Title: Matrix product state evolutions of quantum fields in curved space

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Abstract: <p>The matrix product state (MPS) ansatz makes possible computationally-efficient representations of weakly entangled many-body quantum systems with gapped Hamiltonians near their ground states, notably including massive, relativistic quantum fields on the lattice. No Wick rotation is required to apply the time evolution operator, enabling study of time-dependent Hamiltonians. Using free massive scalar field theory on the 1+1 Robertson-Walker metric as a toy example, I present early efforts to exploit this fact to model quantum fields in curved spacetime. We use the ADM formalism to write the appropriate Hamiltonian witnessed by a particular class of normal observers. Possible applications include simulations of gravitational particle production in the presence of interactions, studies of the slicing-dependence of entanglement production, and inclusion of the expectation of the stress-energy tensor as a matter source in a numerical relativity simulation.</p>

MPS Evolutions of Quantum Fields on Curved Spacetimes

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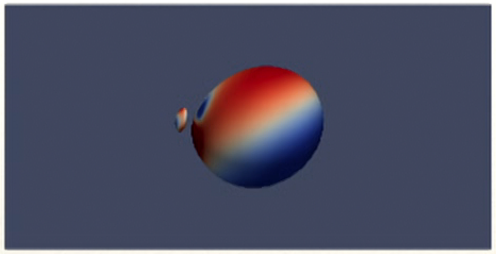
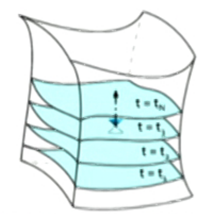
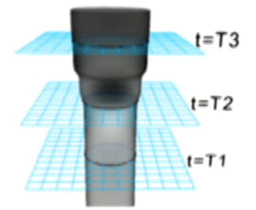
Motivation

- QFT in curved spacetime: a quantum field on a nontrivial metric
  - the gravitational field can excite the quantum field
  - basis for study of Hawking radiation, RQI, inflationary particle production…
- Problems are fundamentally *time-dependent* and often *non-perturbative*.
- Lattice techniques allow non-perturbative calculations. MPS allows time-dependence, including e.g. interactions.
- 3+1 general relativity techniques allow derivation of time-evolution operator.
- By exploiting ideas from both fields, we are producing novel simulations of QFT in curved space.
- Goal one: simulate the excitation of quantum field by gravitational field.

3+1 Formulation of GR

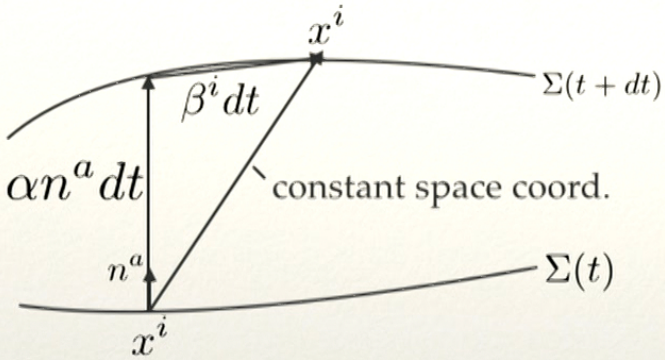






- Divide spacetime up into hypersurfaces, each representing a "moment".
- Describe the 4-geometry in terms of ``ADM" variables available on each slice.
- In ``causally well-behaved" spacetimes the Einstein equations determine the full geometry from any one such slice.
- We can use the same variables to construct time-evolution operators in non-trivial metrics.

ADM Variables



	Minkowski	Rindler	Schwarzschild
α	1	1/x	$\left(1-\frac{2M}{r}\right)^{1/2}$
β^i	0	0	0
γ_{ij}	1	1	$\mathrm{diag}\left(\left(1-\frac{2M}{r}\right)^{-1}, r^2, r^2\sin^2\theta\right)$

$$ds^2 = \alpha^2 dt^2 - \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

- Idea: chop spacetime up into *spacelike hypersurfaces* ("slices").
- The slices are ``moments in time" as perceived by the *normal observers*.
- Distances on slice measured by spatial metric γ_{ij}
- After coordinate time dt:
  - The normal observers measure time αdt. α: "lapse function".
  - They move wrt spatial coordinates by β^i dt. β^i: *"shift vector"*.
  - These are just a coordinate choice.

