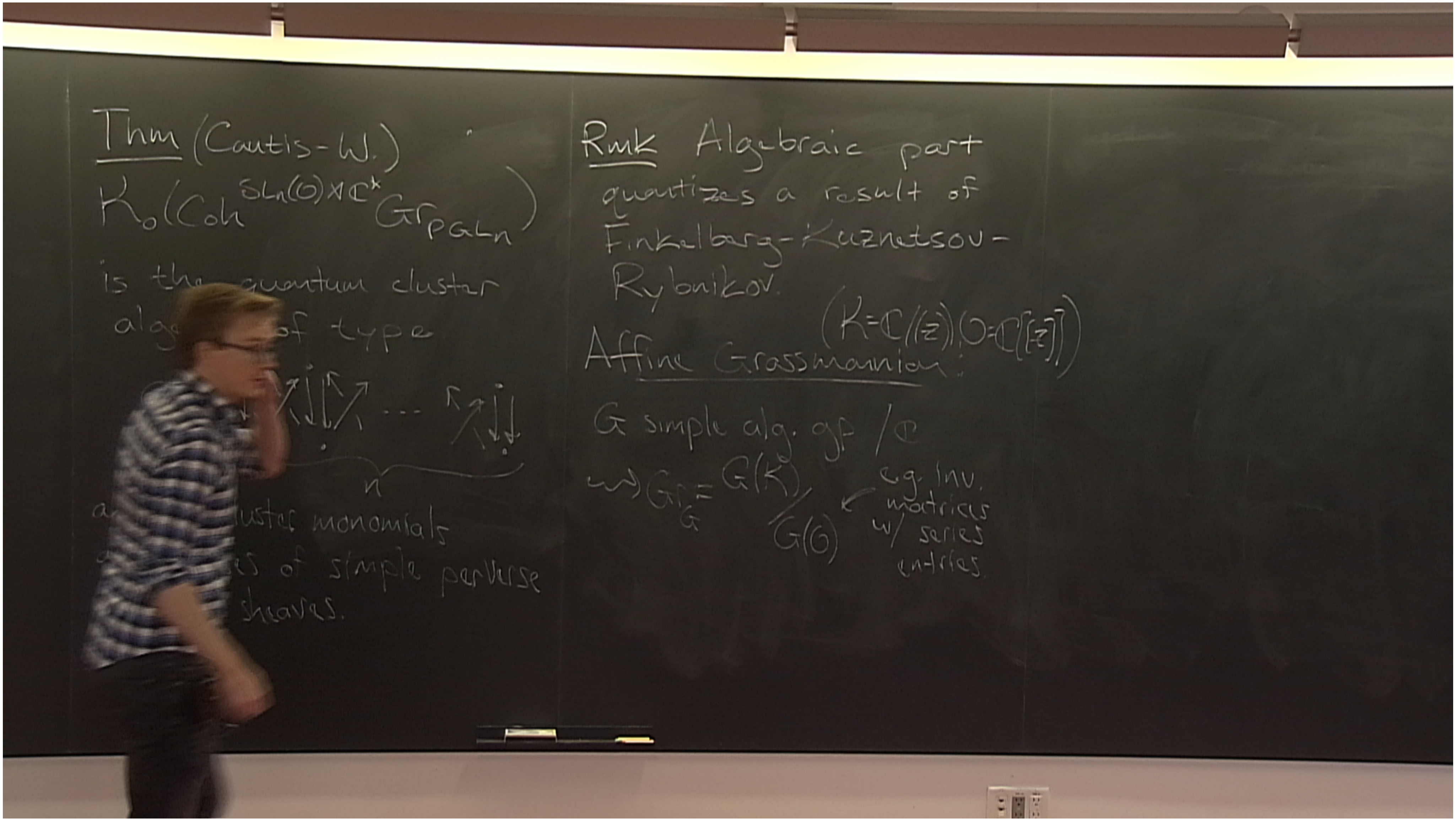
Title: The Coherent Satake Category, Clusters, and Wilson-'t Hooft Operators

Date: Dec 01, 2016 02:00 PM

URL: http://pirsa.org/16120011

Abstract: <p>We discuss recent work showing that in type A_n the category of equivariant perverse coherent sheaves on the affine Grassmannian categorifies the cluster algebra associated to the BPS quiver of pure N=2 gauge theory. Physically, this can be understood as a statement about line operators in this theory, following ideas of Gaiotto-Moore-Neitzke, Costello, and Kapustin-Saulina -- in short, coherent IC sheaves are the precise algebro-geometric counterparts of Wilson-'t Hooft line operators. The proof relies on techniques developed by Kang-Kashiwara-Kim-Oh in the setting of KLR algebras. A key moral is that the appearance of cluster structures is in large part forced by the compatibility between chiral and tensor structures on the category in question (i.e. by formal features of holomorphic-topological field theory). This is joint work with Sabin Cautis.</p>



Thm (Cautis-W.)

$K_0(\mathrm{Coh}^{SL_n(\mathcal{O})\rtimes\mathbb{C}^*}\mathrm{Gr}_{PGL_n})$

is the quantum cluster alg[illegible] of type

a[illegible]uster monomials

[illegible]es of simple perverse sheaves.

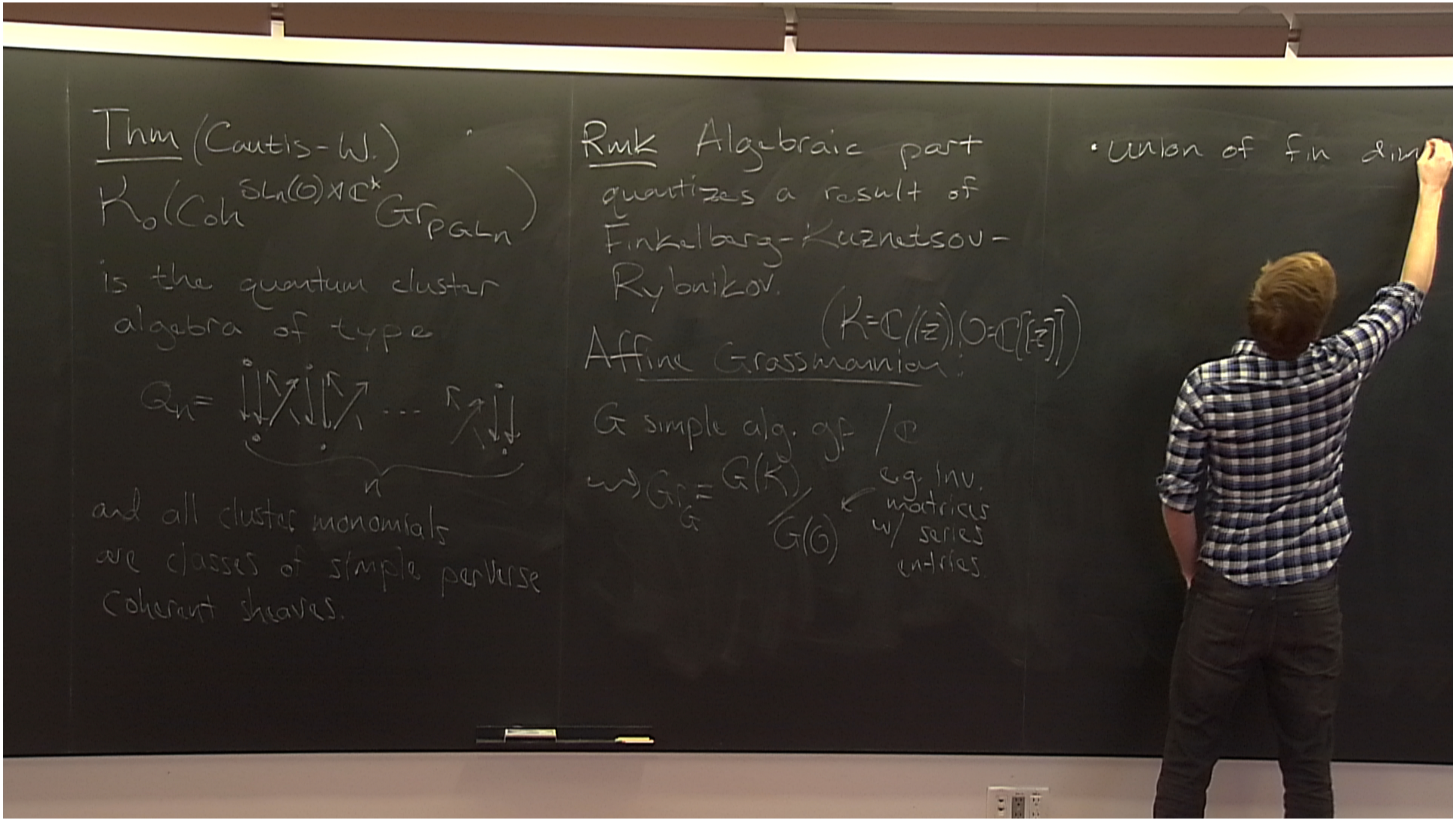
Rmk Algebraic part quantizes a result of Finkelberg-Kuznetsov-Rybnikov.

Affine Grassmannian:

$(K=\mathbb{C}((z)), \mathcal{O}=\mathbb{C}[[z]])$

G simple alg. gp $/\mathbb{C}$

$\rightsquigarrow \mathrm{Gr}_G = G(K)/G(\mathcal{O}) \leftarrow$ e.g. inv. matrices w/ series entries



Thm (Cautis-W.)

$K_0(\mathrm{Coh}^{SL_n(\mathcal{O})\rtimes\mathbb{C}^*}\mathrm{Gr}_{PGL_n})$

is the quantum cluster algebra of type

$Q_n = \underbrace{\qquad\cdots\qquad}_{n}$

and all cluster monomials are classes of simple perverse coherent sheaves.

Rmk Algebraic part quantizes a result of Finkelberg-Kuznetsov-Rybnikov.

