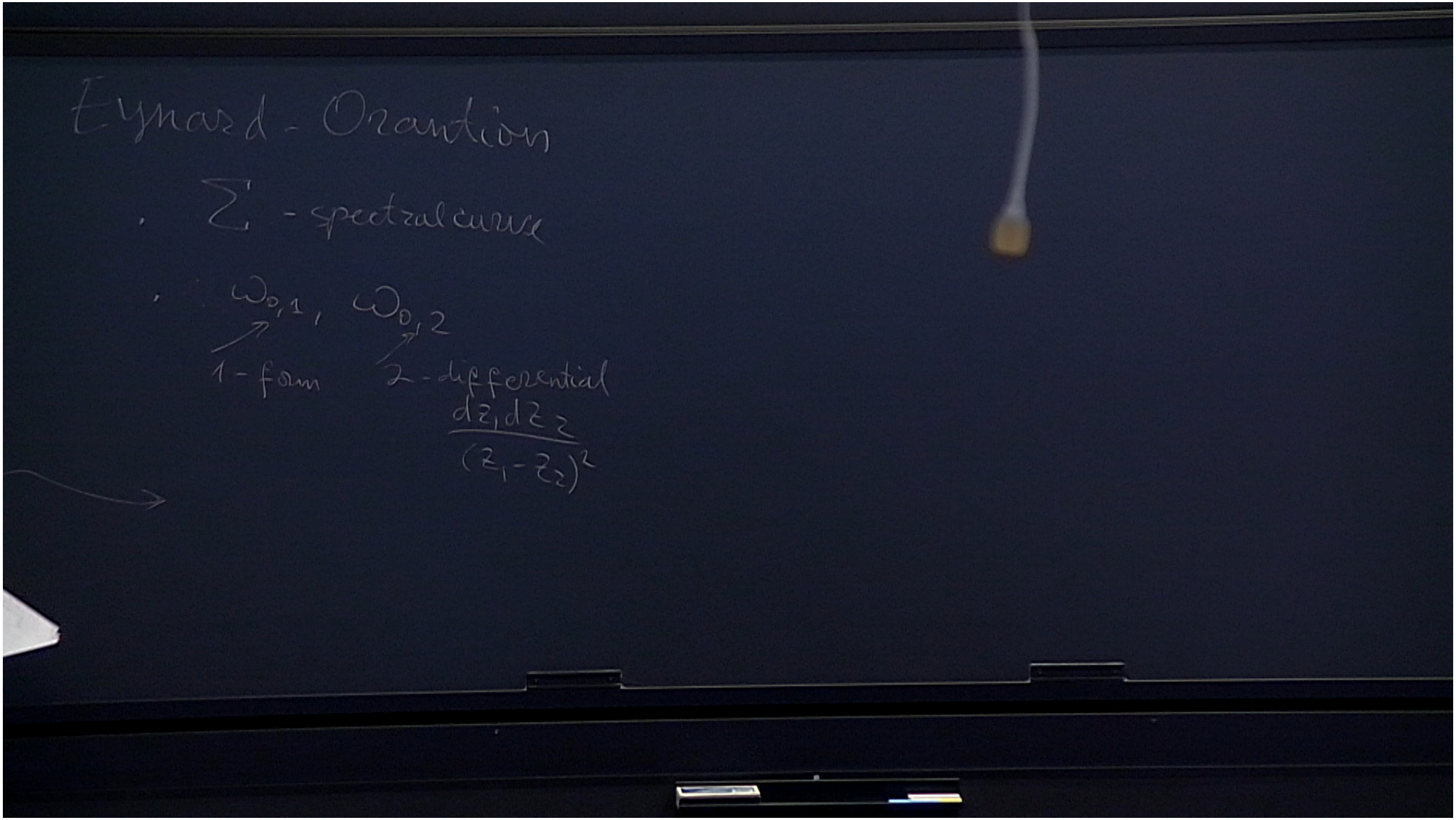
Title: Topological recursion and deformation quantization.

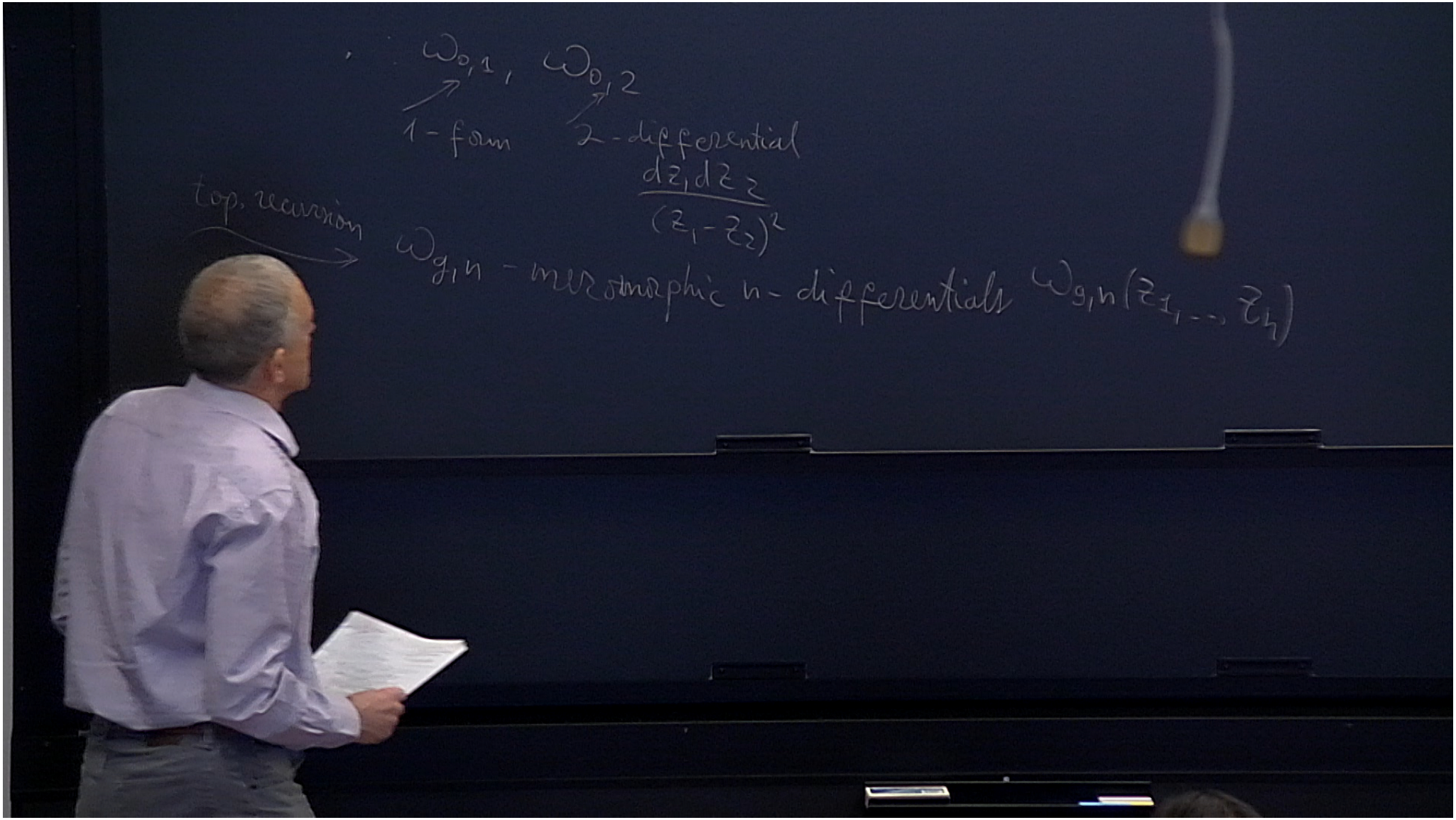
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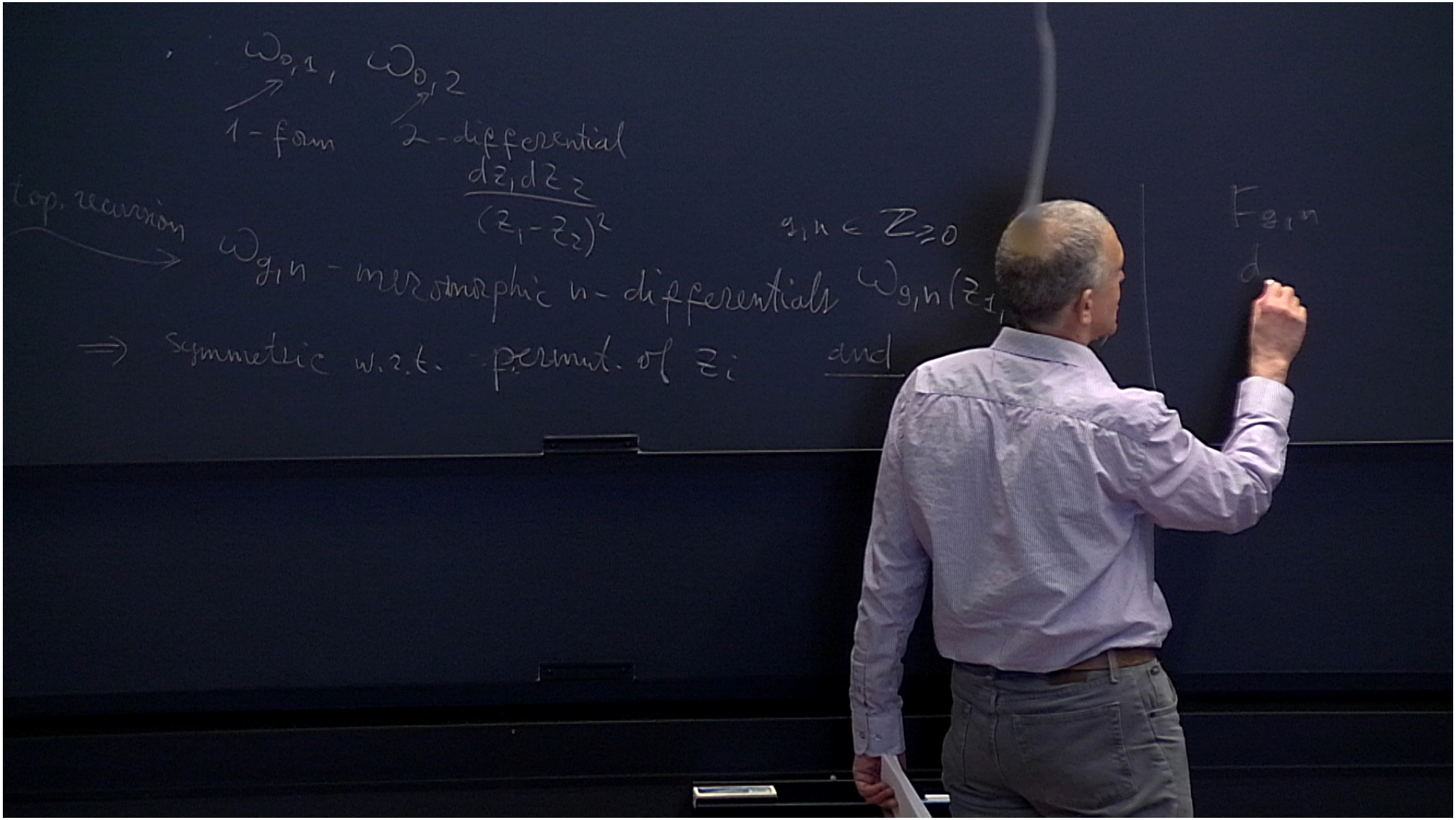
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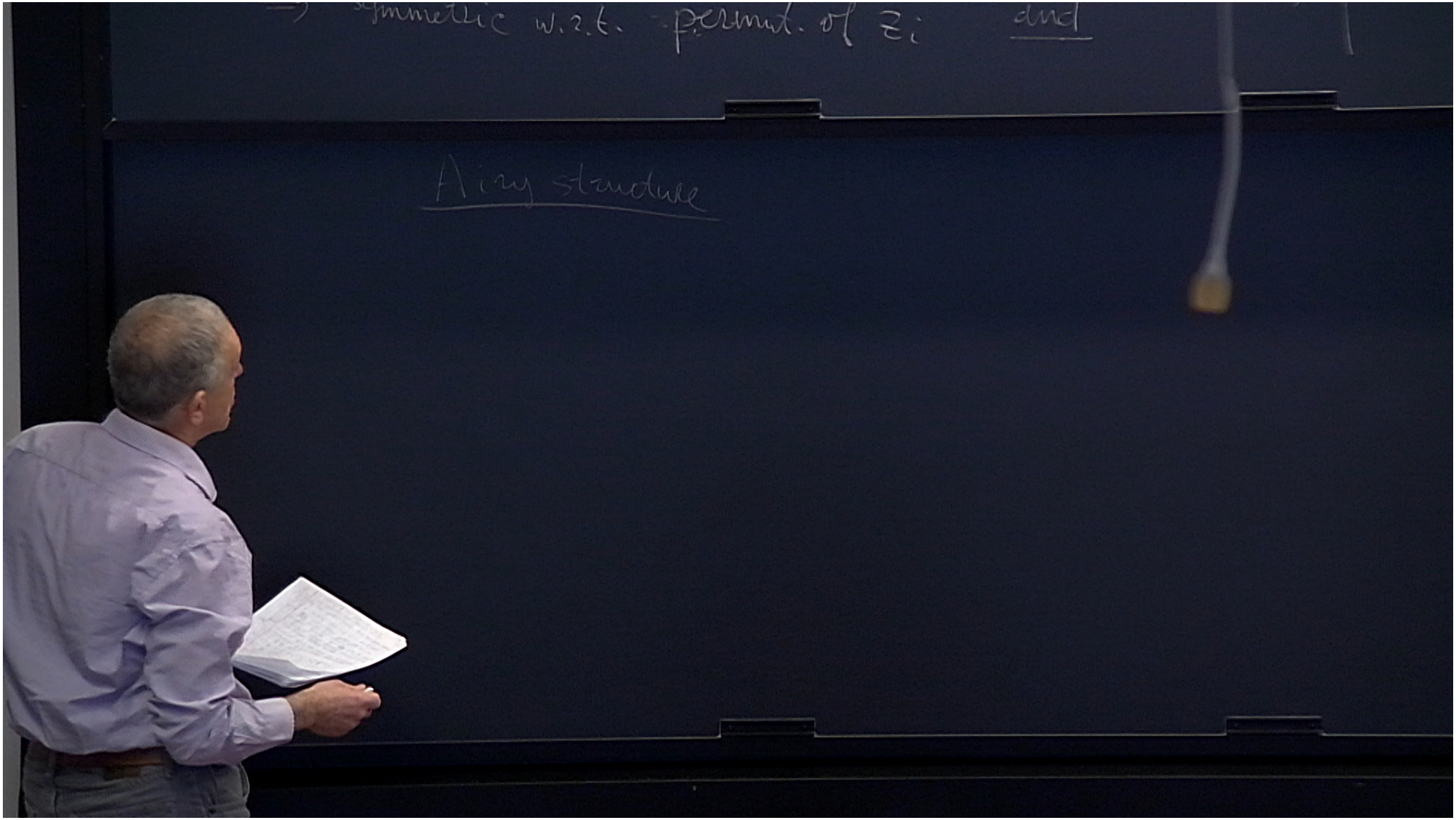
Abstract: About a decade ago Eynard and Orantin proposed a powerful computation algorithm
known as topological recursion. Starting with a ``spectral curve'' and some ``initial data''
(roughly, meromorphic differentials of order one and two) the topological recursion produces by induction
a collection of symmetric meromorphic differentials on the spectral curves parametrized by
pairs of non-negative integers (g,n) (g should be thought of as a genus and n as the number of punctures).
Despite of many applications of the topological recursion (matrix integrals, WKB expansions, TFTs, etc.etc.)
the nature of the recursive relations was not understood.

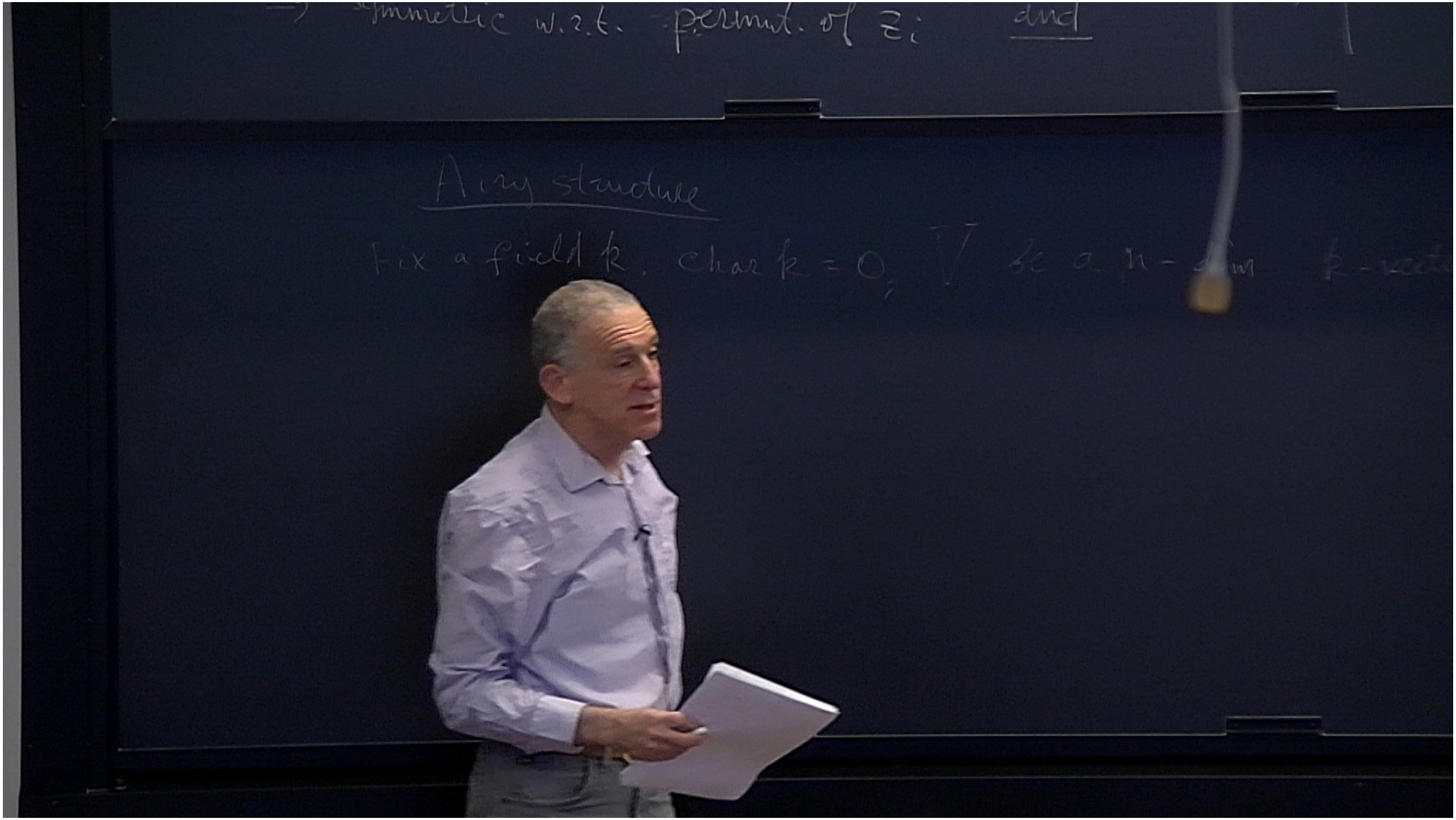
Recently, in a joint work with Maxim Kontsevich we found a simple underlying structure of the recursive relations of Eynard and Orantin. We call it ``Airy structure''. In this talk I am going to define this notion and explain how the recursive relations of Eynard and Orantin follow from the quantization of
a quadratic Lagrangian subvariety in a symplectic vector space.

