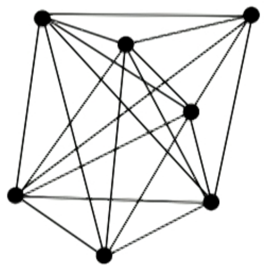
Title: Quantum Fields and Strings Seminar

Date: Dec 05, 2016 10:00 AM

URL: http://pirsa.org/16120019

Abstract:

The SYK model



$$H = \frac{(i)^{q/2}}{q!} \sum_{i_1,\dots i_q} J_{i_1\dots i_q} \chi_{i_1} \cdots \chi_{i_q}$$

$$\overline{J_{i_1\dots i_q}} = 0, \quad \overline{J^2_{i_1\dots i_q}} = (q-1)! \frac{J^2}{N^{q-1}}$$

N Majorana fermions χ_i, $i = 1, \dots N$

- Sachdev, Ye (1993): Fermions in complex representation of $SU(M)$ with two-site interaction.
- Kitaev (2015):
  - Single large N parameter, identical Green function.
  - Suppressed disorder (replica-off-diagonal terms).
  - Black hole behaviour of out-of-time-order four-point function.
- Maldacena, Stanford (2016): Detailed calculations.

Solvable limit

- Restriction to replica-diagonal sector

$$-\beta\overline{F} = \overline{\ln Z} = \lim_{R\to 0}\frac{\overline{Z^R}}{R} \approx \lim_{R\to 0}\frac{e^{-R\beta F_{\rm diag}}}{R} = -\beta F_{diag}$$

$$\overline{Z^R} = \int \mathcal{D}\chi \exp\left(-\frac{1}{2}\sum_{\alpha=1}^{R}\int d\tau\, \chi_i^\alpha \partial_\tau \chi_i^\alpha + \frac{NJ^2}{2q}\sum_{\alpha,\beta}\int d\tau d\tau' \left(\frac{1}{N}\chi_i^\alpha(\tau)\chi_i^\beta(\tau')\right)^q\right)$$

$$G_{\alpha\beta}(\tau,\tau') = -\sum_i \chi_i^\alpha(\tau)\chi_i^\beta(\tau')$$

$$= \int \mathcal{D}\Sigma\mathcal{D}G \exp\left(\frac{N}{2}\sum_{\alpha,\beta}\int d\tau d\tau' \left(\frac{J^2}{q}G_{\alpha\beta}^q - \Sigma_{\alpha\beta}G_{\alpha\beta}\right)\right)\left(\int \mathcal{D}\chi \exp\left(-\frac{1}{2}\sum_\alpha \int d\tau\, \chi^\alpha\partial_\tau\chi^\alpha - \frac{1}{2}\sum_{\alpha\beta}\int d\tau d\tau'\, \Sigma_{\alpha\beta}\chi^\alpha\chi^\beta\right)\right)^N$$

$$\approx \underbrace{\left(\int \mathcal{D}\Sigma\mathcal{D}G \exp\left(N\left(\ln \mathrm{Pf}\,[\delta'(\tau-\tau') + \Sigma(\tau,\tau')] + \frac{1}{2}\int d\tau d\tau' \left(\frac{J^2}{q}G(\tau,\tau')^q - \Sigma(\tau,\tau')G(\tau,\tau')\right)\right)\right)\right)^R}_{Z_{\rm diag} = e^{-\beta F_{\rm diag}}}$$

$$\Sigma_{\alpha\beta} = \Sigma\delta_{\alpha,\beta}$$
$$G_{\alpha\beta} = G\delta_{\alpha,\beta}$$

$$-\int d\tau'\, G(\tau,\tau')\left(\Sigma(\tau',\tau'') + \delta'(\tau',\tau'')\right) = \delta(\tau-\tau'')$$
$$\Sigma(\tau,\tau') = J^2 G(\tau,\tau')^{q-1}$$

- Complete analytic solution for $N \gg \beta J \gg 1$

small quantum fluctuations ($N \gg \beta J$)

analytic solution to saddle-point equations ($\beta J \gg 1$)