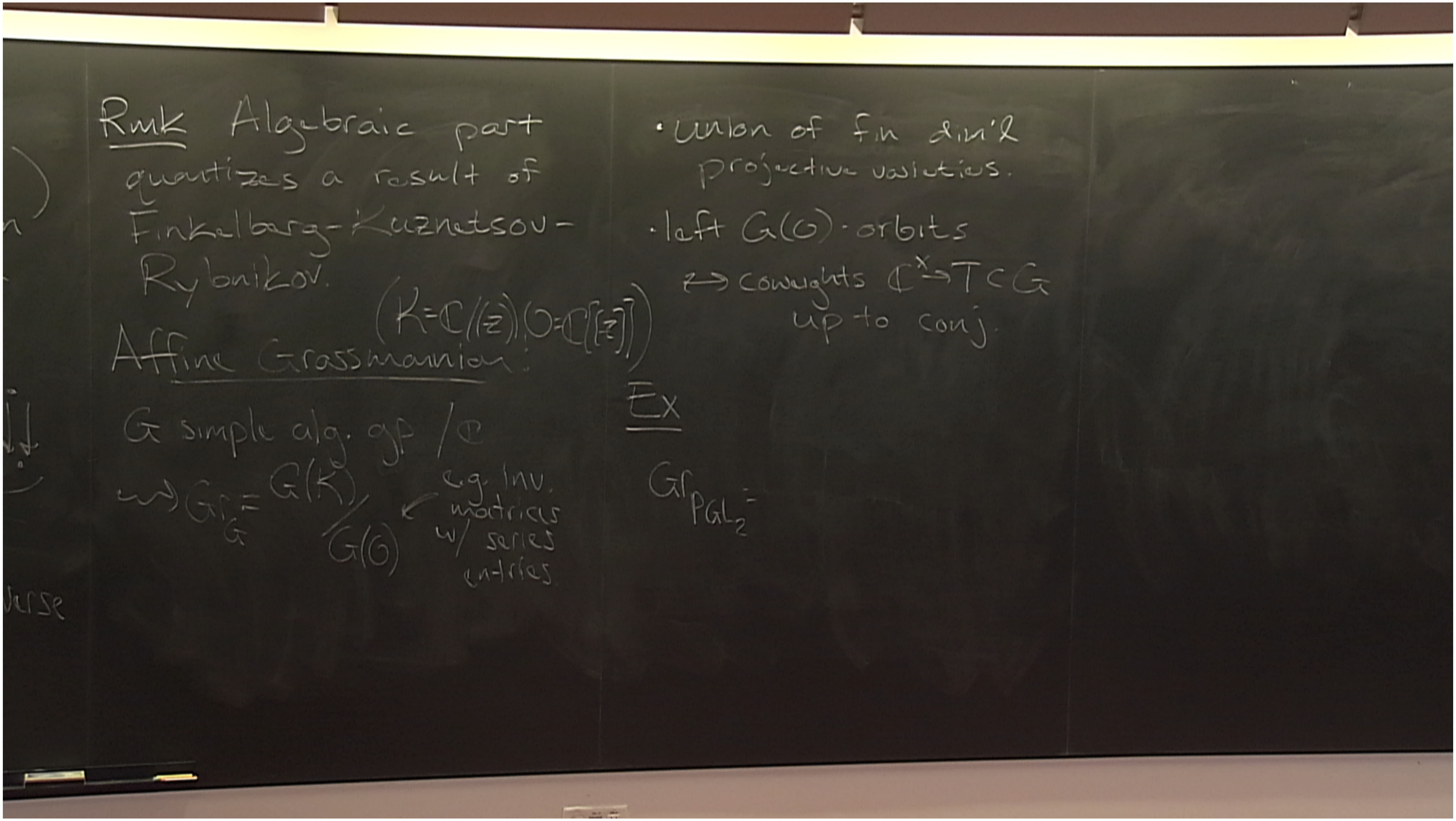
Affine Grassmannian:

$(K = \mathbb{C}((z)), \mathcal{O} = \mathbb{C}[[z]])$

G simple alg. gp / $\mathbb{C}$

$\rightsquigarrow \mathrm{Gr}_G = G(K)/G(\mathcal{O})$ ← e.g. inv. matrices w/ series entries

• union of fin dim



n

verse

<u>Rmk</u> Algebraic part quantizes a result of Finkelberg-Kuznetsov-Rybnikov.

<u>Affine Grassmannian</u>: $\left(K=\mathbb{C}((z)),\ \mathcal{O}=\mathbb{C}[[z]]\right)$

G simple alg. gp / $\mathbb{C}$

$\rightsquigarrow Gr_G = G(K)/G(\mathcal{O})$ ← e.g. inv. matrices w/ series entries

- union of fin dim'l projective varieties.
- left $G(\mathcal{O})$-orbits

  $\leftrightarrow$ coweights $\mathbb{C}^\times \to T \subset G$ up to conj.

<u>Ex</u>

$Gr_{PGL_2} =$

Rmk Algebraic part quantizes a result of Finkelberg-Kuznetsov-Rybnikov.

Affine Grassmannian:

$(K=\mathbb{C}((z)), \mathcal{O}=\mathbb{C}[[z]])$

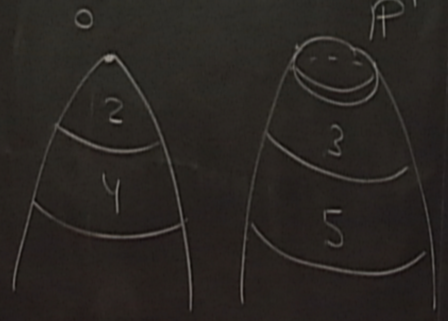
G simple alg. gp / $\mathbb{C}$

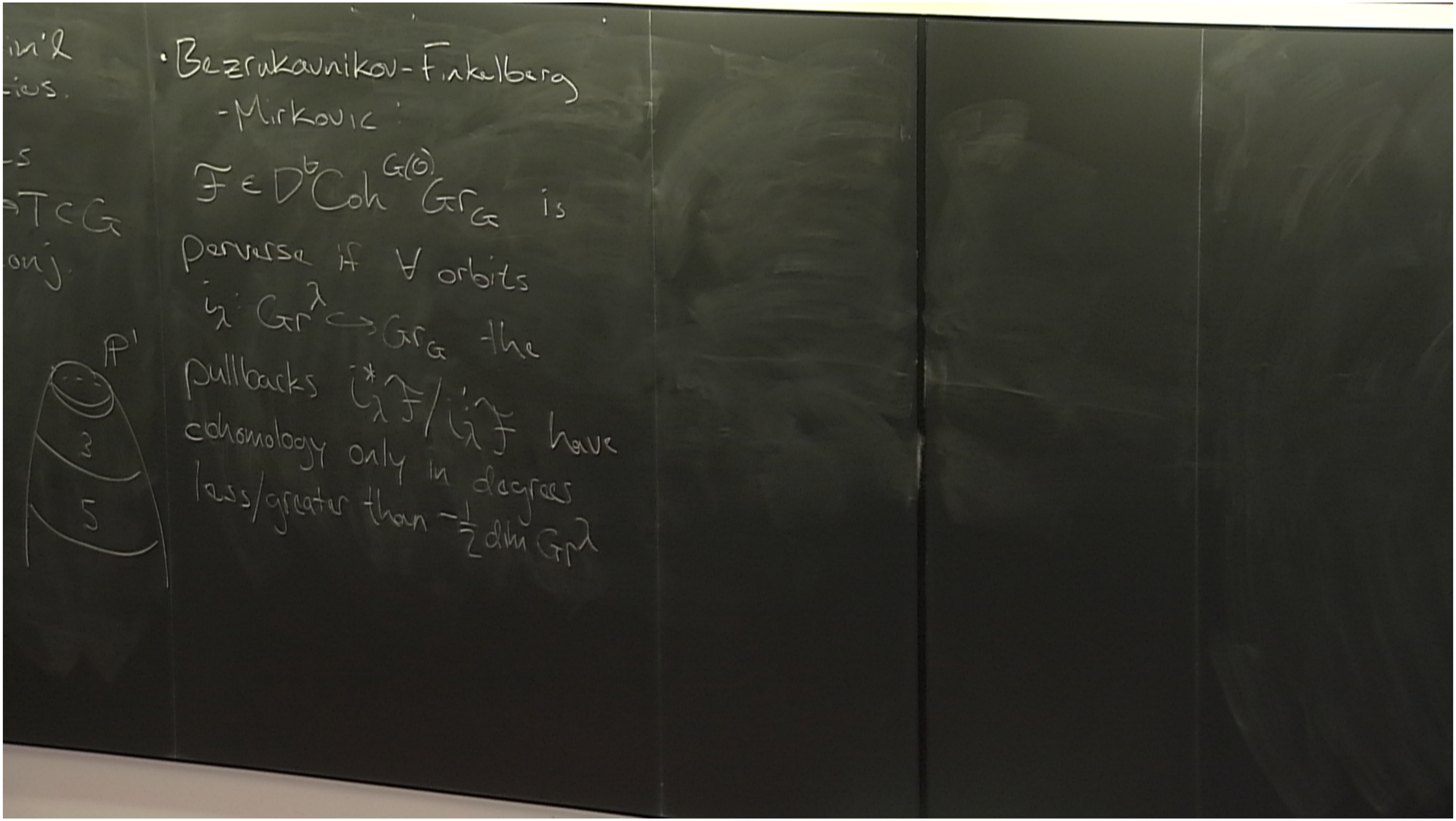
$\rightsquigarrow Gr_G = G(K) / G(\mathcal{O})$ ← e.g. inv. matrices w/ series entries

- union of fin dim'l projective varieties.
- left $G(\mathcal{O})$-orbits $\leftrightarrow$ coweights $\mathbb{C}^\times \to T \subset G$ up to conj.

Ex

$Gr_{PGL_2} =$



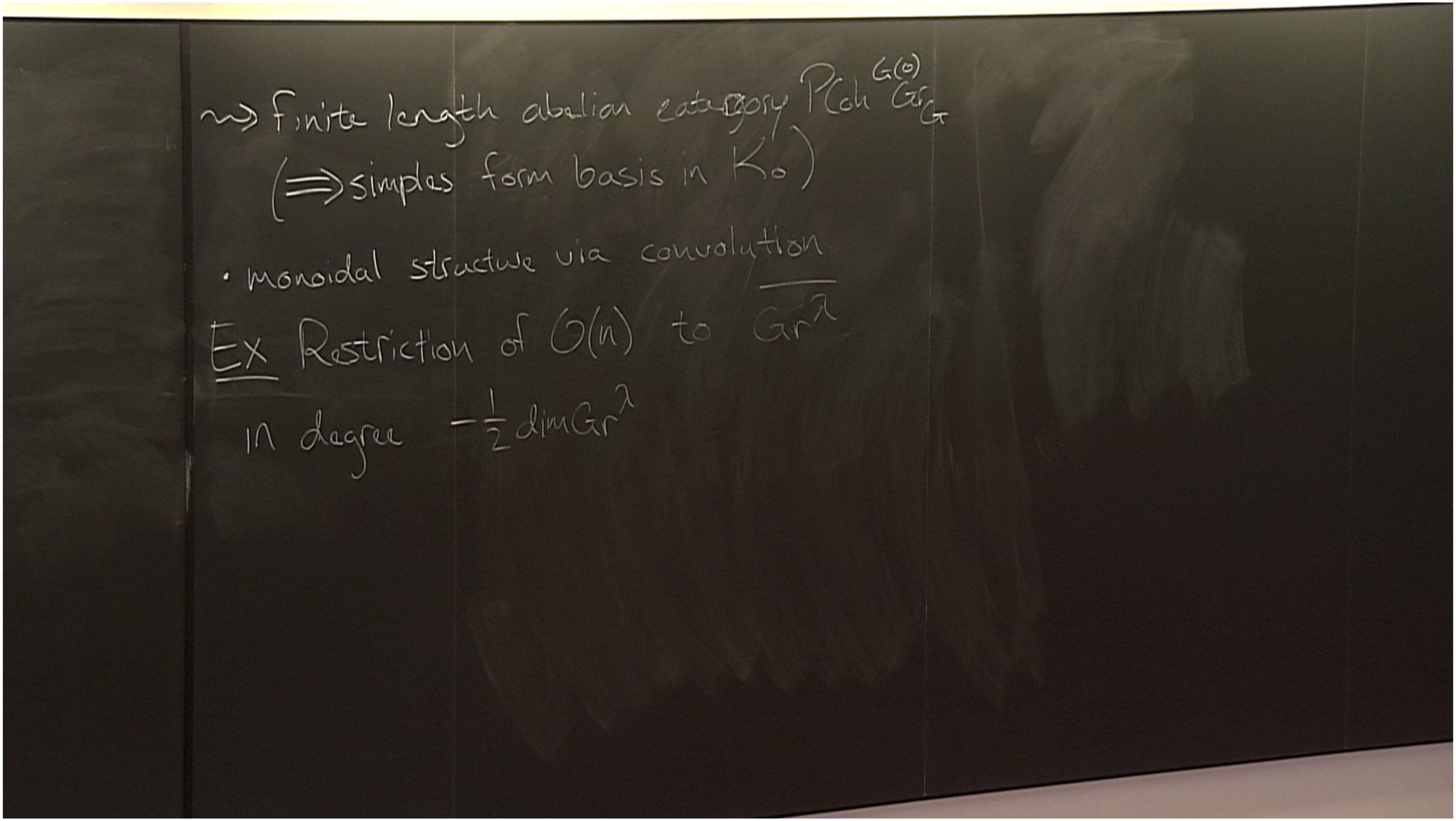


in'l
ies.

s
$T \subset G$
onj.