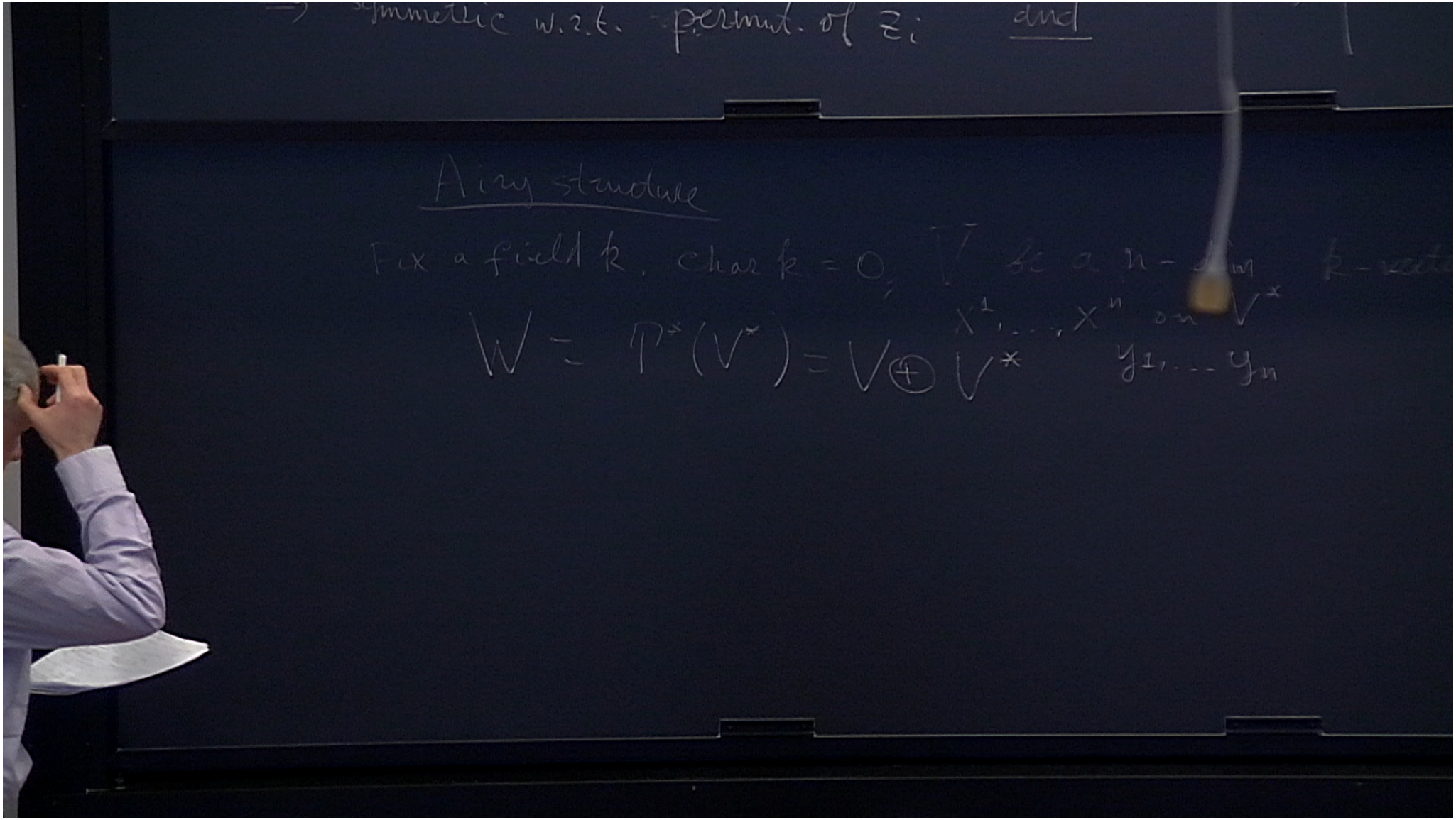

Eynard - Orantion
. Σ - spectral curve
. $\omega_{0,1}$, $\omega_{0,2}$
1 - form
2 - differential
$\frac{dz_1 dz_2}{(z_1 - z_2)^2}$


$\omega_{0,1}$, $\omega_{0,2}$
1-form
2-differential
$\frac{dz_1 dz_2}{(z_1 - z_2)^2}$
top. recursion
$\omega_{g,n}$ - meromorphic n-differentials $\omega_{g,n}(z_1, \ldots, z_n)$


$\omega_{0,1}$, $\omega_{0,2}$
1-form
2-differential
$\frac{dz_1 dz_2}{(z_1-z_2)^2}$
top. recursion
$\omega_{g,n}$ - meromorphic n-differentials
$g,n \in \mathbb{Z}_{\geq 0}$
$\Rightarrow$ symmetric w.r.t. permut. of z_i
and
$F_{g,n}$


symmetric w.r.t. permut. of z_i and
Airy stucture


symmetric w.r.t. permut. of z_i and
Airy structure
Fix a field k, $\mathrm{char}\, k = 0$;


symmetric w.r.t. permut. of z_i and
Airy structure
Fix a field k, $\mathrm{char}\, k = 0$; V be a n-dim k-vect
$x^1, \dots, x^n$ on V^*
$W = T^*(V^*) = V \oplus V^*$
$y_1, \dots y_n$

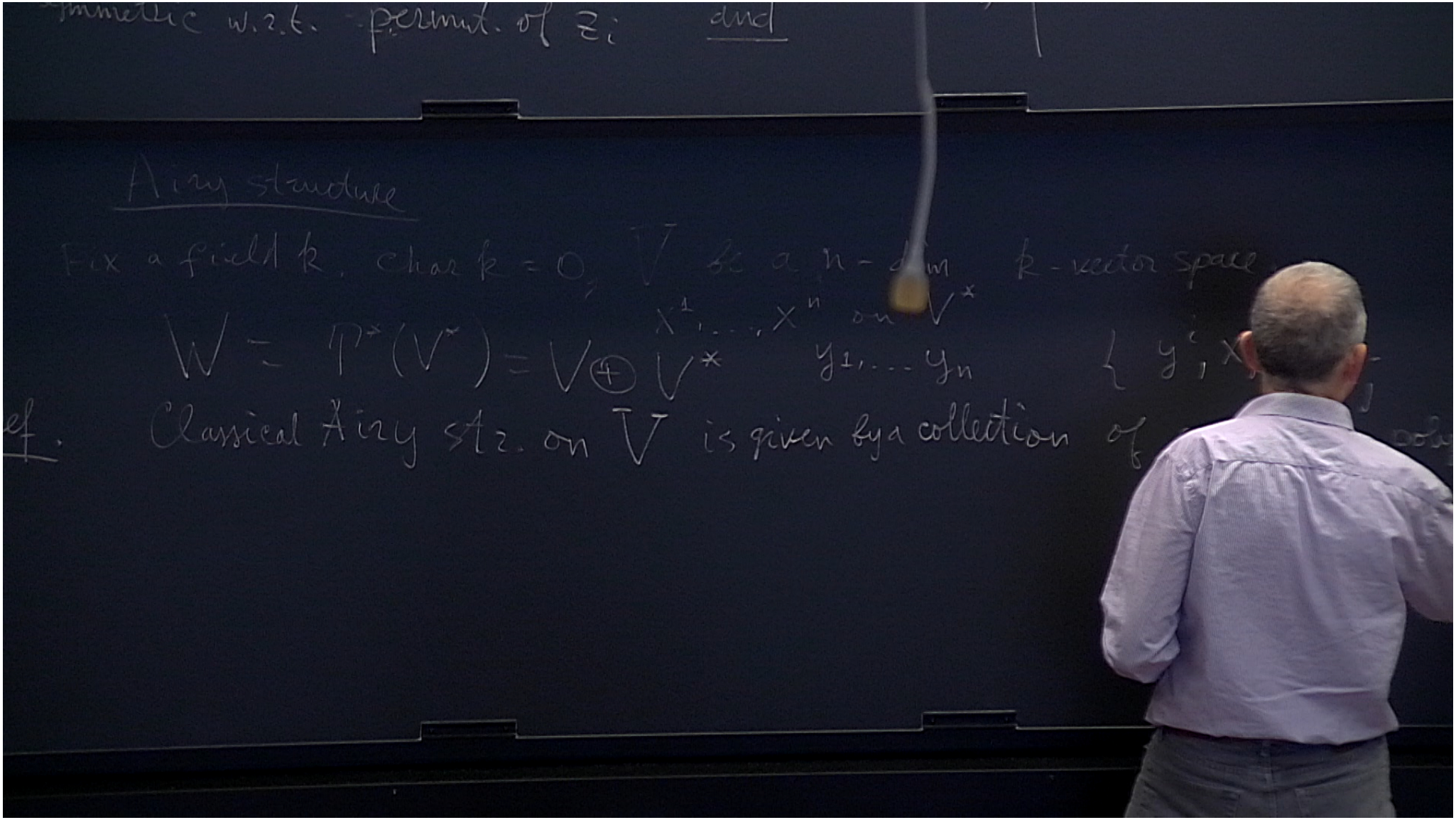
symmetric w.r.t. permut. of z_i and

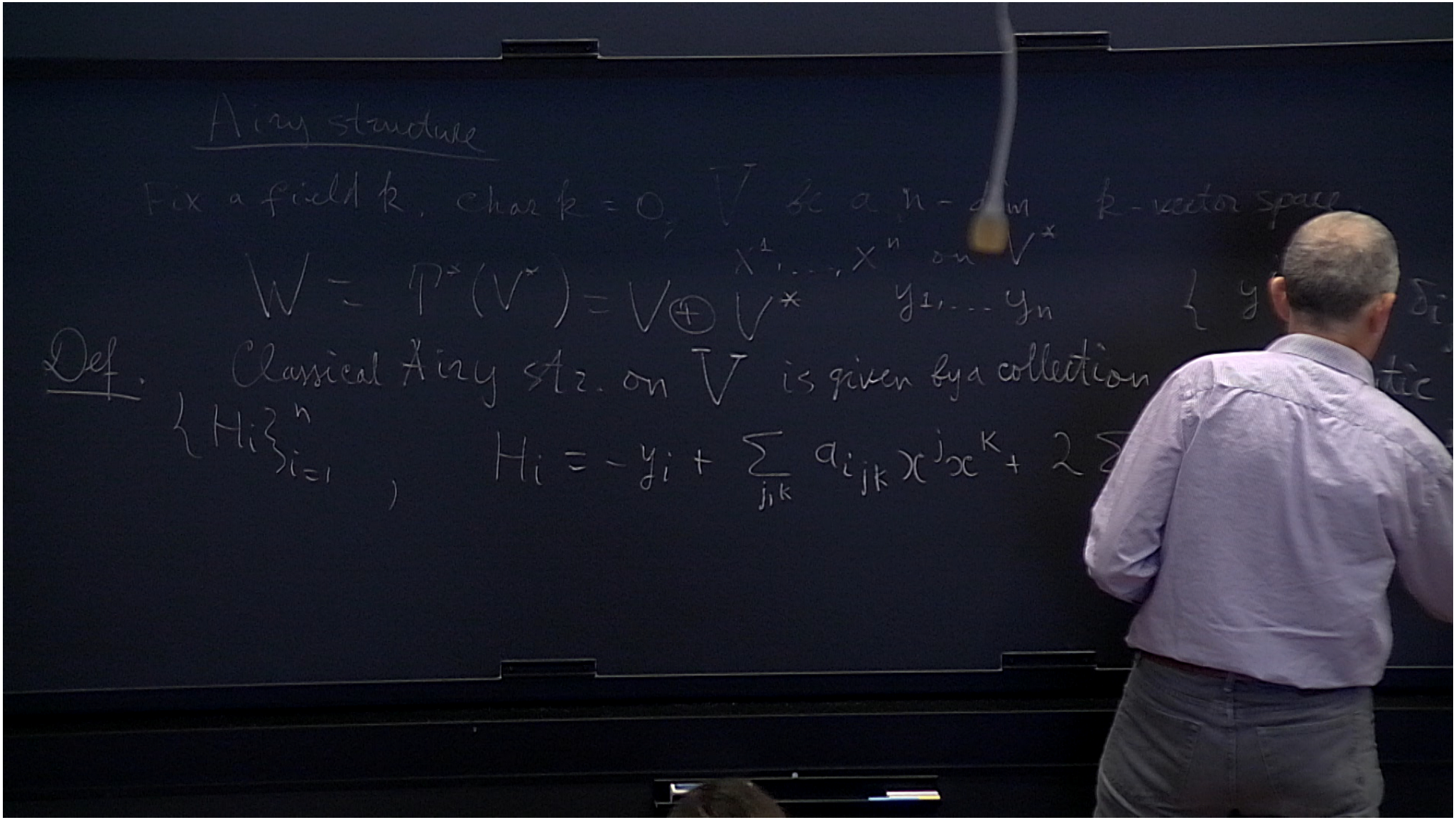
Airy structure

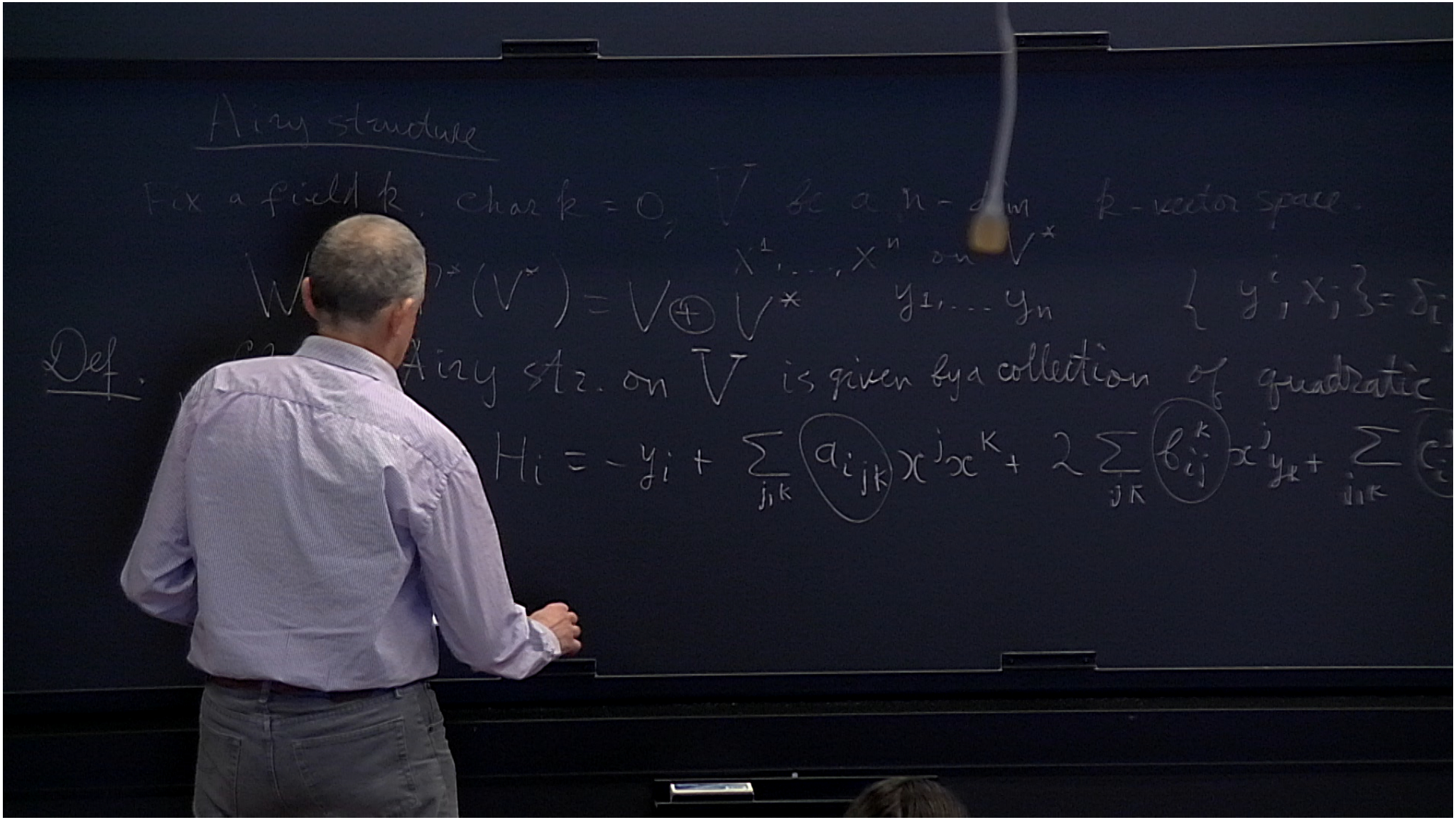
Fix a field k, $\mathrm{char}\, k = 0$. V is a n-dim k-vector space.

$x^1, \dots, x^n$ on V^*

$W = T^*(V^*) = V \oplus V^*$ $\quad y_1, \dots y_n$ $\quad \{ y_i, x_j \} = \delta_{ij}$


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$\{ y^i, x$
Classical Airy str. on V is given by a collection of


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Def. Classical Airy str. on V is given by a collection
{Hi}_{i=1}^n ,
Hi = -y_i + Σ_{j,k} a_{ijk} x^j x^k + 2


Airy structure
Fix a field k, char k = 0, V be a n-dim k-vector space.
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Def: Airy str. on V is given by a collection of quadratic
H_i = -y_i + Σ a_ijk x^j x^K + 2 Σ b_ij^k x^j y_k + Σ

Airy structure

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Def. Classical Airy str. on V is given by a collection of quadratic

$\{H_i\}_{i=1}^n$, $H_i = -y_i + \sum_{j,k} a_{ijk} x^j x^k + 2 \sum_{j,k} b_{ij}^k x^j y_k + \sum_{j,k} c$

s.t. $\bigoplus_{i=1}^n k \cdot H_i$

Airy structure

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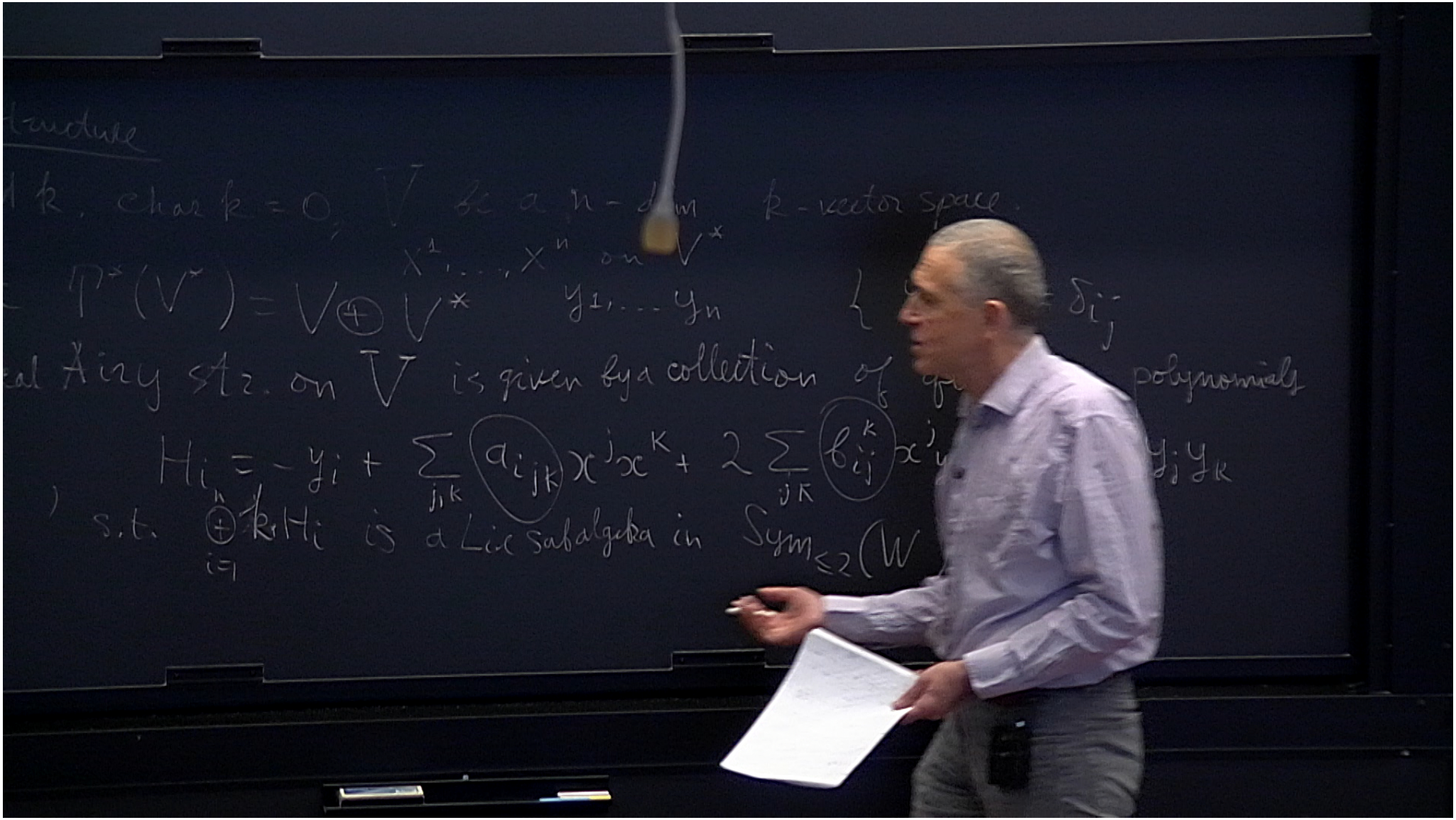
$$W = T^*(V^*) = V \oplus V^* \qquad y_1, \dots y_n \qquad \{y^i, x_j\} = \delta_i$$

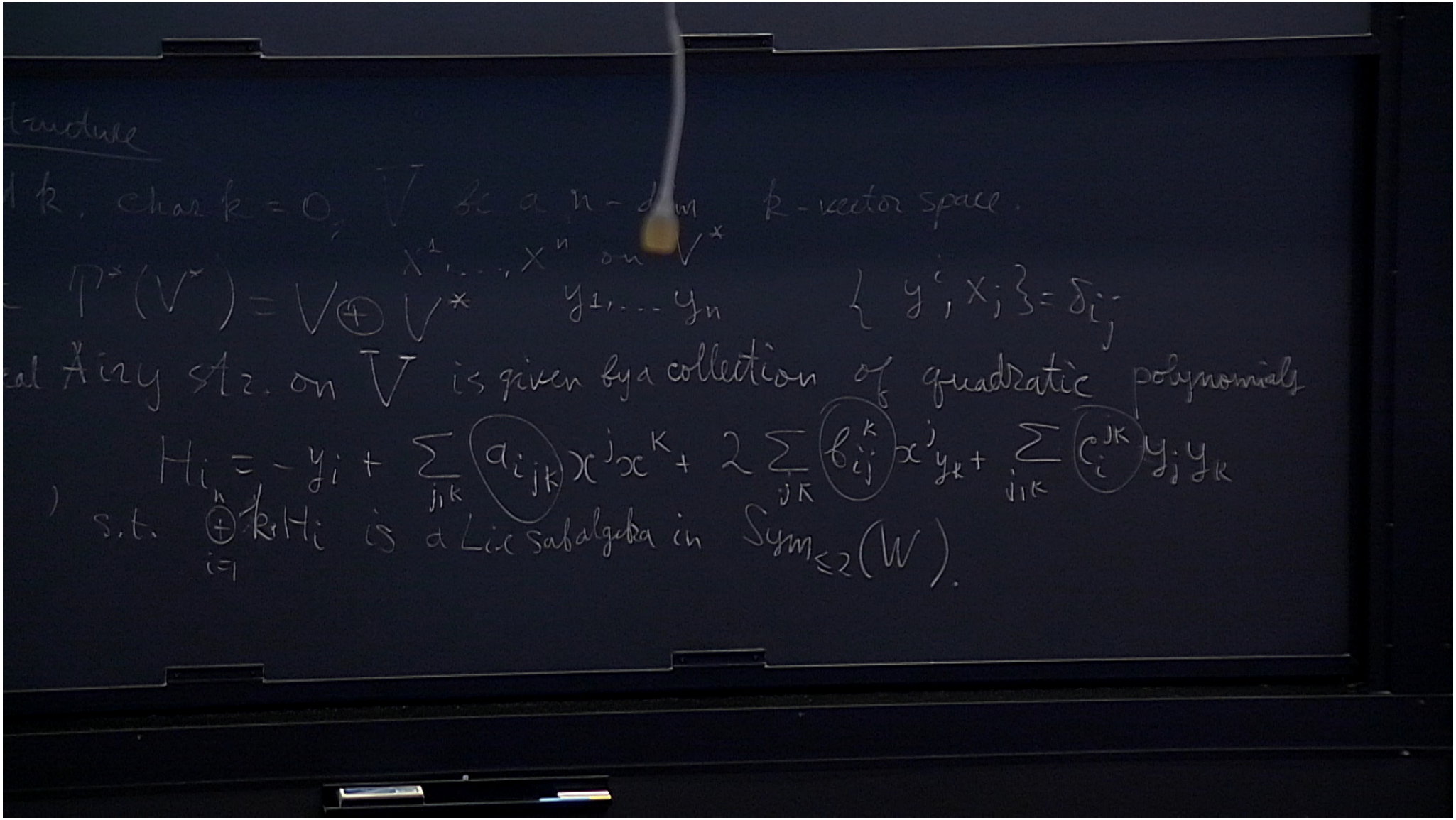
Def. Classical Airy str. on V is given by a collection of quadratic

