Title: Holomorphic Floer quantization, wall-crossing structures and resurgence

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Abstract: <p>1. The notion of wall-crossing structure (as defined by Maxim Kontsevich and myself in arXiv: 1303.3253)
provides the universal framework for description of different types of wall-crossing formulas (e.g. Cecotti-Vafa in 2d or KSWCF in 4d). It also gives
a language and tools for proving algebraicity and analyticity of arising generating series (e.g. for BPS invariants).

2. Holomorphic Floer theory is the Floer theory of a pair of complex Lagrangian subvarieties of a complex symplectic manifold (maybe infinite-dimensional).
This geometry underlies several important topics, both in mathematics and physics. Those include questions about analytic continuation of exponential integrals
(e.g. Feynman integrals),
deformation quantization of holomorphic symplectic manifolds (and related Riemann-Hilbert correspondences), Geometric Langlands correspondence, etc.

3. It was known for a long time that many a priori divergent series (like e.g. formal WKB series for solutions of equations with a small parameter) become analytic or meromorphic functions after taking their Laplace transform
(Borel resummation). This property was called the resurgence property of the divergent series. The relation of the resurgence phenomenon to simplest wall-crossing formulas was realized in the early 90's in the work of French mathematicians (Ecale, Voros, Pham , Malgrange and others).

I plan to discuss how the combination of 1 and 2 can be applied to 3 in a very general situation.
Main idea goes back to our theory of Donaldson-Thomas invariants (arXiv: 0811.2435).
Namely, analyticity of the formal series follows from existence of a global analytic object which is glued from the local ones by means of the formal series.
This analytification of an a priori formal variety is a by-product of the growth estimates on the data of the underlying wall-crossing structure.

In case of the Holomorphic Floer theory (which underlies e.g. the Stokes phenomenon for the WKB solutions of PDE or difference equations) one needs an estimate on the number of pseudo-holomorphic discs
with the boundary on the union of our Lagrangian subvarieties. In practice it often appears as an estimate on the number of gradient lines between two critical points
of the action or potential.</p>

Wall-crossing structure

(stability data on graded Lie algebras)

0811.2435

- $\Gamma \simeq \mathbb{Z}^n$ – "lattice"
- $\mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$ – graded Lie alg. / $\mathbb{C}$
- $Z : \Gamma \longrightarrow \mathbb{C}$ – homomorphism of gps (central charge)
- Collection $a(\gamma) \in \mathfrak{g}_\gamma$, $\gamma \in \Gamma - \{0\}$, $a = (a(\gamma))$

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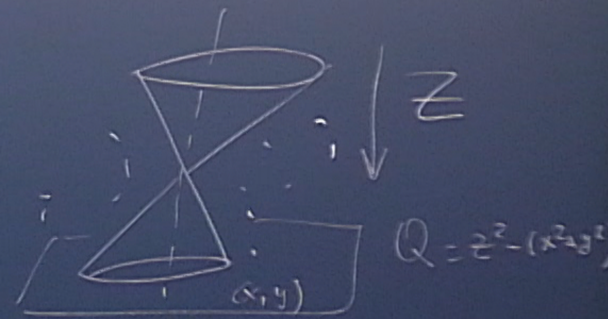

Axiom (Support Property) $\exists$ quadratic form Q s.t. $Q|_{\mathrm{Ker} Z} < 0$ and $Q(\gamma) \geq 0$ if $a(\gamma) \neq 0$

$Q|_{\mathrm{supp}\, a} \geq 0$

Two main examples

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Two main examples

(2d) $\cdot$ $\mathfrak{g} = \mathfrak{gl}(n)$, Γ = root lattice, $\gamma_{ij} = (0, \dots, 1, \dots, -1, \dots, 0)$

$\bigoplus_{(i,j)} \mathbb{C} \cdot E_{ij}$

(z_i), $z_i \neq z_j$ $(i \neq j)$

$$Z(\gamma_{ij}) = z_i - z_j$$

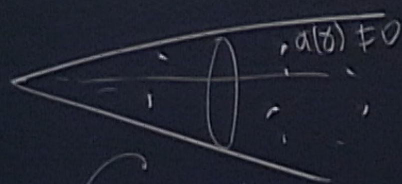
$$a(\gamma_{ij}) = n_{ij} \in \mathbb{Z}$$

