Title: Periods, Motives, and graphical interpretations thereof

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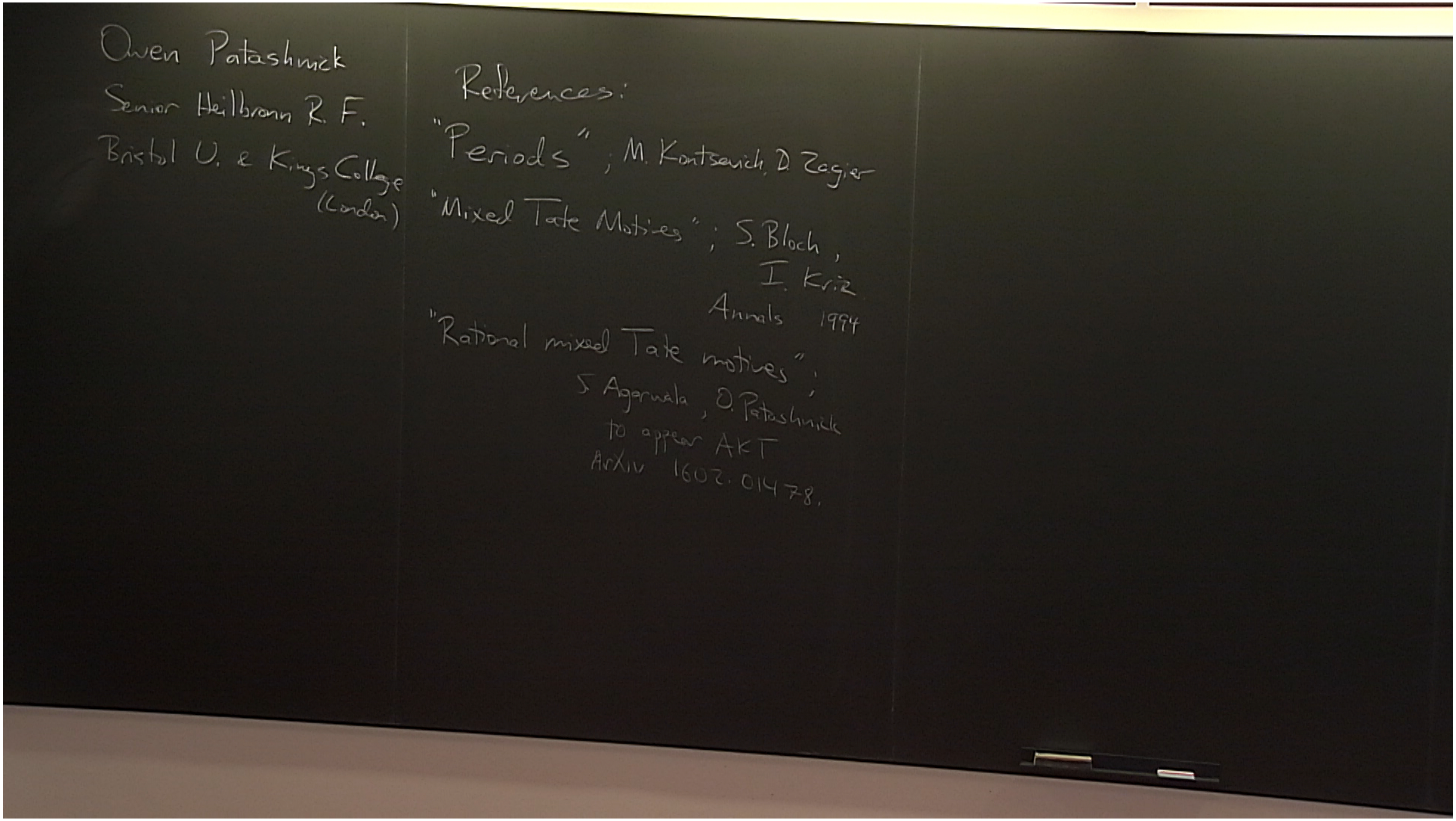
Abstract: <p>Integral values of zeta functions are important not only for what they say about other values of their respective functions, but also for what they say about transcendence degree questions for appropriate extensions of the rationals or other number fields. They also appear in some recent computations relevant to particle physics.
In this talk we will give a quick introduction to the theory of periods and motives, relate said theory to special values of zeta functions, and discuss a graphical definition of the associated category of motives.
Any original work discussed in this talk is joint with Susama Agarwala.</p>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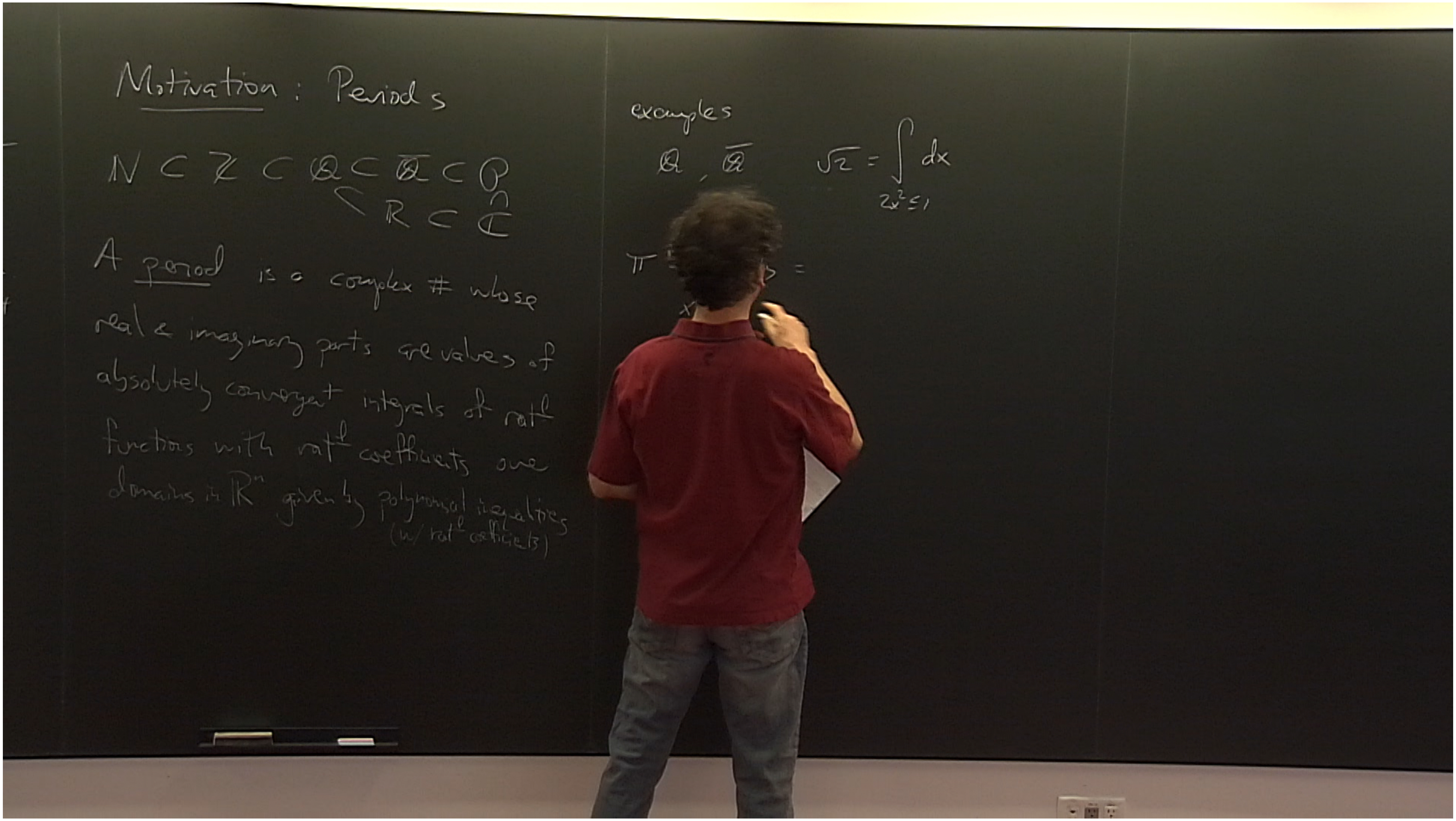
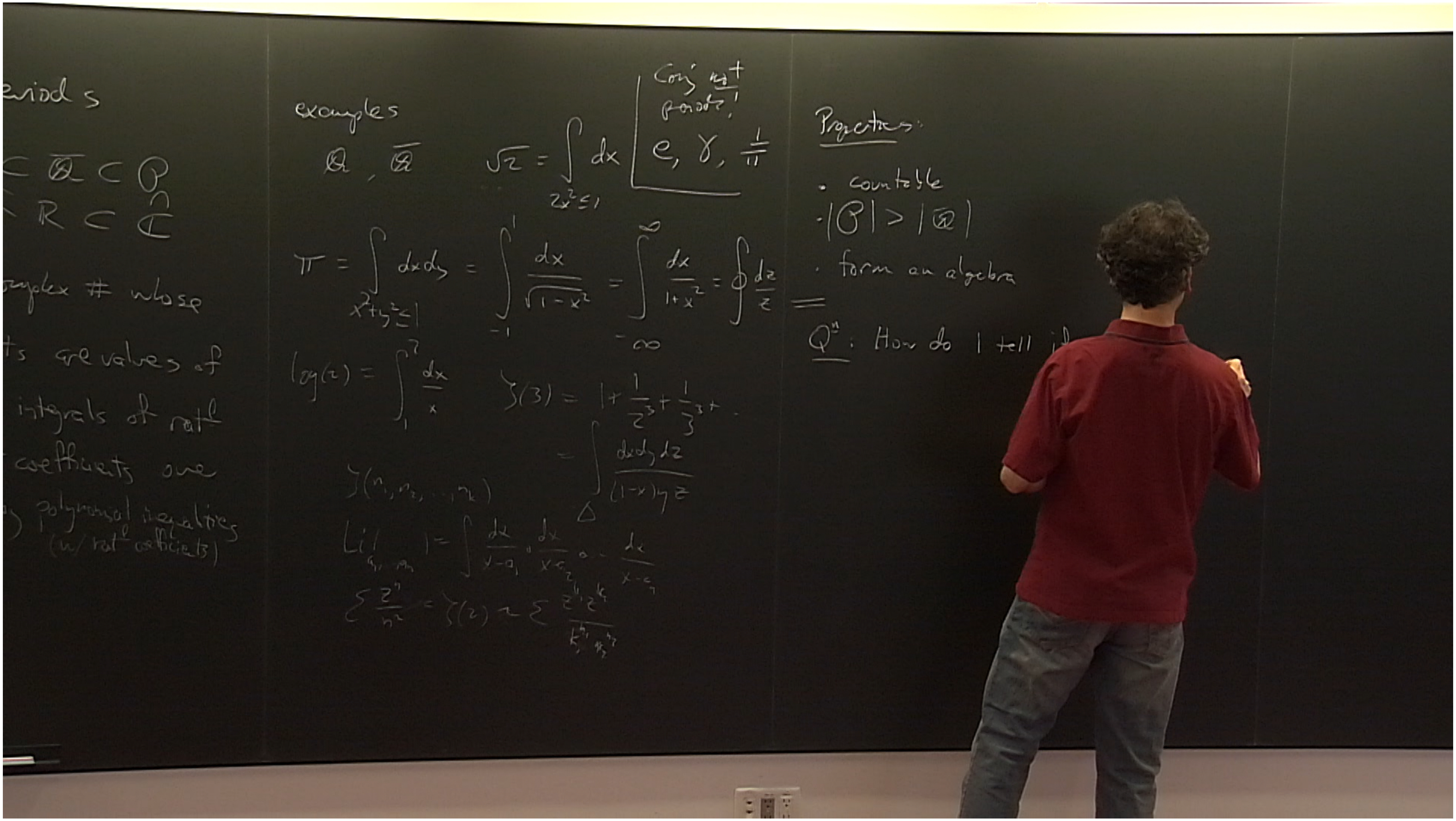
References:
"Periods"; M. Kontsevich, D. Zagier
"Mixed Tate Motives"; S. Bloch,
I. Kriz
Annals 1994
"Rational mixed Tate motives";
S. Agarwala, O. Patashnick
to appear AKT
ArXiv 1602.01478.