Outline

- $\mathrm{Diff}(S^1)$, $\mathrm{PSL}(2,\mathbb{R})$ symmetries in the IR
- Decomposition of fluctuations into irreducible unitary representations of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$
- Conformal time φ;
  the non-linear soft mode $\varepsilon(\varphi)$ and its action
- RG analysis and resulting action for soft mode
- Corresponding dilaton theory

Preview of results

- $O((\beta J)^{-2})$ terms in the action of SYK, written for the soft mode $\varepsilon(\varphi) \sim (\beta J)^{-1}$ and resulting from integrating out least irrelevant perturbation.

$$\frac{S_{\text{soft}}}{N} = -\tilde{\alpha} \int \frac{d\varphi_1}{2\pi} \frac{d\varphi_2}{2\pi} \frac{\varepsilon(\varphi_1)\varepsilon(\varphi_2)}{\varphi_{12}^4} \ln \left(\frac{\varphi_{12}^2}{\varepsilon(\varphi_1)\varepsilon(\varphi_2)} \right) -$$

non-local, dominant in the IR $\varphi_{12} = 2\sin\frac{\varphi_1 - \varphi_2}{2}, \varphi_{12} \gg (\beta J)^{-1}$

$$\frac{\tilde{\alpha}}{4\pi} \int \frac{d\varphi}{2\pi} \underbrace{\left(\varepsilon(\varphi) + 2\varepsilon''(\varphi) - \frac{\varepsilon'(\varphi)^2}{2\varepsilon(\varphi)} \right)}_{S_f(\tau),\ f = e^{i\varphi(\tau)}} + \dots$$

UV regularization of the non-local term

- Dual action in $D = 2$ dilaton gravity

$$S_\phi = -\frac{N}{4\pi} \left(\int d^2z\sqrt{g}\ \left(\phi(R+2) + \tilde{\alpha}\phi^2\right) + 2\int d\varphi \sqrt{g_{\varphi\varphi}} K + \dots \right)$$

(linear term discussed in Maldacena, Stanford, Yang (2016))

Emergent conformal symmetry

For $\beta J \gg 1$

$$-\int d\tau' \, G(\tau,\tau') \left(\Sigma(\tau',\tau'') + \delta'(\tau',\tau'')\right) = \delta(\tau - \tau'')$$

$$\Sigma(\tau,\tau') = J^2 G(\tau,\tau')^{q-1}$$

- Saddle-point equations are $\mathrm{Diff}(S^1)$ -invariant.

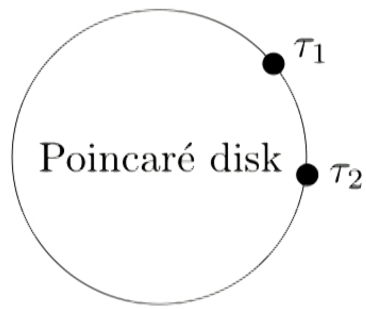
$$\tau \to f(\tau)$$

$$G(\tau,\tau') = (f'(\tau))^{1/q} (f'(\tau'))^{1/q} G(f(\tau), f(\tau'))$$

$$\Sigma(\tau,\tau') = (f'(\tau))^{1-1/q} (f'(\tau'))^{1-1/q} \Sigma(f(\tau), f(\tau'))$$

- Solution is $\mathrm{PSL}(2,\mathbb{R})$ -invariant.

$$G(\tau_1,\tau_2) = -\frac{b^{1/q} \operatorname{sgn}\left(\sin\left(\frac{\pi(\tau_1-\tau_2)}{\beta}\right)\right)}{\left|\frac{\beta J}{\pi} \sin\left(\frac{\pi(\tau_1-\tau_2)}{\beta}\right)\right|^{2/q}}$$



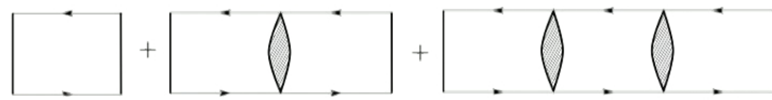
$$\left(z = \sqrt{x} e^{i\varphi}, \; \varphi = \frac{2\pi}{\beta}\tau\right)$$

$$ds^2 = \frac{4}{(1-z\bar{z})^2} dz d\bar{z}, \quad |z| < 1$$

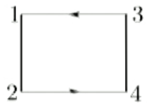
Decomposition of fluctuations (continued)