(4d) Γ, $\langle\,,\,\rangle : \Lambda^2 \Gamma \to \mathbb{Z}$

$T_\Gamma = \operatorname{Hom}(\Gamma, \mathbb{C}^*) \simeq (\mathbb{C}^*)^n$, $n = 2k$

$$(\mathcal{O}(T_\Gamma), \{\,,\,\}) := \mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathbb{C} \cdot x^\gamma$$

$$[x^\gamma, x^\mu] = \pm\langle \gamma, \mu \rangle x^{\gamma+\mu}$$



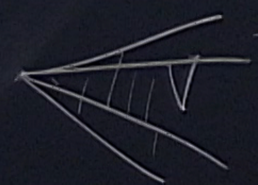
$$C \subset \Gamma\otimes\mathbb{R}\simeq\mathbb{R}^n$$

C - strict cone

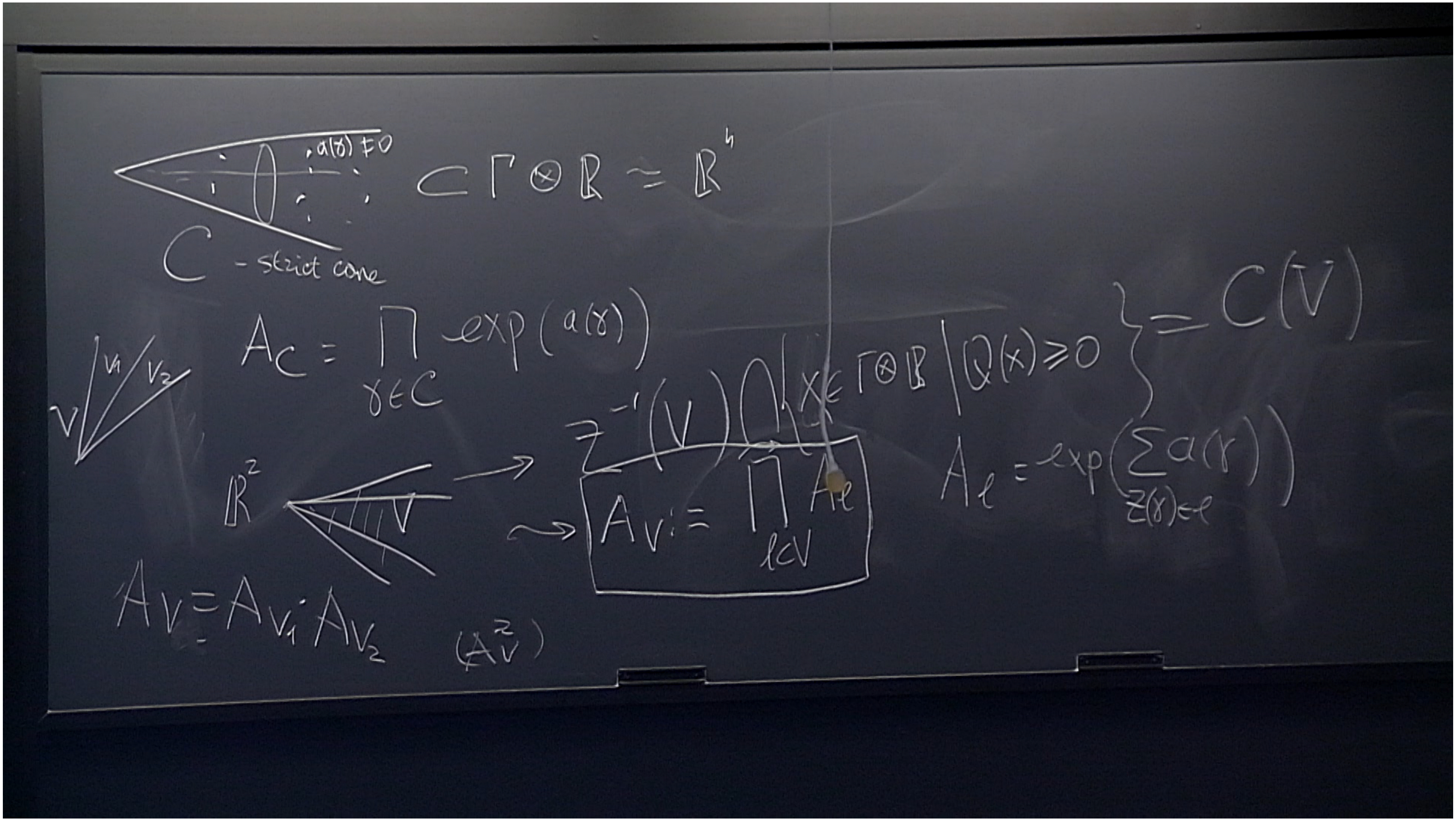
$$A_C = \prod_{\gamma\in C}\exp(a(\gamma))$$