Motivation: Periods
N ⊂ Z ⊂ Q ⊂ Q̄ ⊂ C
⊂ R ⊂ C
A period is a complex # whose real & imaginary parts are values of absolutely convergent integrals of rat'l functions with rat'l coefficients over domains in R^n given by polynomial inequalities (w/ rat'l coefficients)
examples
Q, Q̄
√2 = ∫ dx
2x² ≤ 1
π


eriods
⊂ Q̄ ⊂ P
R ⊂ Ĉ
complex # whose
ts are values of
integrals of ratl
coefficients over
polynomial inequalities
(w/ ratl coefficients)
examples
Q, Q̄
√2 = ∫ dx
2x² ≤ 1
Conj not periods!
e, γ, 1/π
π = ∫ dx dy = ∫ dx/√(1−x²) = ∫ dx/(1+x²) = ∮ dz/z
x²+y² ≤ 1
log(2) = ∫ dx/x
ζ(3) = 1 + 1/2³ + 1/3³ + ...
= ∫ dx dy dz/((1−x)yz)
ζ(n₁, n₂, ..., n_k)
Properties:
· countable
· |P| > |Q̄|
· form an algebra
Q: How do I tell if

…eriods

$\subset \overline{\mathbb{Q}} \subset \mathcal{P}$

$\subset \mathbb{R} \subset \mathbb{C}$

…mplex # whose

…ts are values of

integrals of rat'l

coefficients over

…by polynomial inequalities (w/ rat'l coefficients)

examples

$\mathbb{Q}$, $\overline{\mathbb{Q}}$ $\qquad \sqrt{2} = \int_{2x^2 \le 1} dx$

Conj: not periods!

$e, \gamma, \frac{1}{\pi}$

$$\pi = \int_{x^2+y^2 \le 1} dx\,dy = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \oint \frac{dz}{z}$$

$$\log(2) = \int_1^2 \frac{dx}{x}$$

$$\zeta(3) = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \dots = \int_{\Delta} \frac{dx\,dy\,dz}{(1-x)yz}$$

$$\zeta(n_1, n_2, \dots, n_k)$$

$$Li_{(a_1 \dots a_n)} = \int \frac{dx}{x-a_1} \circ \frac{dx}{x-a_2} \circ \dots \frac{dx}{x-a_n}$$

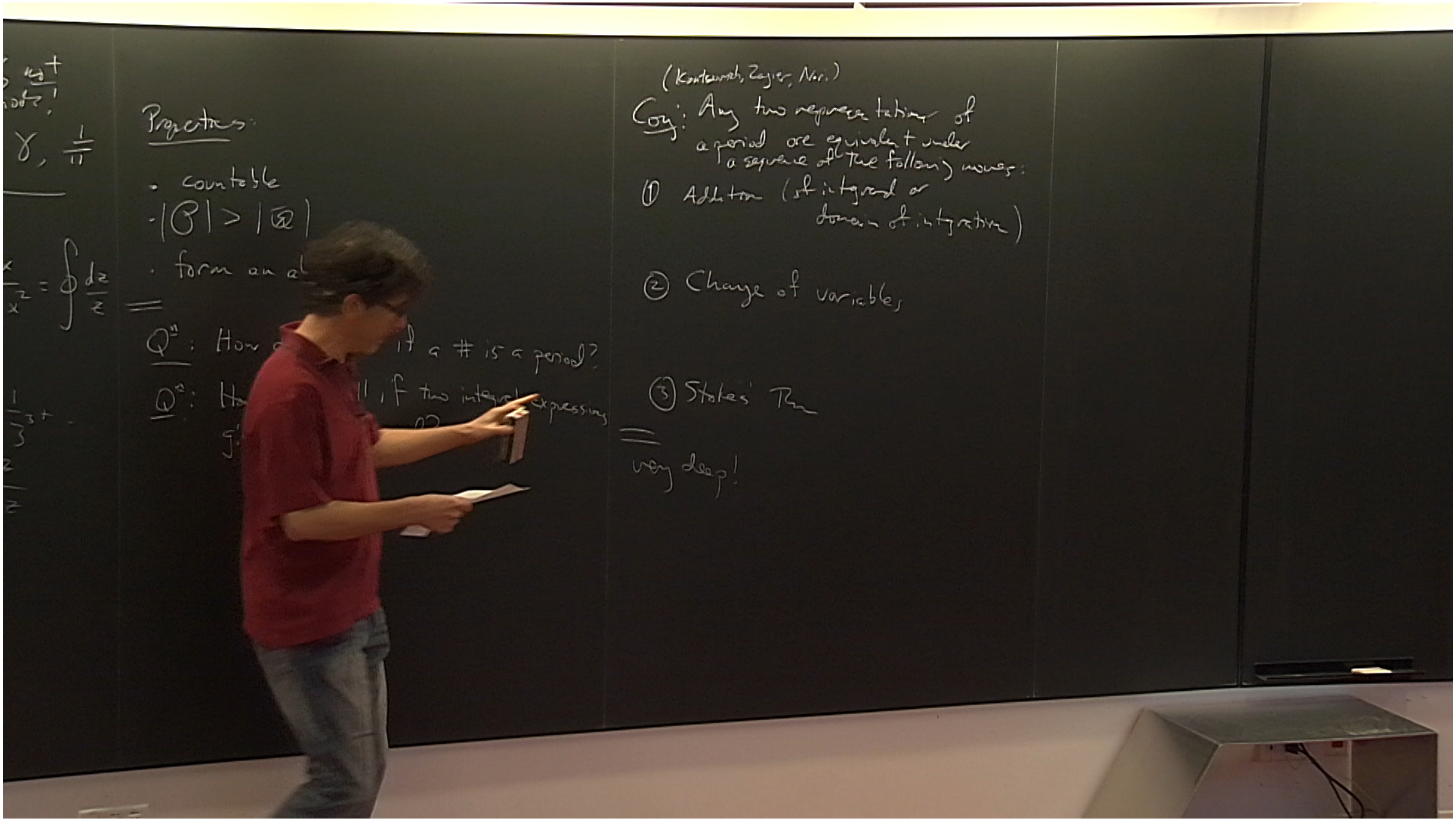
$$\sum \frac{z^n}{n^2} = \zeta(2) \sim \sum \frac{z_1^{k_1} z_2^{k_2}}{k_1^{n_1} k_2^{n_2}}$$