Getting the time-evolution operator: general metric

1. Write down the action:
$$S = \frac{1}{2}\int d^{d+1}x\sqrt{-g}[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2]$$

2. The Lagrangian density is the spatial integrand on some slicing:
$$S = \int dt \int_\Sigma d^d x\mathcal{L}$$

$$\mathcal{L} = \frac{1}{2}\alpha\sqrt{-\gamma}\left\{\frac{\dot{\phi}^2}{\alpha^2} - \frac{2\beta^i}{\alpha^2}\dot{\phi}\partial_i\phi + \left[\frac{\beta^i\beta^j}{\alpha^2} + \gamma^{ij}\right]\partial_i\phi\partial_j\phi - m^2\phi^2\right\}$$

3. Compute the canonical momentum:
$$\pi \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{\sqrt{-\gamma}}{\alpha}\dot{\phi} - \beta^i\partial_i\phi$$

4. Do the Legendre transform:
$$\mathcal{H} \equiv \pi\dot{\phi} - \mathcal{L}$$

$$\mathcal{H} = \frac{\alpha}{2}\sqrt{-\gamma}\left\{\frac{-\pi^2}{\gamma} - \gamma^{ij}\partial_i\phi\partial_j\phi + m^2\phi^2\right\} + \pi\beta^i\partial_i\phi$$

5. Quantize on slices: $[\phi(x,t),\phi(y,t)] = [\pi(x,t),\pi(y,t)] = 0 \qquad [\phi(x,t),\pi(y,t)] = i\delta(x-y)$

3+1 Formulation: Review

- Can write the metric in terms of slice-local quantities.
- This allows us to get time-evolution operators.
- Now we want to use those operators to evolve QFTs.

Representing the QFT: Matrix Product States

- Suppose we can write our Hilbert space as a direct product of smaller Hilbert spaces. Then we can write our state as a sum of product states.

$$H^{ab} = H^a \otimes H^b \qquad |\psi\rangle = \sum_{i_1}^{a} \sum_{i_2}^{b} c_{i_1 i_2} |i_1\rangle \otimes |i_2\rangle$$

- We can express states in the space in terms of Schmidt decomposition (SVD):

$$c_{i_1 i_2} = \sum_{\alpha} \Gamma^{i_1}_{\alpha} \lambda_{\alpha} \Gamma^{i_2}_{\alpha}$$

- To construct a lower-rank approximation to c, drop Schmidt coefs.

Matrix Product States

- Suppose we can write our Hilbert space as a chain of direct products of smaller spaces. Then we can write our state as a sum of product states.

$$H^{dn} = H^d \otimes H^d \otimes H^d \otimes \ldots \qquad |\psi\rangle = \sum_{i_1} \cdots \sum_{i_n} c_{i_1 \ldots i_n} |i_1\rangle \otimes \ldots \otimes |i_n\rangle$$

- Moving from left to right we write down the Schmidt coefs of *each* bipartition.

$$c_{i_1 \ldots i_n} = \sum_{\alpha, \beta, \gamma \ldots} \Gamma^{i_1}_{\alpha} \lambda_{\alpha} \Gamma^{i_2}_{\alpha\beta} \lambda_{\beta} \Gamma^{i_3}_{\beta\gamma} \ldots \Gamma^{i_n}_{\omega}$$

- If the state is weakly entangled, this is an efficient representation.
- Near-ground states of gapped local Hamiltonians often weakly entangled.
- Massive scalar field is a gapped local Hamiltonian.

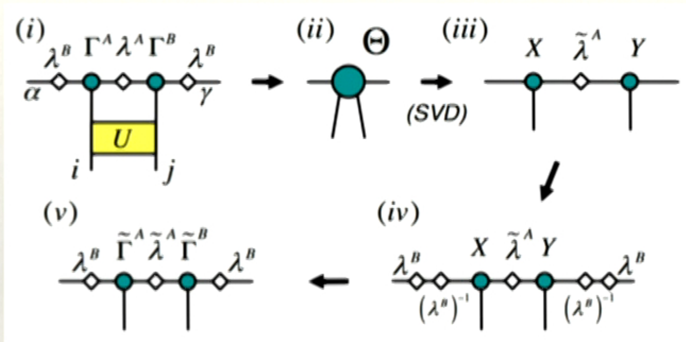
Application to Lattice QFT

$$H = \frac{1}{2}\sum_n \left(\frac{\pi_n^2}{a} + \frac{(\phi_{n+1} - \phi_n)^2}{a} + am^2\phi_n^2 \right)$$

- A nearest-neighbour Hamiltonian!
- Can therefore write time-evolution operator as series of two-site gates.
- ``Time-Evolving Block Decimation" allows us to apply those gates to the MPS.