$$Z^{-1}(V)\cap\{x\in\Gamma\otimes\mathbb{R} \mid Q(x)\geq 0\} = C(V)$$

$\mathbb{R}^2$



$$\rightsquigarrow A_V := \prod_{l\subset V} A_l, \qquad A_l = \exp\Big(\sum_{Z(\gamma)\in l} a(\gamma)\Big)$$


$a(\gamma) \neq 0$
$\subset \Gamma \otimes \mathbb{R} \simeq \mathbb{R}^n$
C - strict cone
$A_C = \prod_{\gamma \in C} \exp(a(\gamma))$
V v_1 v_2
$Z^{-1}(V) \cap \{x \in \Gamma \otimes \mathbb{R} \mid Q(x) \geq 0\} = C(V)$
$\mathbb{R}^2$ V
$A_V := \prod_{\ell \subset V} A_\ell$
$A_\ell = \exp\left(\sum_{Z(\gamma) \in \ell} a(\gamma)\right)$
$A_V = A_{V_1} \cdot A_{V_2}$
(A_V^2)

$\Lambda V_1 \wedge \Lambda V_2 \quad (\Lambda_V^2)$

X - alg. variety $/\mathbb{C}$, $\dim_{\mathbb{C}} X = N$

$\mathbb{C}^N$

$W: X \longrightarrow \mathbb{C}$, regular fnct

. $\mathrm{Crit}(W) = \{x_1, \dots, x_m\}$, $z_i = W(x_i)$ all different

$dW(x_i) = 0$

$$I(\hbar) = \int_\gamma e^{\frac{W(x)}{\hbar}} \mathrm{vol}$$

top degree "volume" form

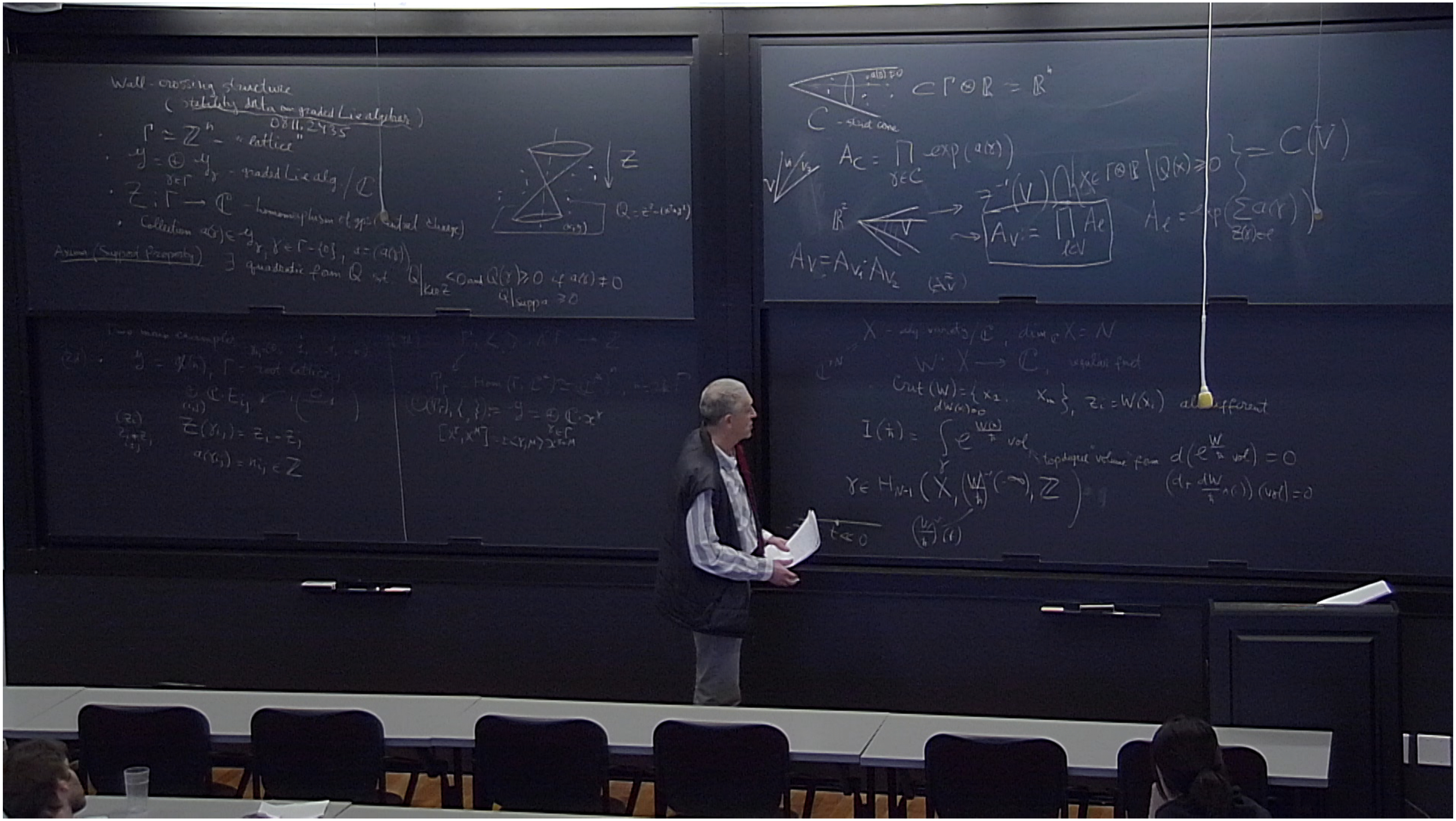
$$d\left(e^{\frac{W}{\hbar}} \mathrm{vol}\right) = 0$$