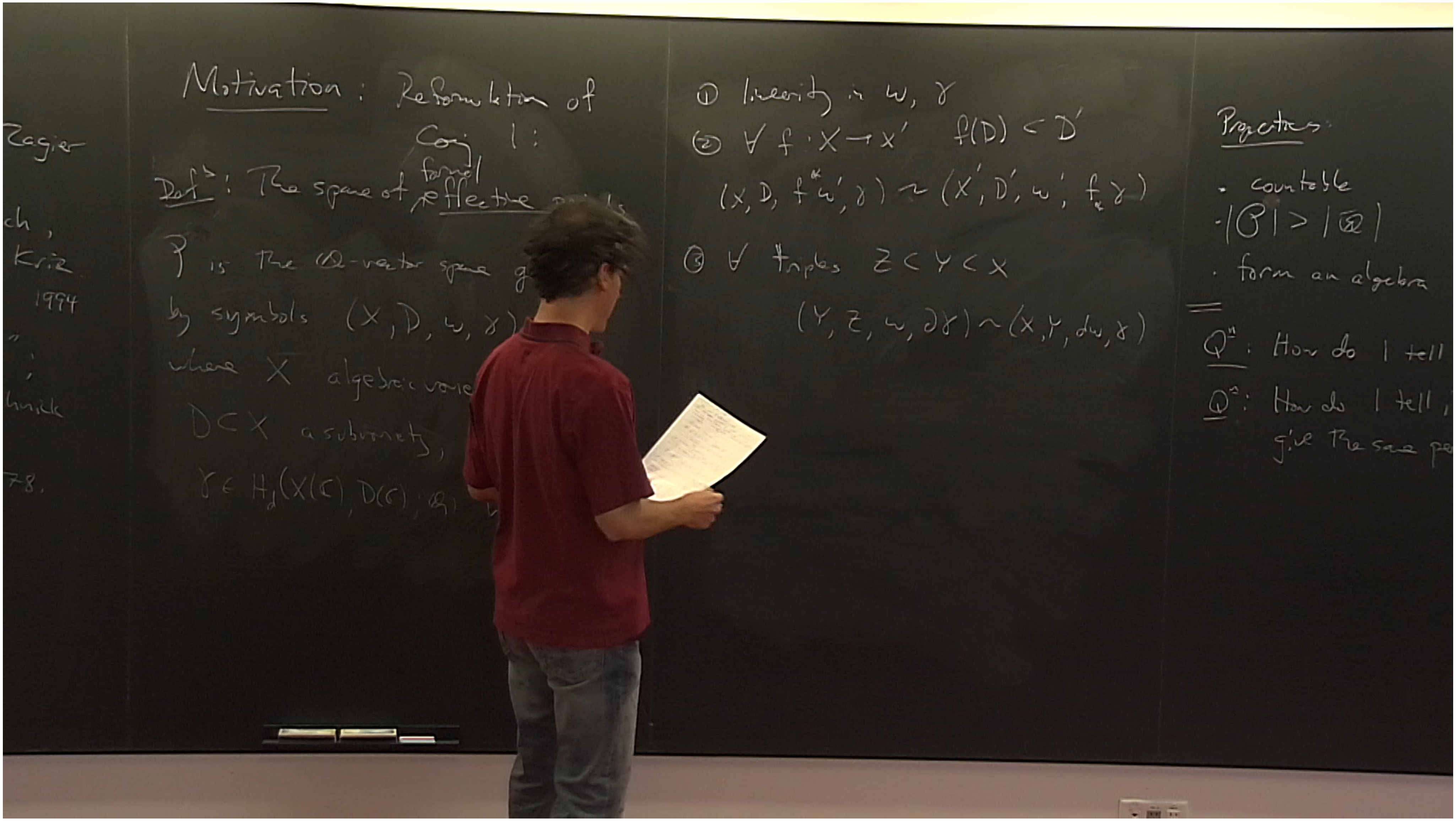
Properties:

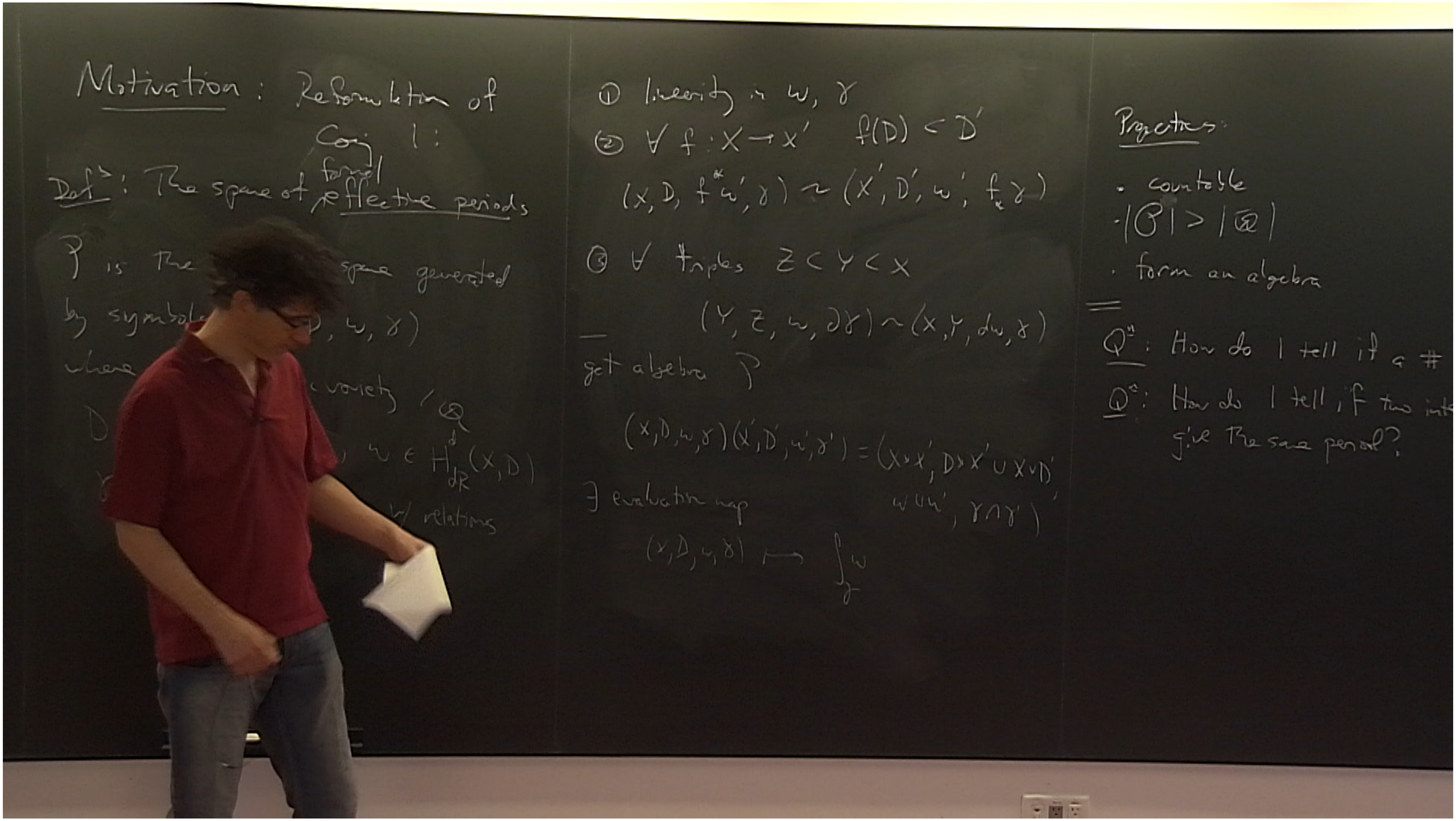
- countable
- $|\mathcal{P}| > |\overline{\mathbb{Q}}|$
- form an algebra

Q^n: How do I tell if a # is a period?

Q^n: How do I tell if two integral expressions give the same period?


Properties:
· countable
· |P| > |Q̄|
· form an a
Q: How … if a # is a period?
Q: How … if two integrals expressing
(Kontsevich, Zagier, Nori)
Conj: Any two representations of a period are equivalent under a sequence of the following moves:
① Addition (of integrand or domain of integration)
② Change of variables
③ Stokes' Thm
very deep!


Motivation: Reformulation of Conj 1:
Def^n: The space of formal effective periods
P is the Q-vector space g
by symbols (X, D, ω, γ)
where X algebraic varie
D ⊂ X a subvariety,
γ ∈ H_d(X(C), D(C); Q)
① linearity in ω, γ
② ∀ f: X → X' f(D) ⊂ D'
(X, D, f*ω', γ) ~ (X', D', ω', f_*γ)
③ ∀ triples Z ⊂ Y ⊂ X
(Y, Z, ω, ∂γ) ~ (X, Y, dω, γ)
Properties:
· countable
· |P| > |Q̄|
· form an algebra
Q^n: How do I tell
Q^n: How do I tell
give the same pe
Zagier
Kriz
1994


Motivation: Reformulation of formal Conj 1:
Def^n: The space of effective periods
P is the ... space generated
by symbols (..., ω, γ)
where ... variety / Q
ω ∈ H^d_dR (X, D)
w/ relations
① linearity in ω, γ
② ∀ f : X → X' f(D) ⊂ D'
(X, D, f*ω', γ) ~ (X', D', ω', f_* γ)
③ ∀ triples Z ⊂ Y ⊂ X
(Y, Z, ω, ∂γ) ~ (X, Y, dω, γ)
get algebra P
(X, D, ω, γ)(X', D', ω', γ') = (X×X', D×X' ∪ X×D', ω∪ω', γ∧γ')
∃ evaluation map
(X, D, ω, γ) ↦ ∫_γ ω
Properties:
· countable
· |C| > |P|
· form an algebra
Q^n: How do I tell if a #
Q^n: How do I tell if two int... give the same period?