- Bezrukavnikov-Finkelberg-Mirkovic:

$\mathcal{F} \in D^b\mathrm{Coh}^{G(\mathcal{O})}\mathrm{Gr}_G$ is perverse if $\forall$ orbits $i_\lambda : \mathrm{Gr}^\lambda \hookrightarrow \mathrm{Gr}_G$ the pullbacks $i_\lambda^* \mathcal{F} / i_\lambda^! \mathcal{F}$ have cohomology only in degrees less/greater than $-\frac{1}{2}\dim \mathrm{Gr}^\lambda$



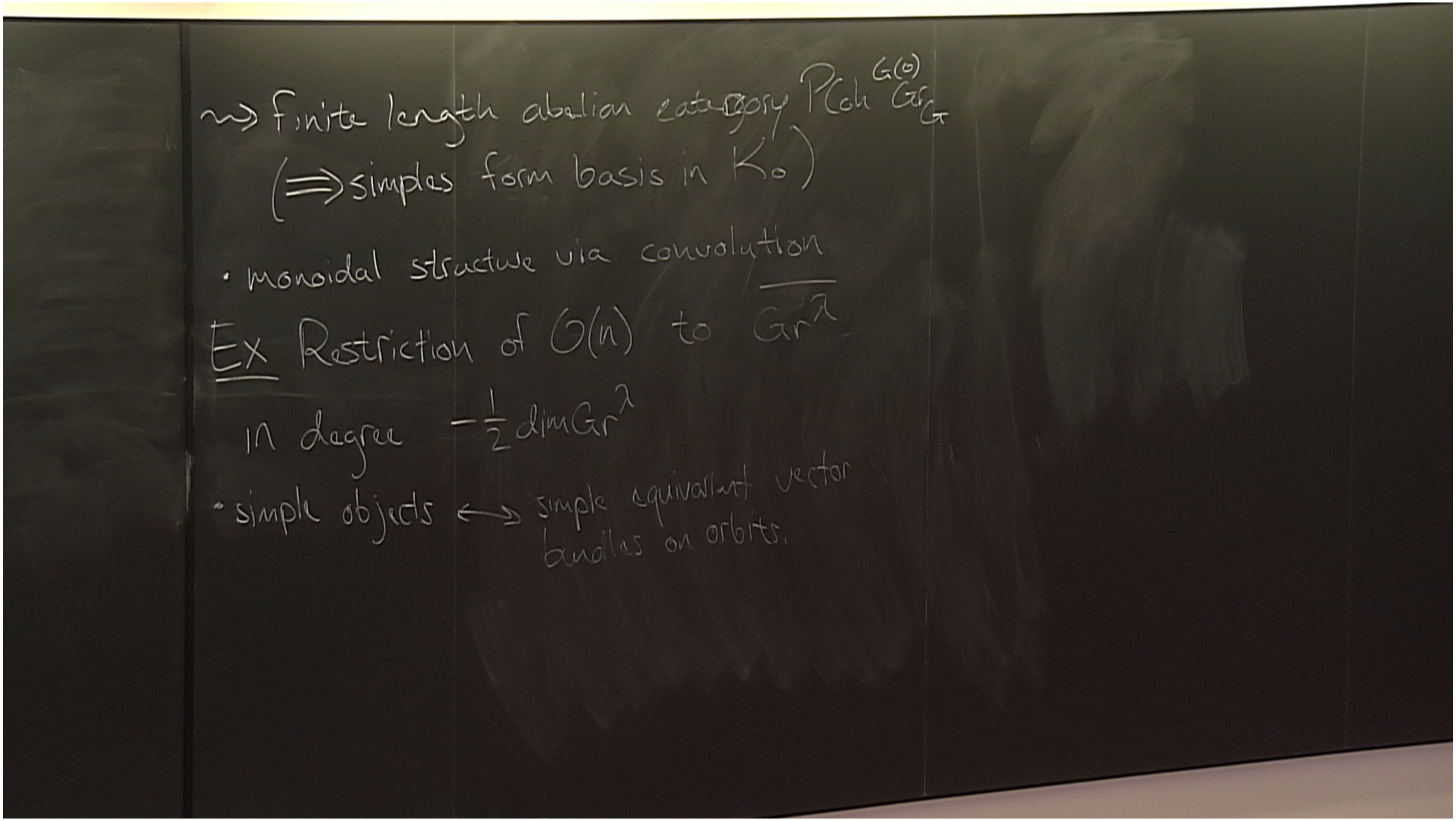
$\rightsquigarrow$ finite length abelian category $PCoh^{G(\mathcal{O})}_{Gr_G}$

($\Longrightarrow$ simples form basis in K_0)

- monoidal structure via convolution

<u>Ex</u> Restriction of $\mathcal{O}(n)$ to $\overline{Gr^\lambda}$

in degree $-\frac{1}{2}\dim Gr^\lambda$



$\rightsquigarrow$ finite length abelian category $\mathrm{PCoh}^{G(\mathcal{O})}_{\mathrm{Gr}_G}$

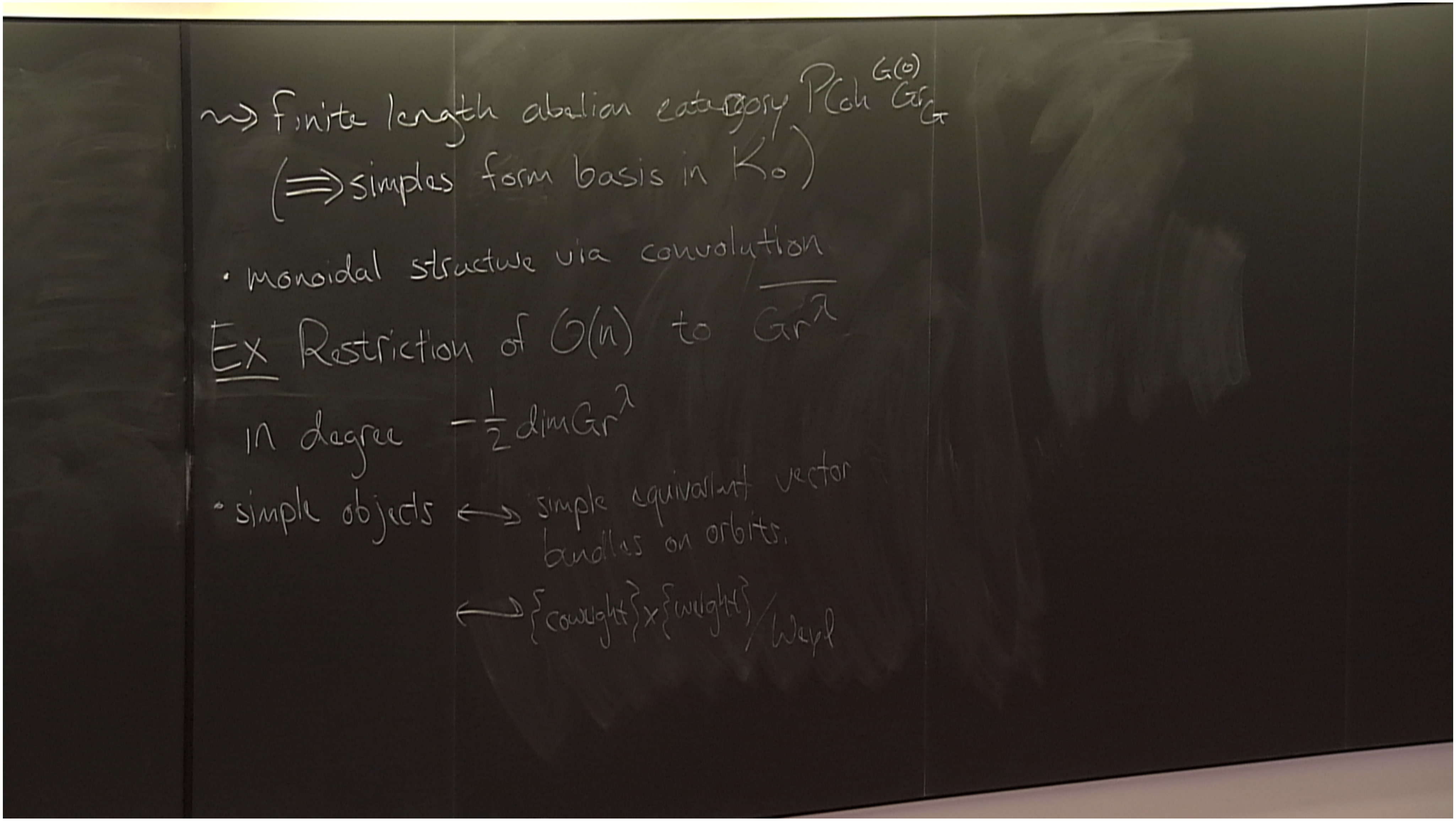
($\Longrightarrow$ simples form basis in K_0)

- monoidal structure via convolution

Ex Restriction of $\mathcal{O}(n)$ to $\overline{\mathrm{Gr}^\lambda}$

in degree $-\frac{1}{2}\dim \mathrm{Gr}^\lambda$

- simple objects $\longleftrightarrow$ simple equivariant vector bundles on orbits.



$\rightsquigarrow$ finite length abelian category $P\mathrm{Coh}^{G(\mathcal{O})} \mathrm{Gr}_G$

($\Longrightarrow$ simples form basis in K_0)