$$\left(d + \frac{dW}{\hbar} \wedge (\cdot)\right)(\mathrm{vol}) = 0$$

$$\gamma \in H_{N-1}\left(X, \left(\frac{W}{\hbar}\right)^{-1}(-\infty), \mathbb{Z}\right)$$

$$\left(\frac{W}{\hbar}\right)^{-1}(t)$$

$-\infty \qquad t \ll 0$


Wall-crossing structure
(stability data on graded Lie algebras)
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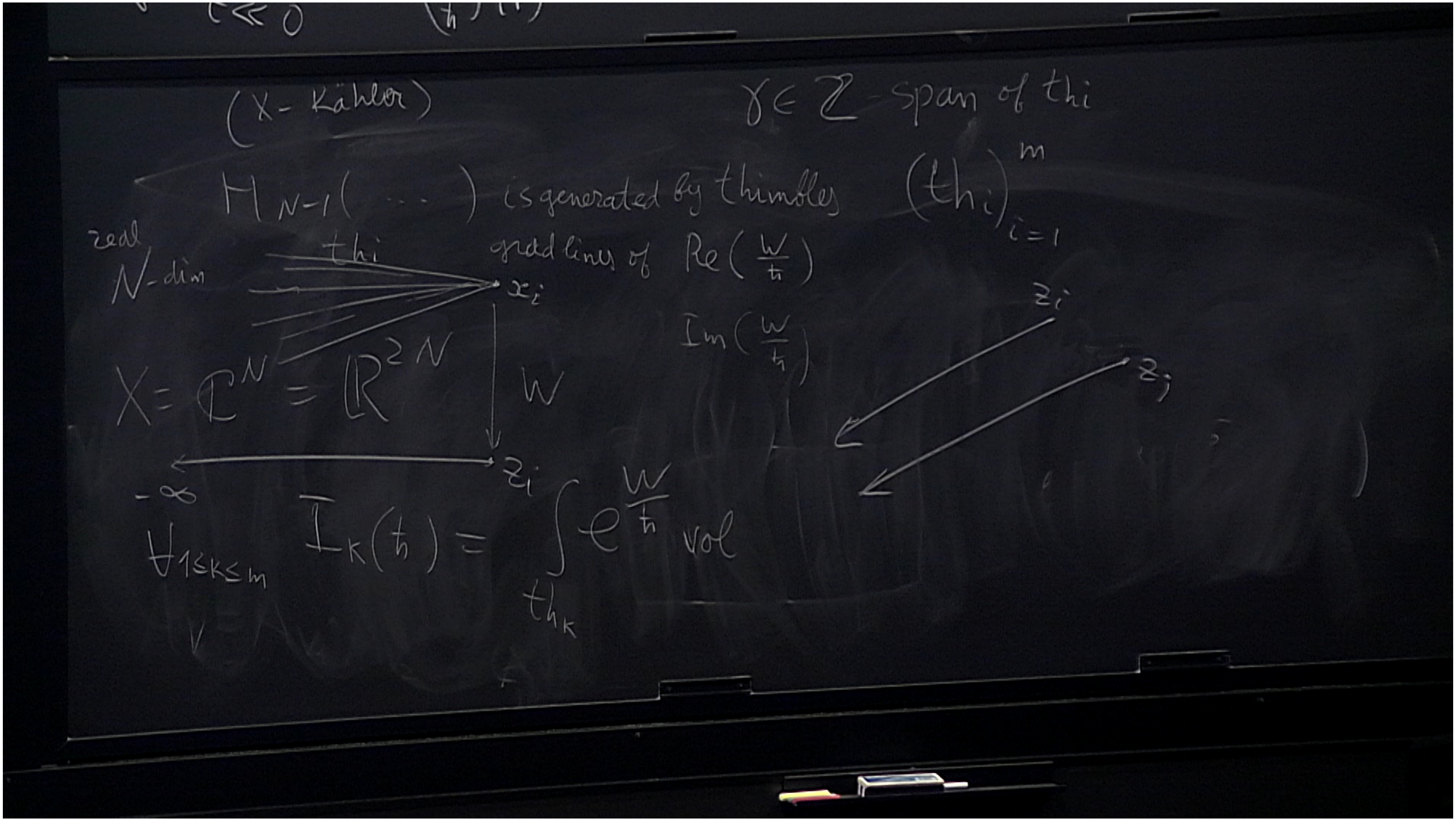
lattice
graded Lie alg
homomorphism of groups (central charge)
Collection
Axiom (Support Property)
∃ quadratic form Q s.t.
C - strict cone
$A_C = \prod_{\gamma \in C} \exp(a(\gamma))$
$A_V = A_{V_1} A_{V_2}$
X - alg variety / ℂ, dim X = N
regular fnct
all different
topological "volume" form
$d(e^{W/\hbar}\,\mathrm{vol}) = 0$