- Quadratic action for fluctuations about IR fixed point + UV perturbation

$$N^{-1}S_2[g] = \frac{1}{2}\langle g|K^{-1}-1|g\rangle - \langle s|g\rangle$$

$$\langle gg\rangle = \frac{K}{1-K} =$$  $+\dots$

$$K(\tau_1,\tau_2;\tau_3,\tau_4) = J^2(q-1)\,|G(\tau_1,\tau_2)|^{\frac{q-2}{2}}\,G(\tau_1,\tau_3)G(\tau_4,\tau_2)\,|G(\tau_3,\tau_4)|^{\frac{q-2}{2}}$$

$=$ 

- K commutes with the Casimir of the Lie algebra $\mathrm{SL}(2,\mathbb{R})$ acting on $\mathcal{F}^{\nu=1/2}_{\lambda=1/2}\otimes_A\mathcal{F}^{\nu=1/2}_{\lambda=1/2}$. The identity over the product space decomposes into irreducible unitary representations of the universal cover $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ labeled by Casimir eigenvalues λ. Each representation consists of a series labeled by eigenvalues of L_0, k.

$$\mathbb{1} = \underbrace{\int_{-\infty}^{\infty} ds\left(\sum_{k\in\mathbb{Z}}\frac{\left|g_{\frac{1}{2}+is,k}\right\rangle\left\langle g_{\frac{1}{2}+is,k}\right|}{\left|g_{\frac{1}{2}+is,k}\right|^2}\right)}_{\text{principal series}} + \sum_{\lambda=2,4,\dots}\left(\underbrace{\sum_{k=\lambda+n,n\in\mathbb{N}}\frac{|g_{\lambda,k}\rangle\langle g_{\lambda,k}|}{|g_{\lambda,k}|^2}}_{\text{positive discrete series}} + \underbrace{\sum_{k=-\lambda-n,n\in\mathbb{N}}\frac{|g_{\lambda,k}\rangle\langle g_{\lambda,k}|}{|g_{\lambda,k}|^2}}_{\text{negative discrete series}}\right)$$

8

Rotation from $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ reps to set of poles

- Discrete series with $\lambda = 2$ have K eigenvalue 1, ie. zero action in the IR, and must be treated non-linearly. I'll describe our formalism for treating the non-linear excitations (soft modes) in the next section.

- The contribution of $\lambda \neq 2$ modes to the four-point function in the IR limit can be written using projectors for $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ reps.

$$\langle g(\tau_1,\tau_2)g(\tau_3,\tau_4)\rangle_{\lambda\neq 2} = \frac{K}{1-K}\left(\int ds\,\Pi_{C^0_{\frac{1}{2}+is}} + \sum_{\lambda=4,6,\dots}\left(\Pi_{D^+_\lambda} + \Pi_{D^-_\lambda}\right)\right)$$

- The contribution can be alternatively decomposed to that over a set of poles by analytic continuation in λ-space (Maldacena, Stanford 2016).

$$\langle g(\hat\varphi_1,\hat\varphi_2)g(\hat\varphi_3,\hat\varphi_4)\rangle_{\lambda\neq 2} \sim \frac{\mathrm{sgn}\left(\sin\hat\varphi_-\sin\hat\varphi'_-\right)}{\sin\hat\varphi_-\sin\hat\varphi'_-}\sum_{I=0,1,\dots}\mathrm{Res}\left[\frac{h-\frac{1}{2}}{\tan(\pi h/2)}\frac{K(h)}{1-K(h)}\chi^h\frac{\Gamma(h)^2}{\Gamma(2h)}F(h,h,2h,\chi)\right]_{h=h_I}$$

$$\hat\varphi = \tau\frac{2\pi}{\beta}\,,\quad \hat\varphi_\pm = \frac{\hat\varphi_1\pm\hat\varphi_2}{2} \qquad K(h_I)=1,$$

double pole at $h_0 = 2$

9

Outline

- $\mathrm{Diff}(S^1)$, $\mathrm{PSL}(2,\mathbb{R})$ symmetries in the IR ✓
- Decomposition of fluctuations into irreps of $\mathrm{PSL}(2,\mathbb{R})$ ✓
- Conformal time $\varphi(\tau)$;
  the non-linear soft mode $\varepsilon(\varphi)$ and its action
- RG analysis and resulting action for soft mode
- Corresponding dilaton theory

The soft mode

- The discrete series with $\lambda = 2,\ K = 1$ correspond to variations in $G(\tau_1, \tau_2)$ due to infinitesimal reparametrizations of time, $\tau \to \tau + \delta\tau$.