Time Evolving Block Decimation (TEBD)

- Suppose we wish to apply a nearest-neighbour operator (e.g. time-evolution) U to an MPS…



- For Hamiltonians composed of two-site ops, this furnishes time-evolution.
- By truncating the Schmidt coefs at each step we maintain an efficient representation - unless the state is in fact too entangled.

Imaginary vs. Real Time Evolution

$$U = \exp(H\tau)$$

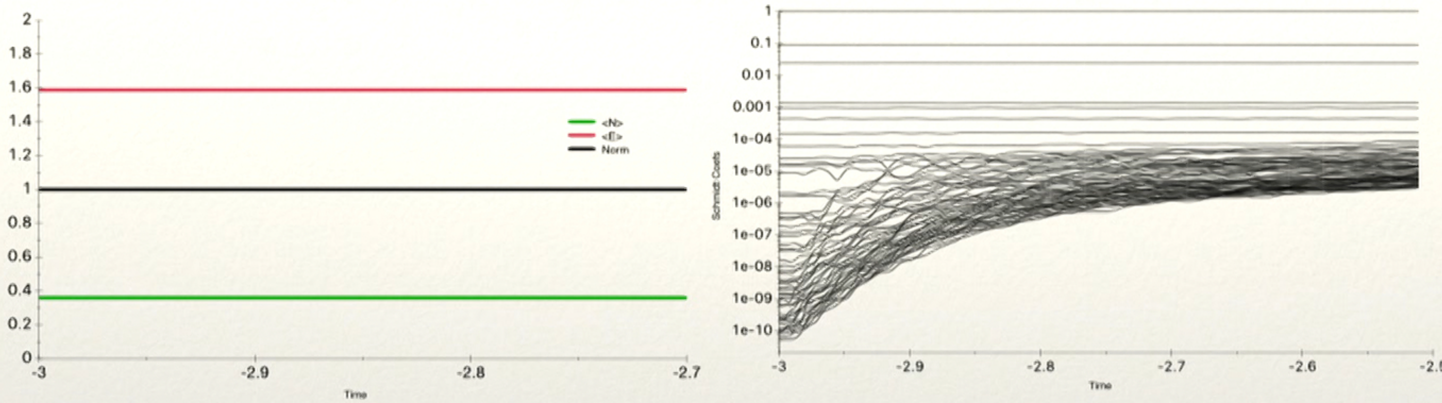
- Brings MPS towards ground state.
- Errors do not compound.

$$U = \exp(-iHt)$$

- Evolves MPS in time.
- New entanglement is generated, so MPS approximation degrades over time.
- Errors *do* compound.

- Need to consider ``short enough" evolutions to keep error acceptable.
- Breakdown is heralded by growth of smallest Schmidt coefs (thus increase of truncation error)

Evolution in Minkowski spacetime



- Mass = lattice spacing = 1, 50 Schmidt coefs, physical dimension 5.
- First we evolve in imaginary time to get vacuum state.
- Then ``evolve'' in real time; should get near-constant behaviour.
- Want coefs to be scale-separated and for the smallest to be numerically small (so adding more would have small effect on output).

Toy problem: RW Metric

Expanding homogeneous universe

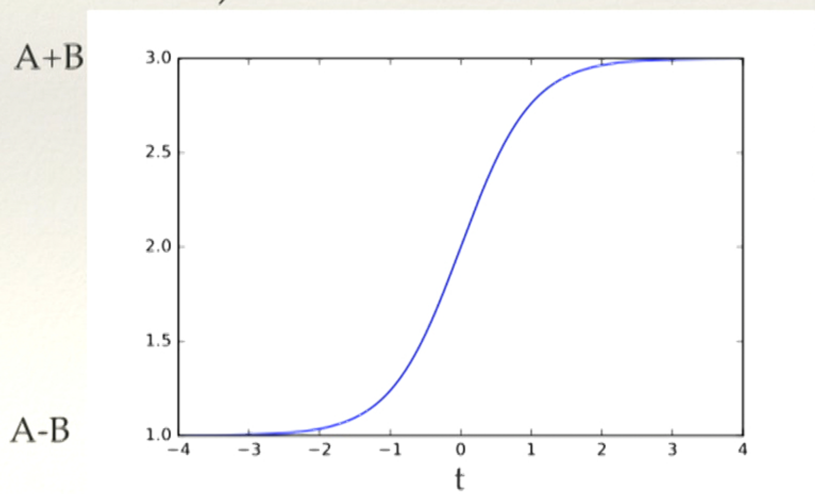
$$ds^2 = \chi(t)[dt^2 - dx^2] \longrightarrow \alpha = \sqrt{\chi(t)} \quad \beta^i = 0$$

$$\gamma_{ij} = \chi(t)\eta_{ij}$$

$$H = \frac{\sqrt{\chi(t)}}{2}\left(\frac{\pi_n^2}{a\chi(t)} + \frac{\chi(t)(\phi_{n+1} - \phi_n)^2}{a} + am^2\phi_n^2\right)$$