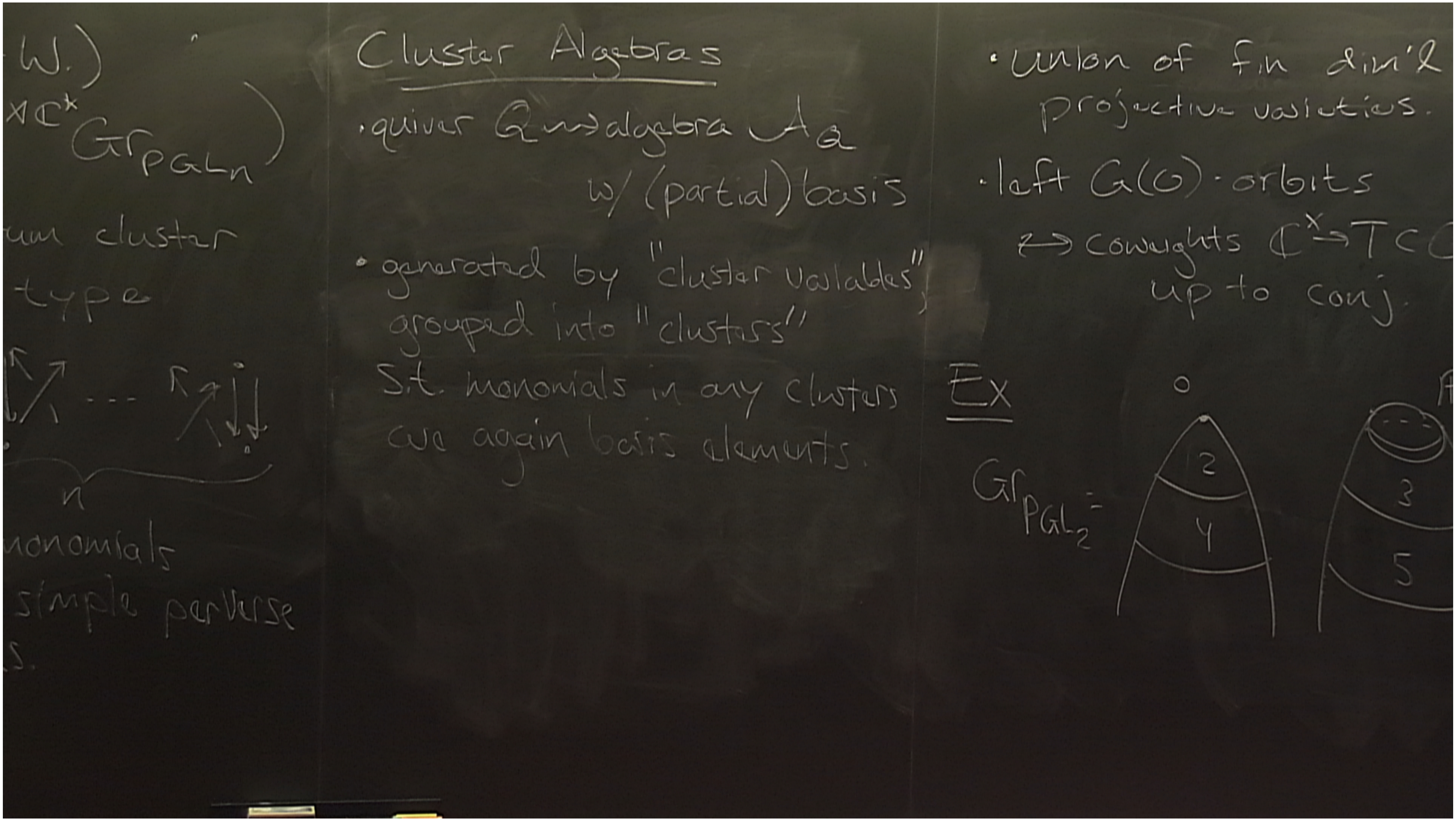
- monoidal structure via convolution

<u>Ex</u> Restriction of $\mathcal{O}(n)$ to $\overline{\mathrm{Gr}^\lambda}$

in degree $-\frac{1}{2}\dim \mathrm{Gr}^\lambda$

- simple objects $\longleftrightarrow$ simple equivariant vector bundles on orbits.

$\longleftrightarrow \{\text{coweight}\} \times \{\text{weight}\} / \text{Weyl}$



W.)

$\times \mathbb{C}^* \mathrm{Gr}_{PGL_n}$)

um cluster

type

$\underbrace{\qquad\qquad\qquad}_{n}$

monomials

simple perverse

s.

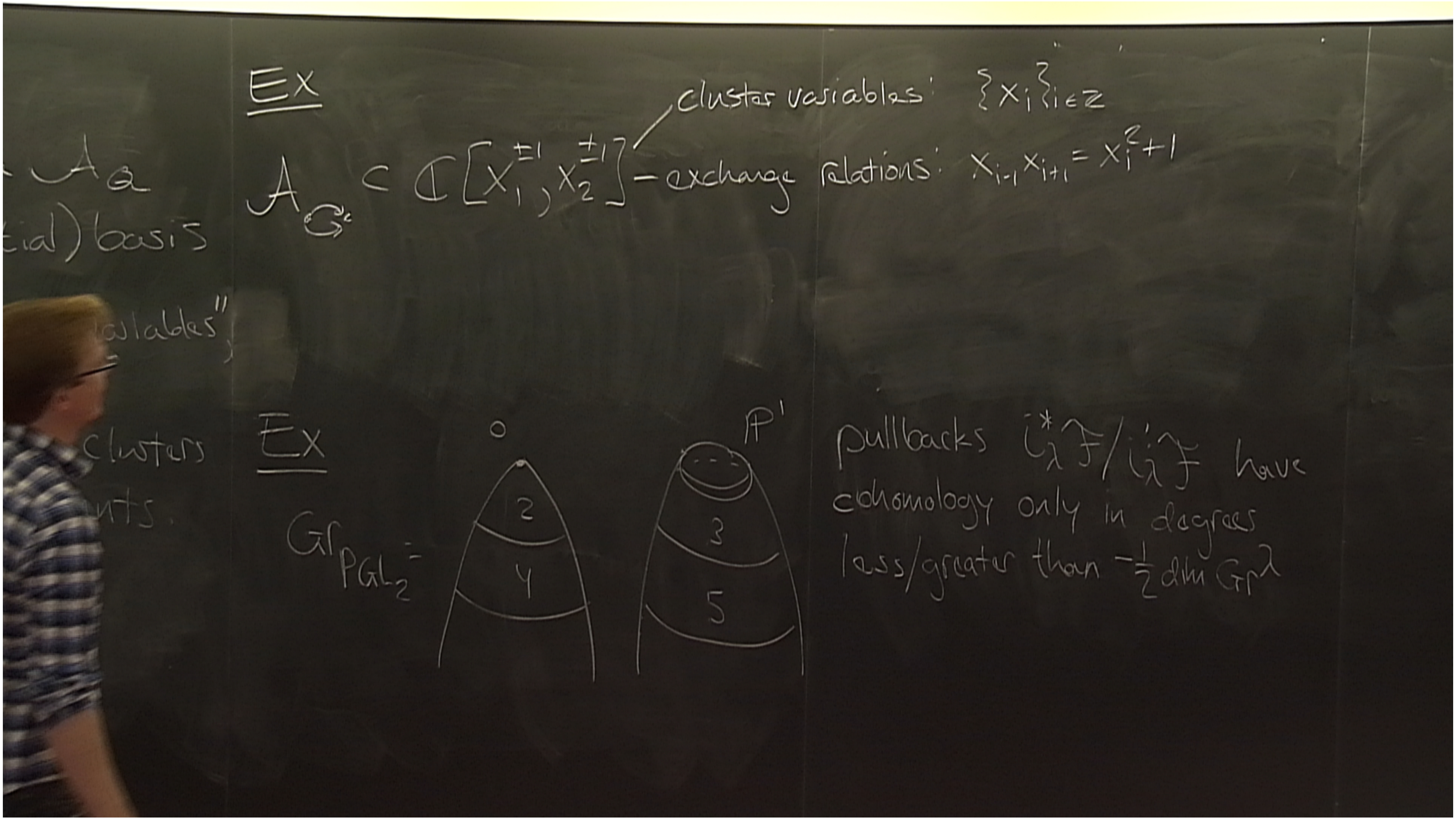
Cluster Algebras

- quiver $Q \rightsquigarrow$ algebra $\mathcal{A}_Q$ w/ (partial) basis
- generated by "cluster variables", grouped into "clusters" s.t. monomials in any clusters are again basis elements.

- union of fin dim'l projective varieties.
- left $G(\mathcal{O})$-orbits $\leftrightarrow$ coweights $\mathbb{C}^\times \to T \subset$ up to conj.

Ex

$\mathrm{Gr}_{PGL_2} =$



$\mathcal{A}_Q$

tial) basis

ariables"

clusters

nts.

Ex

$\mathcal{A}_{\bullet \rightrightarrows \bullet} \subset \mathbb{C}[x_1^{\pm1}, x_2^{\pm1}]$

– cluster variables: $\{x_i\}_{i\in\mathbb{Z}}$

– exchange relations: $x_{i-1}x_{i+1} = x_i^2+1$

Ex

$Gr_{PGL_2} =$

pullbacks $i_\lambda^* \mathcal{F} / i_\lambda^! \mathcal{F}$ have cohomology only in degrees less/greater than $-\frac{1}{2}\dim Gr^\lambda$

$\sim \mathcal{A}_{Q}$

tial) basis

ster variables"

ters"

y clusters

lements.

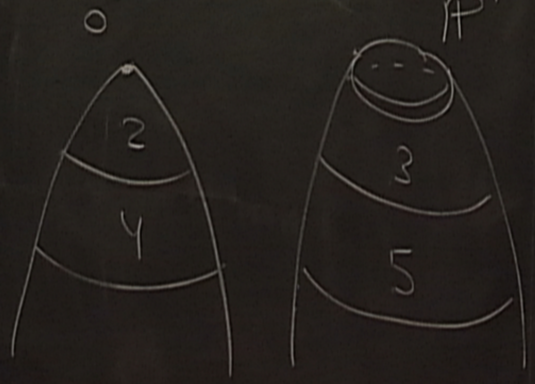
Ex

$\mathcal{A}_{\circlearrowleft} \subset \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$

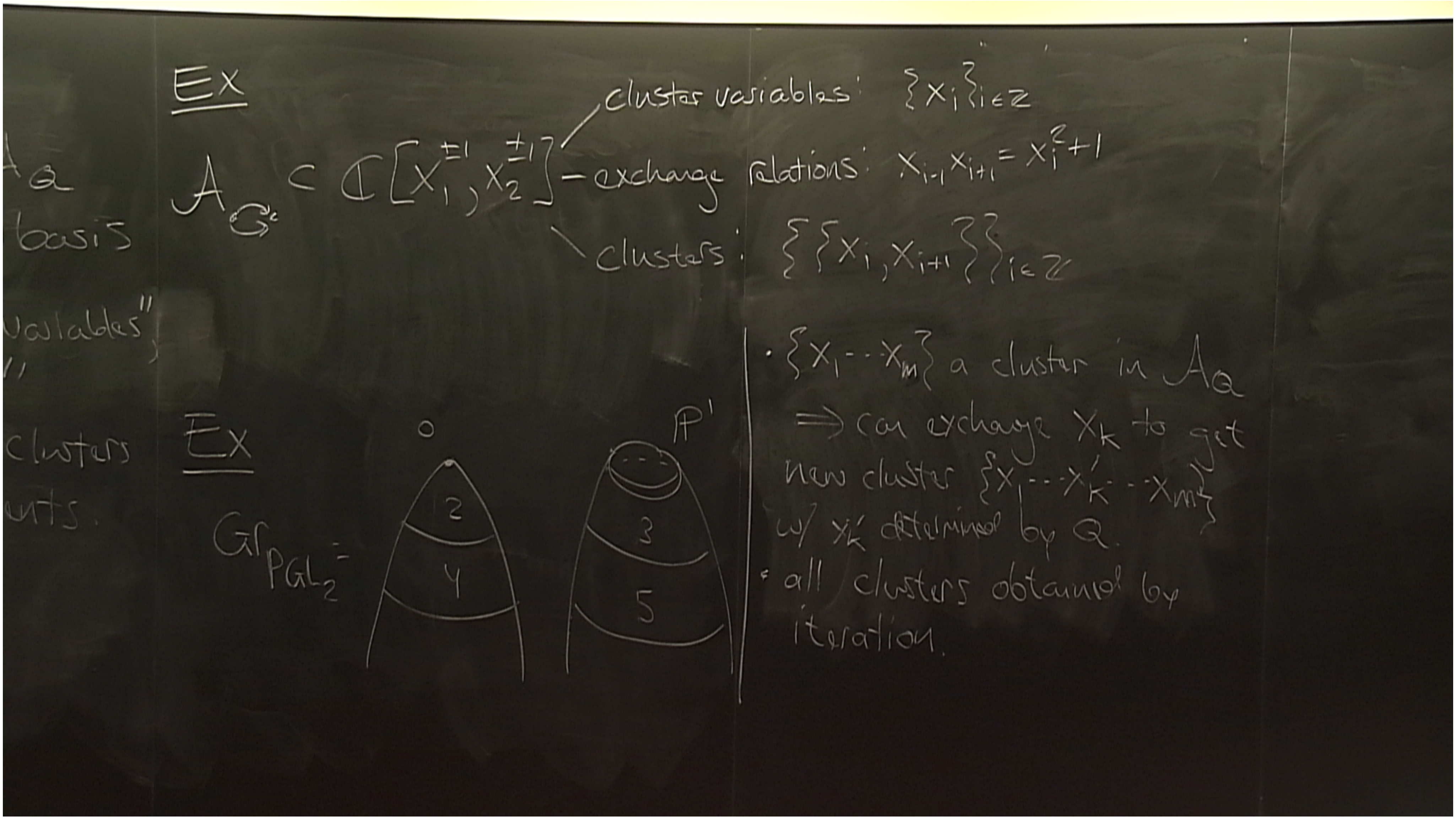
- cluster variables: $\{x_i\}_{i \in \mathbb{Z}}$
- exchange relations: $x_{i-1} x_{i+1} = x_i^2 + 1$
- clusters: $\{\{x_i, x_{i+1}\}\}_{i \in \mathbb{Z}}$

Ex

$Gr_{PGL_2} =$



pullbacks $i_\lambda^* \mathcal{F} / i_\lambda^! \mathcal{F}$ have cohomology only in degrees less/greater than $-\frac{1}{2} \dim Gr^\lambda$



A_Q

basis

variables"

''

clusters

nts.