Reformulation of Conj 1:

...formal effective periods

...tor space generated

(X, D, ω, γ)

...ic variety / $\mathbb{Q}$

...ety, $\omega \in H^d_{dR}(X, D)$

...γ; $\mathbb{Q}$) w/ relations

① linearity in ω, γ

② $\forall f : X \to X'$ $f(D) \subset D'$

$(X, D, f^*\omega', \gamma) \sim (X', D', \omega', f_*\gamma)$

③ $\forall$ triples $Z \subset Y \subset X$

$(Y, Z, \omega, \partial\gamma) \sim (X, Y, d\omega, \gamma)$

get algebra $\mathcal{P}$

$(X, D, \omega, \gamma)(X', D', \omega', \gamma') = (X \times X', D \times X' \cup X \times D', \omega \cup \omega', \gamma \wedge \gamma')$

$\exists$ evaluation map

$(X, D, \omega, \gamma) \longmapsto \int_\gamma \omega$

Conj 1 (reformulated):

The evaluation map is an isom

$\tilde{\mathcal{P}} = \mathcal{P}\{\frac{1}{\pi}\}$ $\qquad \tilde{\mathcal{P}} \xrightarrow{\simeq} \tilde{\mathcal{P}}$

$\tilde{\mathcal{P}} = \mathcal{P}\{\frac{1}{(\mathbb{G}_m, \{1\}, \frac{dx}{x}, S^1)}\}$

(Kontsevich...)

Conj: Any ...

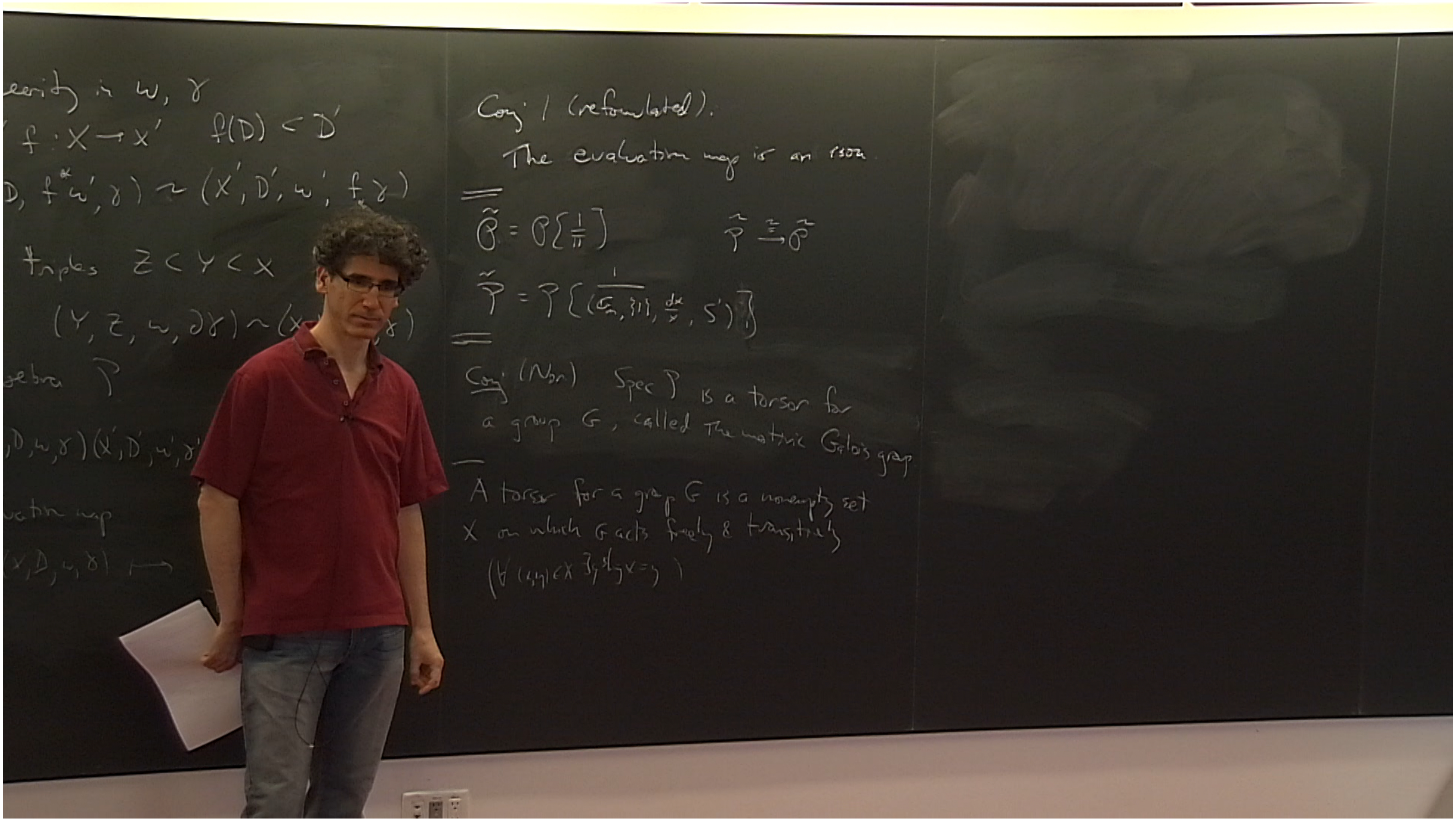
a per... a...

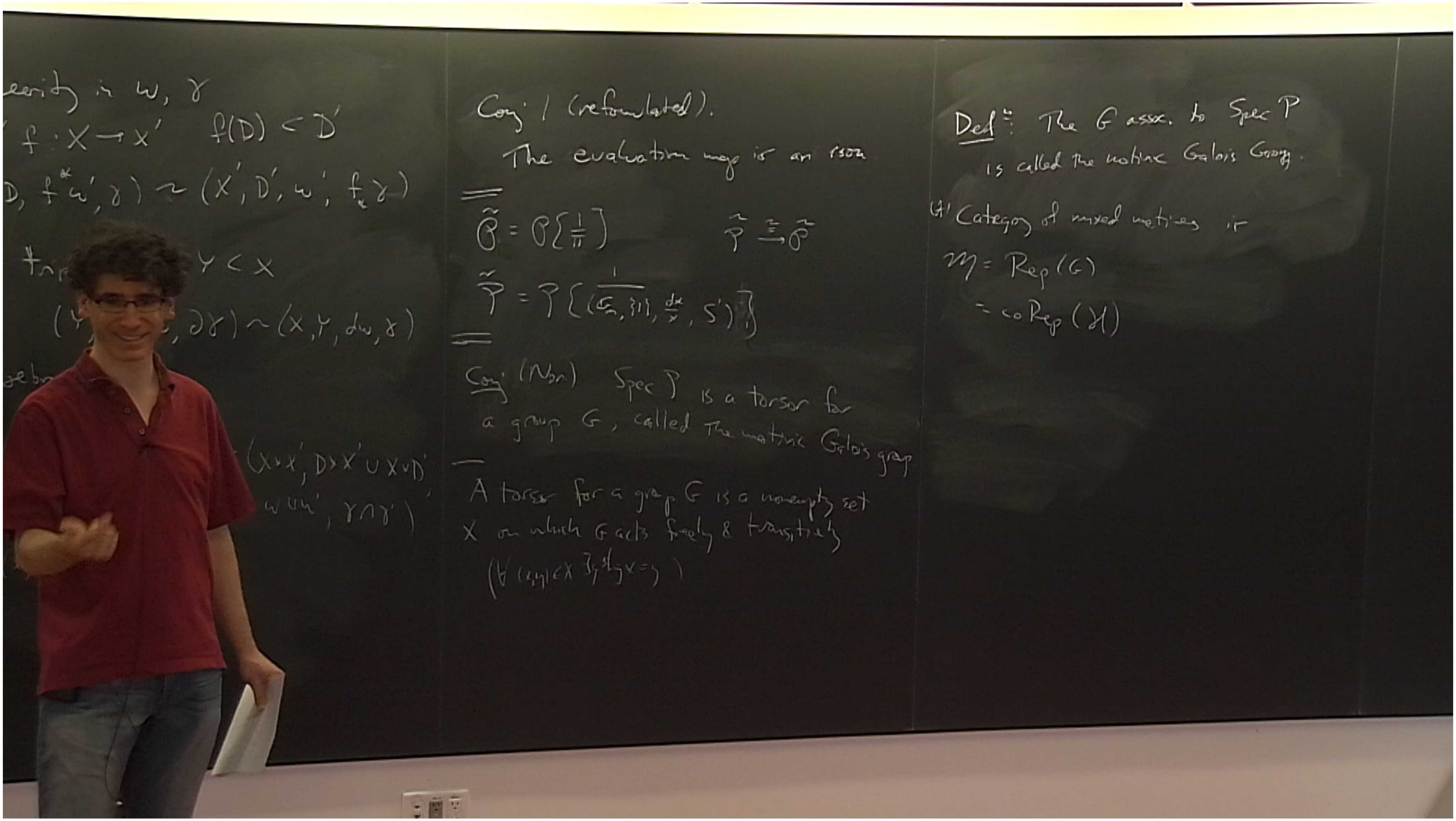
① Additi...

② Chang...

③ Stokes'

very deep!