(X - Kähler)
$\gamma \in \mathbb{Z}$-span of th_i
$H_{N-1}(\dots)$ is generated by thimbles $(th_i)_{i=1}^{m}$
real N-dim
th_i
grad lines of $\mathrm{Re}(\frac{W}{\hbar})$
x_i
$\mathrm{Im}(\frac{W}{\hbar})$
z_i
z_j
$X = \mathbb{C}^N = \mathbb{R}^{2N}$
W
$-\infty$
z_i
$\forall_{1 \le k \le m}$ $I_k(\hbar) = \int_{th_k} e^{\frac{W}{\hbar}} \mathrm{vol}$

$-\infty \quad t \ll 0 \qquad \left(\frac{W}{\hbar}\right)'(t) \qquad W = z_i + \sum t_i^2$

(X - Kähler) $\qquad \gamma \in \mathbb{Z}$ - span of th_i

$H_{N-1}(\ \dots\ )$ is generated by thimbles $(th_i)_{i=1}^{m}$

real N-dim $\quad th_i$ grad lines of $\mathrm{Re}\left(\frac{W}{\hbar}\right)$, $\mathrm{Im}\left(\frac{W}{\hbar}\right)$ $\quad x_i$

$X = \mathbb{C}^N = \mathbb{R}^{2N}$

$\downarrow W$

$-\infty \longleftarrow z_i$

$\frac{z_i}{\hbar}$, $\frac{z_j}{\hbar}$

$\forall_{1 \le k \le m} \left\{ I_k(\hbar) = \int_{th_k} e^{\frac{W}{\hbar}} \mathrm{vol} \right\}$

$e^{\frac{z_i}{\hbar}}(\ \dots\ )$

order $\frac{z_i}{\hbar}$ by Im

$\longleftarrow z_1$

$\longleftarrow z_2$

$\longleftarrow$

$\longleftarrow z_m$

Axiom (Support Property) $\exists$ quadratic form Q s.t. $Q|_{\mathrm{Ker}\, Z} < 0$ and $Q(\gamma) \geq 0$ if $a(\gamma) \neq 0$

$Q|_{\mathrm{supp}\, a} \geq 0$

Basis $(th_i)_1^m$ does not jump if we stay outside of Stokes rays $\mathrm{Arg}\, \hbar = \mathrm{Arg}(z_i - z_j)$