Ex

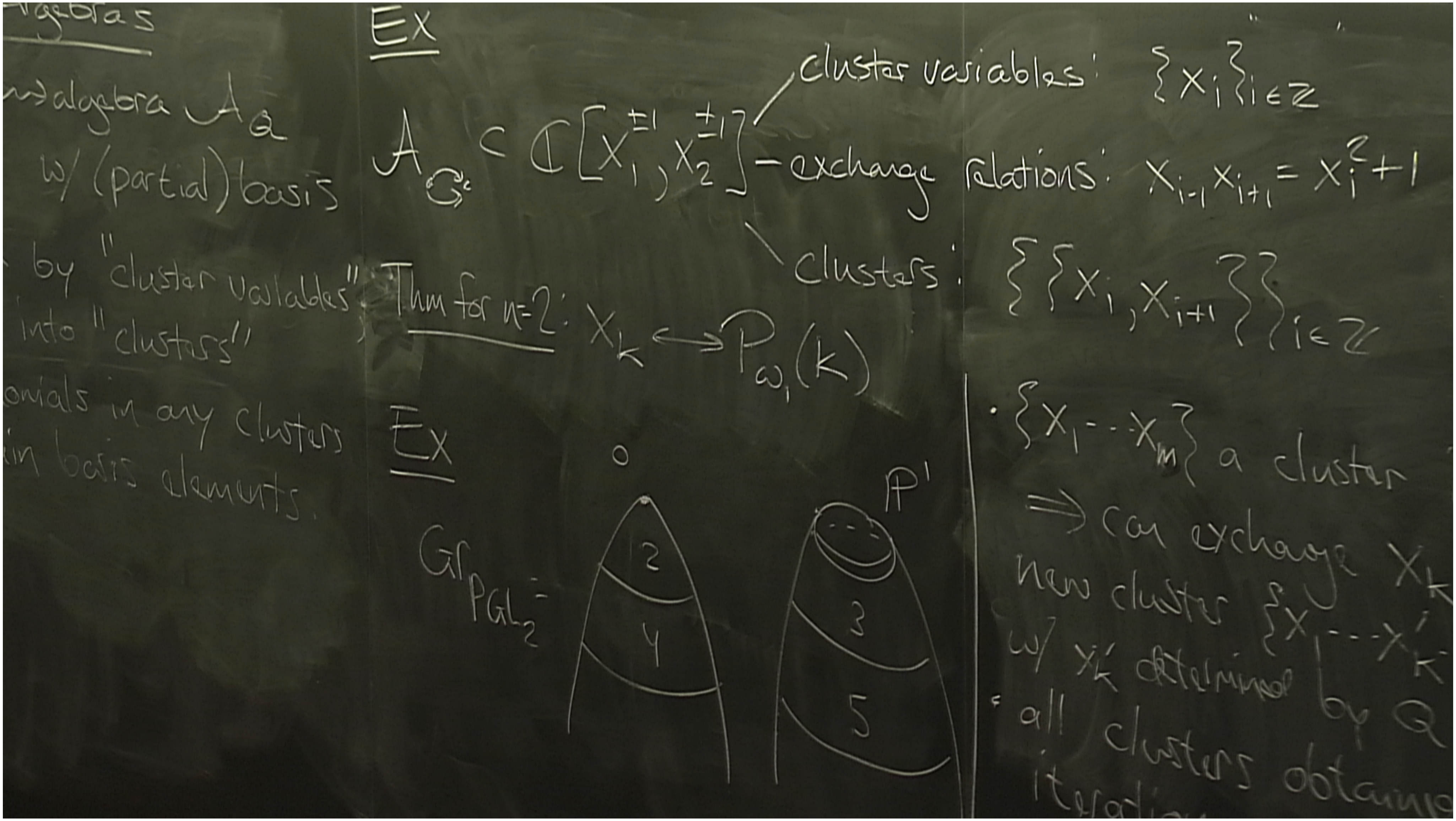
$$\mathcal{A}_{Q} \subset \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$$

- cluster variables: $\{x_i\}_{i \in \mathbb{Z}}$
- exchange relations: $x_{i-1}x_{i+1} = x_i^2+1$
- clusters: $\{\{x_i, x_{i+1}\}\}_{i \in \mathbb{Z}}$

Ex

$Gr_{PGL_2} =$

- $\{x_1 \cdots x_m\}$ a cluster in $\mathcal{A}_Q$
  $\Rightarrow$ can exchange x_k to get new cluster $\{x_1 \cdots x_k' \cdots x_m\}$ w/ x_k' determined by Q.
- all clusters obtained by iteration.



lgebras

⇝ algebra $\mathcal{A}_Q$

w/ (partial) basis

by "cluster variables"

into "clusters"

omials in any clusters

in basis elements.

Ex

$\mathcal{A}_{\circlearrowleft} \subset \mathbb{C}[x_1^{\pm1}, x_2^{\pm1}]$

- cluster variables: $\{x_i\}_{i\in\mathbb{Z}}$
- exchange relations: $x_{i-1}x_{i+1} = x_i^2+1$
- clusters: $\{\{x_i, x_{i+1}\}\}_{i\in\mathbb{Z}}$

Thm for n=2: $x_k \longleftrightarrow P_{\omega_1}(k)$

Ex

Gr_{PGL_2}:

0

2

4

$\mathbb{P}^1$

3

5

- $\{x_1 \cdots x_m\}$ a cluster

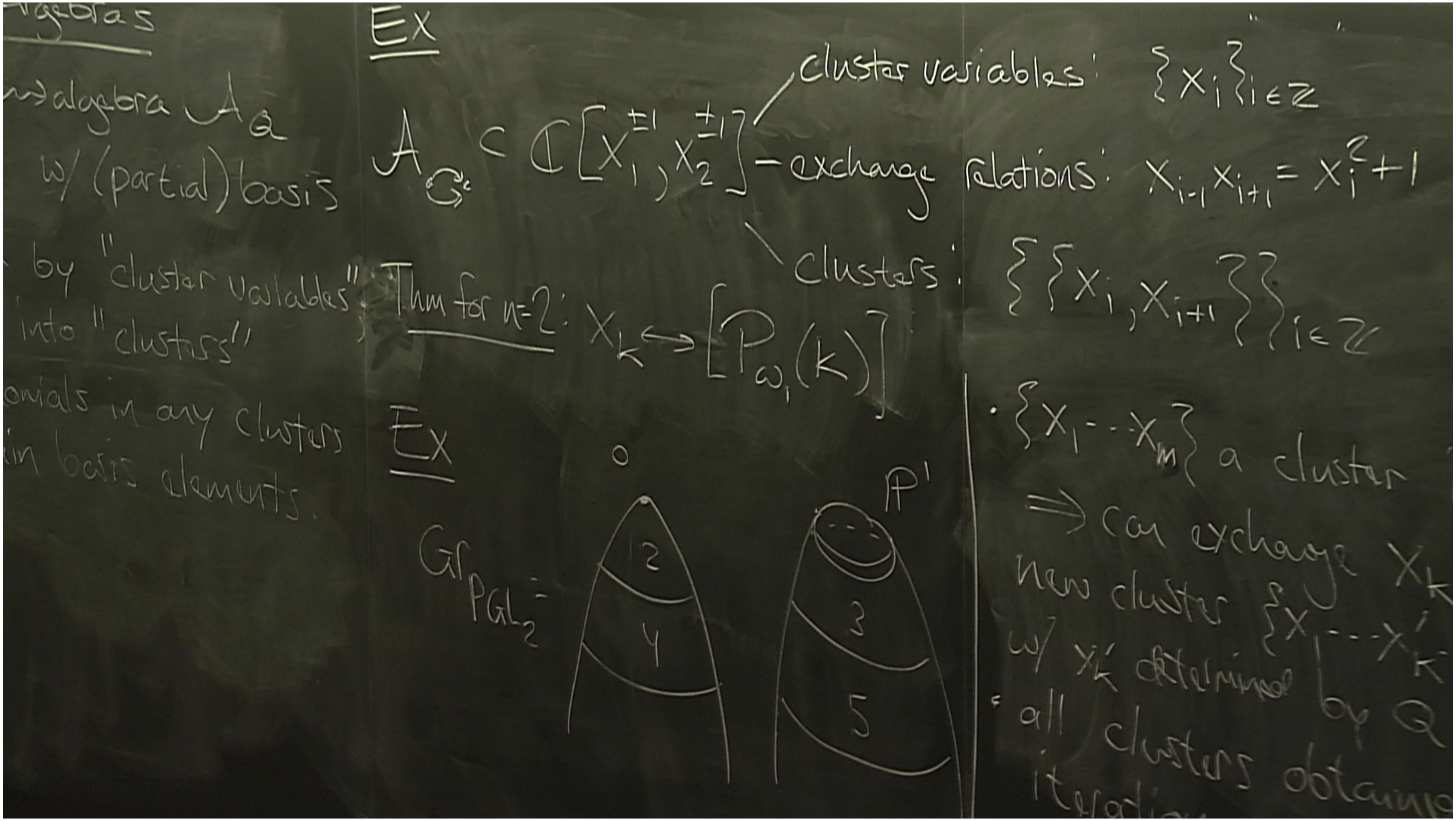
$\Rightarrow$ can exchange x_k

new cluster $\{x_1 \cdots x_k' \cdots$

w/ x_k' determined by Q

- all clusters obtaine

iteratio



Algebras

$\rightsquigarrow$ algebra $\mathcal{A}_Q$

w/ (partial) basis

by "cluster variables"

into "clusters"

onials in any clusters

in basis elements.