Conj 1 (reformulated).
The evaluation map is an isom.
Conj (Nori) Spec P is a torsor for a group G, called the motivic Galois group
A torsor for a group G is a nonempty set X on which G acts freely & transitively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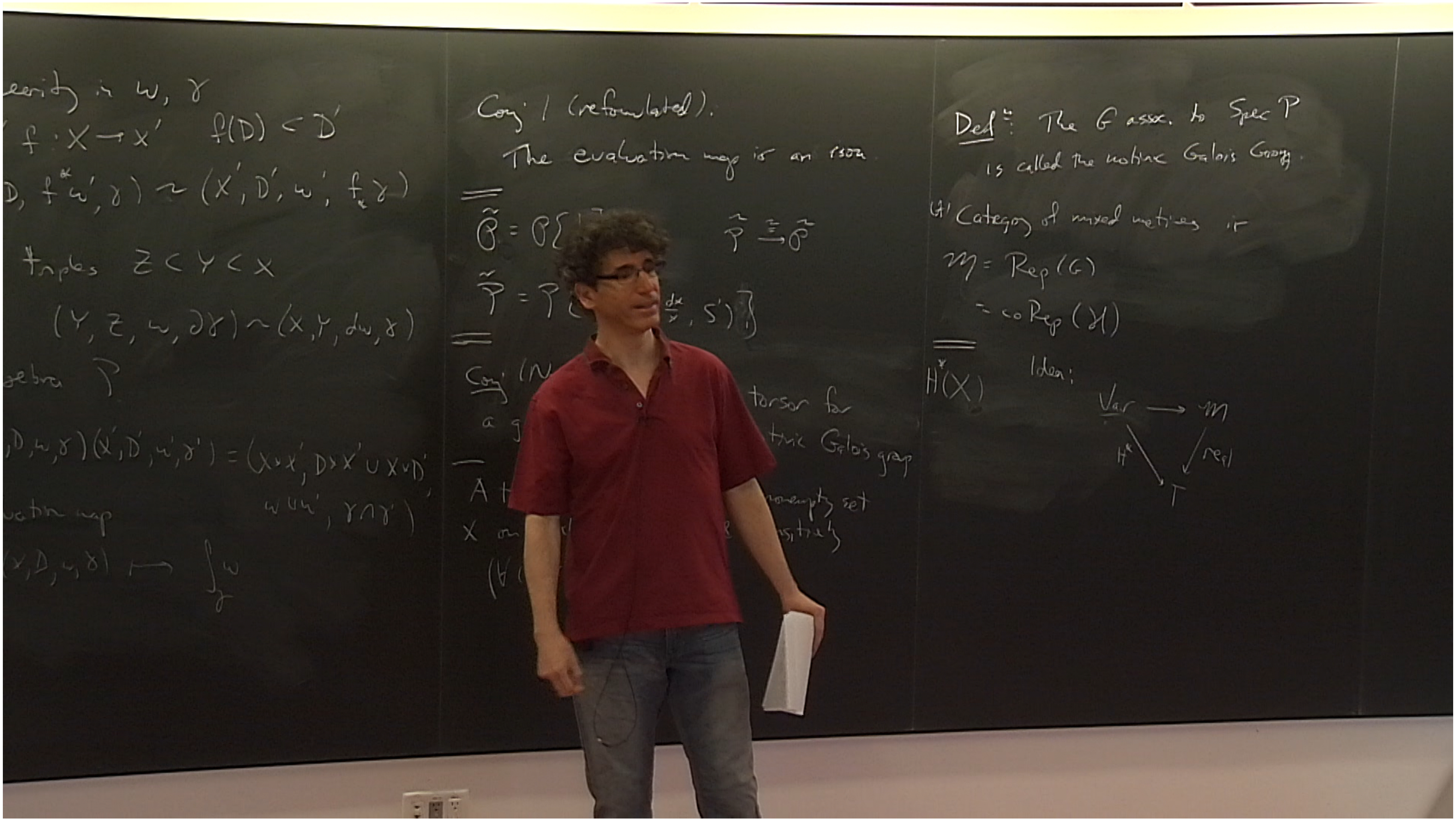
The evaluation map is an isom
$\tilde{\mathcal{P}} = \mathcal{P}[\frac{1}{\pi}]$
$\tilde{P} \xrightarrow{\simeq} \tilde{\mathcal{P}}$
Conj (Nori) Spec P is a torsor for a group G, called the motivic Galois group
A torsor for a group G is a nonempty set X on which G acts freely & transitively
Def: The G assoc. to Spec P is called the motivic Galois Group.
Category of mixed motives is
$\mathcal{M} = \mathrm{Rep}(G)$
$= \mathrm{coRep}(\mathcal{H})$
$f: X \to X'$ $f(D) \subset D'$


Conj 1 (reformulated).
The evaluation map is an isom
Def: The G assoc. to Spec P
is called the motivic Galois Group.
Category of mixed motives is
M = Rep(G)
= coRep(H)
Idea:
Var → M
H*
real
T
H*(X)
triples Z < Y < X
(Y, Z, ω, ∂γ) ~ (X, Y, dω, γ)
torsor for
tivic Galois group
nonempty set
transitively

Conj 1 (reformulated).

The evaluation map is an isom

$\tilde{\mathcal{P}} = \mathcal{P}[\frac{1}{\pi}]$ $\quad \tilde{\mathcal{P}} \xrightarrow{\cong} \tilde{\mathcal{P}}$

$\tilde{\mathcal{P}} = \mathcal{P}\left[\frac{1}{(\mathbb{G}_m, \{1\}, \frac{dx}{x}, S')}\right]$

Conj (Nori) Spec $\mathcal{P}$ is a torsor for a group G, called the motivic Galois group

A torsor for a group G is a nonempty set X on which G acts freely & transitively

$(\forall (x,y) \in X \; \exists ! g \text{ s/t } gx = y)$

Def: The G assoc. to Spec $\mathcal{P}$ is called the motivic Galois Group.

(A) Category of mixed motives is

$\mathcal{M} = \mathrm{Rep}(G)$

$= \mathrm{coRep}(\mathcal{H})$

$H^*(X)$

Idea:

$\mathrm{Per}(M) \ni |G(M)|$

$$\begin{tikzcd} \underline{Var} \arrow[r] \arrow[dr, "H^*"'] & \mathcal{M} \arrow[d, "real"] \\ & T \end{tikzcd}$$

semisimple subcat - pure motives

Zagier

ch,
Kr...

G is too hard. Let's look at simpler subcategories.

Mixed Tate motives - generated by $\langle H^2(\mathbb{P}^1) \rangle$ $\overset{\shortparallel}{\mathbb{Q}(-1)}$

Pure Tate motives $\quad$ Rep $\mathbb{G}_m$

mixed Tate $\quad$ Rep G_T

Conj: G_T = free abelian gr gen by $\{\log p \mid p = \text{prime}\}$ & $\{\zeta(2n+1) \quad n \in \mathbb{N}\}$ $/\mathbb{Q}$

$\mathbb{P}^1 \supset pt$

$\mathbb{P}^n \supset \mathbb{P}^{n-1} \supset \mathbb{P}^{n-2} \supset \dots \supset pt$

$\mathbb{Q}_{n,j} \quad \mathbb{Q}_{\zeta_1}$

① linearity in ω, γ

② $\forall f: X \to X' \quad f(D) \subset D'$

$(X, D, f^*\omega', \gamma) \sim (X', D', \omega', f_*\gamma)$

③ $\forall$ triples $Z \subset Y \subset X$

$(Y, Z, \omega, \partial\gamma) \sim (X, Y, d\omega, \gamma)$

get algebra $\mathcal{P}$

$(X, D, \omega, \gamma)(X', D', \omega', \gamma') = (X \times X', D \times X' \cup X \times D', \omega \wedge \omega', \gamma \wedge \gamma')$

$\exists$ evaluation map

$(X, D, \omega, \gamma) \longmapsto \int_\gamma \omega$

Conj / (reformulated)

The evaluation

$\tilde{\mathcal{P}} = \mathcal{P}[\frac{1}{\pi}]$

$\tilde{\mathcal{P}} = \mathcal{P}\{(\frac{1}{(\mathbb{G}_m, \{1\}, \frac{dx}{x}, \dots}$