Choose

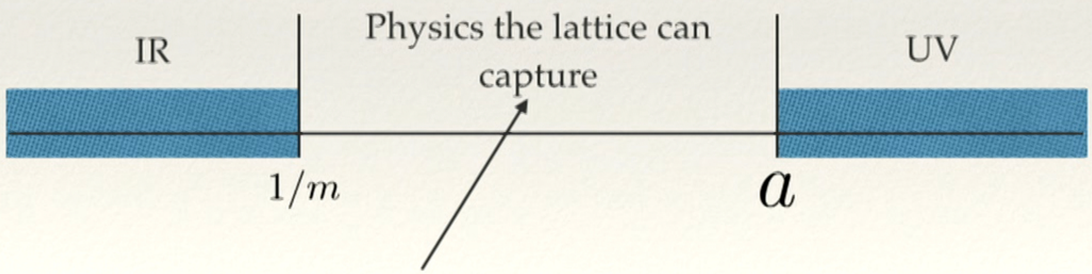
$$\chi(t) = A + B\tanh(Ct)$$



RW Metric

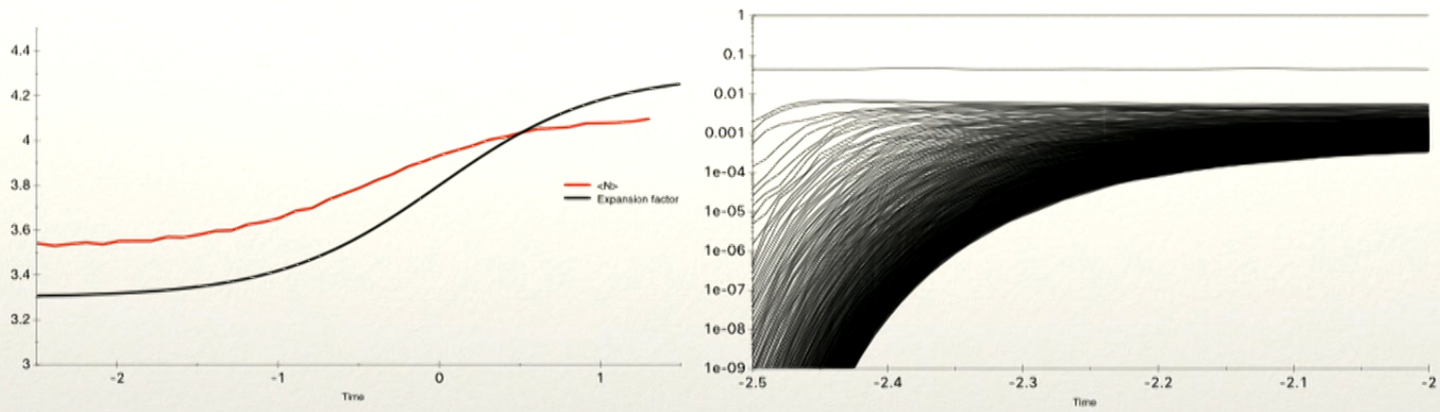
$$H = \frac{\sqrt{\chi(t)}}{2}\left(\frac{\pi_n^2}{a\chi(t)} + \frac{\chi(t)(\phi_{n+1}-\phi_n)^2}{a} + am^2\phi_n^2\right) \qquad \chi(t) = A + B\tanh(Ct)$$

- Expectation: slow enough expansion/big enough mass = adiabatic evolution; no field excitation (``particles'').
- Interested in adiabatic regime if we wish to ``create particles''.

