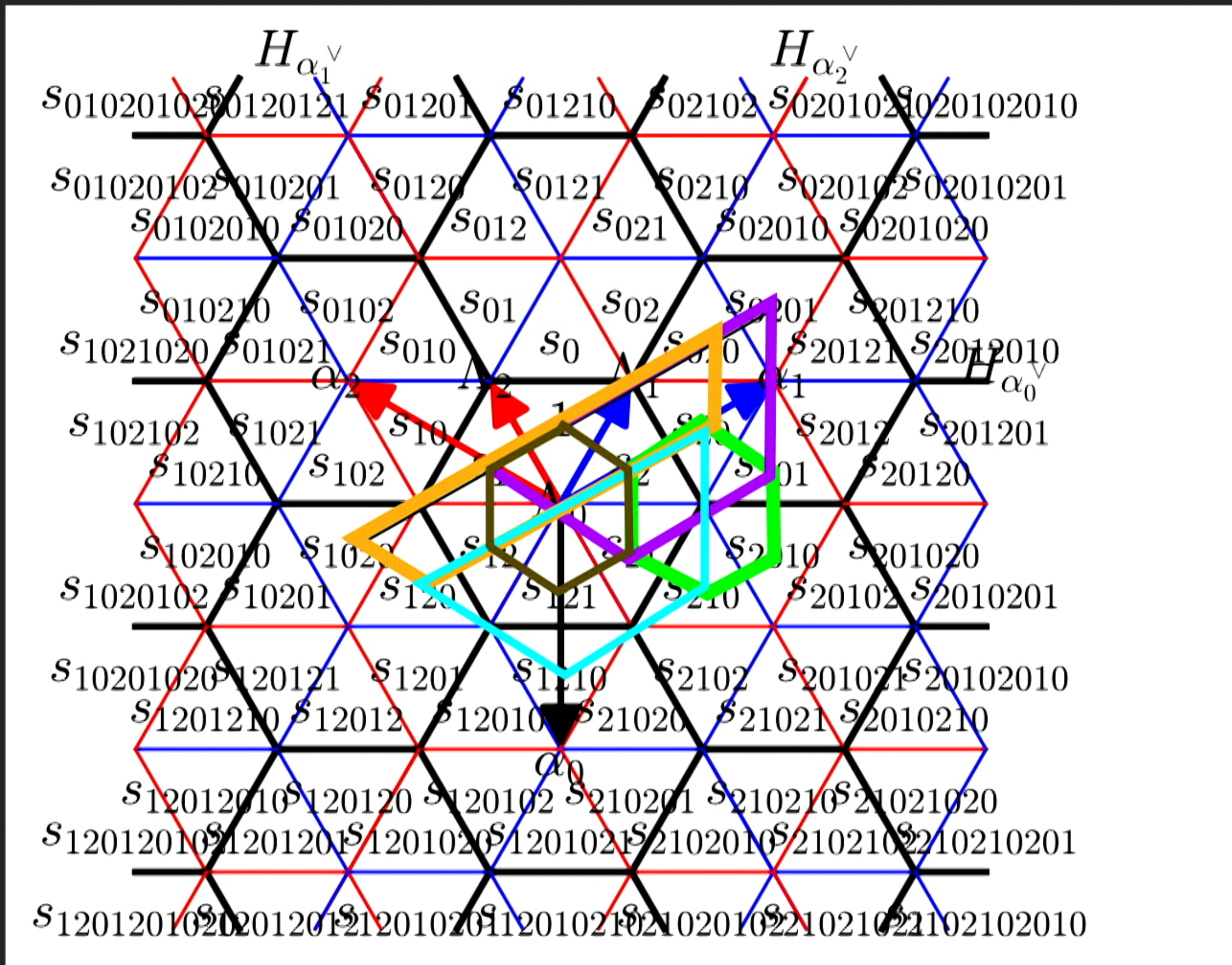
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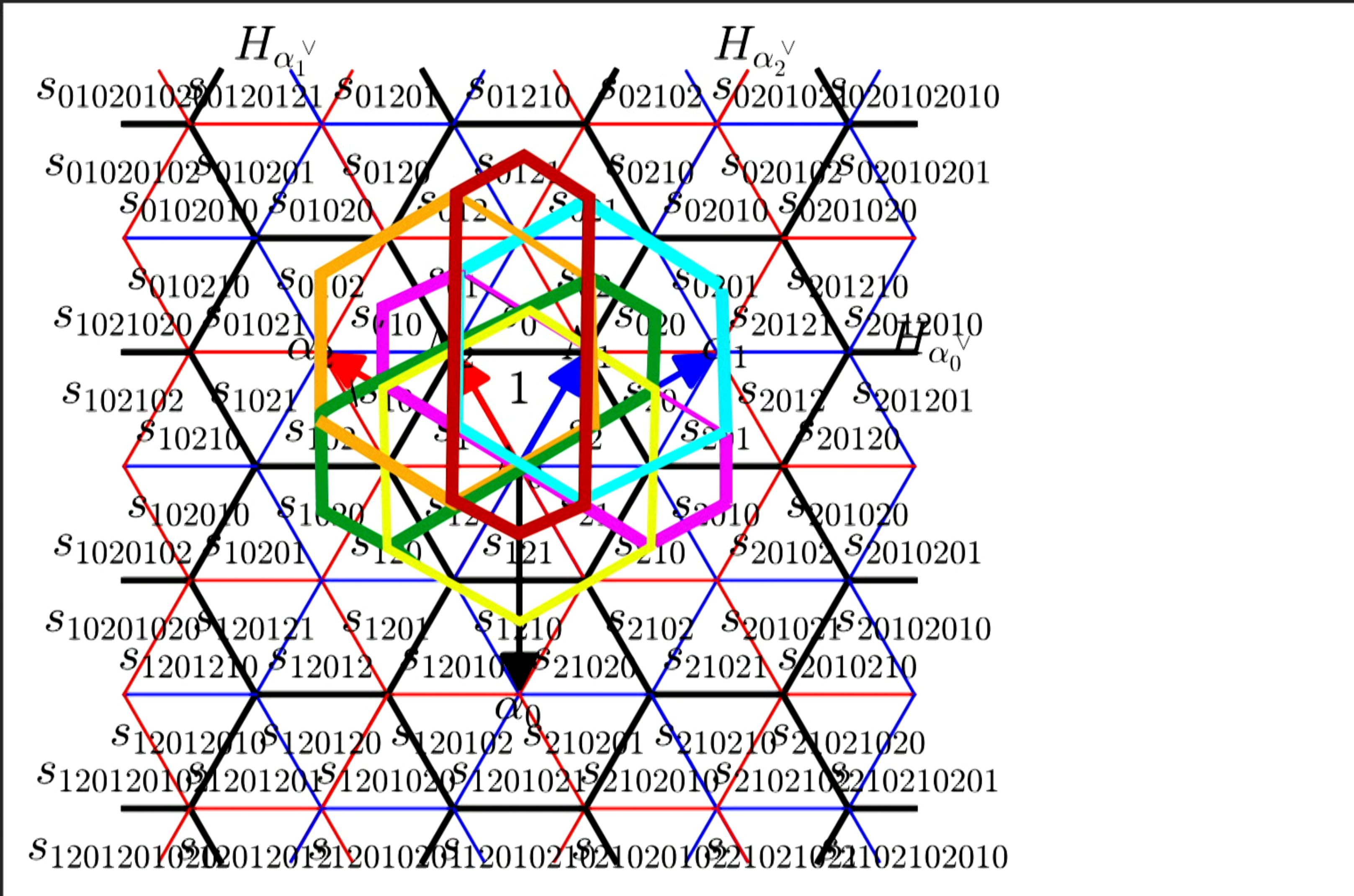