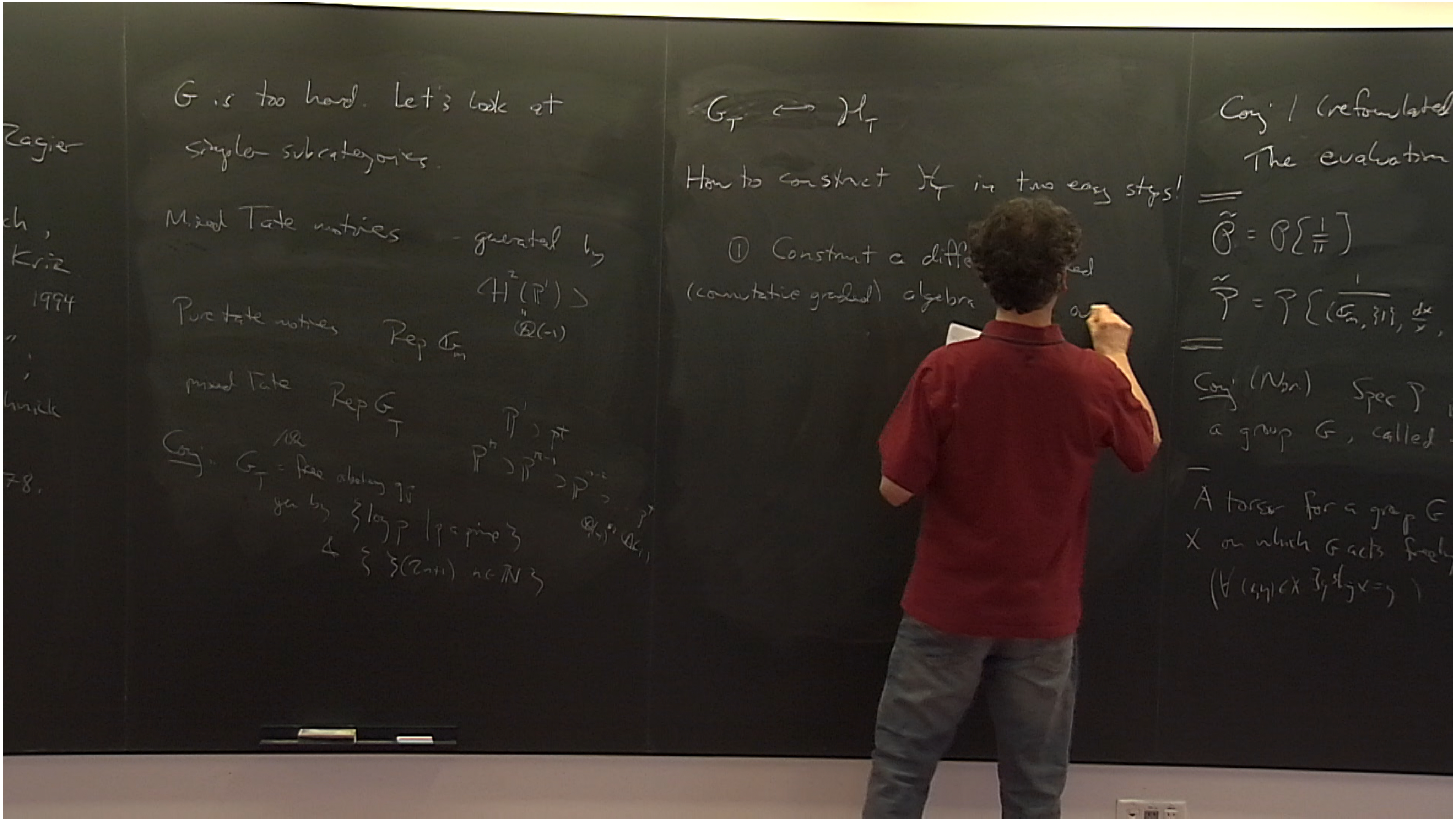
Conj (Kon) $\quad$ Spec $\mathcal{P}$

a group G, called

A torsor for a group G

X on which G acts freely

$(\forall (x,y) \in X \; \exists! g \; gx = y)$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Mixed Tate motives - generated by
Pure tate motives Rep $\mathbb{G}_m$
mixed Tate Rep G_T
Conj: G_T = free algebra
gen by {log p | p prime}
& {ζ(2n+1) n ∈ ℕ}
$G_T \longleftrightarrow \mathcal{H}_T$
How to construct $\mathcal{H}_T$ in two easy steps!
① Construct a diffe
(commutative graded) algebra
Conj (reformulated)
The evaluation
Spec P
a group G, called
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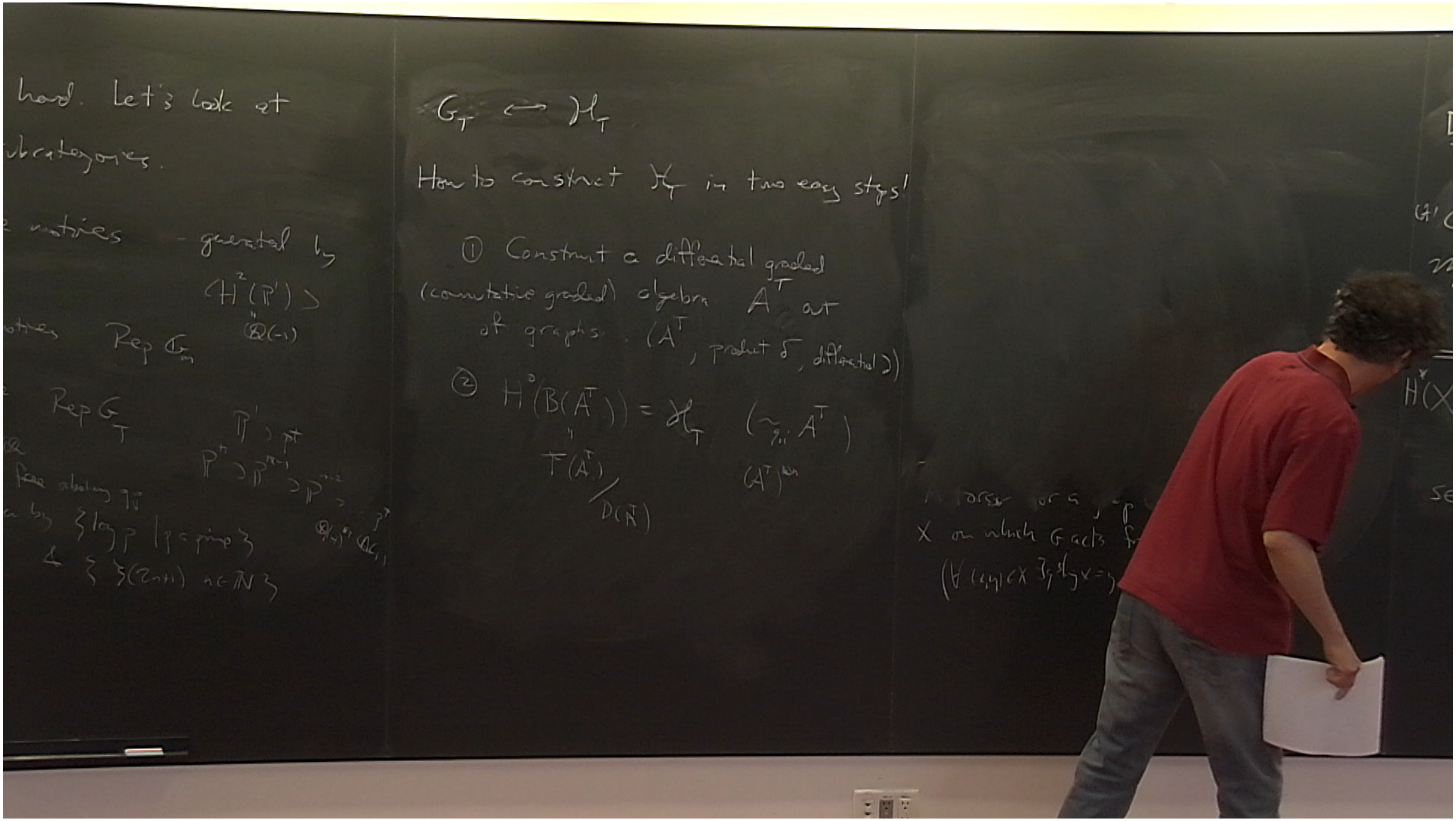
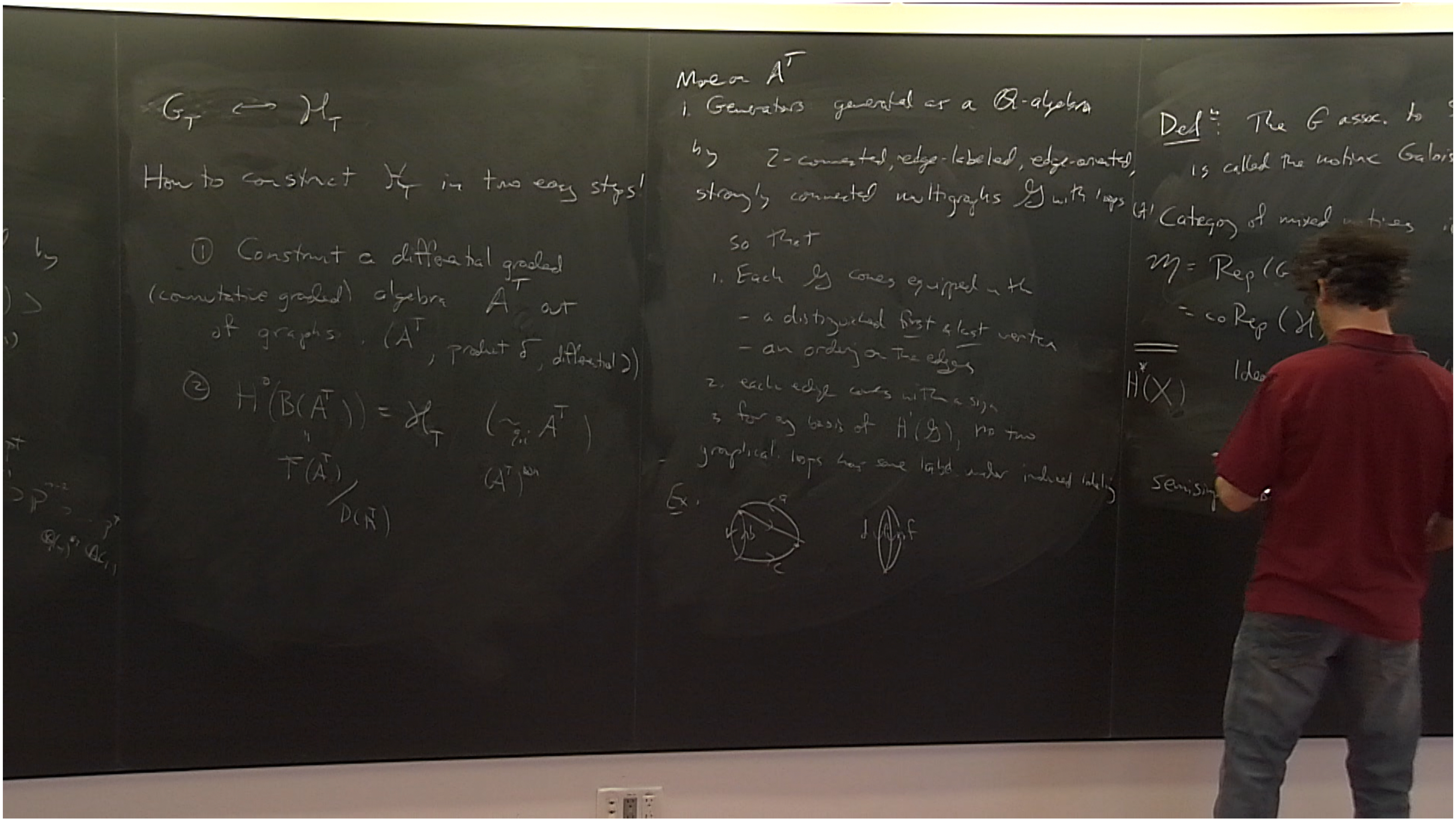
X on which G acts freely


hard. Let's look at
subcategories.
matrices - generated by
$\langle H^2(\mathbb{P}^1) \rangle$
$\mathbb{Q}(-1)$
Rep $\mathbb{G}_m$
Rep G_T
$\mathbb{P}^1 \supset pt$
$\mathbb{P}^n \supset \mathbb{P}^{n-1} \supset \mathbb{P}^{n-2} \supset$
free algebra gen by $\{\log p \mid p = \text{prime}\}$
& $\{\zeta(2n+1) \mid n \in \mathbb{N}\}$
$G_T \longleftrightarrow \mathcal{H}_T$
How to construct $\mathcal{H}_T$ in two easy steps!
① Construct a differential graded (commutative graded) algebra A^T out of graphs. (A^T, product δ, differential ∂)
② $H^0(B(A^T)) = \mathcal{H}_T$ ($\sim_{q.i.} A^T$)
$T(A^T)$
$D(A^T)$
$(A^T)^{\mathrm{ab}}$
X on which G acts
$(\forall (x,y) \in X \; \exists g \; gx = y)$