$\mathbb{C}^*_\hbar$

$th_i \longmapsto th_i$

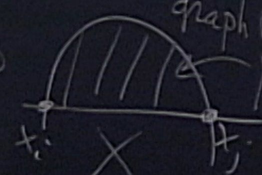
$th_j \longmapsto th_j + n_{ij}\, th_i, \quad n_{ij} \in \mathbb{Z}$

1) $\#$ grad lines for $\mathrm{Re}\left(\frac{W}{\hbar}\right)$

2)

3) $T^*X \supset$ graph $\left(d\left(\frac{W}{\hbar}\right)\right)$

pseudo-holom. discs



$\mathbb{C}^*_\hbar - \{\text{Stokes rays}\}$

· Local system of Γ : $\Gamma_\hbar = \mathrm{Ker}\left(\mathbb{Z}^m \to \mathbb{Z}\right)$ = root lattice A_{m-1}

$(a_1, \dots, a_m) \mapsto \sum_1^m a_i$

· $g_\hbar = \mathrm{End}\left(\bigoplus_{i=1}^{m} H_N\left(X, \left(\frac{W}{\hbar}\right)^{-1}(-\infty), \mathbb{Z}\right) \otimes \mathbb{C} \right)$

· $Z_\hbar : \Gamma \to \mathbb{C}$, $Z(\gamma_{ij}) = \dfrac{z_i - z_j}{\hbar}$

· $a(\gamma_{ij}) = n_{ij} E_{ij}$ $\quad i \quad j$

$\underbrace{\begin{pmatrix} 1 & n_{i_1 j_1} & \\ & 1 & \\ & & 1 \end{pmatrix}}_{A''} \underbrace{\begin{pmatrix} 1 & & n_{i_2 j_2} \\ & 1 & \\ & & 1 \end{pmatrix}}_{B''} \underbrace{\begin{pmatrix} 1 & & \\ & 1 & n_{i_3 j_3} \\ & & 1 \end{pmatrix}}_{C''} = C \begin{pmatrix} 1 & & n_{i_2 j_2} + n_{i_1 j_1} n_{i_3 j_3} \\ & \ddots & \\ & & 1 \end{pmatrix} A$

Walls: $\frac{z_{i_1}}{\hbar}$ $\frac{z_{i_2}}{\hbar}$ $\frac{z_{i_3}}{\hbar}$

$$\mathcal{I}_k(\hbar) = \frac{e^{-\frac{Z_k}{\hbar}}}{(2\pi\hbar)^{N/2}}\, I_k(\hbar) \underset{\hbar\to 0}{\sim} c_{k0} + \hbar\, c_{k1} + \hbar^2(\cdots) \in \mathbb{C}[[\hbar]]$$