Ex

$\mathcal{A}_{\circlearrowleft} \subset \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$

- cluster variables: $\{x_i\}_{i \in \mathbb{Z}}$
- exchange relations: $x_{i-1} x_{i+1} = x_i^2 + 1$
- clusters: $\{\{x_i, x_{i+1}\}\}_{i \in \mathbb{Z}}$

Thm for $n=2$: $x_k \leftrightarrow [P_{\omega_1}(k)]$

Ex

Gr_{PGL_2}:

- $\{x_1 \cdots x_m\}$ a cluster

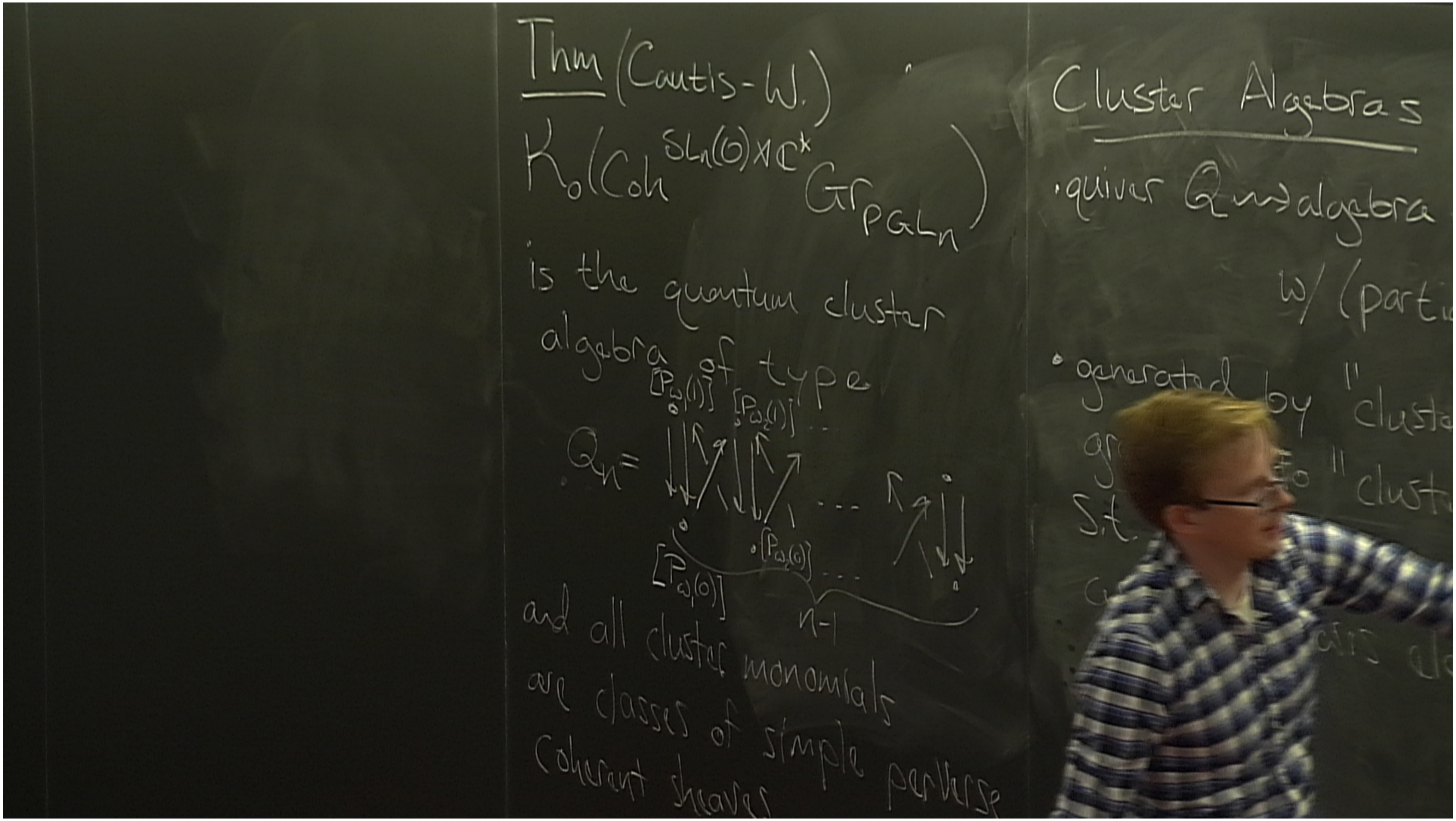
$\Rightarrow$ can exchange x_k

new cluster $\{x_1 \cdots x_k'$

w/ x_k' determined by Q

- all clusters obtaine

iteratio



Thm (Cautis-W.)

$K_0(\mathrm{Coh}^{SL_n(\mathcal{O})\rtimes\mathbb{C}^*}\mathrm{Gr}_{PGL_n})$

is the quantum cluster algebra of type

$Q_n =$

and all cluster monomials are classes of simple perverse coherent sheaves

Cluster Algebras

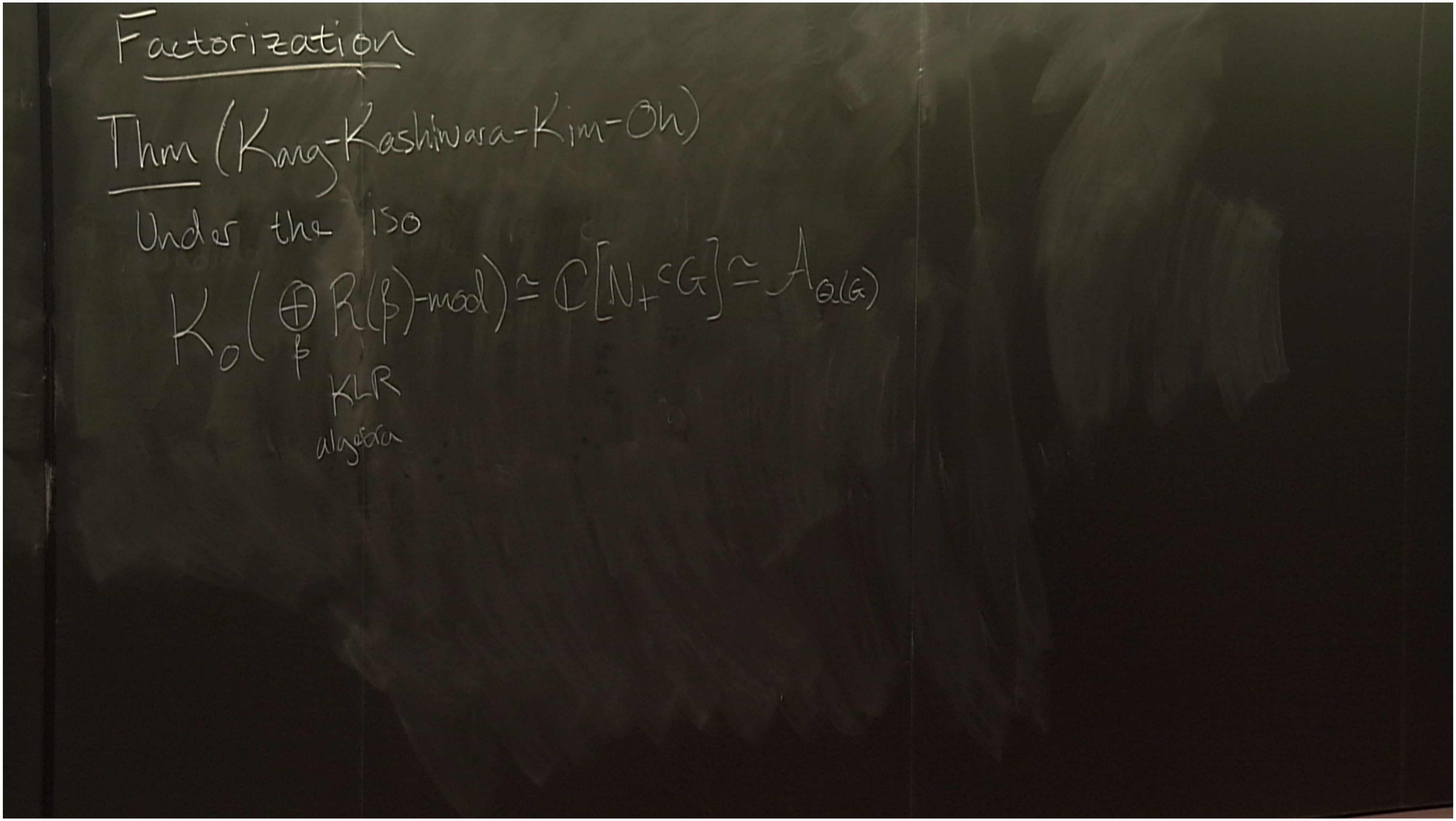
· quiver Q ⇝ algebra

w/ (parti

· generated by "clust

gr "clust

S.t.

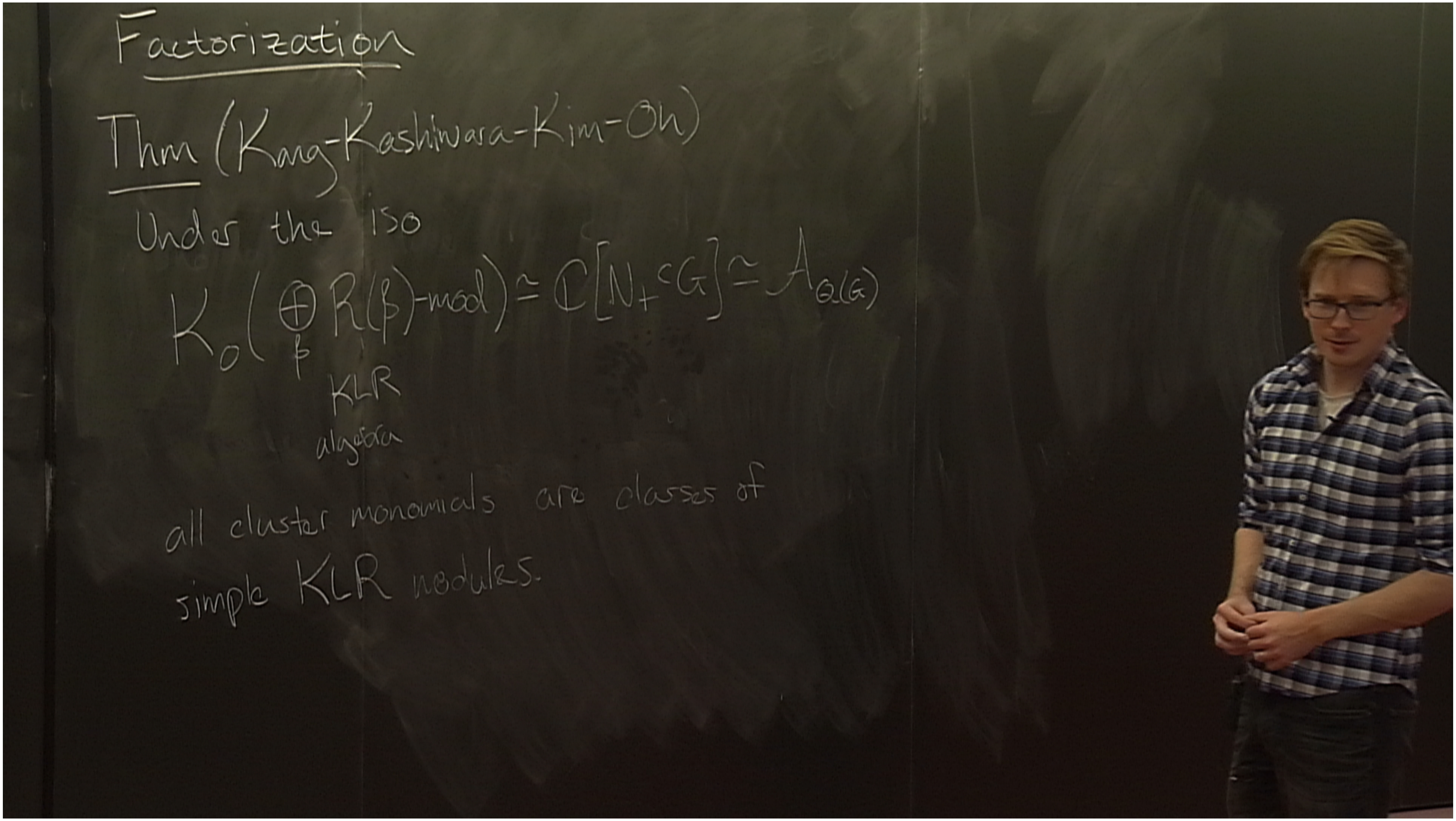


Factorization

Thm (Kang-Kashiwara-Kim-Oh)