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$T(A^T)$
$D(R^T)$
$(A^T)^{ab}$
More on A^T
1. Generators generated as a $\mathbb{Q}$-algebra
by 2-connected, edge-labeled, edge-oriented,
strongly connected multigraphs $\mathcal{G}$ with loops
so that
1. Each $\mathcal{G}$ comes equipped with
- a distinguished first & last vertex
- an ordering on the edges
2. each edge comes with a sign
3. for any basis of $H^1(\mathcal{G})$, no two
graphical loops have same label under induced labeling
Ex.
Def: The G assoc. to
is called the motivic Galois
Category of mixed motives
$\mathcal{M} = \mathrm{Rep}(G$
$= \mathrm{coRep}(\mathcal{H}$
$H^*(X)$
Idea
Semisimp

$\mathcal{H}_T$

...ruct $\mathcal{H}_T$ in two easy steps!

...ruct a differential graded (graded) algebra A^T out ...graphs. (A^T, product δ, differential ∂)

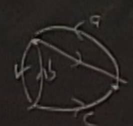
$H(A^T)) = \mathcal{H}_T \quad (\sim_{?} A^T)$

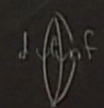
$F(A^T) / D(A^T) \quad (A^T)^{?}$

More on A^T

1. Generators generated as a $\mathbb{Q}$-algebra by 2-connected, edge-labeled, edge-oriented, strongly connected multigraphs $\mathcal{G}$ with loops so that
   1. Each $\mathcal{G}$ comes equipped with
      - a distinguished first & last vertex
      - an ordering on the edges
   2. each edge comes with a sign
   3. for any basis of $H^1(\mathcal{G})$, no two graphical loops have same label under induced labeling

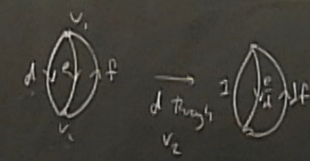
Ex.



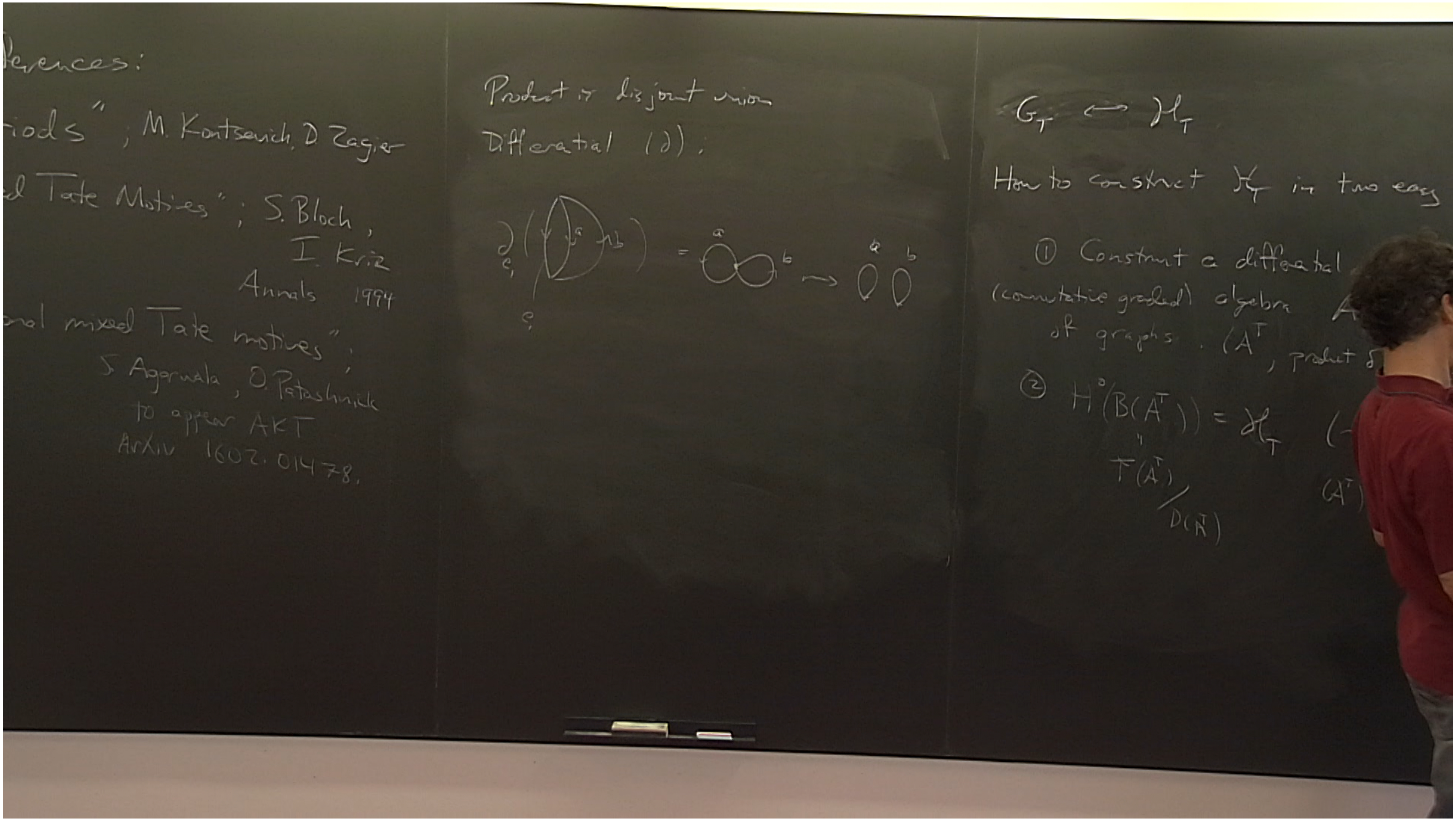


Relations:

1. Pushing a label through a vertex



2. Alternating projection under $\Sigma_n \propto (\mathbb{Z}/2\mathbb{Z})^2$ — permutation of labels — $a \mapsto \frac{1}{a}$
3. Global orientation reversing