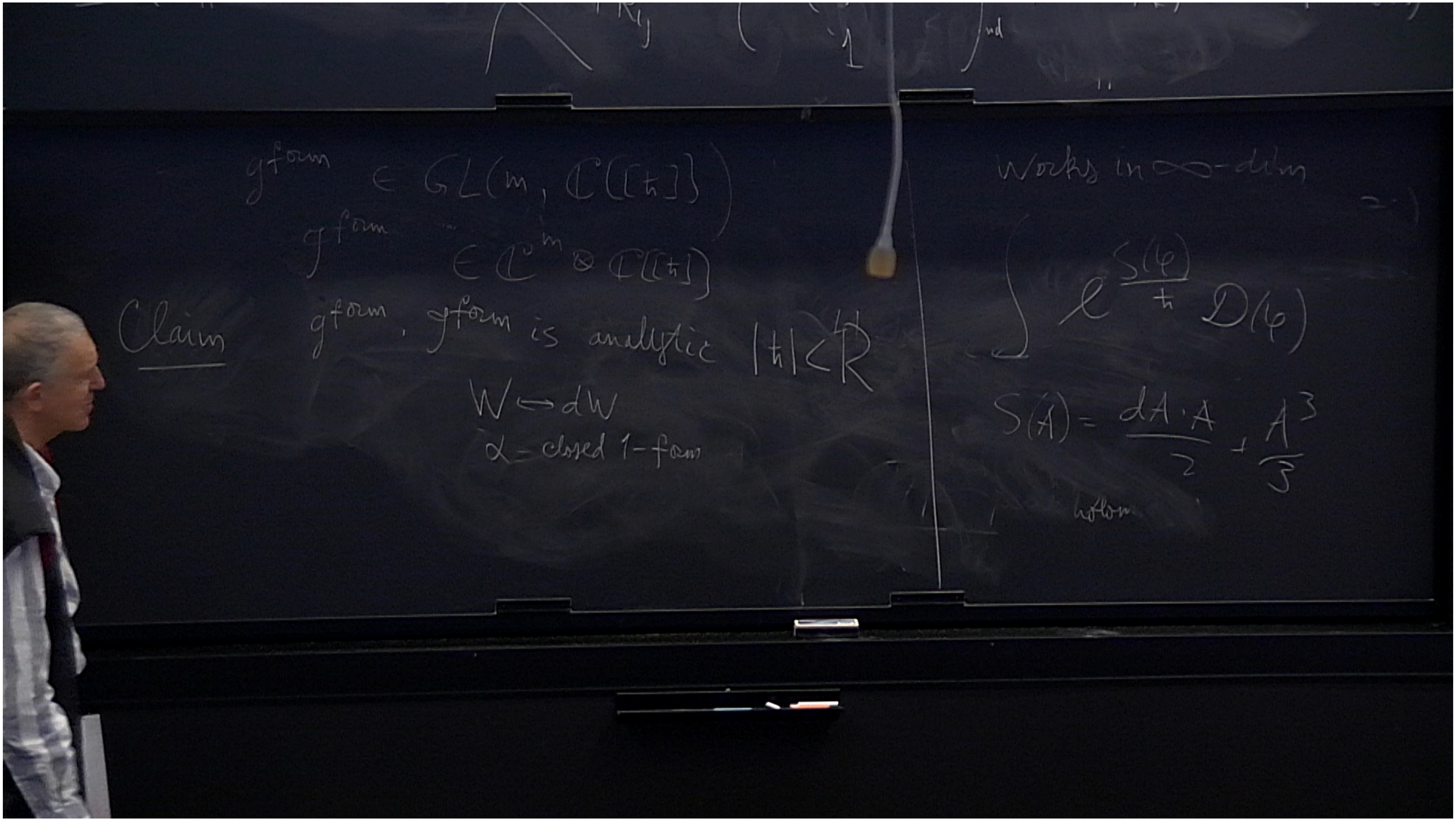
$$Z_k \longleftrightarrow Z_\ell$$

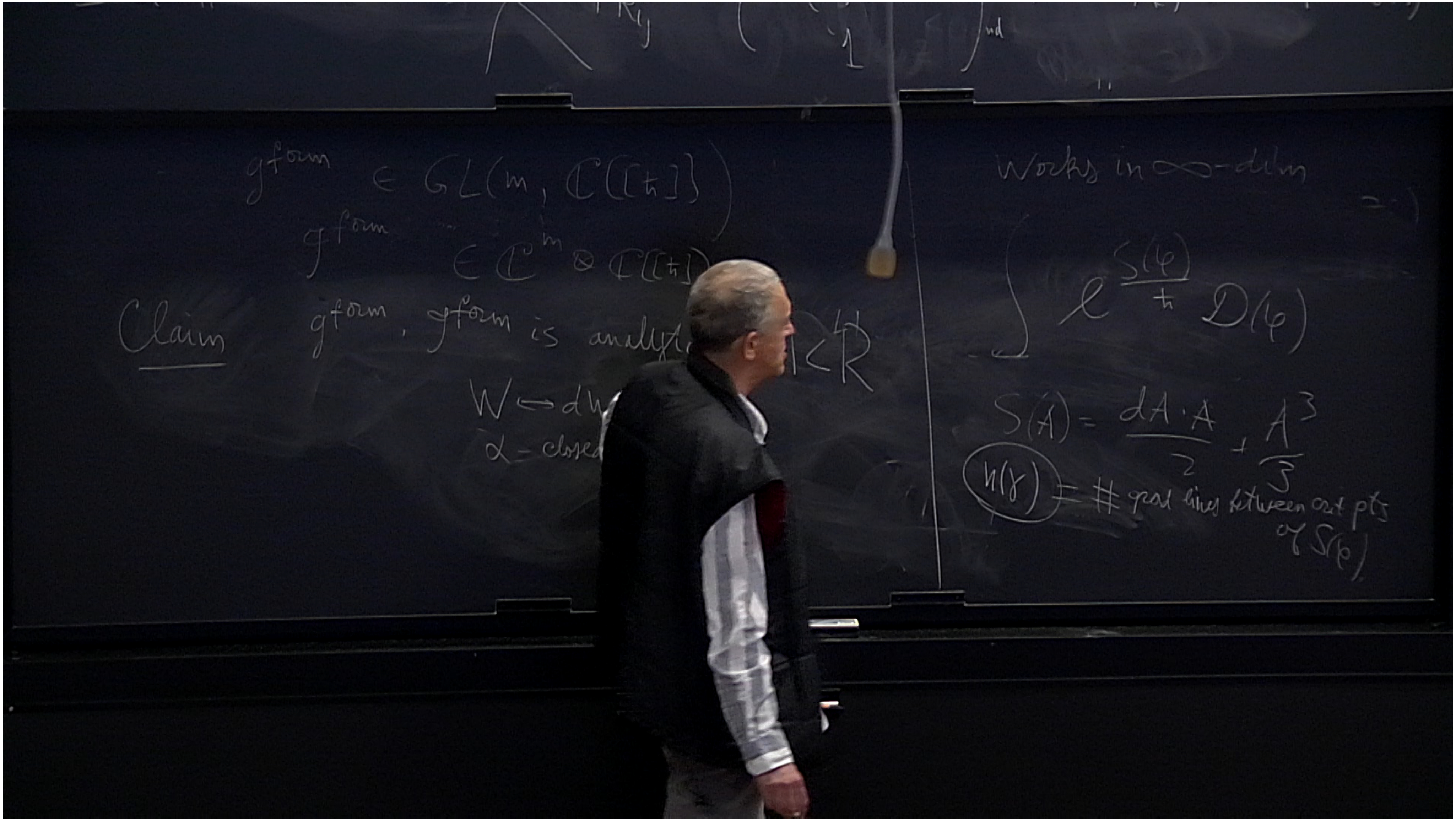
$$\begin{cases} \Delta \mathcal{I}_k = 0 \\ \Delta \mathcal{I}_\ell = n_{k\ell}\, e^{\frac{Z_k - Z_\ell}{\hbar}}\, \mathcal{I}_k \end{cases} \Longrightarrow \text{RH problem depends on } \{n_{ij}\},\ \{Z_i\}$$