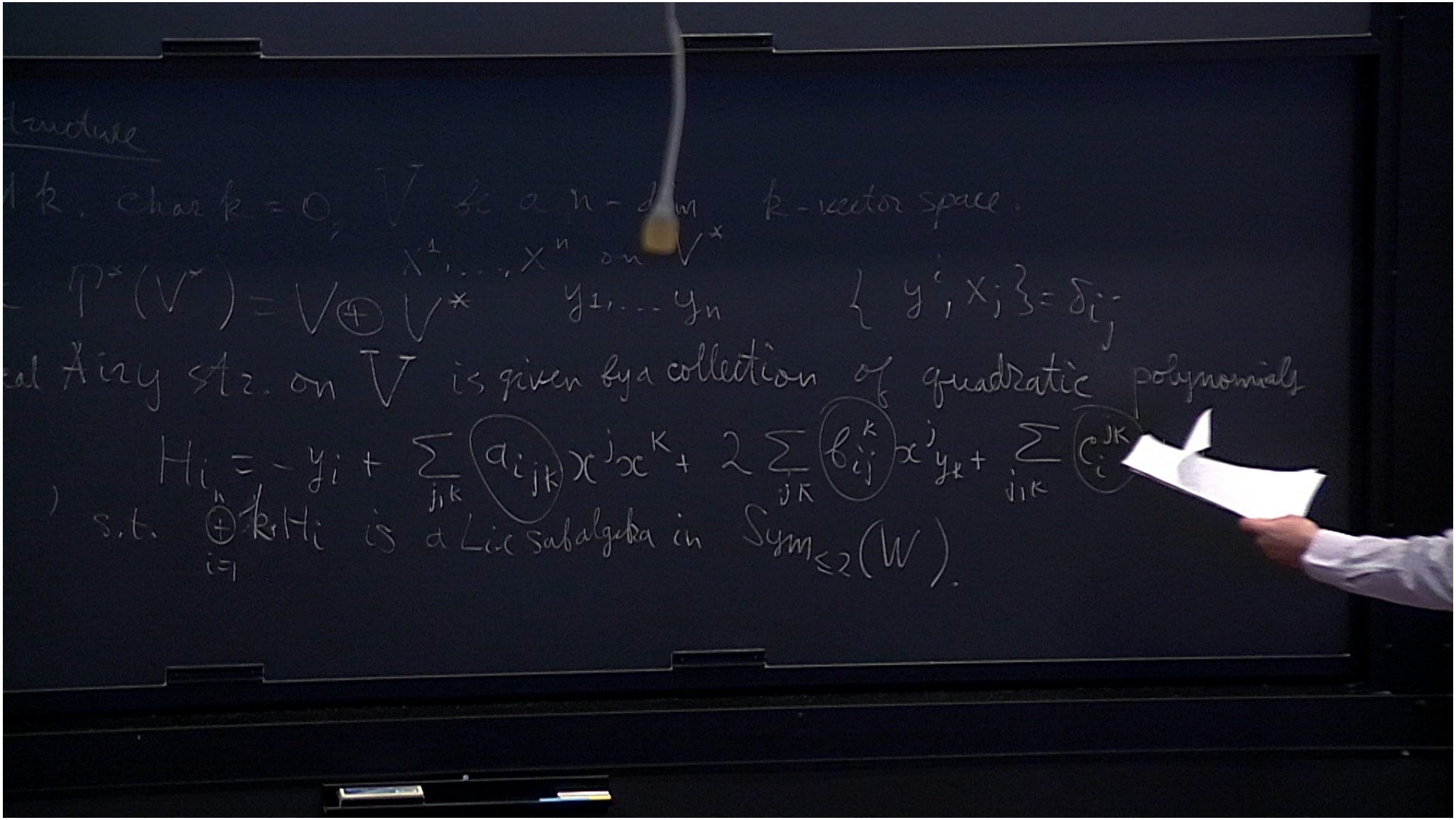
$\{H_i\}_{i=1}^n$,

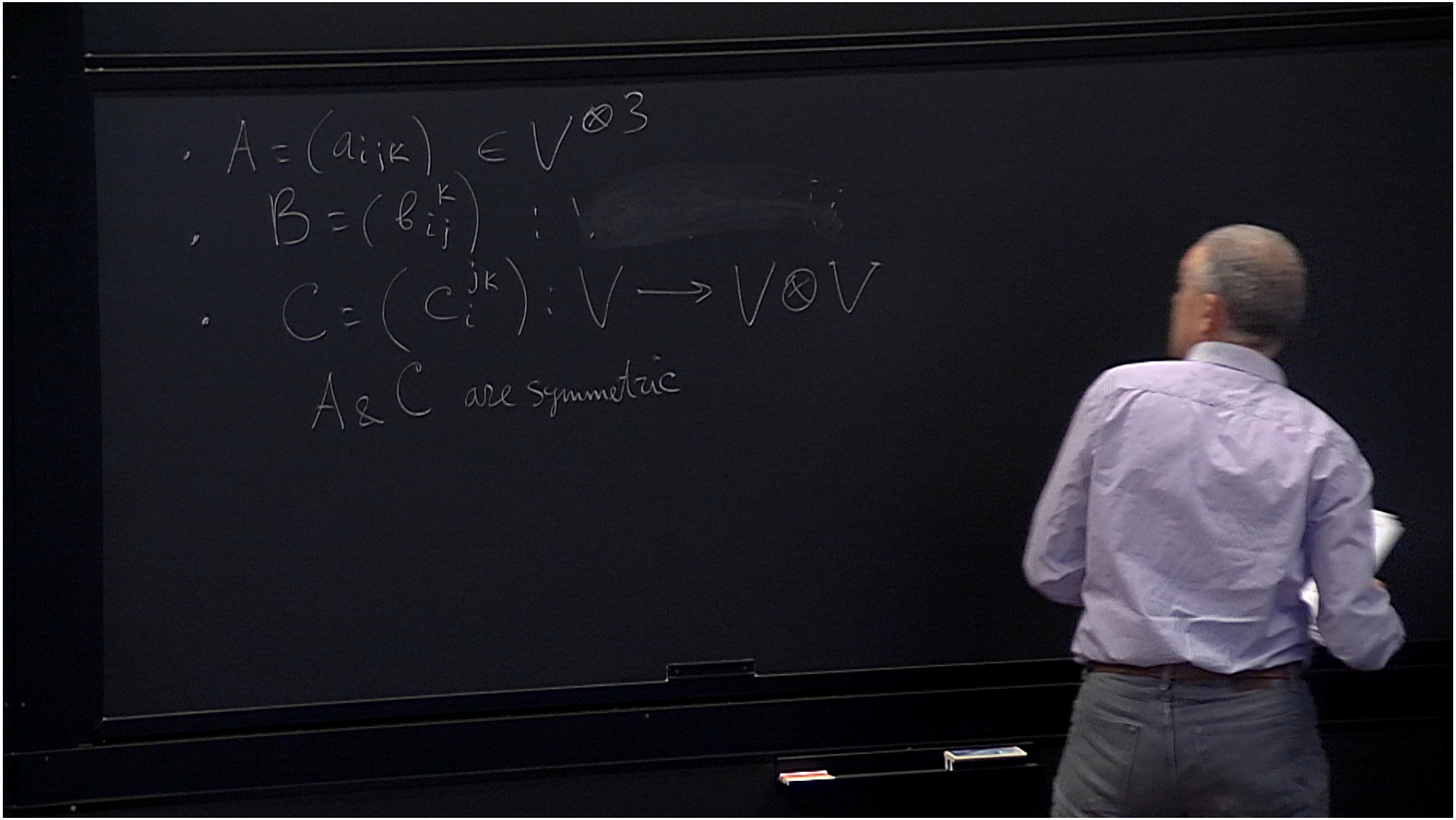
$$H_i = -y_i + \sum_{j,k} a_{ijk} x^j x^k + 2 \sum_{j,k} b_{ij}^k x^j y_k + \sum_{j,k} c$$

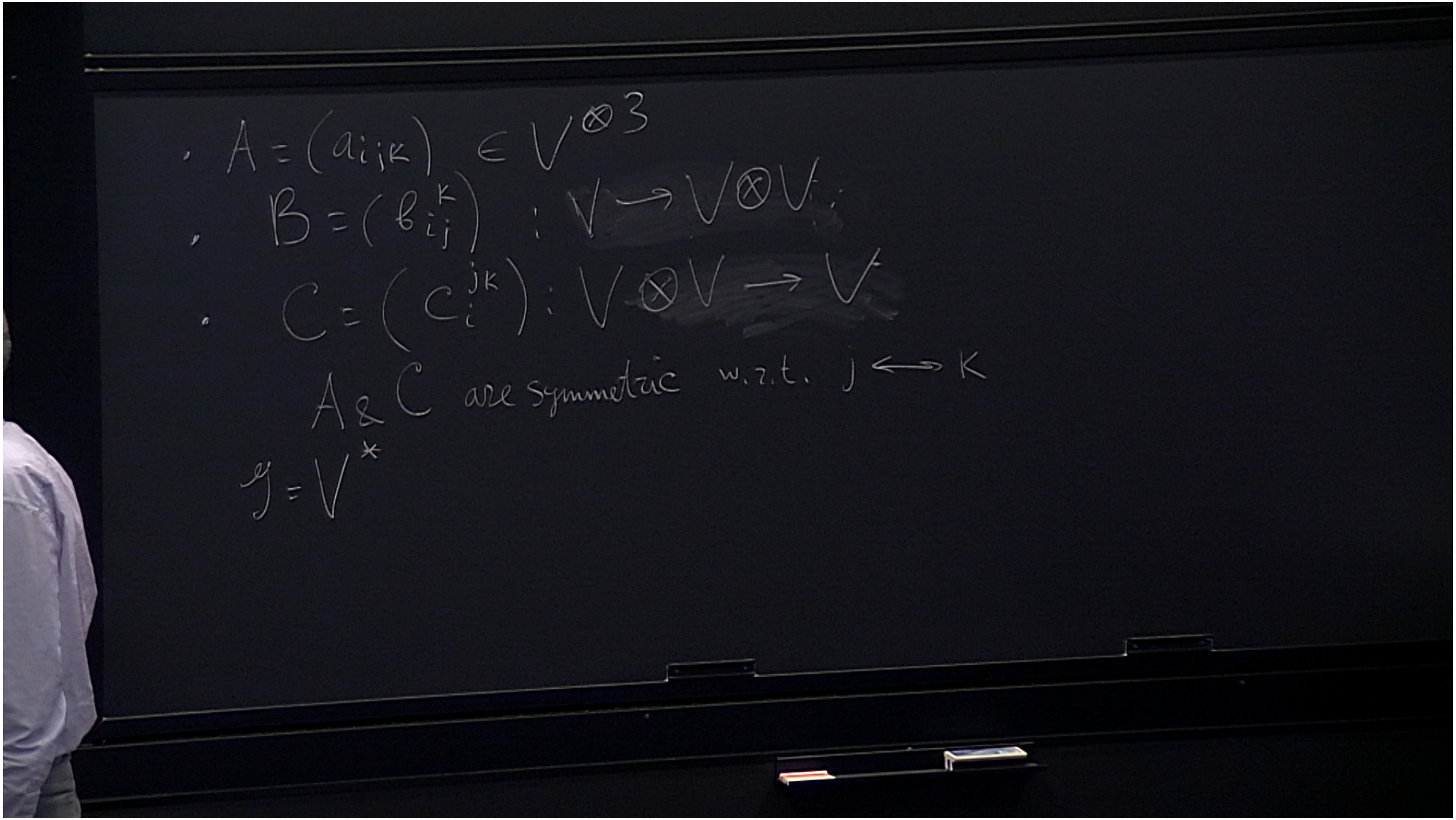
s.t. $\bigoplus_{i=1}^n k \cdot H_i$ is a Lie subalgebra in $\mathrm{Sym}_{\leq 2}(W)$.

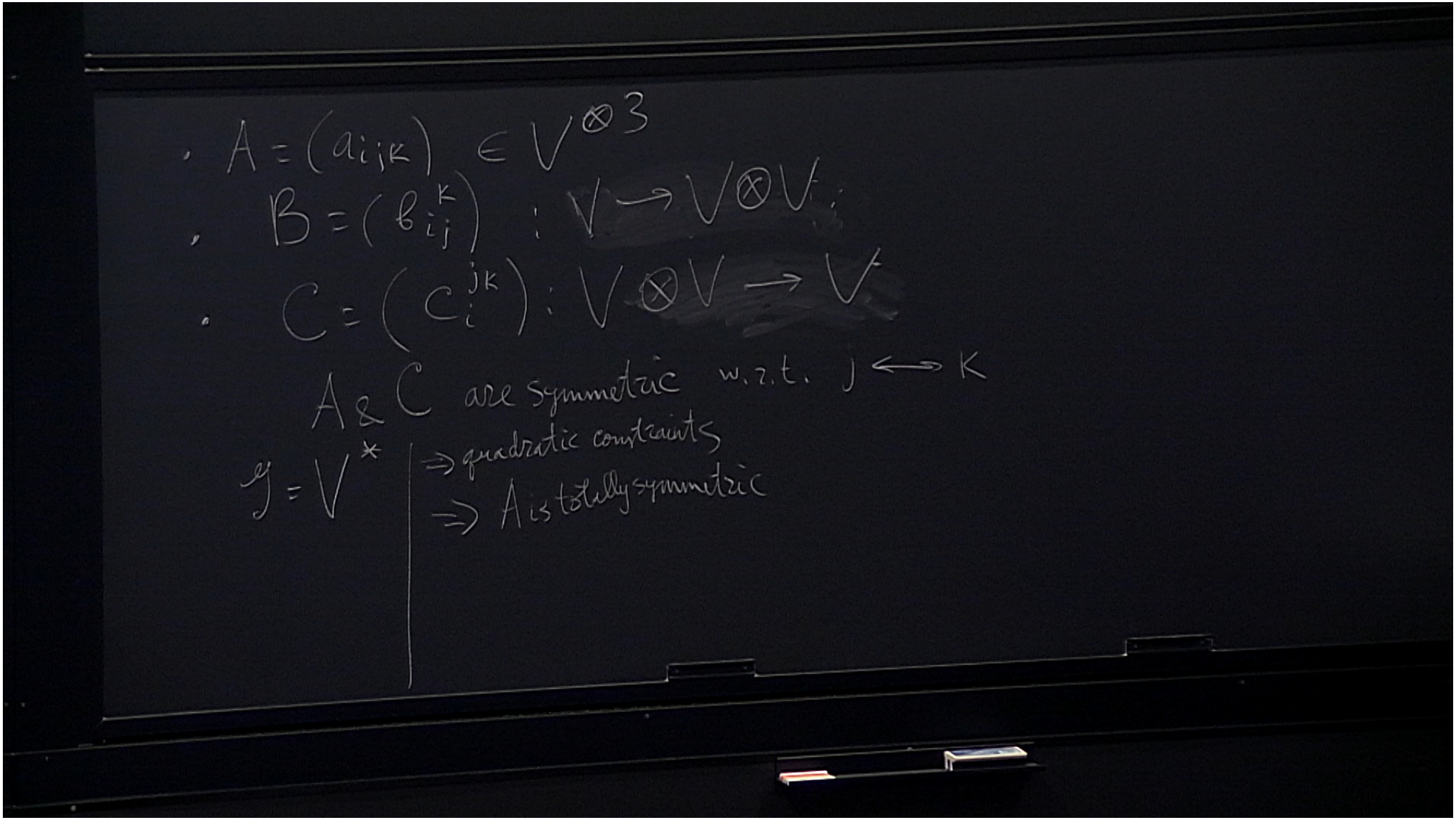

structure
k, char k = 0, V be a n-dim k-vector space.
x^1, ..., x^n on V*
T*(V) = V ⊕ V*
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structure
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$A = (a_{ijk}) \in V^{\otimes 3}$
$B = (b_{ij}^{k})$:
$C = (c_{i}^{jk}) : V \longrightarrow V \otimes V$
A & C are symmetric


$A = (a_{ijk}) \in V^{\otimes 3}$
$B = (b_{ij}^{\ k}) : V \to V \otimes V$
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$\Rightarrow$ quadratic constraints
$\Rightarrow$ A is totally symmetric

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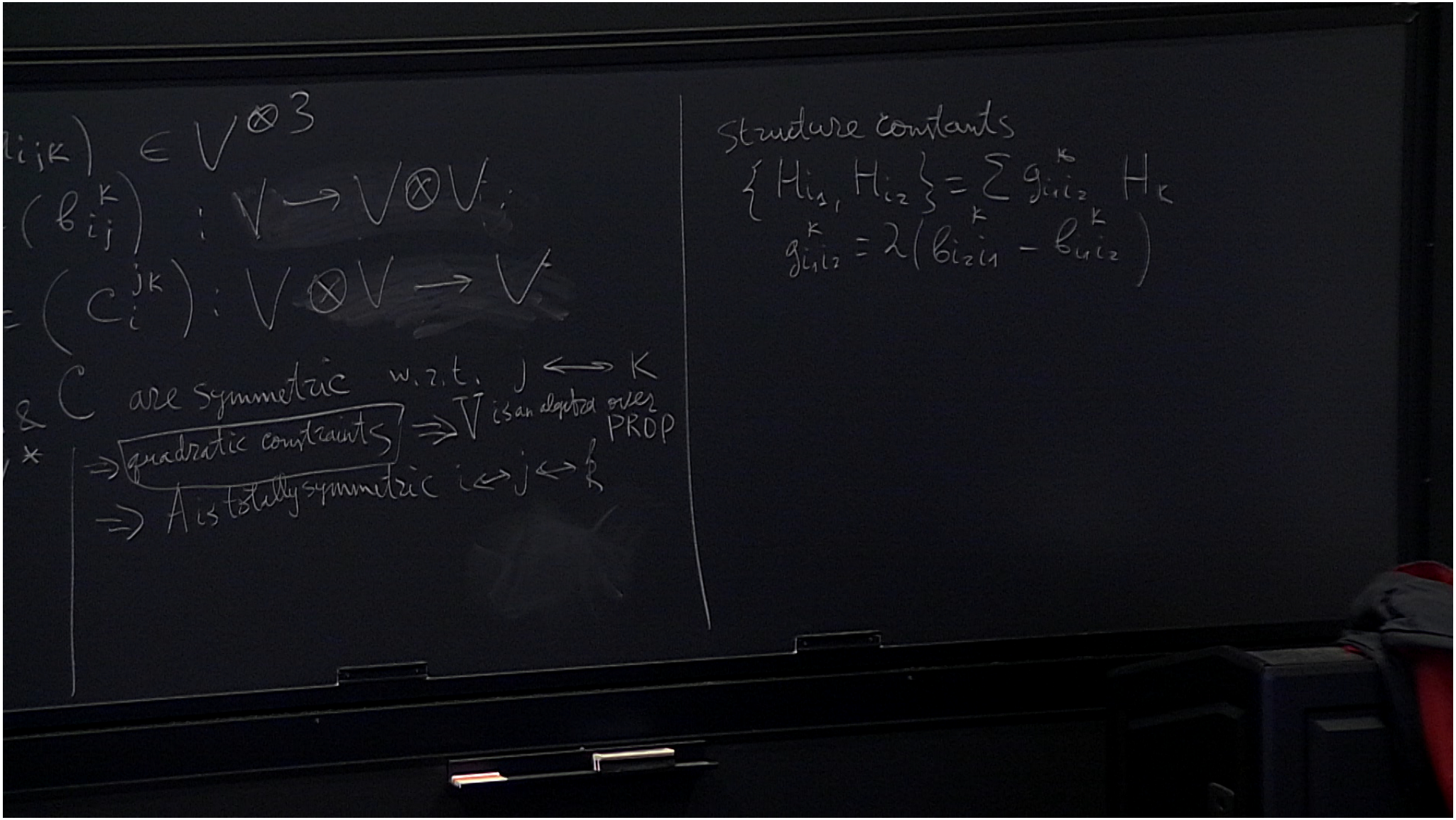
$\Rightarrow$ quadratic constraints $\Rightarrow \bar{V}$ is an algebra over PROP