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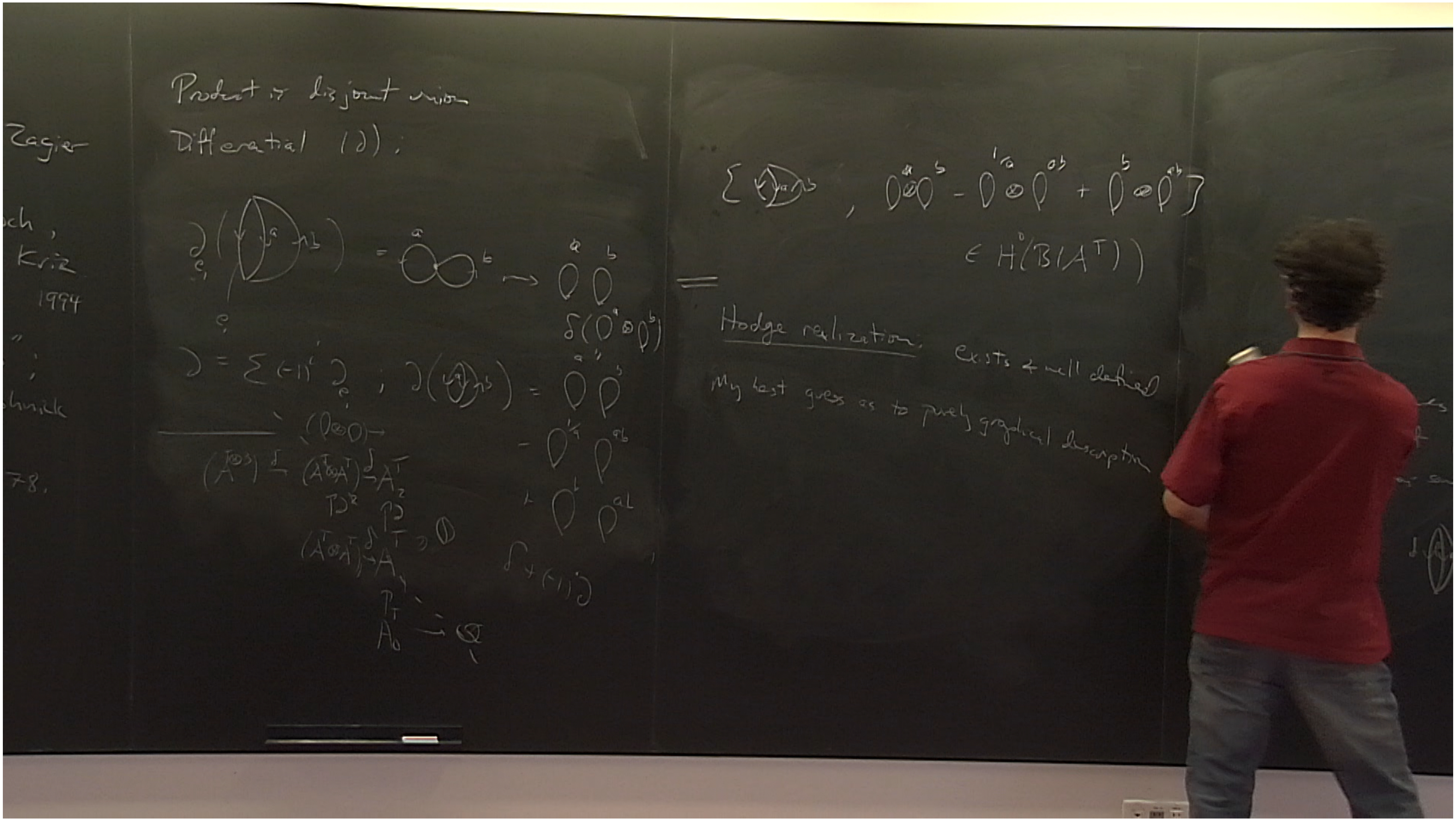
Zagier
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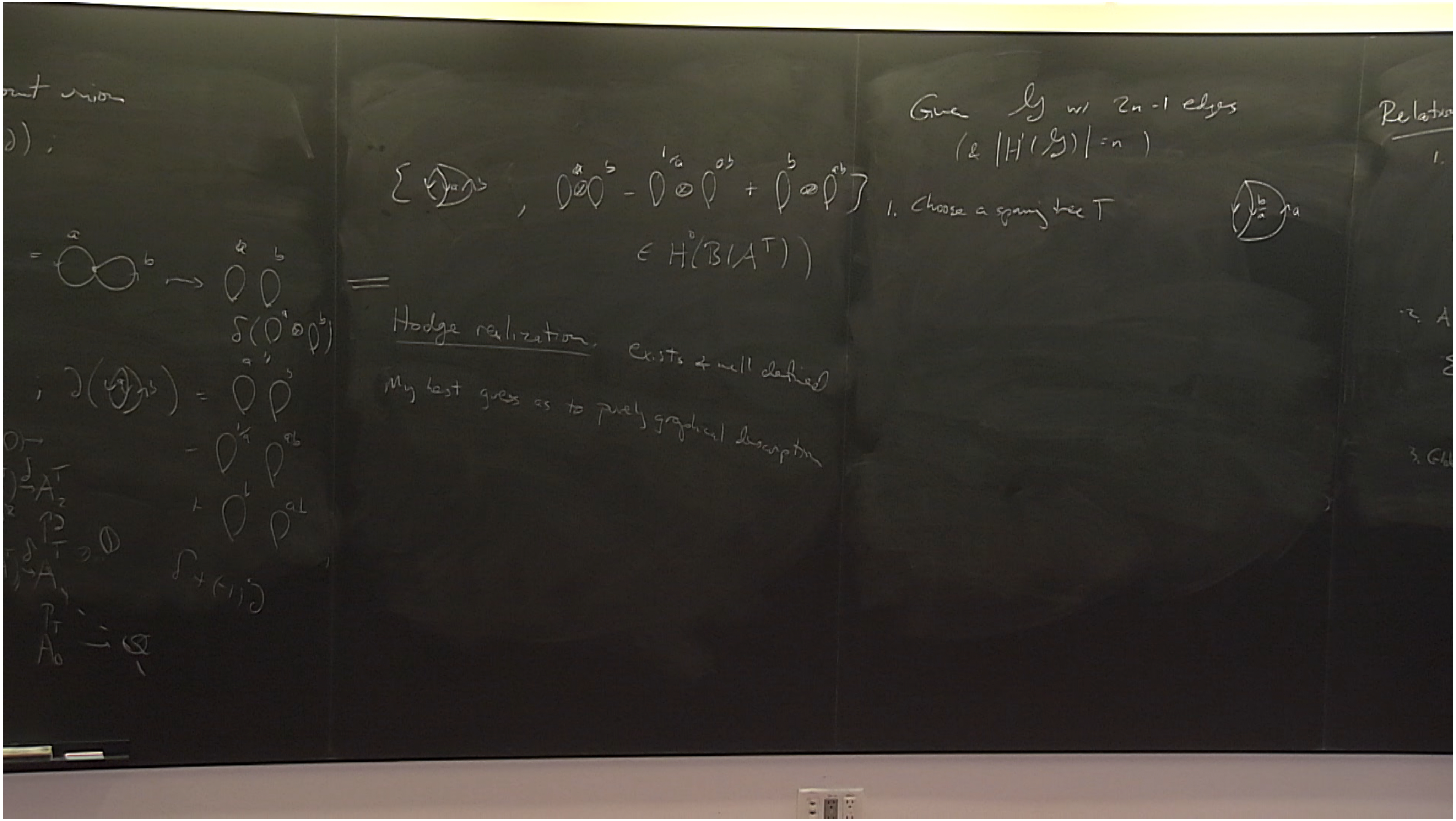
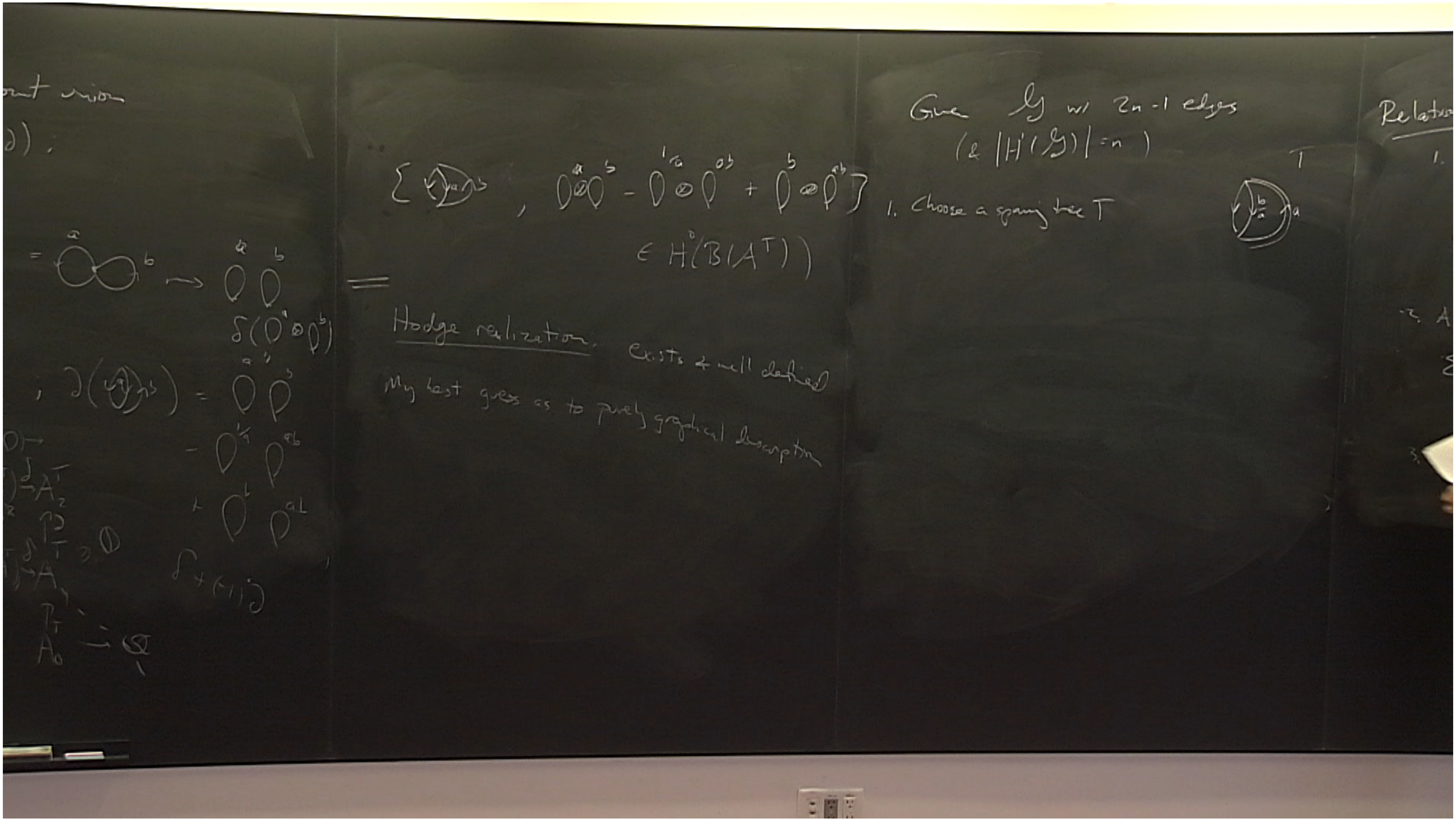
1994
Product is disjoint union
Differential (∂):
Hodge realization: exists & well defined
My best guess as to purely graphical description


Given 𝒢 w/ 2n-1 edges
(& |H¹(𝒢)| = n)
1. Choose a spanning tree T
Hodge realization: exists & well defined
My best guess as to purely graphical description
Relations


Given $\mathcal{G}$ w/ 2n-1 edges
(& $|H^1(\mathcal{G})| = n$)
1. Choose a spanning tree T
T
Relations
$\in H^0(B(A^T))$
Hodge realization: exists & well defined
My best guess as to purely graphical description