- In fact, $\mathrm{Diff}(S^1)$-invariance at the IR fixed point implies a submanifold of near-extremal configurations in (Σ, G)-space parametrized by non-linear functions $f(\tau) = e^{i\varphi(\tau)}$

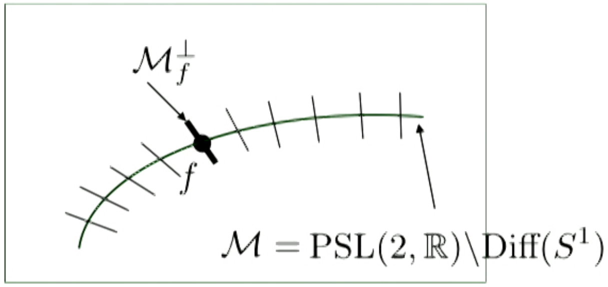
$$G_f(\tau_1, \tau_2) = f'(\tau_1)^\Delta f'(\tau_2)^\Delta G_0(f(\tau_1, \tau_2))$$
$$\Sigma_f(\tau_1, \tau_2) = f'(\tau_1)^{1-\Delta} f'(\tau_2)^{1-\Delta} \Sigma_0(f(\tau_1, \tau_2))$$

$G_0(\tau_1, \tau_2)$: Saddle-point solution at zero temperature.

$\varphi(\tau)$: Conformal time.

- The maps $f,\ L \circ f$ where $L \in \mathrm{PSL}(2, \mathbb{R})$ define the same Green function. Thus, the submanifold in question is $\mathcal{M} = \mathrm{PSL}(2, \mathbb{R}) / \mathrm{Diff}(S^1)$.

An effective action for the soft mode



- Having identified the non-linear degree of freedom f, or the soft mode, it is natural to integrate over the linear modes $g_{\lambda\neq 2}$ at fixed f to obtain an effective action for f.
- For each point on the manifold, we define a perpendicular direction $\mathcal{M}_f^{\perp}$ and integrate over linear modes in it using the saddle-point approximation.

$$S_{\mathrm{soft}}(f) = \mathrm{extremum}\left\{\beta F(\Sigma, G) : (\Sigma, G) \in \mathcal{M}_f^{\perp}\right\}$$

- By construction, S_{soft} is $\mathrm{PSL}(2,\mathbb{R})$-invariant. It is also natural to demand that it be diffeomorphism-covariant, i.e. covariant as we traverse $\mathcal{M}$.

Soft mode as covariant UV cutoff

- Instead of f, we often use

$$\boxed{\varepsilon(\varphi) = J^{-1}\frac{d\varphi(\tau)}{d\tau}} \sim \frac{1}{\beta J}$$

- ε can be thought of the UV cutoff J^{-1} transformed to conformal time.
- Later we will see that J^{-1} should in fact transform with conformal dimension -1 in order for S_{soft} to be diffeormorphism-covariant.
- Note there is a piece in ε that is non-zero right at the saddle-point, i.e. purely due to finite temperature and distinct from $\mathrm{PSL}(2,\mathbb{R})/\mathrm{Diff}(S^1)$ degrees of freedom.

$$\hat{\varphi} = \frac{2\pi\tau}{\beta}, \quad \hat{\varepsilon} = \frac{2\pi}{\beta J}$$

13

Outline

- $\mathrm{Diff}(S^1)$, $\mathrm{PSL}(2,\mathbb{R})$ symmetries in the IR ✓
- Decomposition of fluctuations into irreps of $\mathrm{PSL}(2,\mathbb{R})$ ✓
- Conformal time $\varphi(\tau)$;
  the non-linear soft mode $\varepsilon(\varphi)$ and its action ✓
- RG analysis and resulting action for soft mode
- Corresponding dilaton theory