Under the iso

$$K_0\Big(\underset{\beta}{\bigoplus} \underset{\substack{\text{KLR} \\ \text{algebra}}}{R(\beta)}\text{-mod}\Big) \simeq \mathbb{C}[N_+ \subset G] \simeq A_{Q(G)}$$



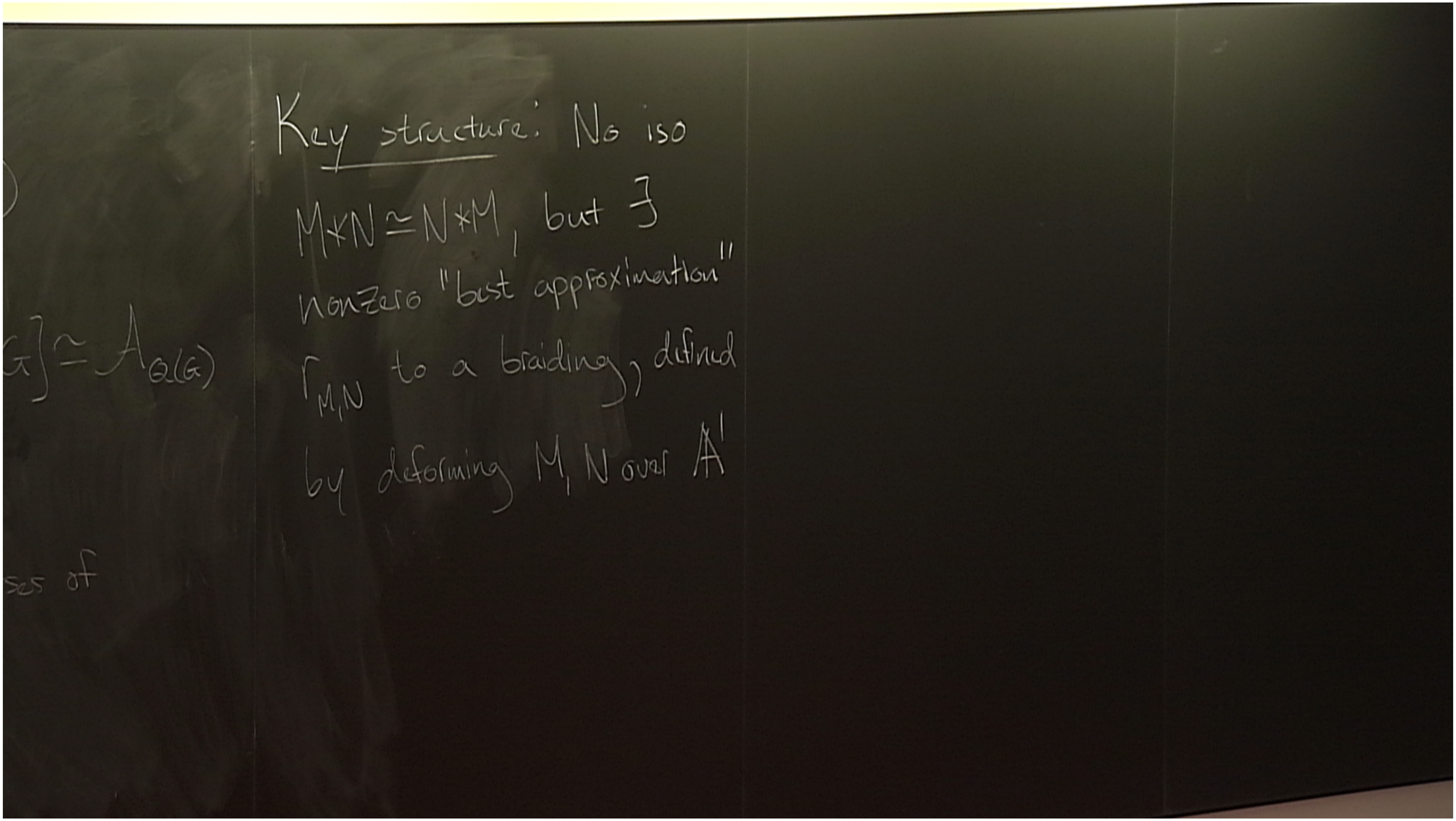
Factorization

Thm (Kang-Kashiwara-Kim-Oh)

Under the iso

$$K_0\Big(\underbrace{\bigoplus_{\beta} R(\beta)}_{\substack{\text{KLR} \\ \text{algebra}}}\text{-mod}\Big) \simeq \mathbb{C}[N_+ \subset G] \simeq \mathcal{A}_{Q(G)}$$

all cluster monomials are classes of simple KLR modules.



$G] \simeq A_{Q(G)}$

ses of

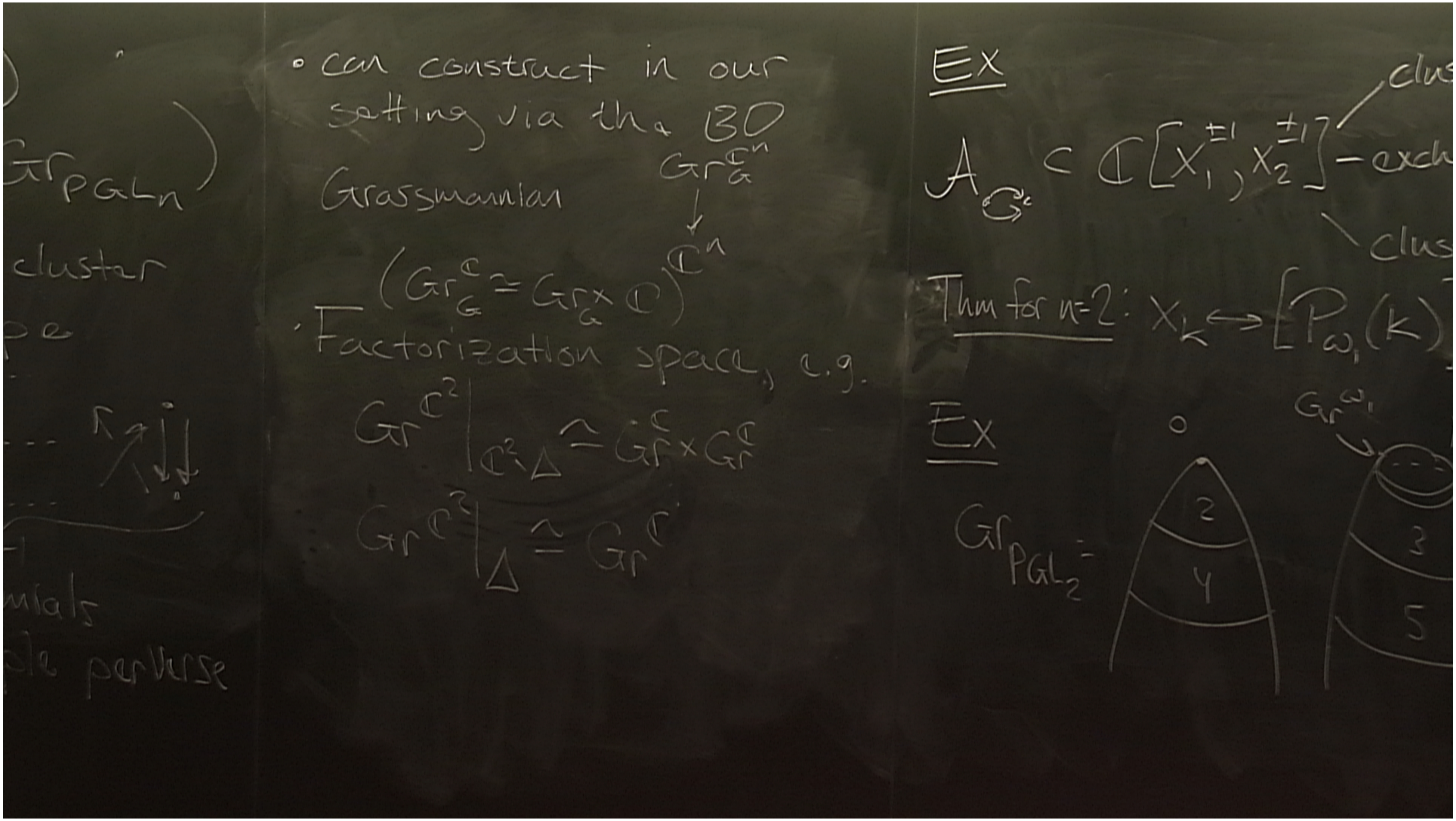
Key structure: No iso

$M * N \simeq N * M$, but $\exists$

nonzero "best approximation"

$r_{M,N}$ to a braiding, defined

by deforming M, N over $\mathbb{A}^1$



Gr_{PGL_n})

cluster

pe

mials

ple perverse

- can construct in our setting via the BD Grassmannian $Gr^{\mathbb{C}^n}_G \to \mathbb{C}^n$

$(Gr^{\mathbb{C}}_G \cong Gr_G \times \mathbb{C})$

- Factorization space, e.g.

$$Gr^{\mathbb{C}^2}\big|_{\mathbb{C}^2 \setminus \Delta} \cong Gr^{\mathbb{C}} \times Gr^{\mathbb{C}}$$

$$Gr^{\mathbb{C}^2}\big|_{\Delta} \cong Gr^{\mathbb{C}}$$

Ex

$\mathcal{A}_{\circlearrowright} \subset \mathbb{C}[x_1^{\pm1}, x_2^{\pm1}]$ — exch