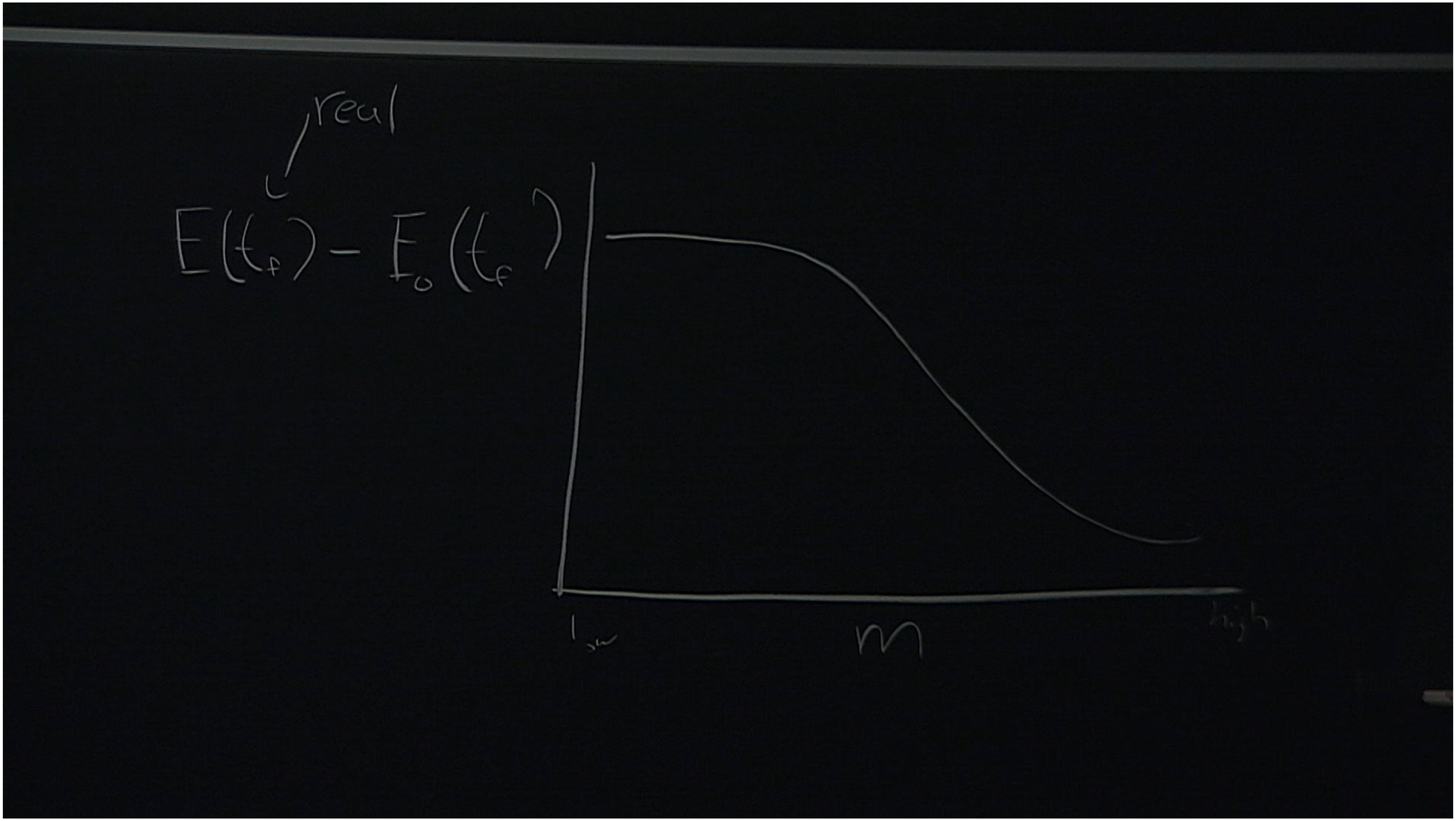


The wider this region:
-The more excitations we can model (more physics)
-The more excitations we *must* model (more compute time)

Evolution in RW spacetime



- Mass = 0.1, Lattice spacing = 1, maxchi = 200, d = 6.
- First we evolve in imaginary time at a fixed spatial slice to get initial vacuum state.
- Then evolve in real time with time-dependent Hamiltonian. The initial vacuum state is not stationary.
- Want coefs to be scale-separated and for the smallest to be numerically small (so adding more would have small effect on output).
- Again, need more Schmidt coefs and/or physical dimension.


real
E(t_f) - E_0(t_f)
low
m
high

Where are we?