Renormalization group analysis

- Recall UV perturbation to IR fixed point

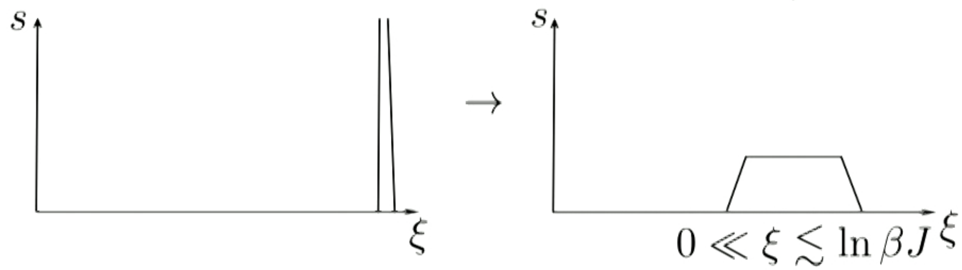
$$N^{-1}S = -\ln \mathrm{Pf}\,[\delta'(\tau-\tau') + \Sigma(\tau,\tau')] - \frac{1}{2}\int d\tau d\tau' \left(\frac{J^2}{q}G(\tau,\tau')^q - \Sigma(\tau,\tau')G(\tau,\tau')\right)$$

$$= -\ln \mathrm{Pf}\left[\tilde{\Sigma}(\tau,\tau')\right] - \frac{1}{2}\int d\tau d\tau' \left(\frac{J^2}{q}G(\tau,\tau')^q - \tilde{\Sigma}(\tau,\tau')G(\tau,\tau')\right) + \frac{1}{2}\int d\tau d\tau'\,\sigma(\tau,\tau')G(\tau,\tau')$$

$$\tilde{\Sigma} = \Sigma + \sigma, \;\; \sigma = \delta'(\tau-\tau')$$

- Idea: replace singular, intractable perturbation with smooth perturbations supported in the UV region of the RG parameter

$$\xi(|\sin\hat{\varphi}_-|) = -\ln|\sin\hat{\varphi}_-| \qquad \left(\hat{\varphi}_\pm = \frac{1}{2}(\hat{\varphi}\pm\hat{\varphi}')\right)$$



such that their effect in the IR is identical to the original perturbation.

Ansatz for UV perturbations

- Let us consider the quadratic action for fluctuations about the IR fixed point with some smooth UV perturbation s in $\mathcal{F}^{\nu=1/2}_{\lambda=1/2} \otimes_A \mathcal{F}^{\nu=1/2}_{\lambda=1/2}$

$$N^{-1} S_2[g] = \frac{1}{2} \langle g | K^{-1} - 1 | g \rangle - \langle s | g \rangle$$

- Equation of motion determining IR response in $\lambda \neq 2$ sector:

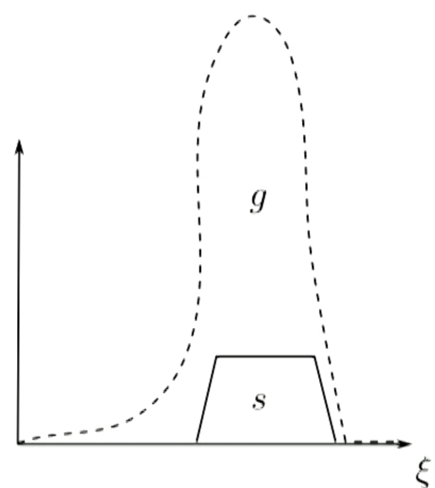
$$\left(K^{-1} - 1\right) |g\rangle_{\lambda \neq 2} = |s\rangle_{\lambda \neq 2}$$

- One can invert it to obtain the response

$$|g\rangle_{\lambda \neq 2} = \frac{1}{K^{-1} - 1} |s\rangle_{\lambda \neq 2}$$

- The response will not spill over into the IR unless s is approximately (i.e. up to UV regularizations) an eigenfunction of K with

$$K(h) = 1.$$



16

Ansatz for UV perturbations (continued)