$\Rightarrow A$ is totally symmetric $i \leftrightarrow j \leftrightarrow k$

Structure constants

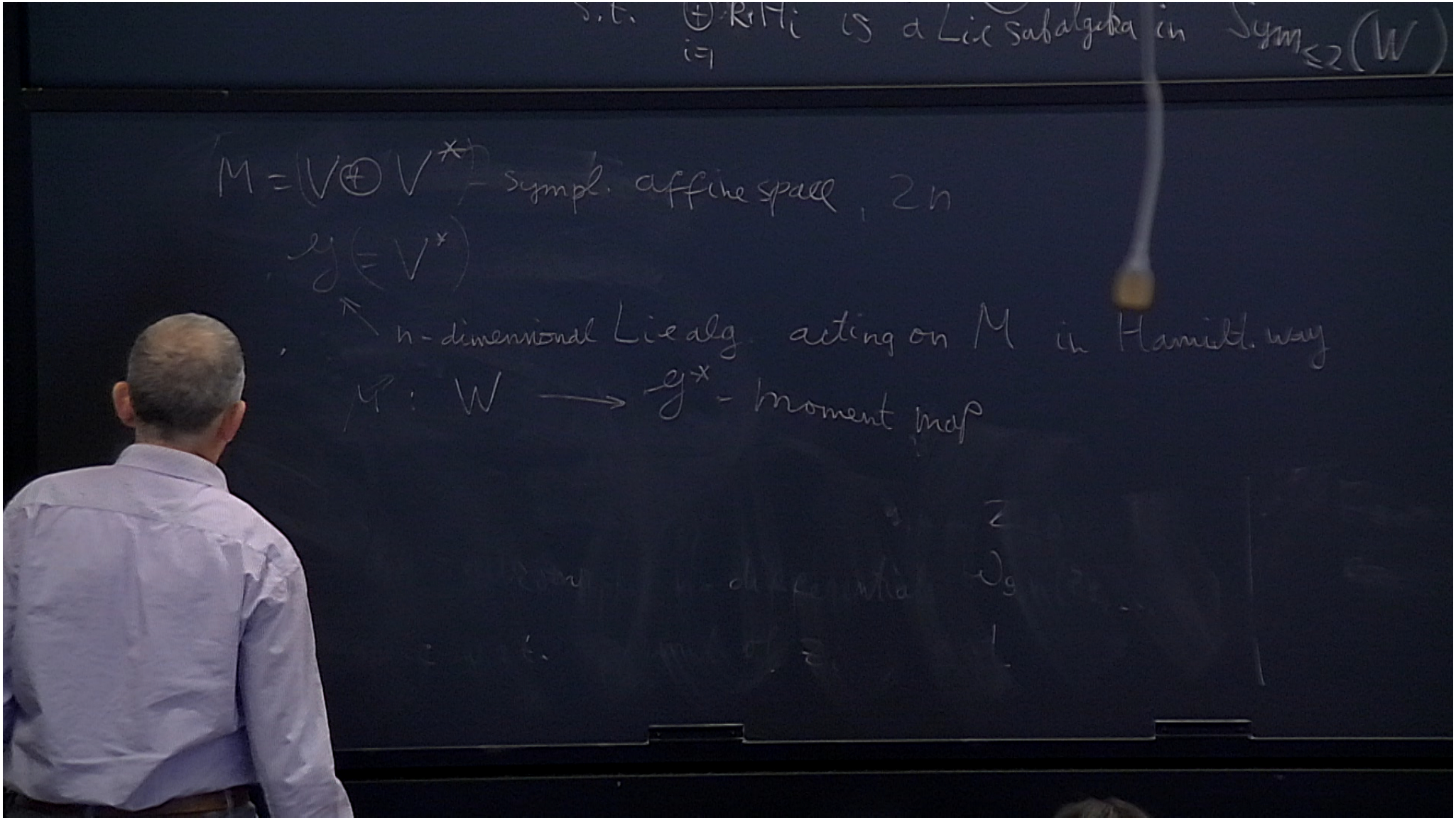
$$\{H_{i_1}, H_{i_2}\} = \sum g_{i_1 i_2}^{K} H_K$$

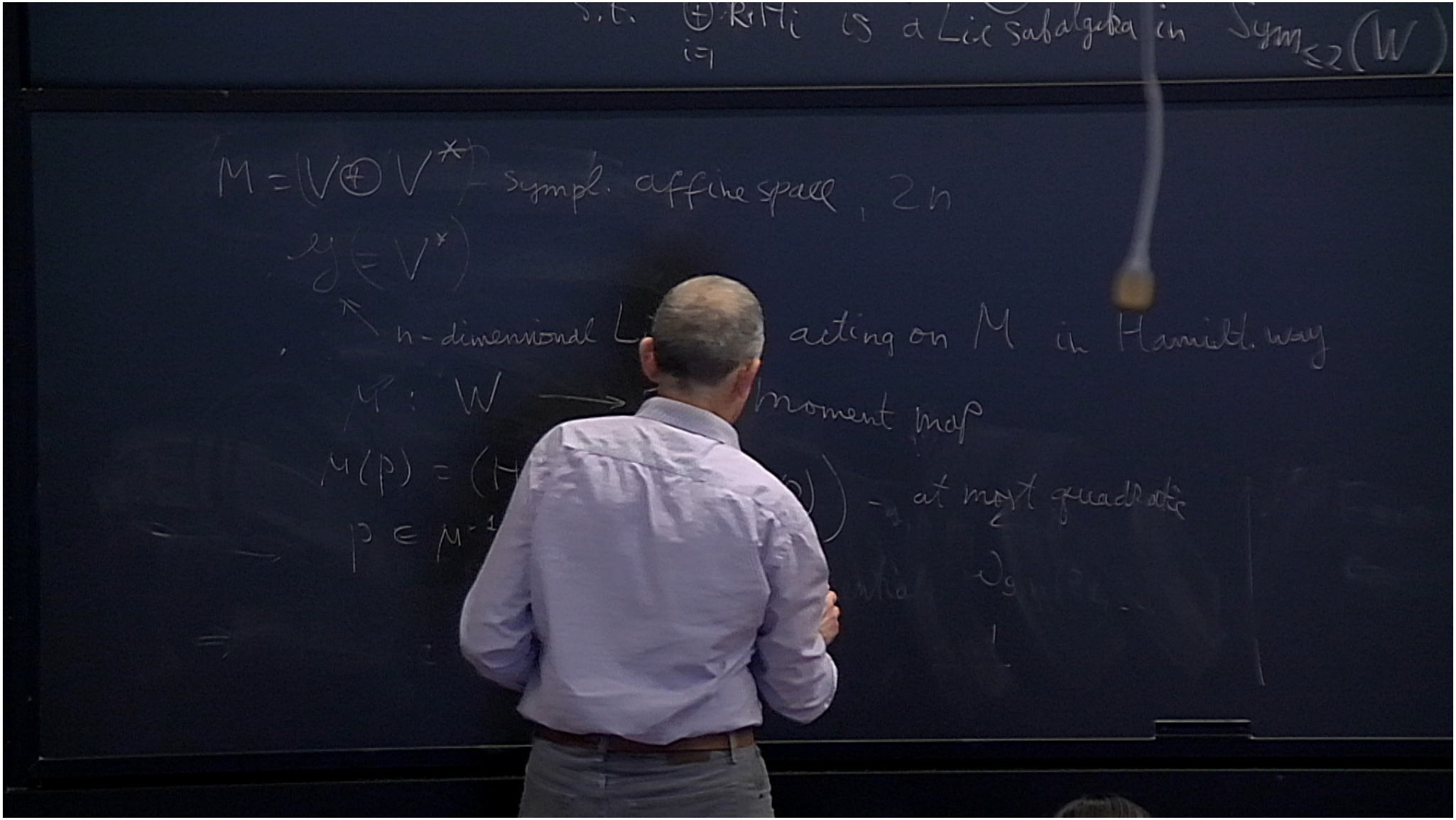
$$g_{i_1 i_2}^{K} = 2\left(b_{i_2 i_1}^{K} - b_{i_1 i_2}^{K}\right)$$

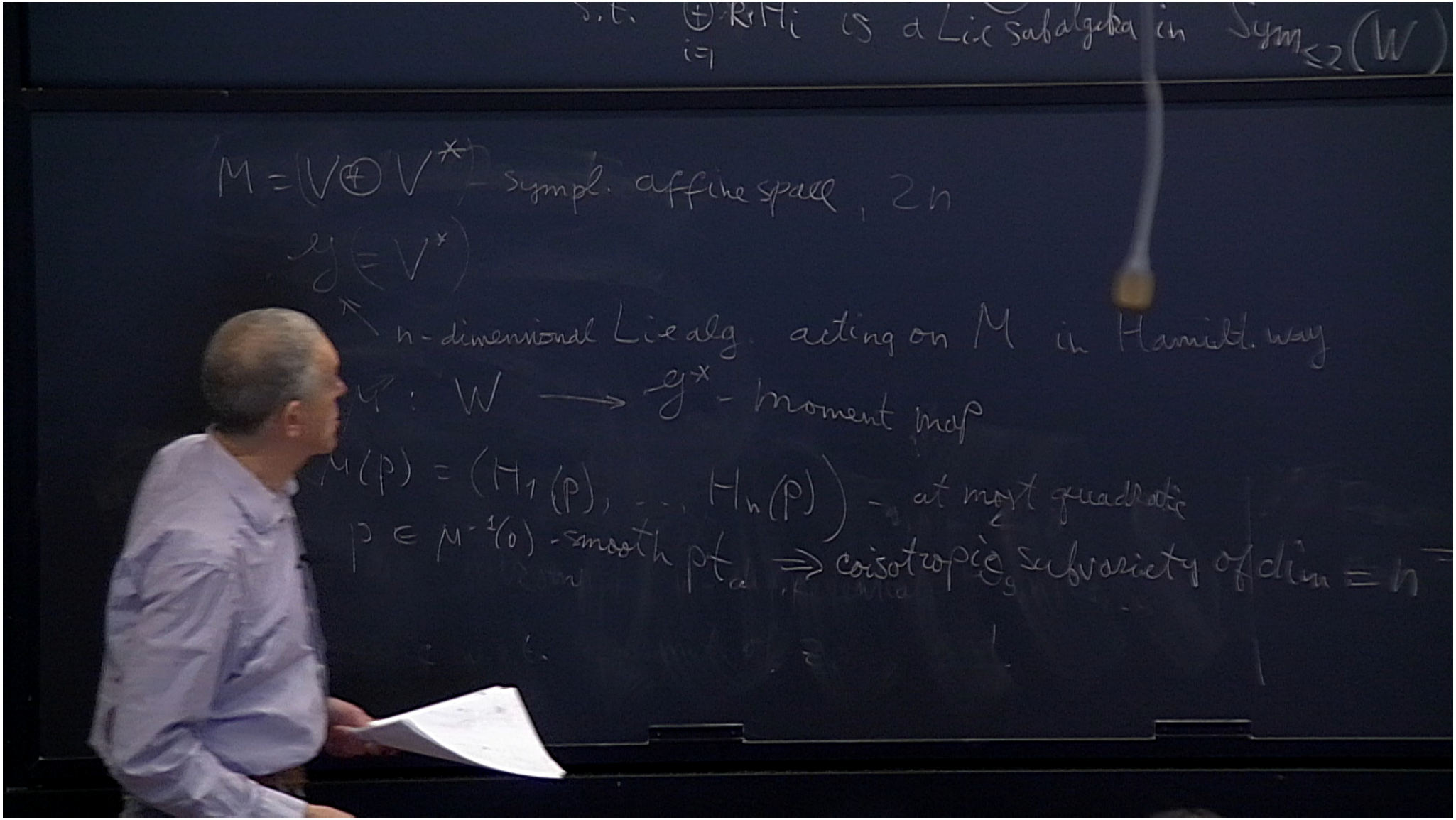

$(b_{ij}^{k}) : V \to V \otimes V$
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quadratic constraints $\Rightarrow \bar{V}$ is an algebra over PROP
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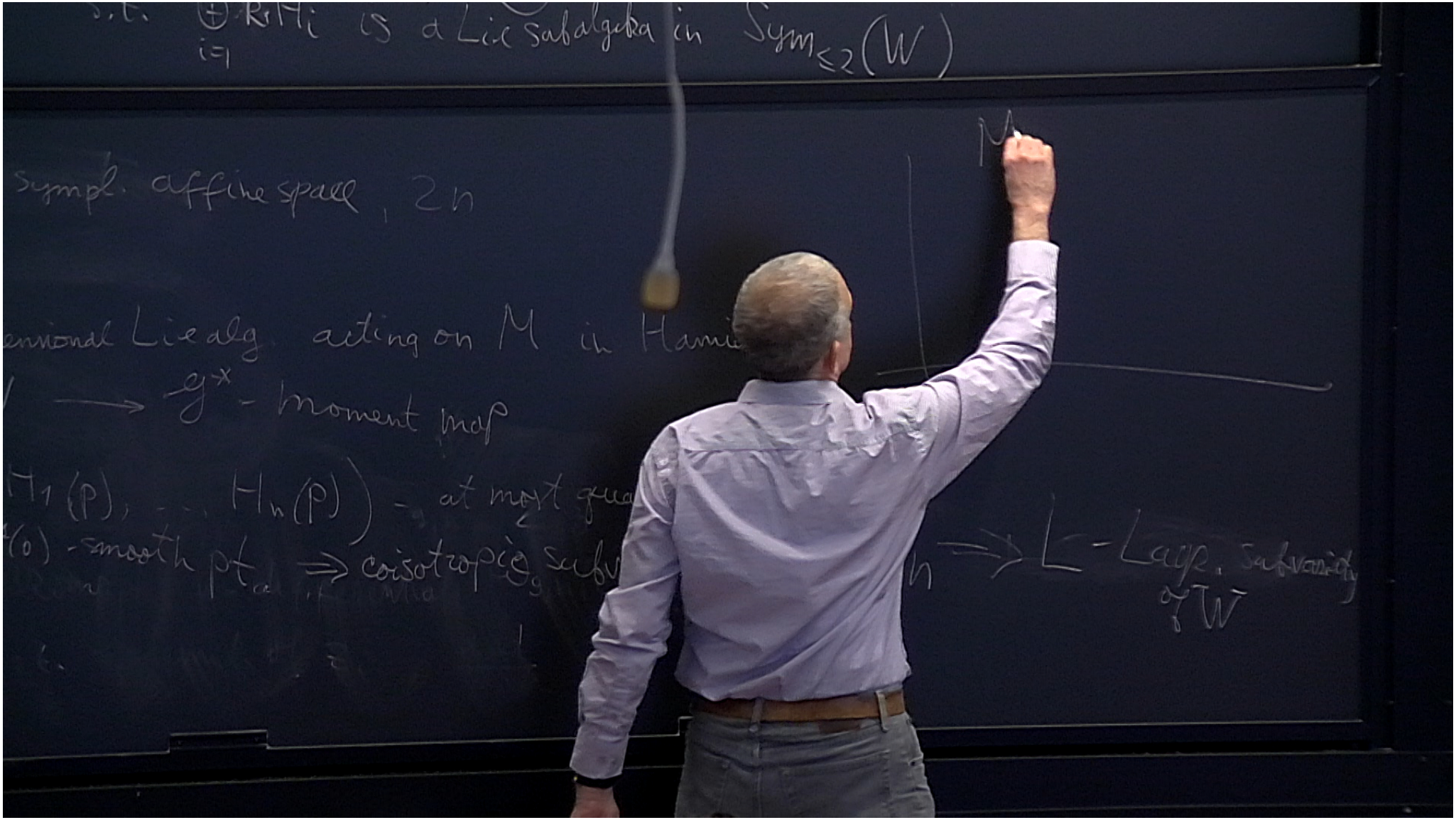
s.t. $\bigoplus_{i=1} R_i H_i$ is a Lie subalgebra in $\mathrm{Sym}_{\leq 2}(W)$

$M = V \oplus V^*$ sympl. affine space, $2n$


$\bigoplus_{i=1} \mathbb{R} H_i$ is a Lie subalgebra in $\mathrm{Sym}_{\leq 2}(W)$
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$\mathfrak{g} (= V^*)$
n-dimensional Lie alg acting on M in Hamilt. way
$\mu : W \longrightarrow \mathfrak{g}^*$ - moment map


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moment map
$\mu(p) = (H$
$p \in \mu^{-1}$
$))$
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s.t. $\bigoplus_{i\geq 1} \mathbb{R}H_i$ is a Lie subalgebra in $\mathrm{Sym}_{\leq 2}(W)$
$M = (V \oplus V^*)$ sympl. affine space, $2n$
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$\mu : W \longrightarrow \mathfrak{g}^*$ - moment map
$\mu(p) = (H_1(p), \ldots, H_n(p))$ - at most quadratic
$p \in \mu^{-1}(0)$ - smooth pt $\Rightarrow$ coisotropic subvariety of dim $= n$


$\bigoplus_{i=1} \mathbb{R}H_i$ is a Lie subalgebra in $Sym_{\leq 2}(W)$
sympl. affine space, $2n$
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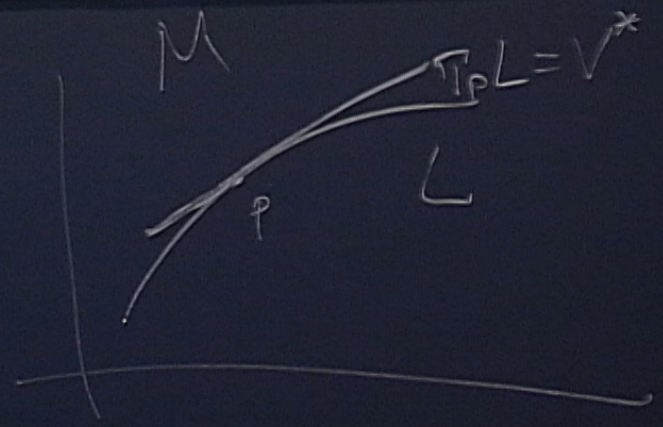
affine space, $2n$

Lie alg. acting on M in Hamilt. way

$\longrightarrow g^*$ - moment map

$H_n(p))$ - at most quadratic

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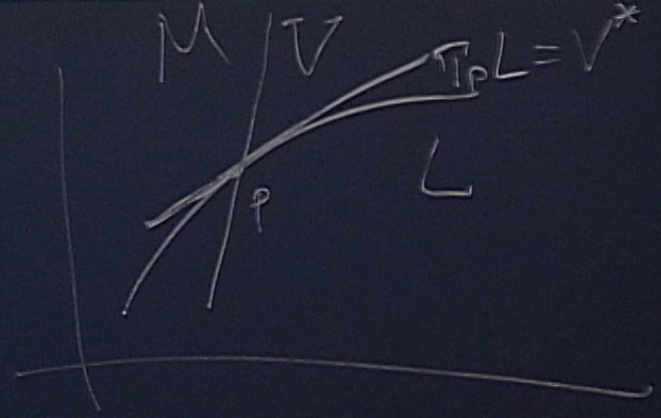
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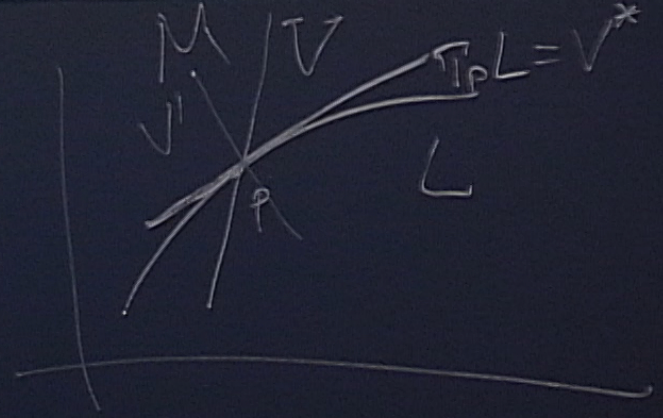
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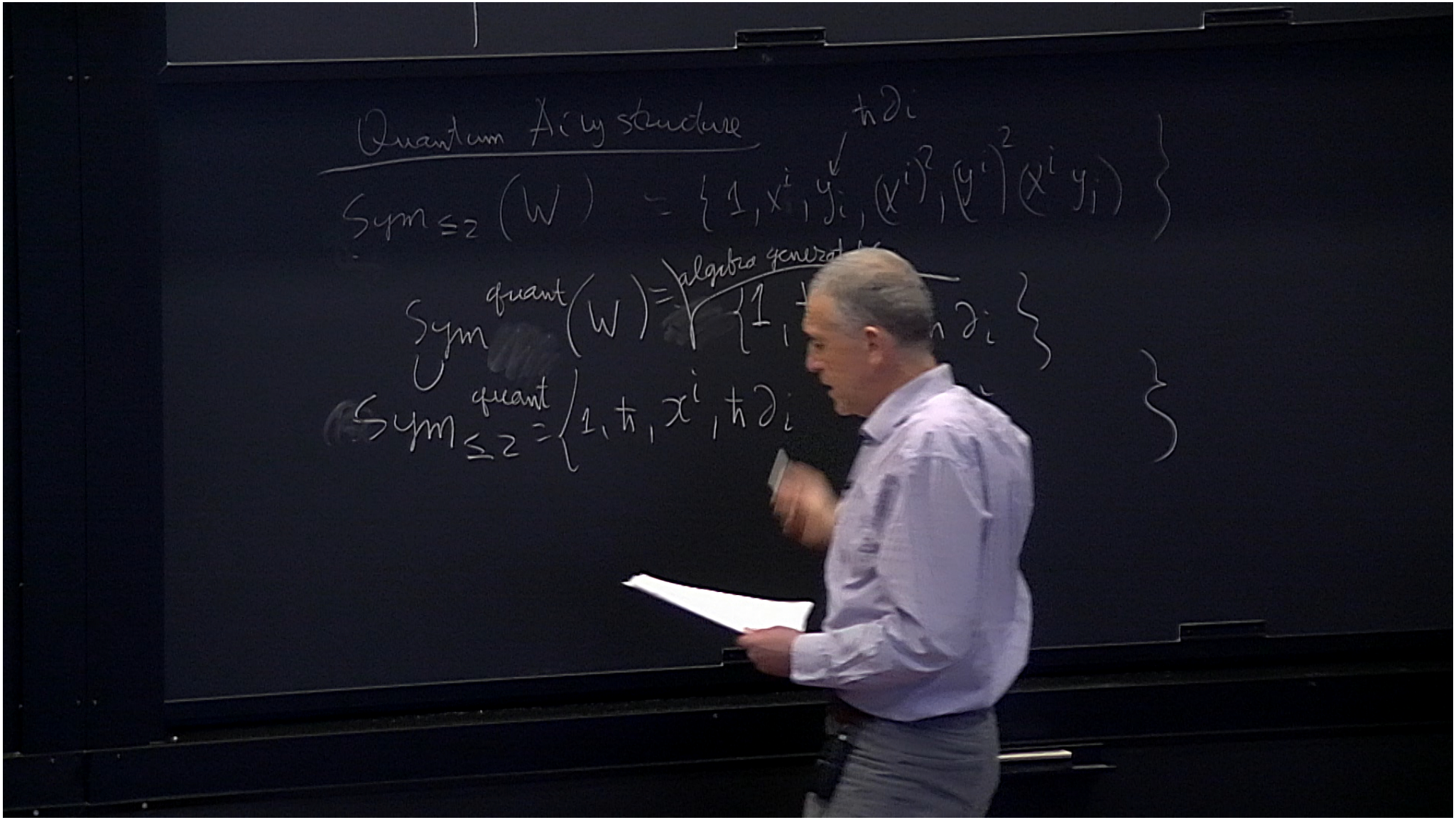
Lie alg. acting on M in Hamilt. way