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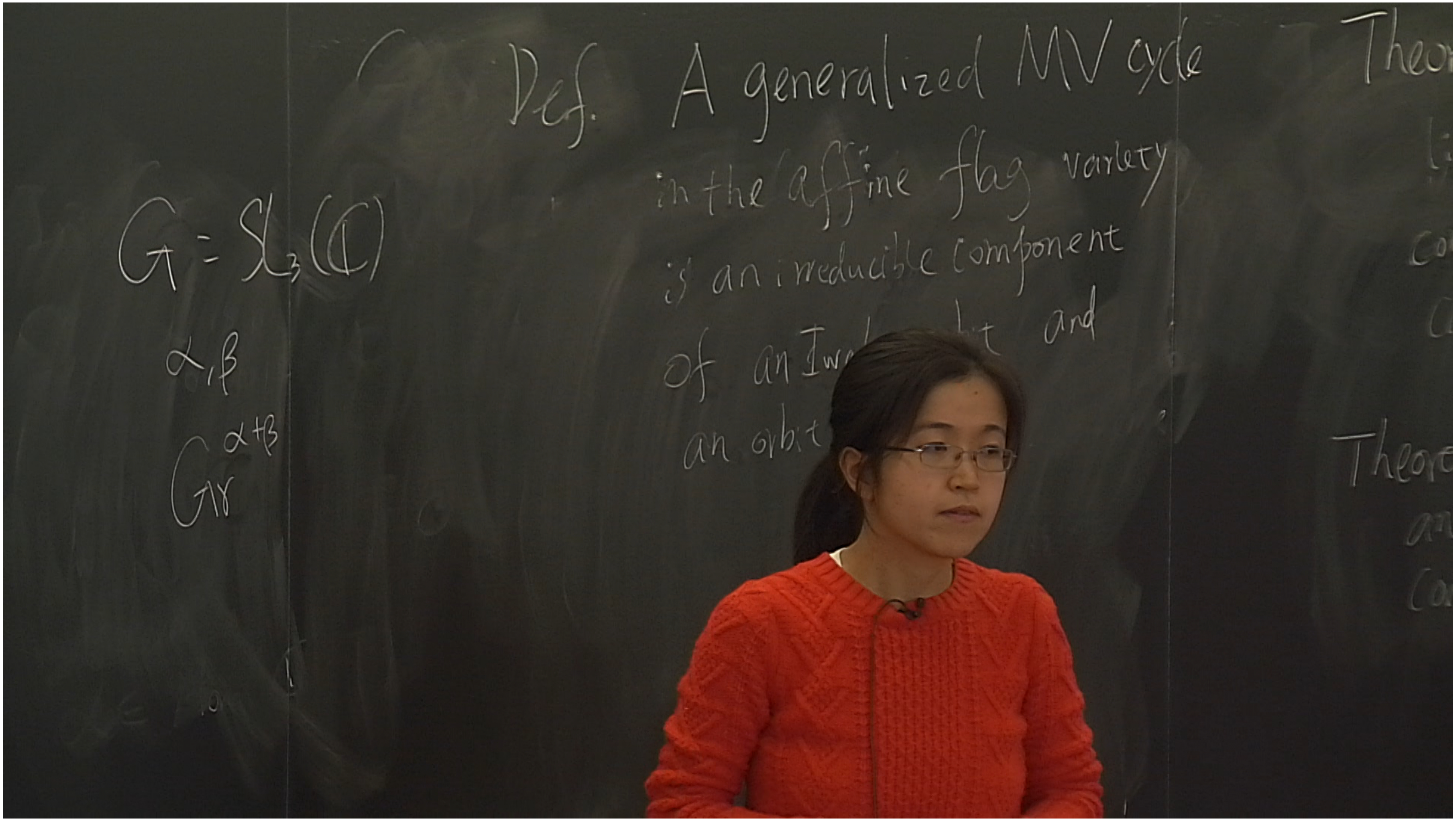
Def. A generalized MV cycle
in the affine flag variety
is an irreducible component
of an Iw
and
an orbit
G = SL3(C)
α, β
Theor
Theor