- Thus we pose the ansatz for a family of UV perturbations

$$s(\hat{\varphi}_+, \hat{\varphi}_-) = \underbrace{e^{-im\hat{\varphi}_+}}_{\text{mode of centre of coordinates}} \underbrace{\text{sign}(\sin\hat{\varphi}_-)\,|\sin\hat{\varphi}_-|^{-h}}_{\text{non-normalizable eigenfunction of } K} \underbrace{u(|\sin\hat{\varphi}_-|)}_{\text{window function}}$$

- The window function is a `black box' for UV regularization, and simultaneously i) makes s normalizable and ii) regulates its $K = 1$ eigenvalue. For our calculations we only need to use the lowest-order approximation

$$u(|\hat{\varphi}|) = (\ln\hat{\varphi}_u/\hat{\varphi}_l)^{-1}\left(\theta(\hat{\varphi}_u - |\hat{\varphi}|) - \theta(\hat{\varphi}_l - |\hat{\varphi}|)\right)\ , \qquad \hat{\varphi}_u\hat{\varphi}_l \propto (\beta J)^{-2}$$

- The least irrelevant perturbation is given by the above with $I = 0, h_I = 2.$

Effective action in the presence of perturbation

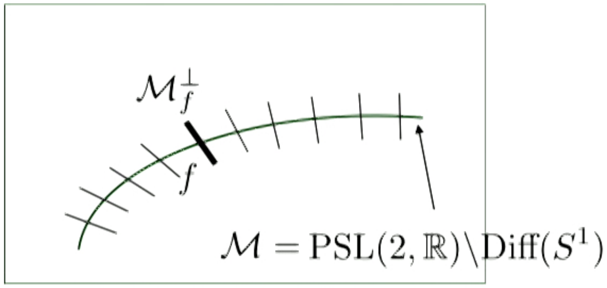
- General form of least irrelevant perturbation:

$$s(\hat{\varphi}_+, \hat{\varphi}_-) = \sum_m a_m e^{-im\hat{\varphi}_+} \mathrm{sign}(\sin\hat{\varphi}_-)\, |\sin\hat{\varphi}_-|^{-2} u(|\sin\hat{\varphi}_-|)$$

- Integration defining $S_{\rm soft}$

Among $\mathrm{PSL}(2,\mathbb{R})$ representations, $g_{\lambda=2}$ generate translations along $\mathcal{M}$ while $g_{\lambda\neq 2} \equiv g_\perp$ lie perpendicular to $\mathcal{M}$. Thus

$$e^{-S_{\rm soft}} \sim \int \mathcal{D}g_\perp\, e^{-NS_2} = e^{N\langle s|g_{\lambda=2}\rangle} \int \mathcal{D}g_\perp\, e^{-N\left(\frac{1}{2}\langle g_\perp|K^{-1}-1|g_\perp\rangle - \langle s|g_\perp\rangle\right)}$$



18

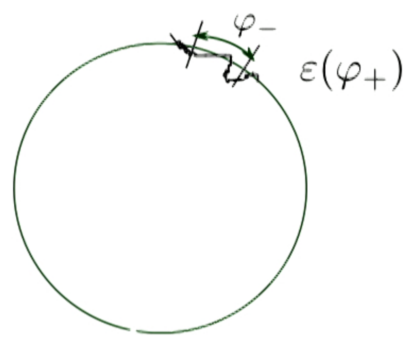
The Schwarzian from the coupling of s to $g_{\lambda=2}$

- The term $\langle s|g_f\rangle$ coincides with the linear term in the expansion of the Schwarzian

$$S_f(\tau) = \frac{f'''(\tau)}{f'(\tau)} - \frac{3}{2}\left(\frac{f''(\tau)}{f'(\tau)}\right)^2 = 6 \lim_{\tau_1\to\tau_2}\left(\frac{J^2}{b}\,|G_f(\tau_1,\tau_2)|^q - (\tau_1-\tau_2)^{-2}\right)$$

in $G_f - G|_{\rm MF}$.

- The modulating function $a(\hat{\varphi}_+)$ appearing in the perturbation, by dimensional analysis is proportional to the UV cutoff $J^{-1} = \varepsilon(\tau_+)$.