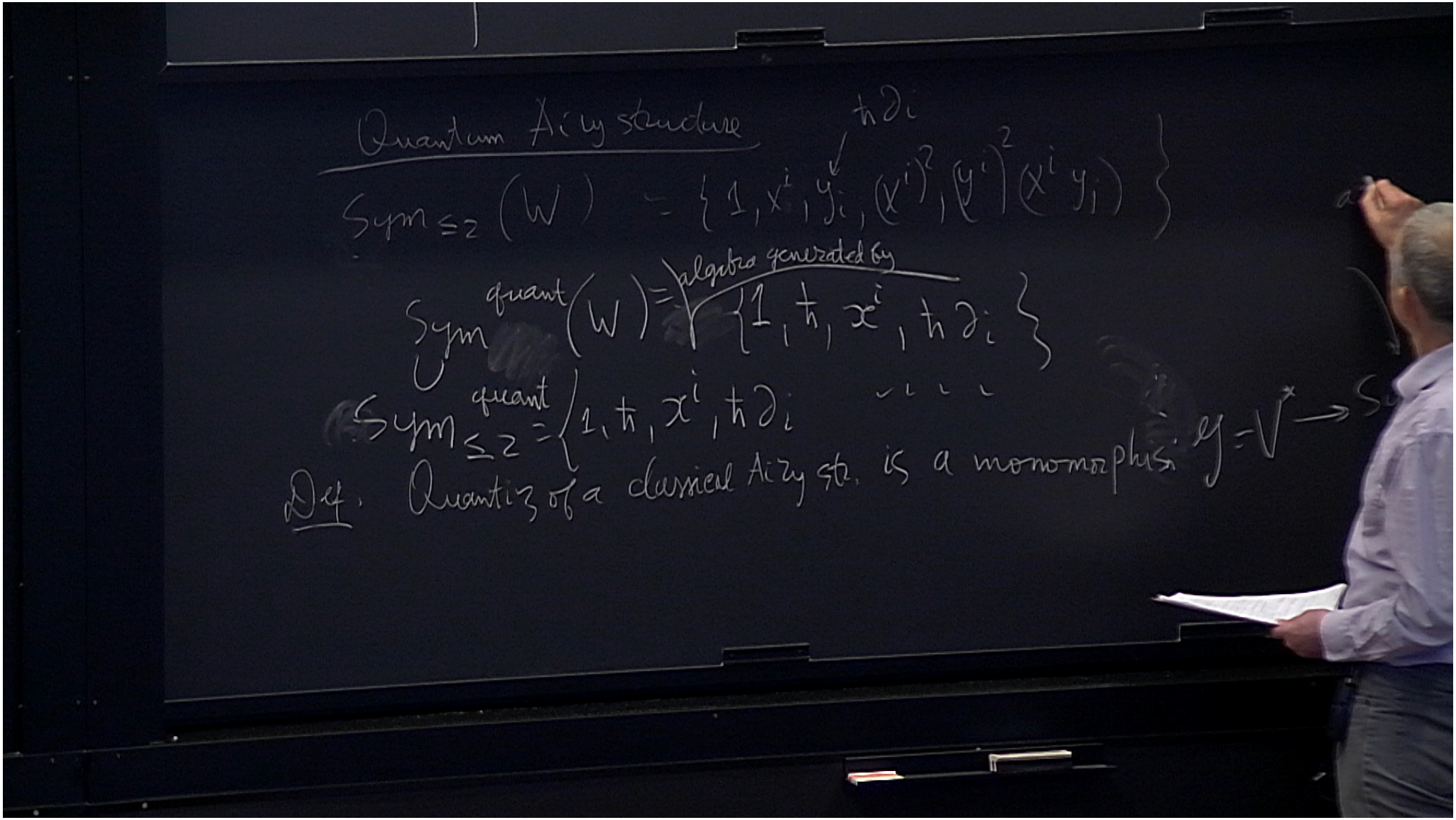
$\rightarrow \mathfrak{g}^*$ - moment map

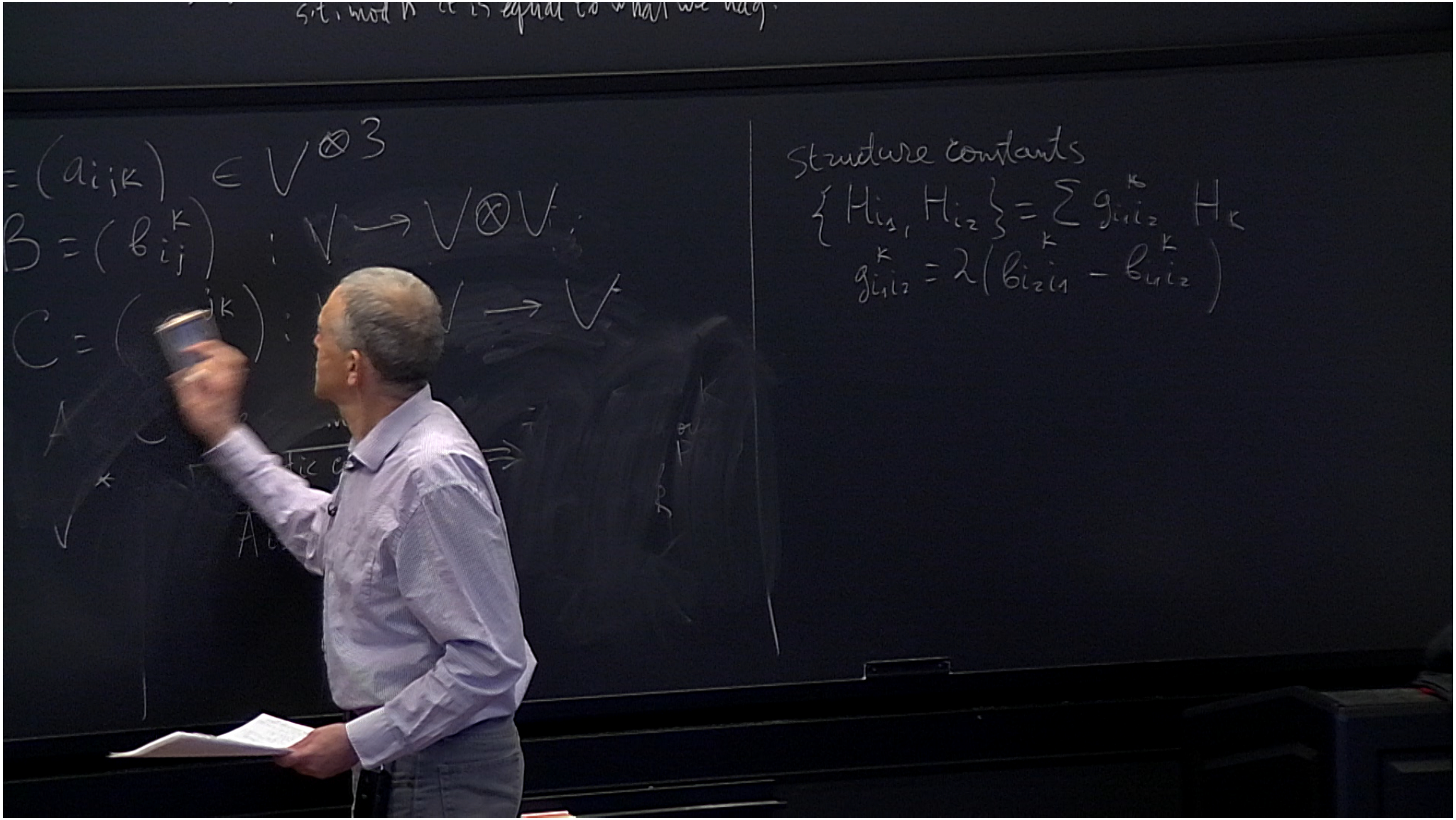
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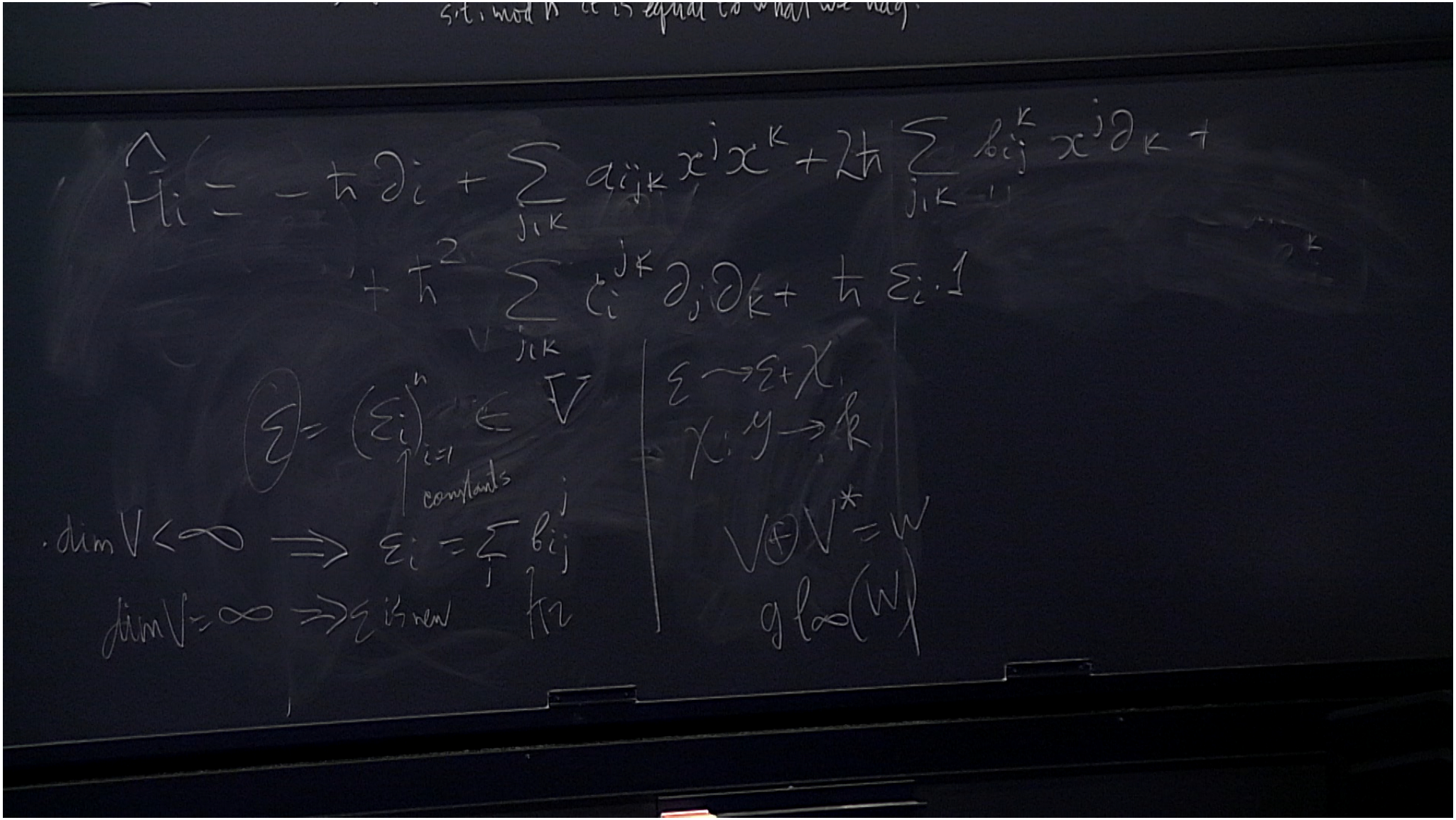
smooth pt $\Rightarrow$ coisotropic subvariety of $\dim = n \Rightarrow$ L - Lagr. subvariety of W

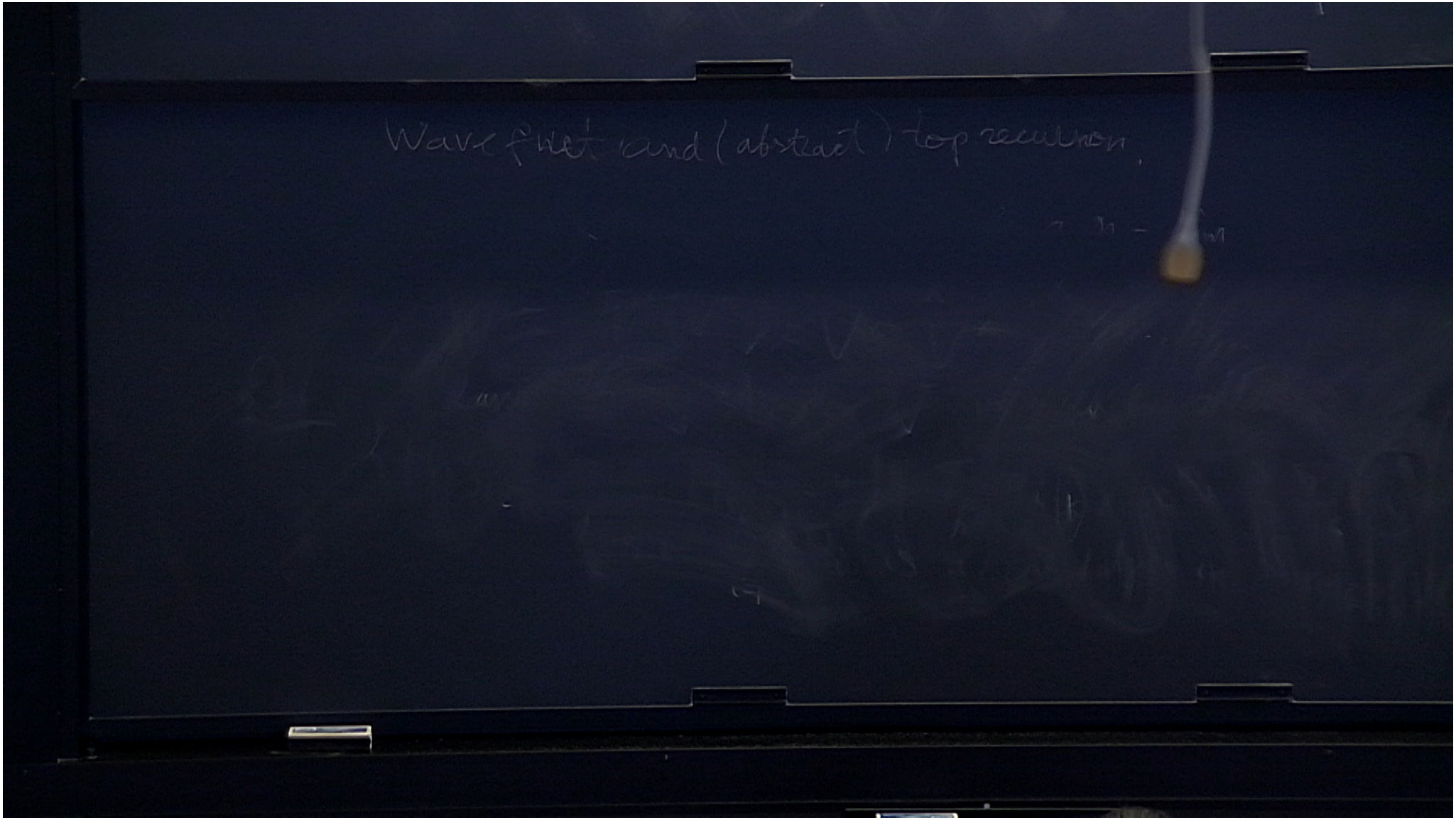


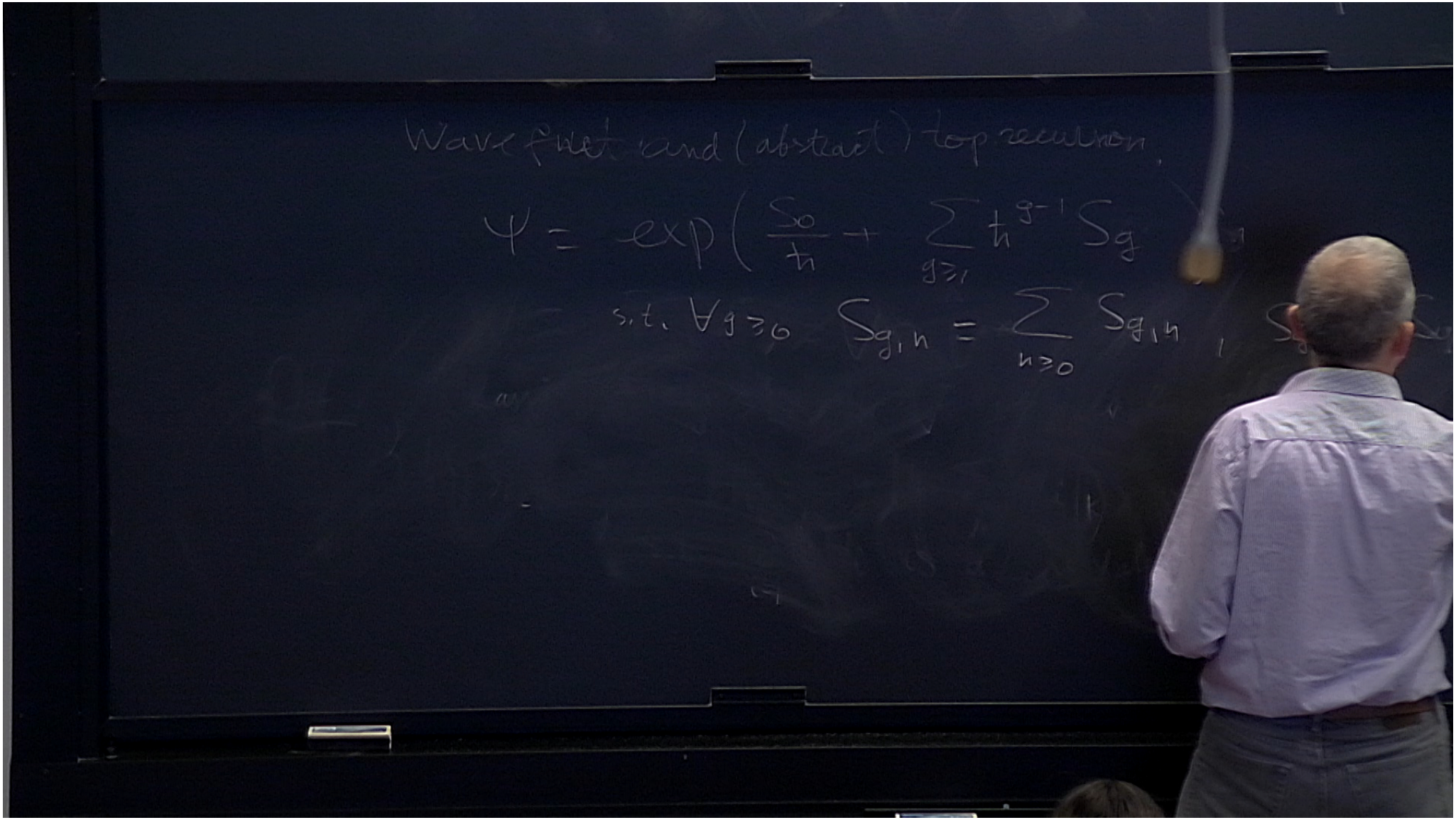

Quantum Acity structure
$\hbar\partial_i$
$Sym_{\leq 2}(W) = \{1, x^i, y_i, (x^i)^2, (y^i)^2, (x^i y_i)\}$
$Sym^{quant}(W) = \{1, \hbar, \ldots, \hbar\partial_i\}$
algebra
$Sym^{quant}_{\leq 2} = \{1, \hbar, x^i, \hbar\partial_i$

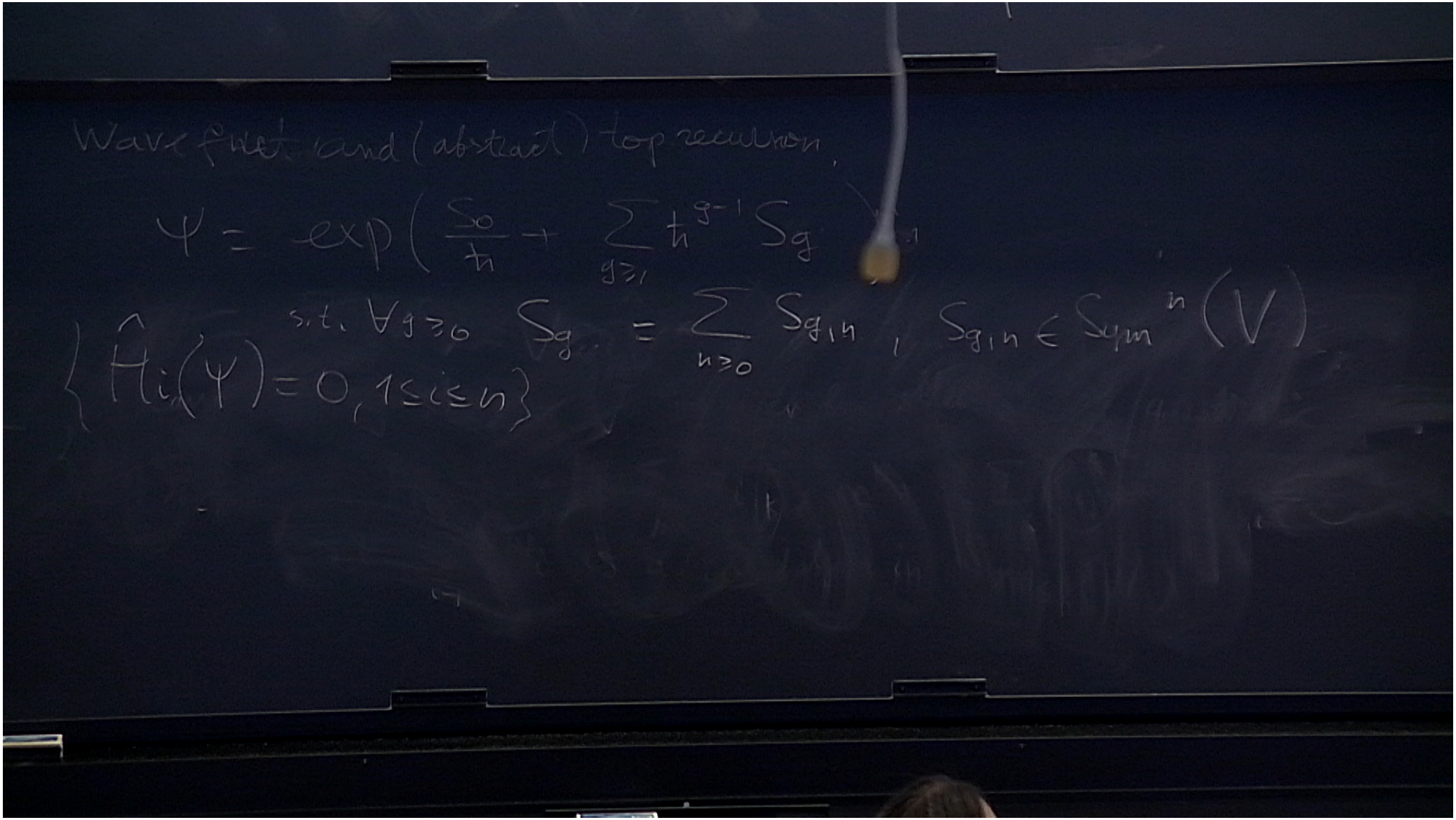

Quantum Airy structure
$\hbar\partial_i$
$Sym_{\leq 2}(W) = \{1, x^i, y_i, (x^i)^2, (y^i)^2, (x^i y_i)\}$
$Sym^{quant}(W) =$ algebra generated by $\{1, \hbar, x^i, \hbar\partial_i\}$
$Sym^{quant}_{\leq 2} = \{1, \hbar, x^i, \hbar\partial_i$
Def. Quantization of a classical Airy str. is a monomorphism