$\Longrightarrow$ can glue holomorphic bundle on $\mathbb{C}_\hbar$

R_{ij}

$$\begin{pmatrix} 1 & & n_{ij}\, e^{\frac{Z_i - Z_j}{\hbar}} \\ & \ddots & \\ & & 1 \end{pmatrix}_{nd} \Longrightarrow GL(m, \mathbb{C})\text{-valued fnct } g(\hbar)$$


g^form ∈ GL(m, ℂ[[ħ]])
g^form ∈ ℂ^m ⊗ ℂ[[ħ]]
Claim g^form, g^form is analytic |ħ| < R
W ↔ dW
α = closed 1-form
Works in ∞-dim
∫ e^{S(φ)/ħ} D(φ)
S(A) = dA·A/2 + A^3/3


Claim
Works in ∞-dim
S(A) = dA·A/2 + A^3/3
grad lines between crit pts of S(φ)

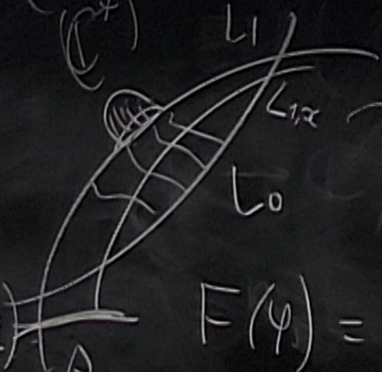
More general (geom) framework.

$(M, \omega^{2,0})$ - $\mathbb{C}$-sympl

$(\mathbb{C}^*)^2$

$\omega^{2,0} = d\lambda$

$\lambda|_{L_i} = df_i$



$x \in B$

$HF_\hbar(L_0, L_{1,x})$

$\frac{\omega^{2,0}}{\hbar}$

2d / 1d

$L_{1,x} \rightsquigarrow \mathcal{X} = \{\varphi : [0,1] \to M,\ \varphi(0) \in L_0,\ \varphi(1) \in L_1\}$

$\eta \overset{\text{often}}{=} dF$

$F(\varphi) = \int_{[0,1]} \varphi^* \lambda + f_1(\varphi(1)) - f_0(\varphi(0))$

$\exists$ closed 1-form $\eta = (p_{\mathcal{X}} : \mathcal{X} \times [0,1] \to \mathcal{X})_* \circ ev^*(\omega^{2,0})$

$(\varphi, t) \to \varphi(t)$