Non-local action from integration over $g_\perp$

- The action at the saddle-point of integral over $g_\perp$ can be expressed as an effective action for source function s.

$$N^{-1} S_{\text{saddle}} \overset{\text{eom}}{=} -\frac{1}{2}\left\langle s|g^\perp\right\rangle = -\frac{1}{2}\left\langle s|\Gamma^\perp|s\right\rangle$$
$$= -\frac{1}{2}\int d\tau_+ d\tau_- d\tau'_+ d\tau'_-\ s(\tau_+,\tau_-)\Gamma^\perp(\tau_+,\tau_-;\tau'_+,\tau'_-)s(\tau'_+,\tau'_-)$$
$$g^\perp(\tau_+,\tau_-) = \int d\tau'_+ d\tau'_- \Gamma^\perp(\tau_+,\tau_-;\tau'_+,\tau'_-) s(\tau'_+,\tau'_-)$$

response function

- At scales much greater than $(\beta J)^{-1}$, we can integrate over the UV support of s in τ_-, τ'_- the least irrelevant, double pole of $\Gamma^\perp$ to obtain an effective action for the modulating function of , s

$$S_{\text{eff}} \sim \int d\hat{\varphi}_+ d\hat{\varphi}'_+ \frac{(\beta J)^{-2}}{\sin^4 \frac{\Delta\hat{\varphi}_+}{2}} \ln\left(\frac{\sin^2 \frac{\Delta\hat{\varphi}_+}{2}}{(\beta J)^{-2}}\right)$$

J has acquired conformal dimension -1 in the IR!

- In conformal time, we have the non-local term in the soft action

$$S_{\text{soft,nl}} \sim \int d\varphi d\varphi' \frac{\varepsilon(\varphi)\varepsilon(\varphi')}{\sin^4 \frac{\Delta\varphi}{2}} \ln\left(\frac{\sin^2 \frac{\Delta\varphi}{2}}{\varepsilon(\varphi)\varepsilon(\varphi')}\right)$$

Local terms from cutoff-dependence of integral

Outline

- $\mathrm{Diff}(S^1)$, $\mathrm{PSL}(2,\mathbb{R})$ symmetries in the IR ✓
- Decomposition of fluctuations into irreps of $\mathrm{PSL}(2,\mathbb{R})$ ✓
- Conformal time $\varphi(\tau)$;
  the non-linear soft mode $\varepsilon(\varphi)$ and its action ✓
- RG analysis and resulting action for soft mode ✓
- Corresponding dilaton theory

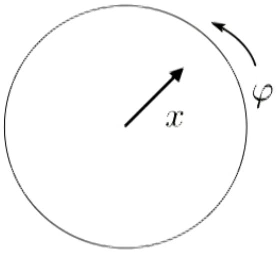
Dual action in dilaton gravity

- Both non-local and local terms in the soft action arise from integrating out quadratic fluctuations in a bulk theory with a quadratic term in the dilaton,

$$S_\phi = -\frac{N}{4\pi}\left(\int d^2z\sqrt{g}\,\left(\phi(R+2)+\tilde{\alpha}\phi^2\right)+2\int d\varphi\sqrt{g_{\varphi\varphi}}K+\dots\right)$$

where ϕ and g satisfy boundary conditions

$$g_{\varphi\varphi}|_{\partial D} = \varepsilon(\varphi)^{-2}\ , \quad \phi|_{\partial D} = 1$$

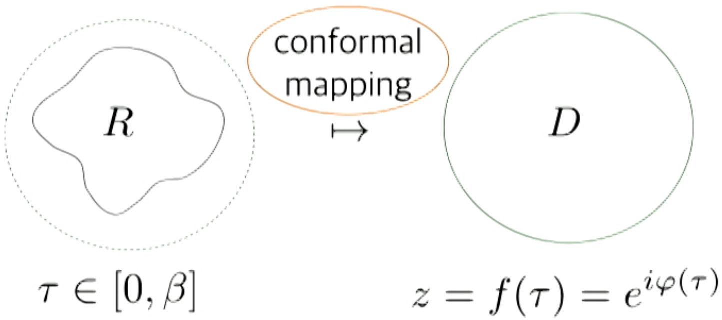


$$|z|<1,\ \ z=\sqrt{x}e^{i\varphi}$$

- Local terms, including the Schwarzian, arise from the extrinsic curvature term + integration of the on-shell bulk action near the boundary. In other words, from the UV regularization of the non-local term.