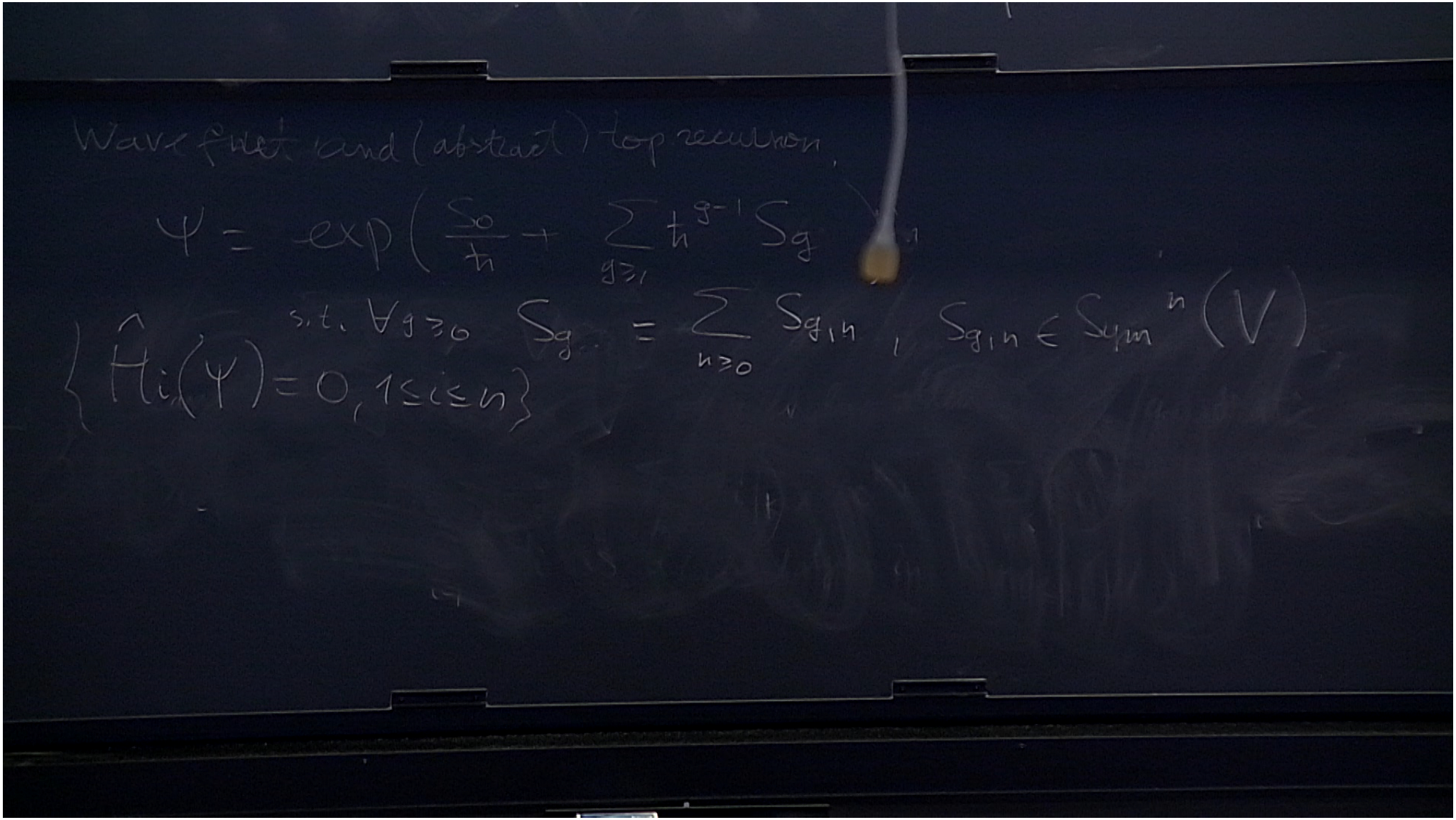

Structure constants
$\{H_{i_1}, H_{i_2}\} = \sum g^{k}_{i_1 i_2} H_k$
$g^{k}_{i_1 i_2} = 2(b^{k}_{i_2 i_1} - b^{k}_{i_1 i_2})$

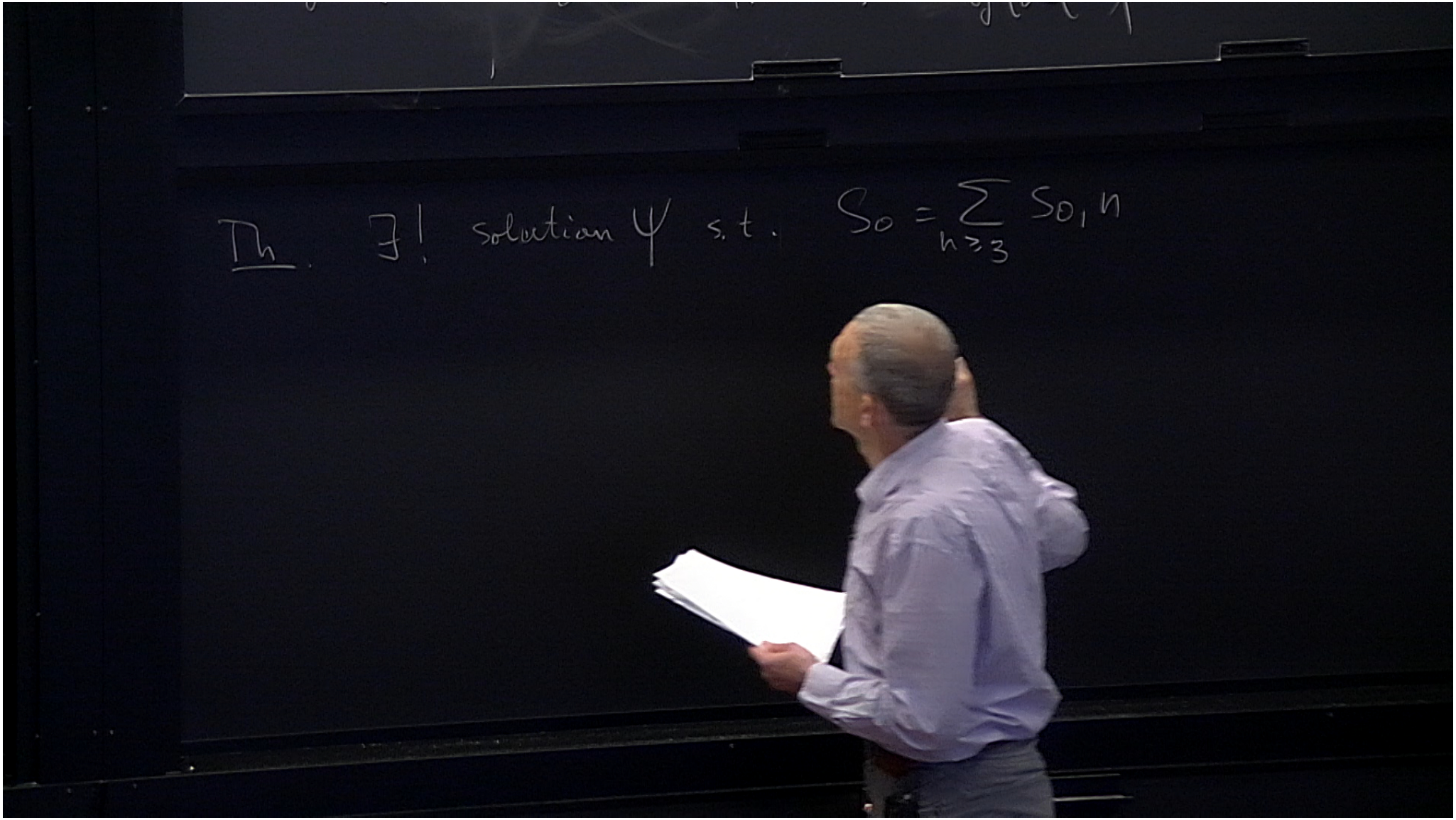

$\hat{H}_i = -\hbar\partial_i + \sum_{j,k} a_{ijk} x^j x^k + 2\hbar \sum_{j,k} b_{ij}^k x^j \partial_k +$
$+ \hbar^2 \sum_{j,k} c_i^{jk} \partial_j \partial_k + \hbar\, \varepsilon_i \cdot 1$
$\xi = (\xi_i)_{i=1}^n \in V$
constants
$\dim V < \infty \Rightarrow \xi_i = \sum_j b_{ij}^j$
$\dim V = \infty \Rightarrow \xi$ is new
$\xi \to \xi + \chi$
$\chi, y \to k$
$V \oplus V^* = W$
$gl_\infty(W)$

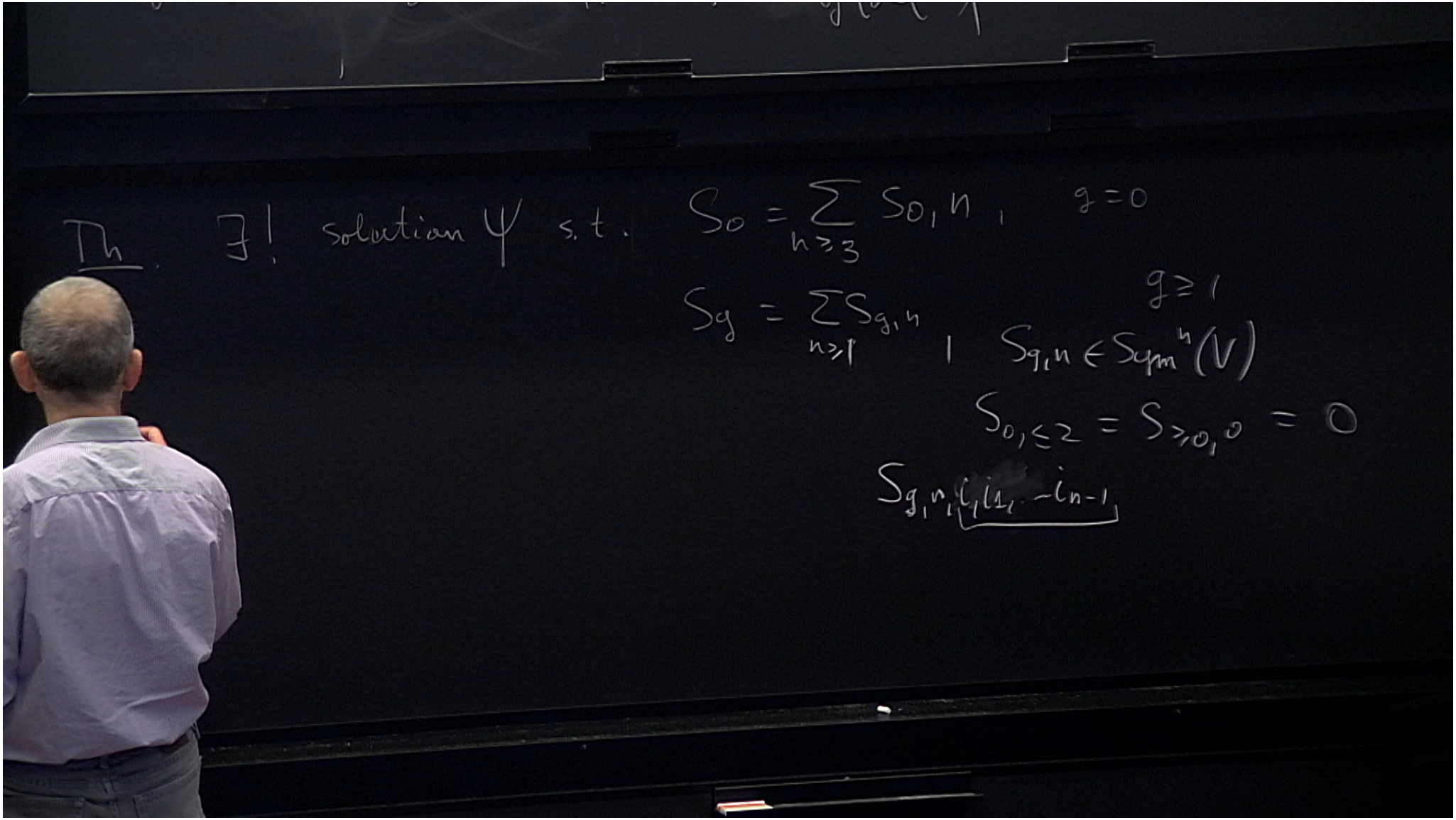

Wave front and (abstract) top reaction.

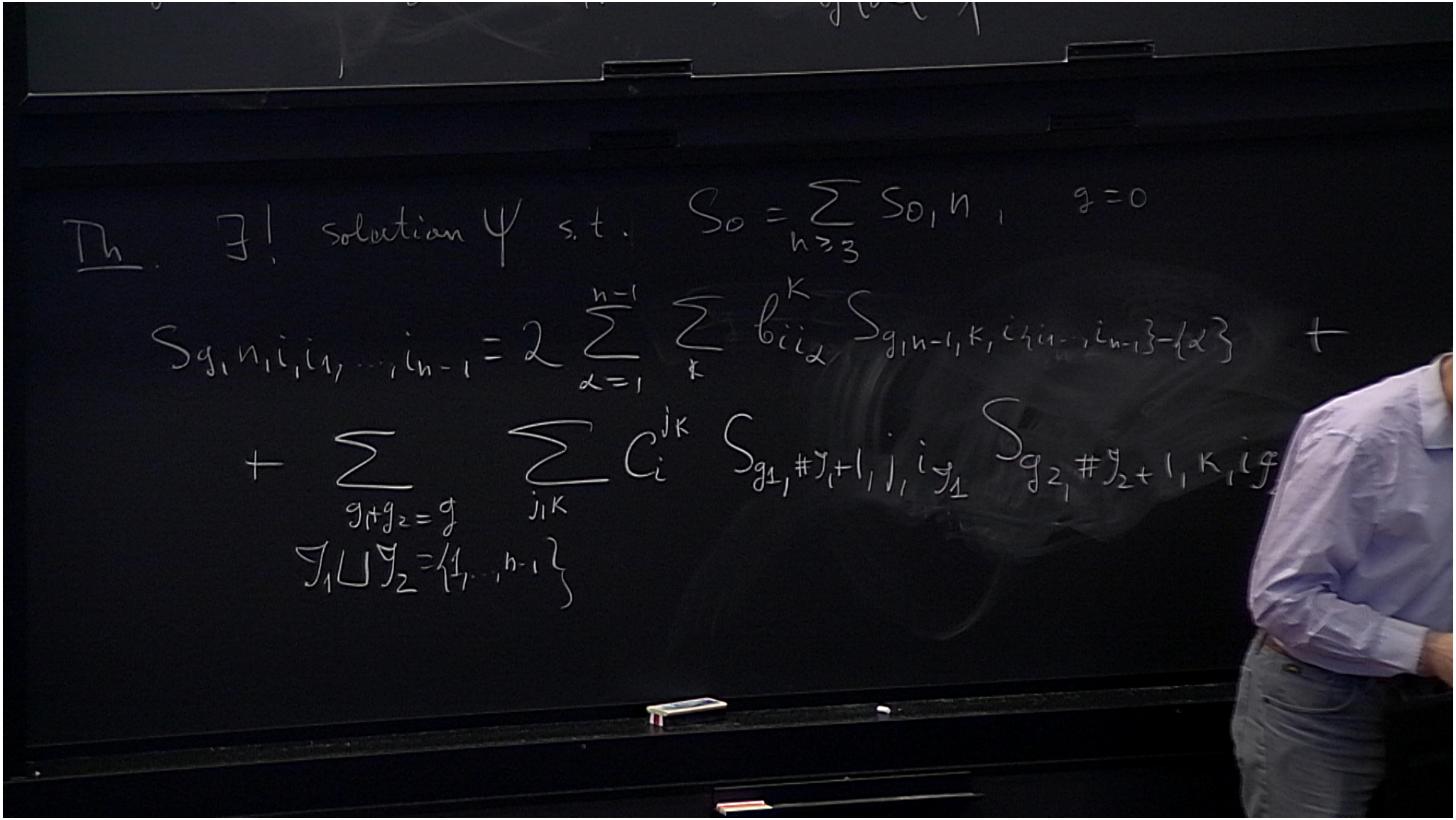

Wavefunction and (abstract) top secursion.
$\Psi = \exp\left(\frac{S_0}{\hbar} + \sum_{g\geq 1} \hbar^{g-1} S_g\right.$
s.t. $\forall g \geq 0 \quad S_{g,n} = \sum_{n\geq 0} S_{g,n}$


Wavefunction and (abstract) top recursion.
$\Psi = \exp\left(\frac{S_0}{\hbar} + \sum_{g\geq 1} \hbar^{g-1} S_g\right)$
$\{\hat{H}_i(\Psi) = 0, 1 \leq i \leq n\}$
s.t. $\forall g \geq 0$ $S_g = \sum_{n\geq 0} S_{g,n}$, $S_{g,n} \in Sym^n(V)$


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Th. ∃! solution Ψ s.t. $S_0 = \sum_{n \geq 3} S_{0,n}$


Th. ∃! solution Ψ s.t. $S_0 = \sum_{n\geq 3} S_{0,n}$, $g=0$
$S_g = \sum_{n\geq 1} S_{g,n}$ $g\geq 1$
$S_{g,n} \in \mathrm{Sym}^n(V)$
$S_{0,\leq 2} = S_{\geq 0,0} = 0$
$S_{g,n,i,i_1,\dots,i_{n-1}}$

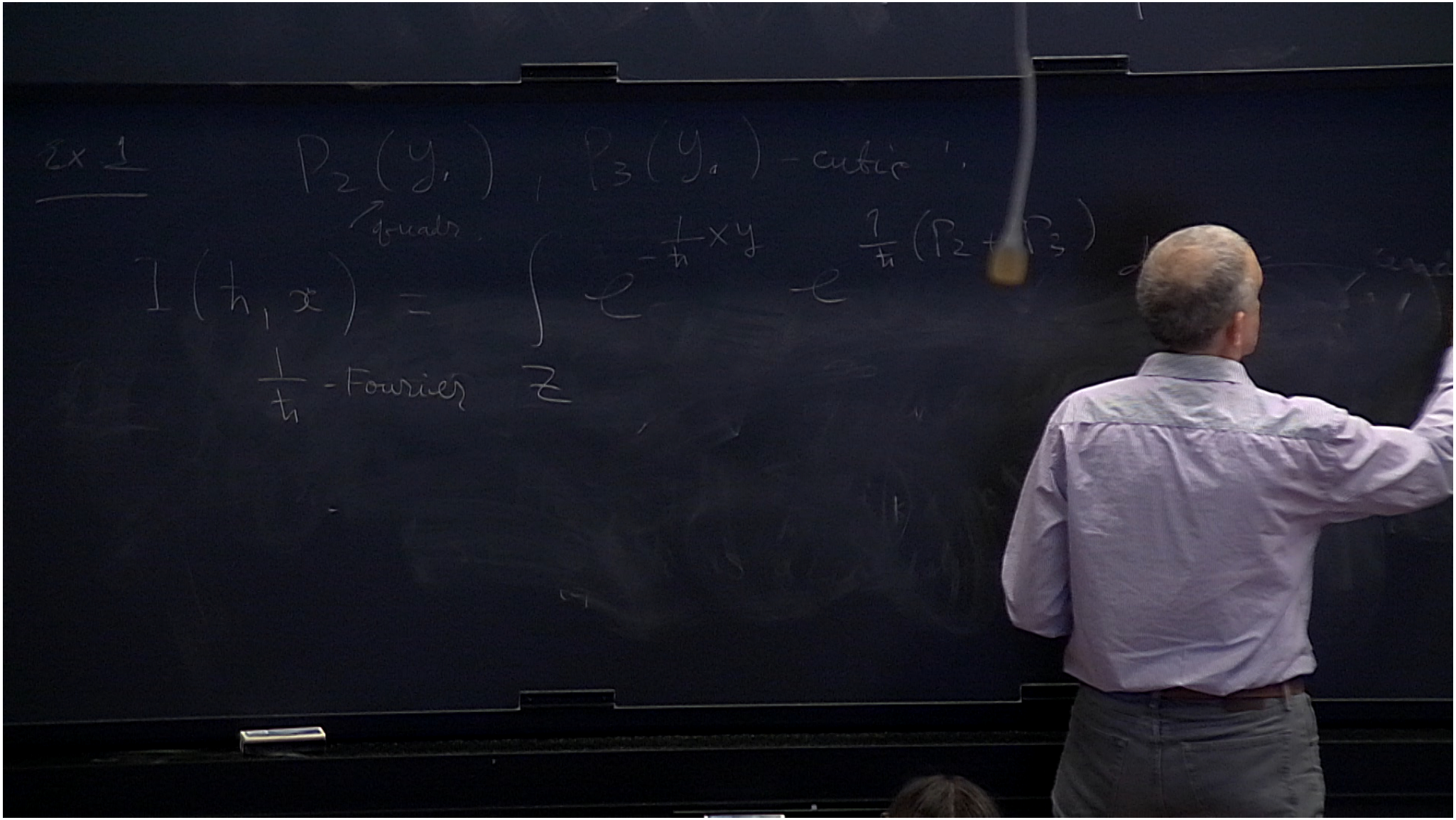

Th. ∃! solution Ψ s.t. $S_0 = \sum_{n\geq 3} S_{0,n}$, $g=0$
$S_{g,n,i,i_1,\dots,i_{n-1}} = 2\sum_{\alpha=1}^{n-1} \sum_{k} b^{k}_{i i_\alpha} S_{g,n-1,k,i_1,\dots,i_{n-1}-\{\alpha\}} +$
$+ \sum_{g_1+g_2=g} \sum_{j,k} C_i^{jk} S_{g_1,\#J_1+1,j,i_{J_1}} S_{g_2,\#J_2+1,k,i_{J_2}}$
$J_1 \sqcup J_2 = \{1,\dots,n-1\}$