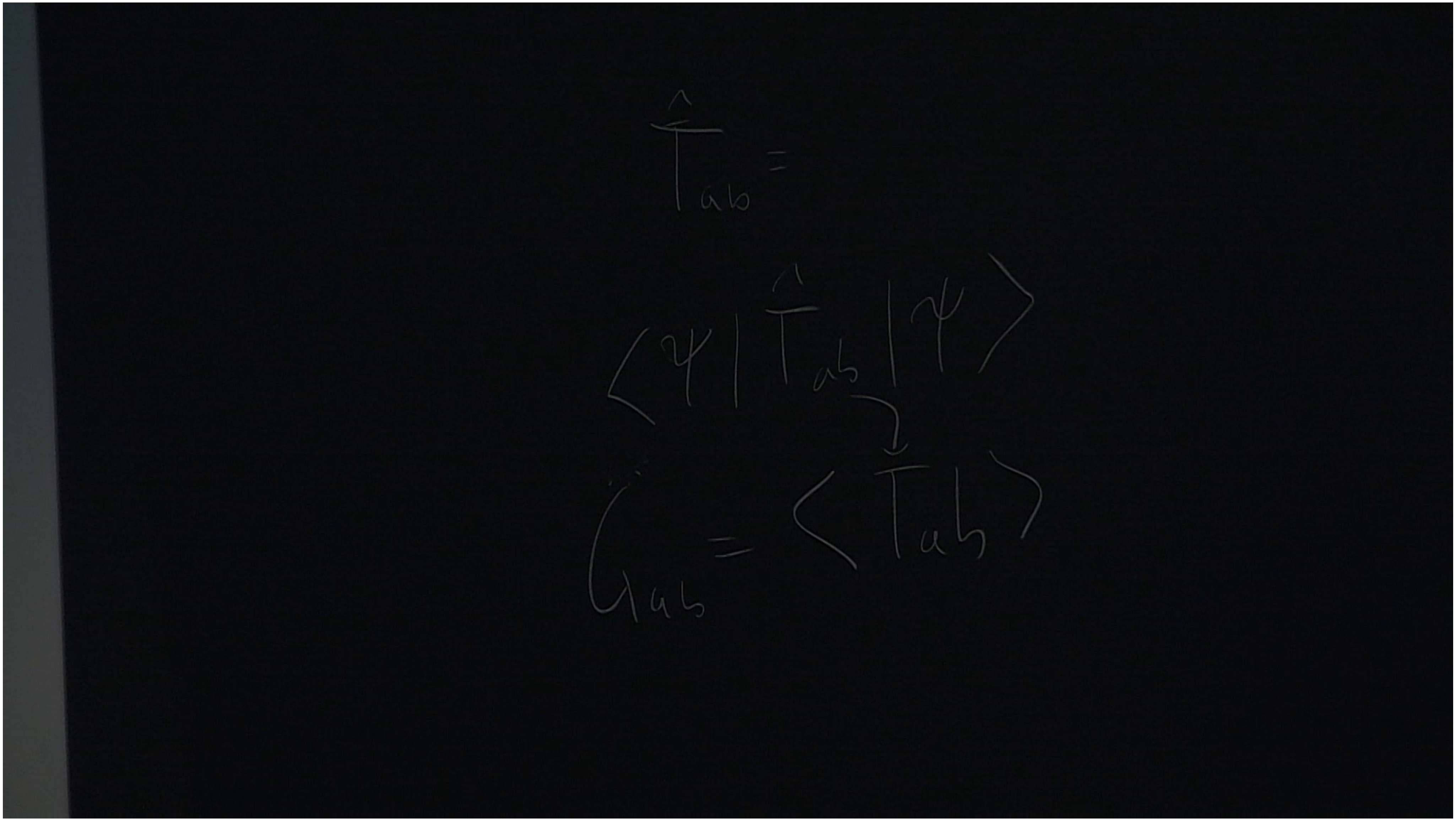
- Have successfully developed code capable of performing RW spacetime evolutions.
- Now exploring parameter space to determine where we get well-resolved non-adiabatic behaviour.

Future Directions

- Complete study of RW real scalar field:
  - What regimes can we efficiently simulate?
  - Extract spectrum and compare to analytic theory.
- Gauge choices for favourable entanglement production?
- Interaction terms?
- Unruh effect?
- Use stress-energy expectation as NR matter source.

Conclusion

- We are unifying MPS with 3+1 GR techniques to produce time-dependent, non-perturbative computations of QFT in curved spacetime.
- A code to do this for the RW universe is now complete.
- Using it to map out parameter space in terms of simulation expense and physics.
- Much to explore: slice-dependence of entanglement production, interactions, gravitational back-action…


$\hat{T}_{ab} =$
$\langle \psi | \hat{T}_{ab} | \psi \rangle$
$G_{ab} = \langle T_{ab} \rangle$