Conformal gauge

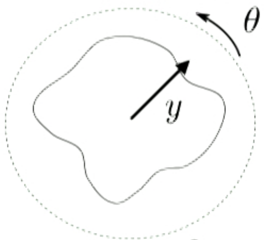
- After mapping to conformal time, the bulk configurations we would like to integrate over, those with total boundary length β and $\phi|_{\partial D} = 1$ are characterized by $\widetilde{\text{Diff}}\text{-}S^1$ invariant boundary conditions.



- Thus we are free to fix the gauge of the metric to be conformal to the unit metric. As we are working on the disk there are no remaining moduli.

- For purposes of integrating on-shell action, it is convenient to work on the irregular subset R of the Poincare disk, rather than on D with a deformed metric. On R we may still assume the metric is conformal.

Integrating the on-shell action



$$w = \sqrt{y} e^{i\theta} \qquad g_{\tau\tau}|_{\partial R} = J^2 \ , \quad \phi|_{\partial R} = 1$$

- Letting $g_{w\bar{w}} = 2e^{2\rho}$, the equations of motion for the dilaton and metric are

$$\nabla^2 \rho = 1 - \tilde{\alpha}\rho \ , \quad \nabla^2 \phi = 2(\phi - \frac{\tilde{\alpha}}{2}\phi^2)$$

- Using the equation of motion for the dilaton in the action,

$$S_{\text{on-shell}} = -\frac{N}{4\pi}\tilde{\alpha} \int_R d^2 w \sqrt{g}\, \phi^2$$

- To obtain the on-shell action at $O(\varepsilon^2)$, it is sufficient to use lowest order solution satisfying $\nabla^2 \phi = 2\phi$ on the Poincare disk.

  eigenvalue equation for Casimir of $\mathrm{PSL}(2,\mathbb{R})$

- Using the boundary-condition-satisfying bulk-to-boundary propagator

$$\phi(w) = \int \frac{d\varphi}{2\pi} \varepsilon(\varphi) \frac{(1 - w\bar{w})^2}{(1 - we^{-i\varphi})^2 (1 - \bar{w}e^{i\varphi})^2} + O(\varepsilon^2)$$

and integrating out bulk coordinates, we obtain over the boundary the exact non-local term found in SYK.

Results

- $O((\beta J)^{-2})$ terms in the action of SYK, written for the soft mode $\varepsilon(\varphi) \sim (\beta J)^{-1}$ and resulting from integrating out least irrelevant perturbation.

$$\frac{S_{\rm soft}}{N} = -\tilde{\alpha} \int \frac{d\varphi_1}{2\pi}\frac{d\varphi_2}{2\pi}\frac{\varepsilon(\varphi_1)\varepsilon(\varphi_2)}{\varphi_{12}^4} \ln\left(\frac{\varphi_{12}^2}{\varepsilon(\varphi_1)\varepsilon(\varphi_2)}\right) -$$

non-local, dominant in the IR $\quad \varphi_{12} = 2\sin\frac{\varphi_1-\varphi_2}{2}, \varphi_{12} \gg (\beta J)^{-1}$

$$\frac{\tilde{\alpha}}{4\pi}\int \frac{d\varphi}{2\pi}\underbrace{\left(\varepsilon(\varphi) + 2\varepsilon''(\varphi) - \frac{\varepsilon'(\varphi)^2}{2\varepsilon(\varphi)}\right)}_{S_f(\tau),\ f = e^{i\varphi(\tau)}} + \ldots$$

UV regularization of the non-local term

- Dual action in $D=2$ dilaton gravity

$$S_\phi = -\frac{N}{4\pi}\left(\int d^2z\sqrt{g}\ \left(\phi(R+2) + \tilde{\alpha}\phi^2\right) + 2\int d\varphi\sqrt{g_{\varphi\varphi}}K + \ldots\right)$$

(linear term discussed in Maldacena, Stanford, Yang (2016))

Further directions

- Can we identify matter fields in the bulk? Quantify the dissipation due to the matter fields of the soft mode.
- Find a consistent Hilbert space from quantizing the Schwarzian +non-local action.
- Can we confirm the existence of strings in SYK?