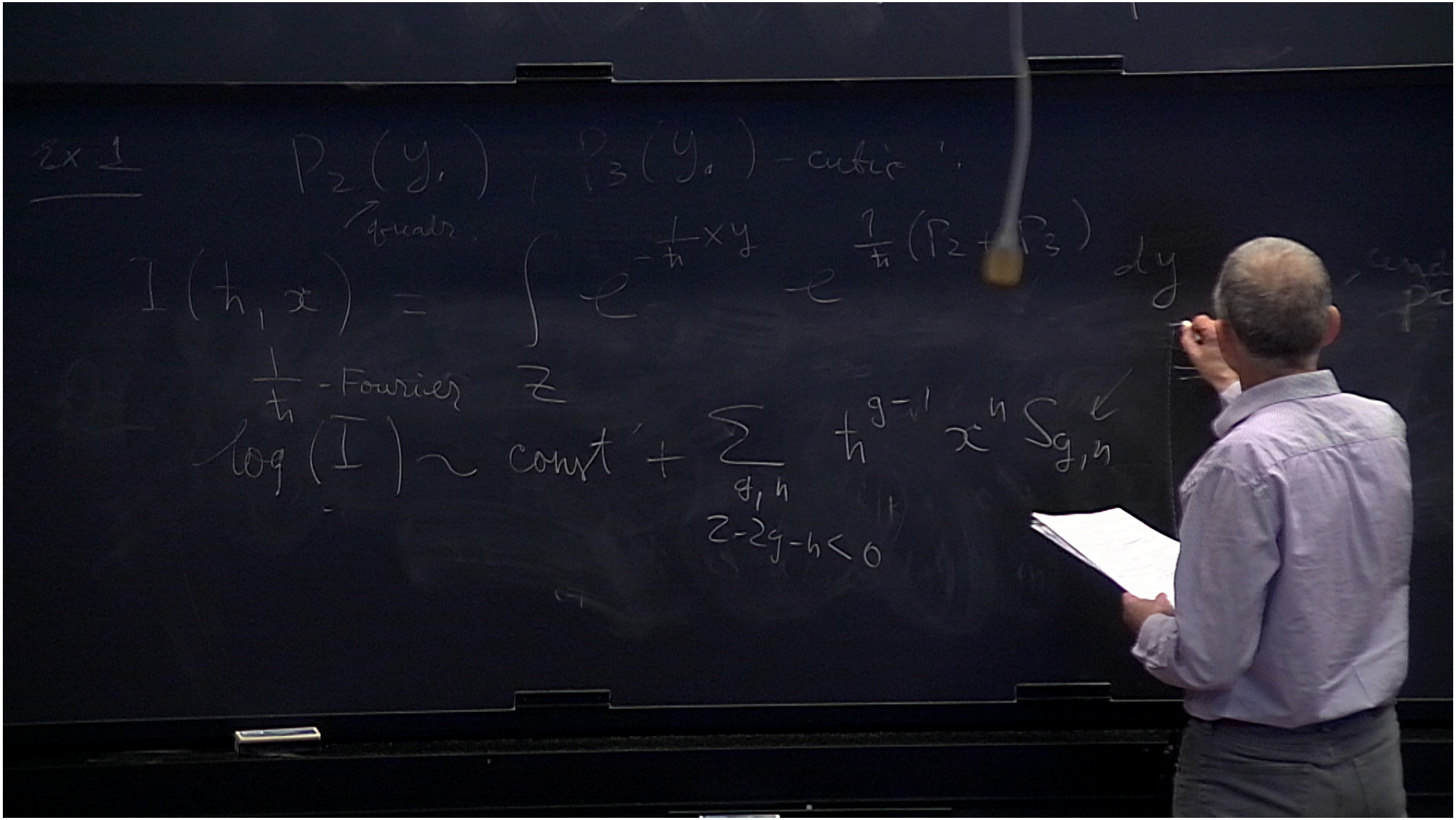
$\exists!$ solution Ψ s.t. $S_0 = \sum_{n\geq 3} S_{0,n}$, $g=0$

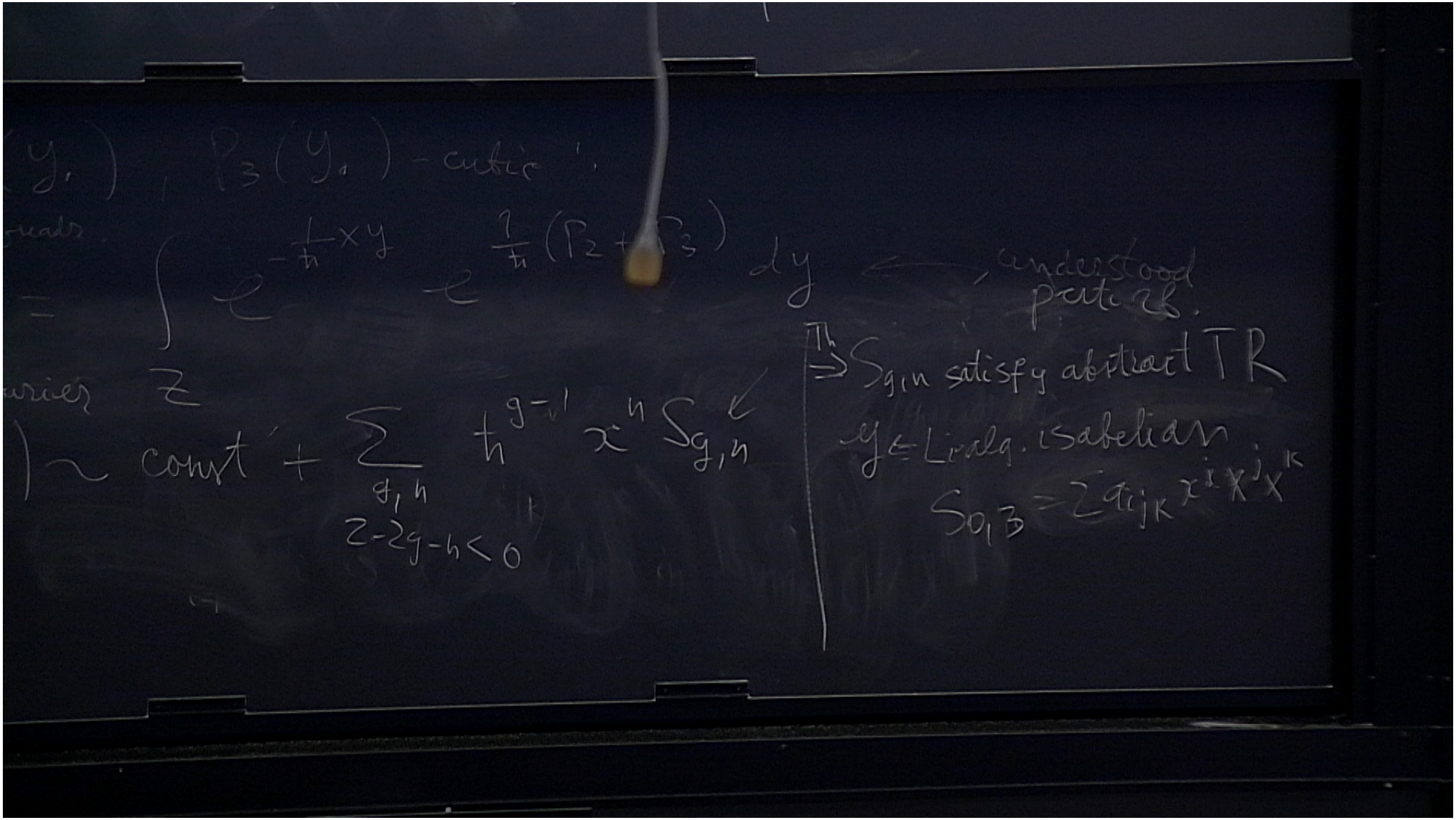
$$S_{g,n,i,i_1,\dots,i_{n-1}} = 2\sum_{\alpha=1}^{n-1}\sum_{k} b_{i i_\alpha}^{k} S_{g,n-1,k,i_1,\dots,i_{n-1}-\{\alpha\}} +$$

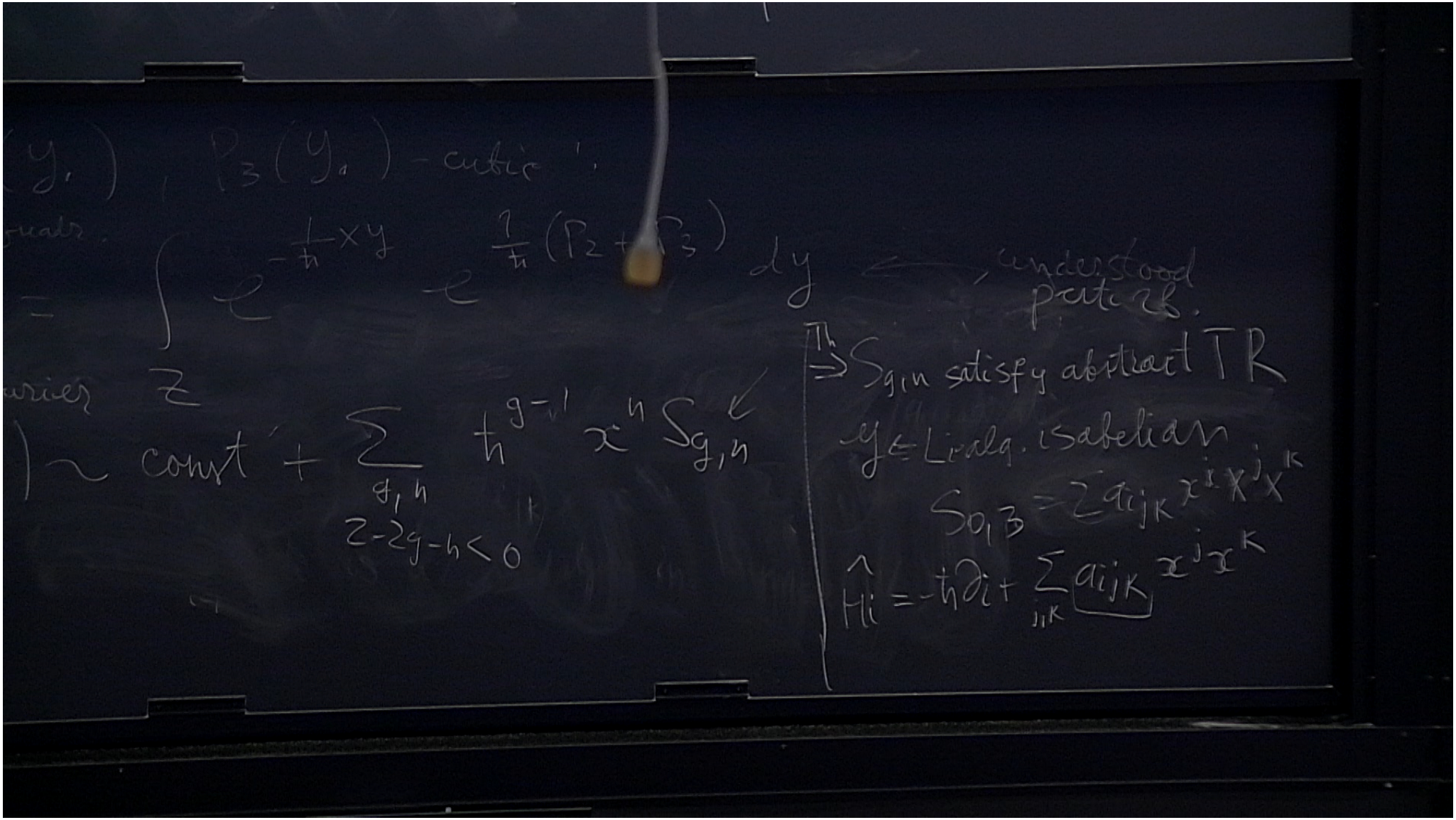
$$+ \sum_{g_1+g_2=g}\ \sum_{j,k} C_i^{jk} S_{g_1,\#J_1+1,j,i_{J_1}} S_{g_2,\#J_2+1,k,i_{J_2}} +$$

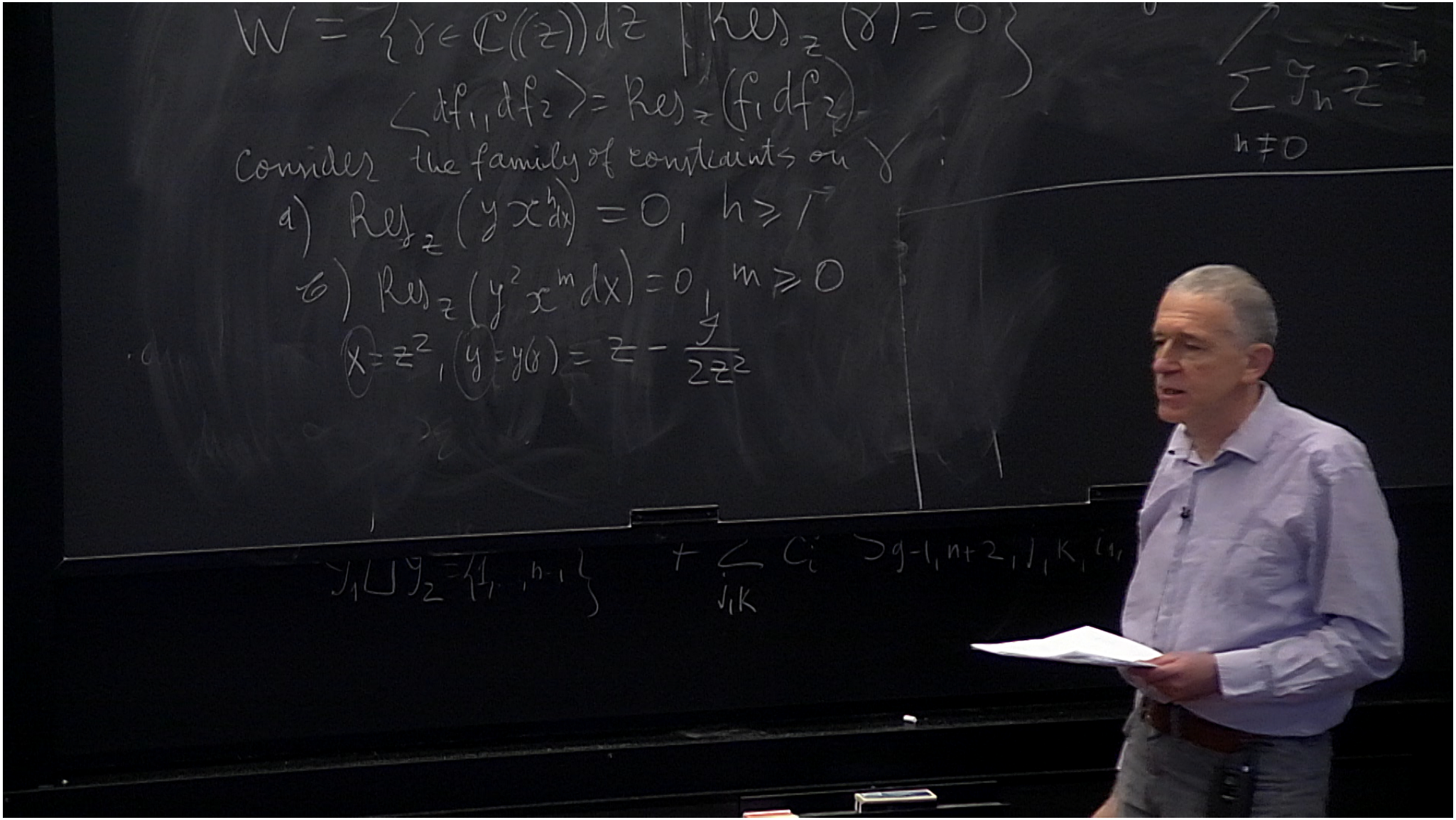
$$J_1 \sqcup J_2 = \{1,\dots,n-1\} \qquad + \sum_{j,k} C_i^{jk} S_{g-1,n+2,j,k,i_1,\dots i_{n-1}}$$

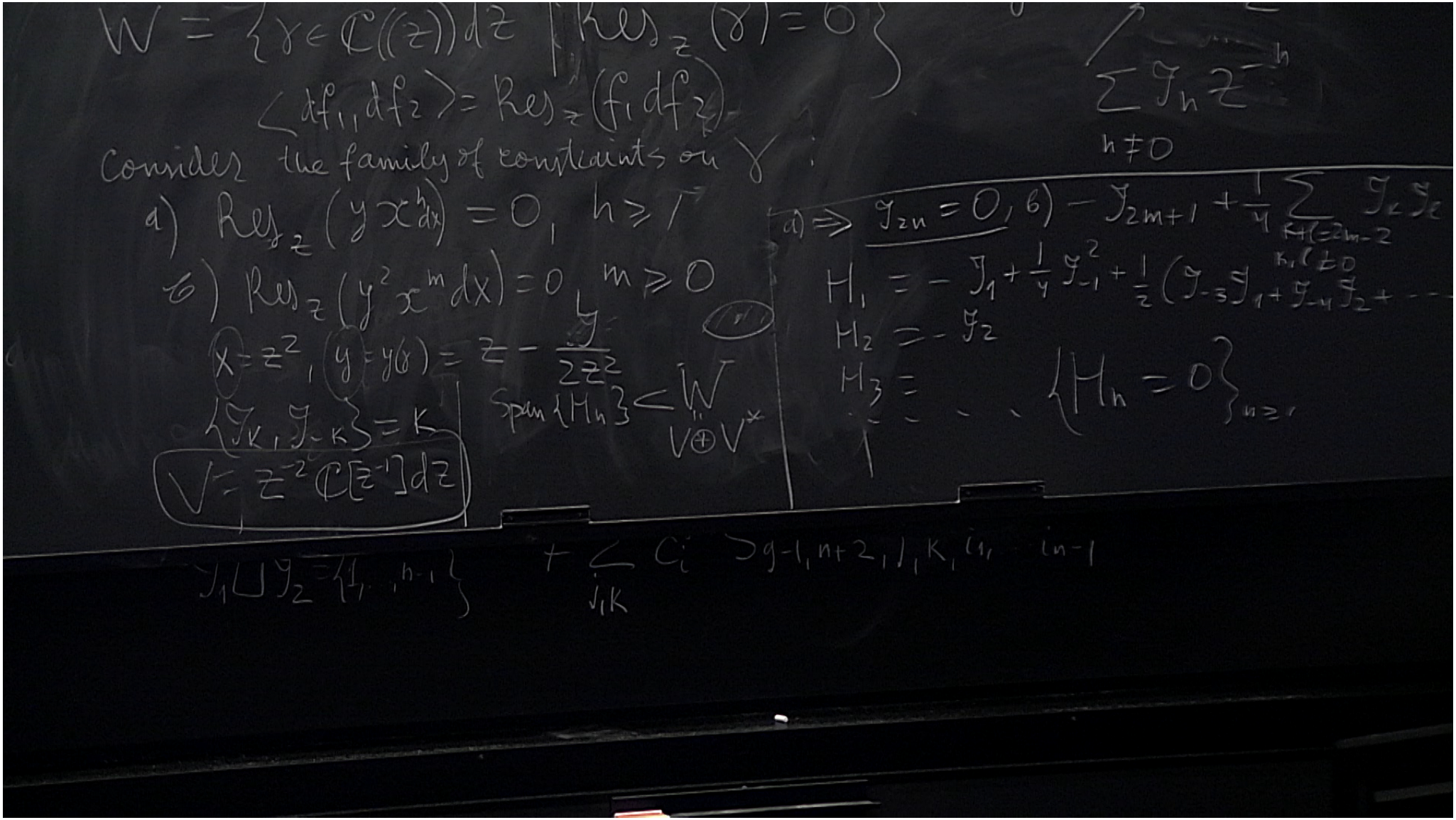

2x1
$P_2(y_1)$, $P_3(y_1)$ - cubic
quadr.
$I(t, x) = \int e^{-\frac{1}{t}xy} \; e^{\frac{1}{t}(P_2 + P_3)}$
$\frac{1}{t}$ - Fourier z

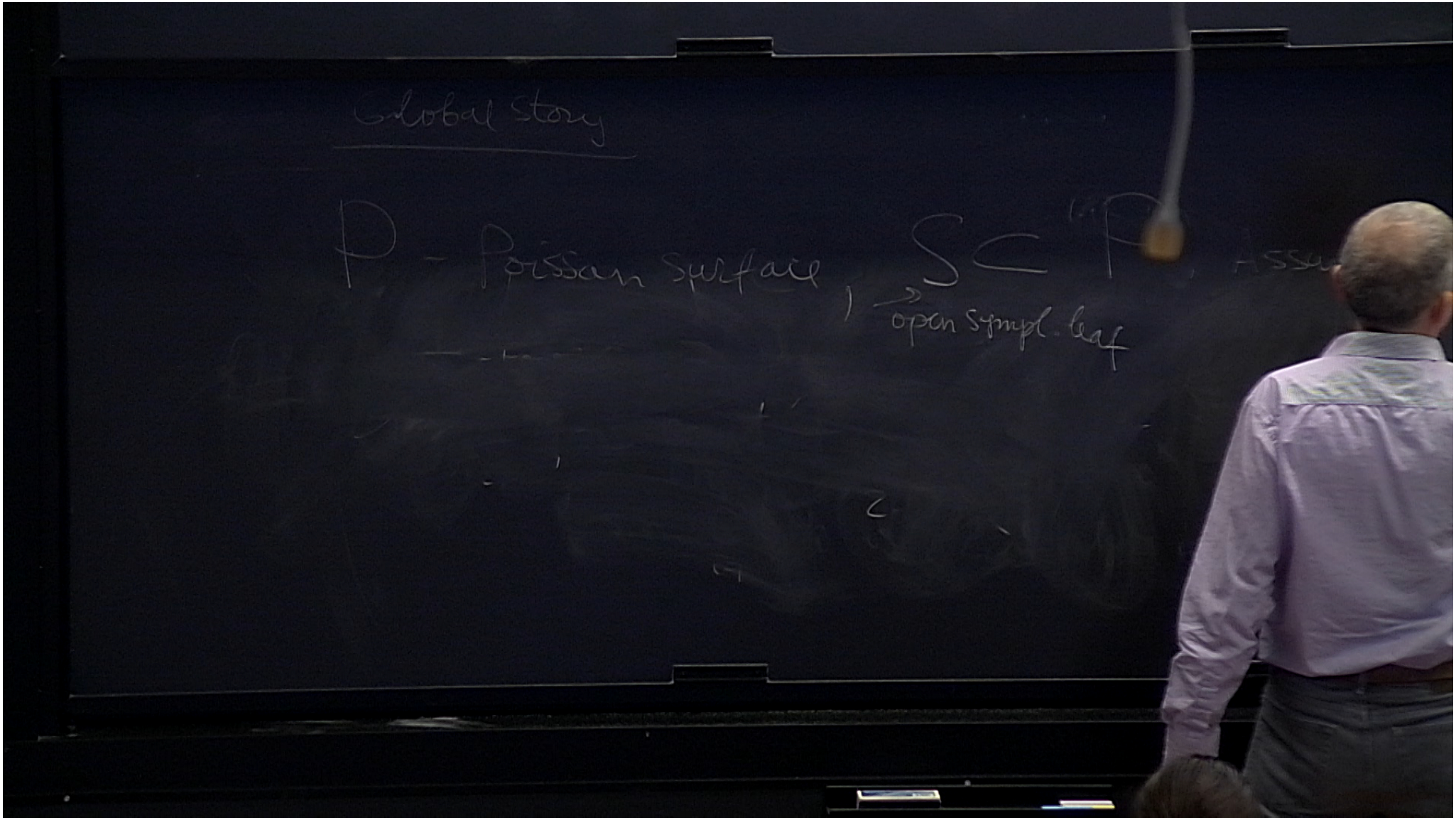

Ex 1
$P_2(y_1)$, $P_3(y_1)$ - cubic
quadr.
$I(\hbar, x) = \int e^{-\frac{1}{\hbar}xy} e^{\frac{1}{\hbar}(P_2+P_3)} dy$
$\frac{1}{\hbar}$ - Fourier
$\log(I) \sim \text{const} + \sum_{\substack{g,n \\ 2-2g-n<0}} \hbar^{g-1} x^n S_{g,n}$


(y_1), $P_3(y_1)$ - cubic :
$= \int e^{-\frac{1}{\hbar}xy} \; e^{\frac{1}{\hbar}(P_2+P_3)} \, dy$
understood
Z
$\sim$ const $+ \sum_{g,n} \hbar^{g-1} x^n S_{g,n}$
$2-2g-n<0$
$\Rightarrow S_{g,n}$ satisfy abstract TR
$y \in$ Lie alg. isabelian
$S_{0,3} = \sum a_{ijk} x^i x^j x^k$


$P_3(y_0)$ - cubic
$\int e^{-\frac{1}{\hbar}xy} \; e^{\frac{1}{\hbar}(P_2+P_3)} \, dy$
understood
$Z \sim \text{const} + \sum_{\substack{g,n \\ 2-2g-n<0}} \hbar^{g-1} x^n S_{g,n}$
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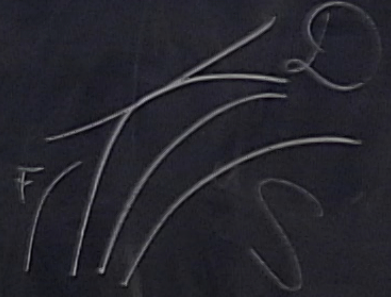

$W = \{\gamma \in \mathbb{C}((z))dz \mid \mathrm{Res}_z(\gamma) = 0\}$
$\langle df_1, df_2 \rangle = \mathrm{Res}_z(f_1 df_2)$
Consider the family of constraints on γ:
a) $\mathrm{Res}_z(y x^h dx) = 0, \quad h \geq 1$
b) $\mathrm{Res}_z(y^2 x^m dx) = 0, \quad m \geq 0$
$x = z^2, \quad y = y(0) = z - \frac{1}{2z^2}$
$\sum_{h \neq 0} \gamma_h z^{-h}$


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$x = z^2, \quad y = y(z) = z - \frac{y}{2z^2}$
$\{J_k, J_{-k}\} = k$
$V = z^{-2}\mathbb{C}[z^{-1}]dz$
$\mathrm{Span}\{H_n\} \subset W \subset V \oplus V^*$
$\sum_{n \neq 0} J_n z^{-n}$
a) $\Rightarrow$ $J_{2n} = 0$, b) $-J_{2m+1} + \frac{1}{4}\sum_{\substack{k+l=2m-2 \\ k,l \neq 0}} J_k J_l$
$H_1 = -J_1 + \frac{1}{4}J_{-1}^2 + \frac{1}{2}(J_{-3}J_1 + J_{-4}J_2 + \cdots$
$H_2 = -J_2$
$H_3 = \cdots$
$\{H_n = 0\}_{n \geq 1}$
$J_1 \sqcup J_2 = \{1, \ldots, n-1\}$
$+ \sum_{J_1 \sqcup J_2} C_i \, \ldots g-1, n+2, J_1, K, i_1, \ldots i_{n-1}$


Global Story
P - Poisson Structure,
open sympl. leaf

Global story

- P – Poisson surface, $S \subset P$, assume $\mathcal{D} = P - S$ → open sympl. leaf
- F – foliation of P, s.t. F is tangent to $\mathcal{D}$



Global story

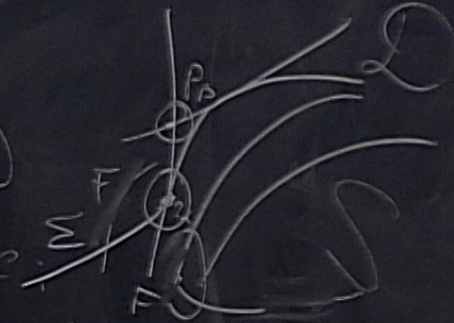
P – Poisson surface, $S \subset P$, assume $\mathcal{D} = P - S$ (open sympl. leaf)

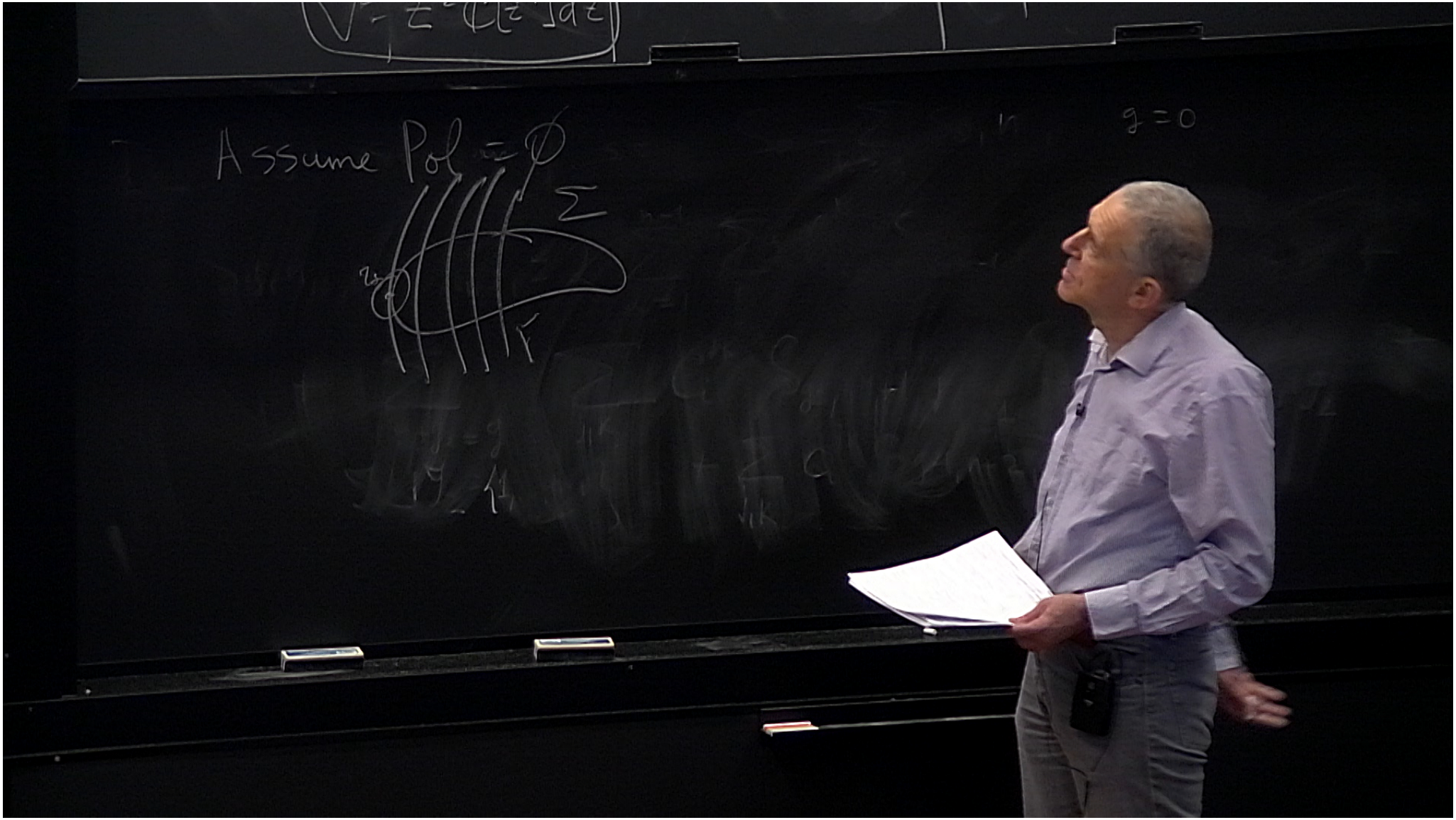
$\mathcal{F}$ – foliation of P, s.t. $\mathcal{F}$ is tangent to $\mathcal{D}$

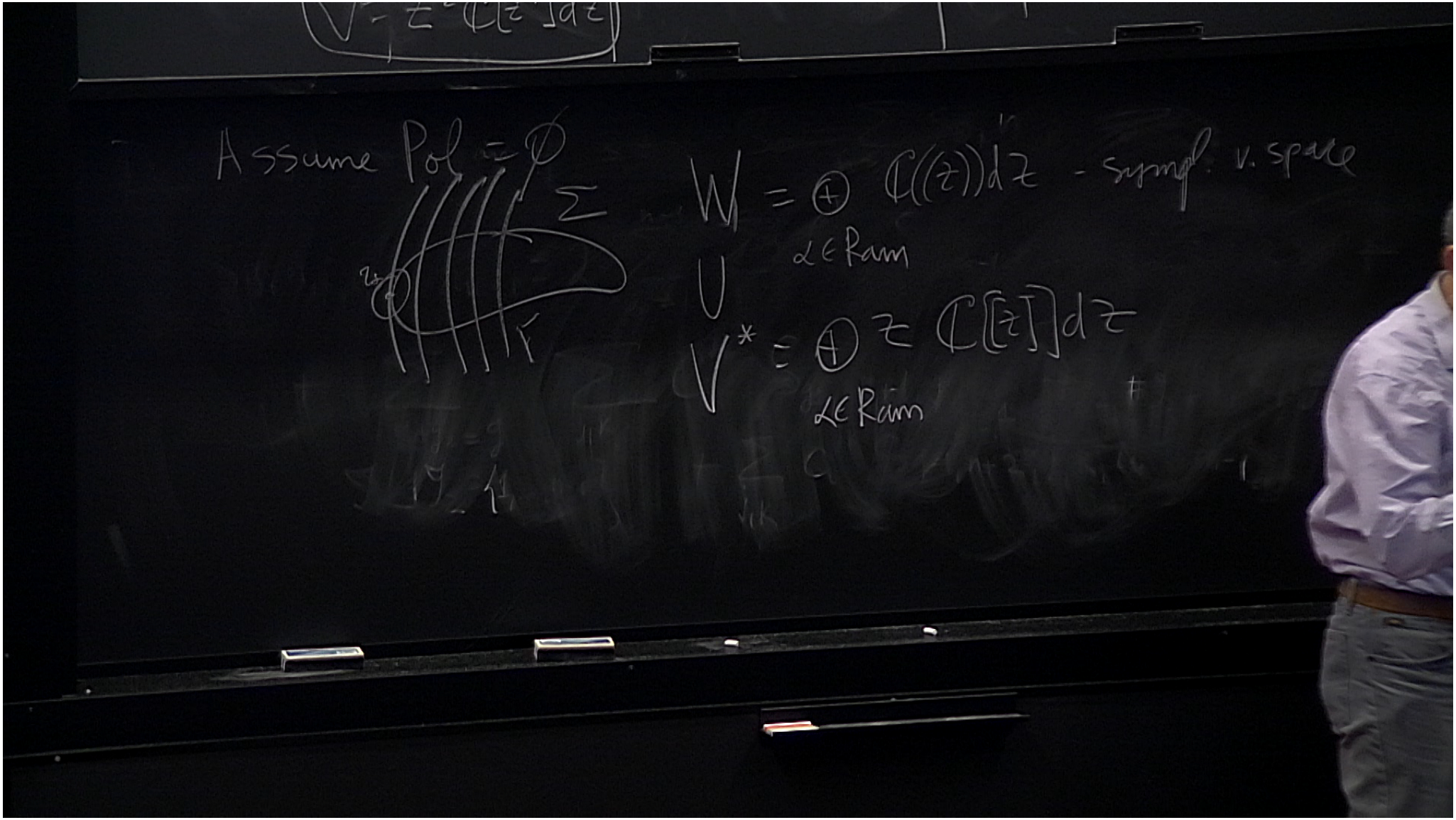
Def. $\Sigma \subset P$ is called $\mathcal{F}$-transversal S.C. if

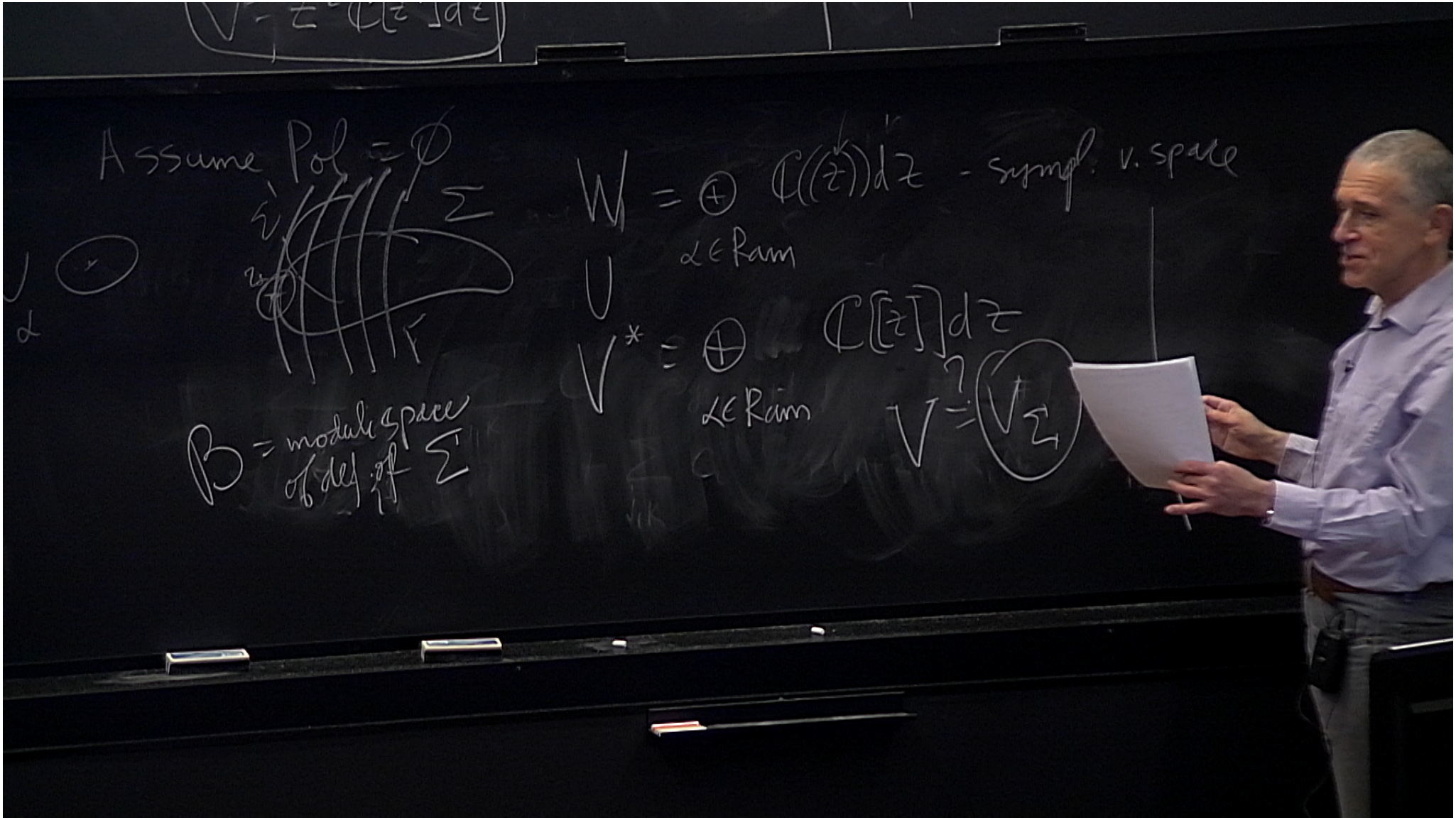
a) Σ is smooth compact

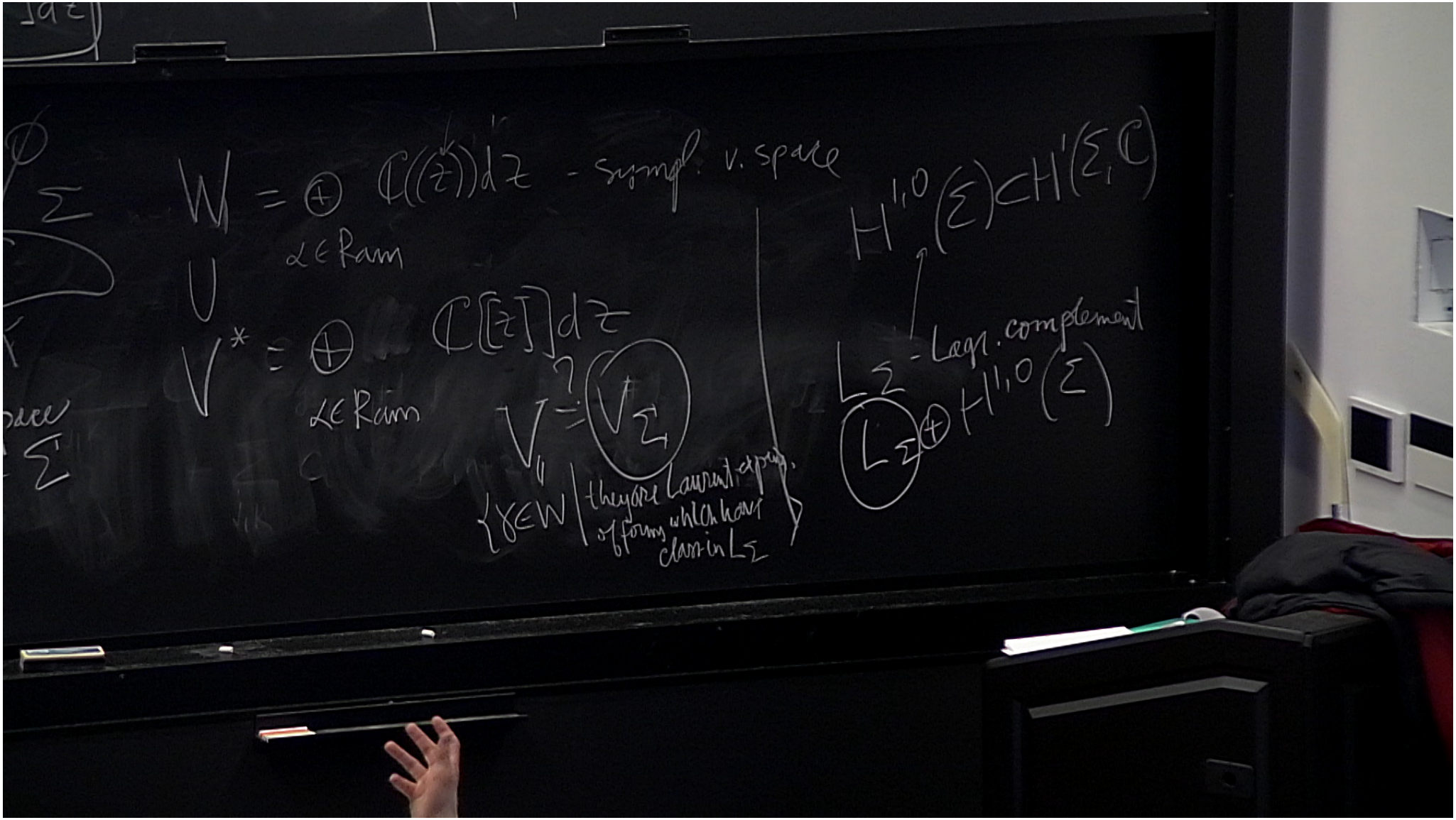
b) Σ is tgt to $\mathcal{F}$ at $\{\gamma_\alpha\}_{\alpha \in Ram}$, $\Sigma \cap \mathcal{D} = \{p_\beta\}_{\beta \in Pol}$, intersection is transversal, tgcy of order 1 ($x = y^2$)




Assume Pol = ∅
Σ
g = 0


Assume Pol = ∅
Σ
W = ⊕ ℂ((z))dz - sympl. v. space
α∈Ram
U
V* = ⊕ z ℂ[[z]]dz
α∈Ram


Assume Pol = ∅
Σ
F
B = moduli space of def of Σ
W = ⊕ ℂ((z)) dz - sympl. v. space
α ∈ Ram
V* = ⊕ ℂ[[z]] dz
α ∈ Ram
V ≅ V_Σ


$W = \bigoplus_{\alpha \in Ram} \mathbb{C}((z))dz$ - symp. v. space
U
$V^* = \bigoplus_{\alpha \in Ram} \mathbb{C}[[z]]dz$
$V \overset{?}{=} V_\Sigma$
$\{\xi \in W \mid$ the are Laurent exp. of forms which have class in $L_\Sigma\}$
$H^{1,0}(\Sigma) \subset H^1(\Sigma, \mathbb{C})$
L_Σ - Lagr. complement
$L_\Sigma \oplus H^{1,0}(\Sigma)$