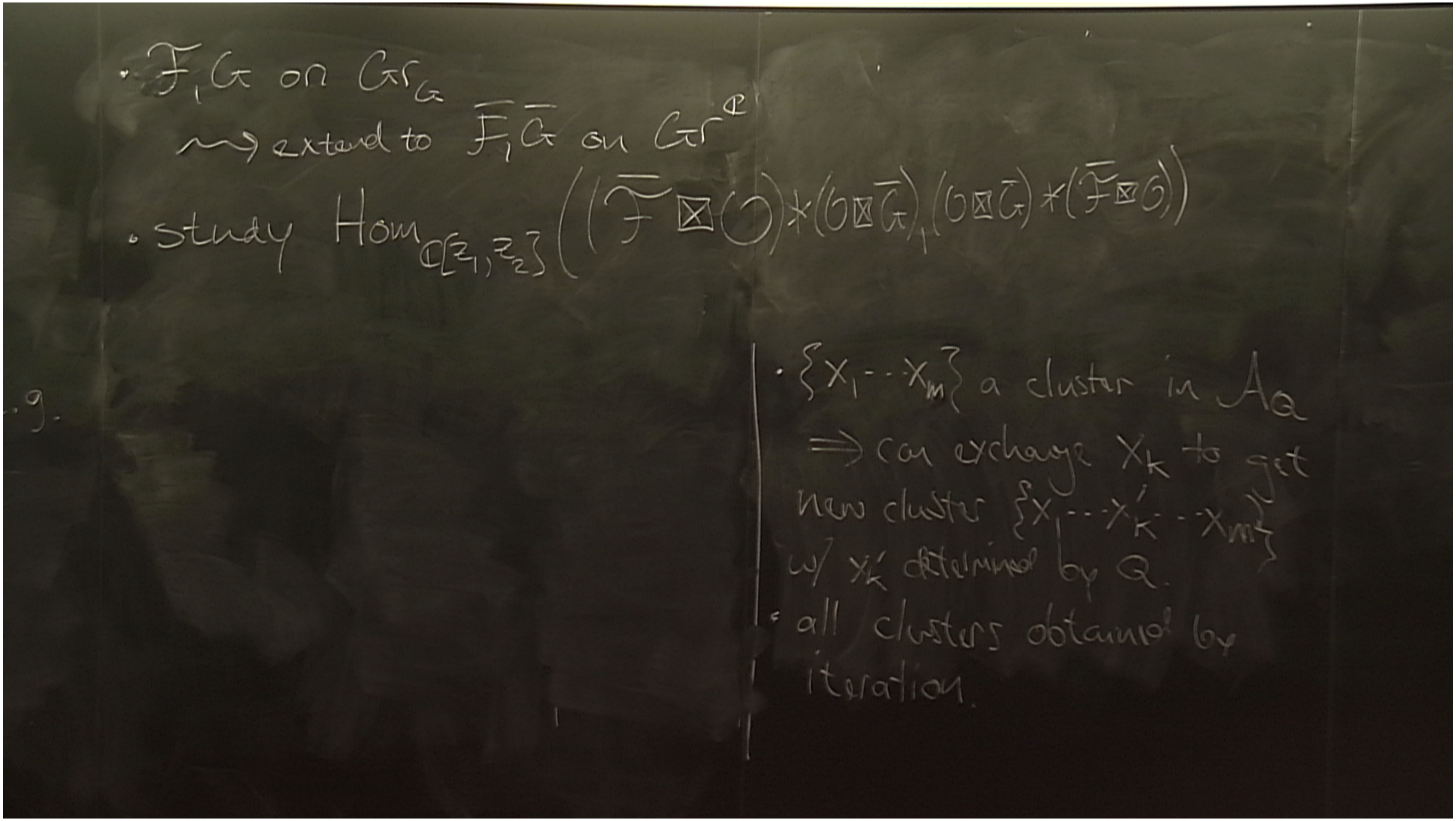
clus

clus

Thm for n=2: $x_k \leftrightarrow [P_{\omega_1}(k)$

Ex

$Gr_{PGL_2} =$



- $\mathcal{F}, G$ on Gr_G

  ⇝ extend to $\bar{\mathcal{F}}, \bar{G}$ on $Gr^{\mathbb{C}}$

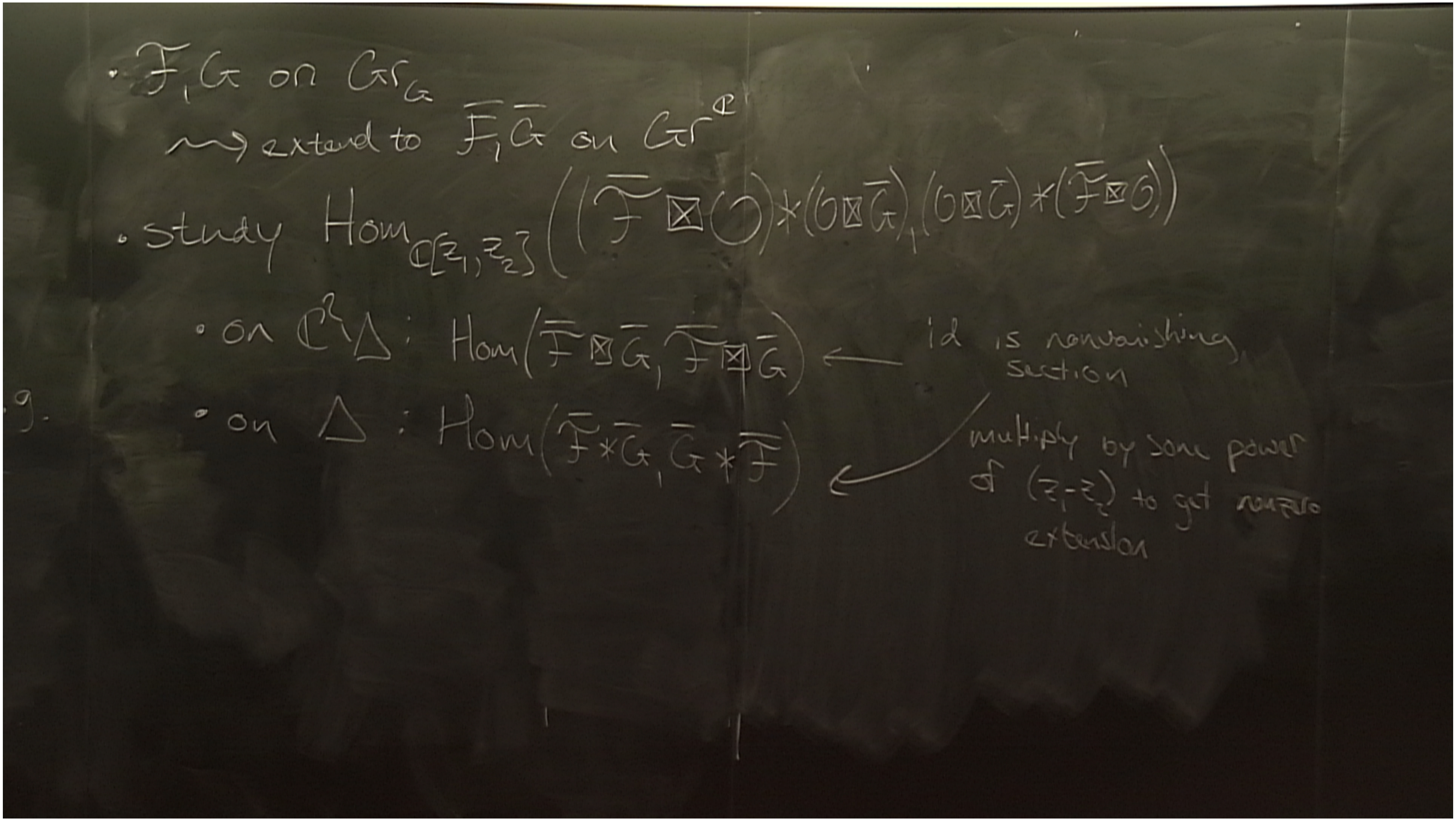
- study $\mathrm{Hom}_{\mathbb{C}[z_1,z_2]}\left((\bar{\mathcal{F}} \boxtimes \mathcal{O}) * (\mathcal{O} \boxtimes \bar{G}), (\mathcal{O} \boxtimes \bar{G}) * (\bar{\mathcal{F}} \boxtimes \mathcal{O})\right)$

.g.

- $\{x_1 \cdots x_m\}$ a cluster in $\mathcal{A}_Q$

  $\Rightarrow$ can exchange x_k to get new cluster $\{x_1 \cdots x_k' \cdots x_m\}$ w/ x_k' determined by Q.

- all clusters obtained by iteration.



- $\mathcal{F}, \mathcal{G}$ on Gr_G

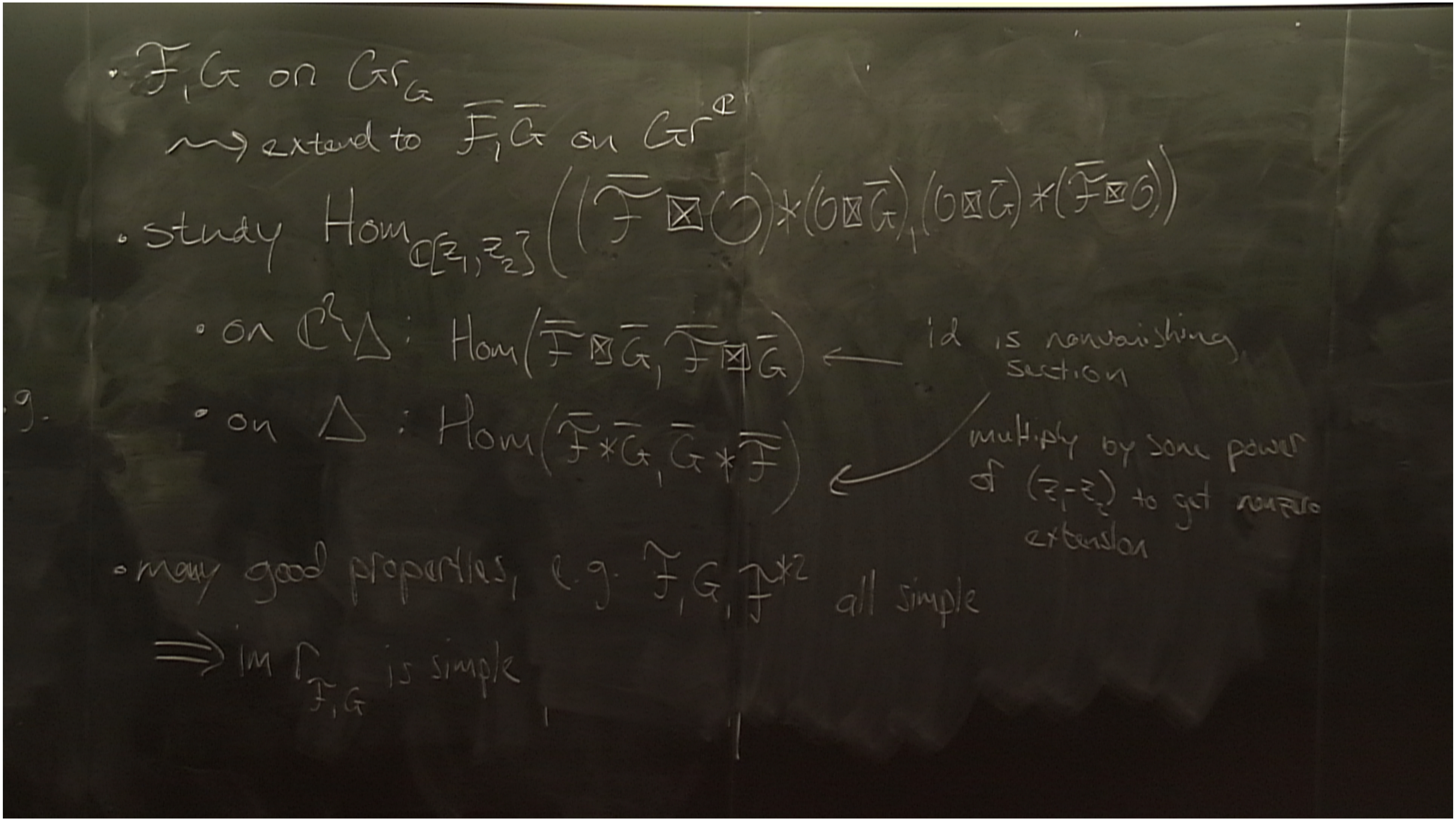
  $\rightsquigarrow$ extend to $\overline{\mathcal{F}}, \overline{\mathcal{G}}$ on $Gr^{\mathbb{C}}$

- study $\mathrm{Hom}_{\mathbb{C}[z_1, z_2]}\left((\overline{\mathcal{F}} \boxtimes \mathcal{O}) * (\mathcal{O} \boxtimes \overline{\mathcal{G}}), (\mathcal{O} \boxtimes \overline{\mathcal{G}}) * (\overline{\mathcal{F}} \boxtimes \mathcal{O}) \right)$

  - on $\mathbb{C}^2 \setminus \Delta$: $\mathrm{Hom}(\overline{\mathcal{F}} \boxtimes \overline{\mathcal{G}}, \overline{\mathcal{F}} \boxtimes \overline{\mathcal{G}}) \longleftarrow$ id is nonvanishing section

g.

  - on Δ: $\mathrm{Hom}(\overline{\mathcal{F}} * \overline{\mathcal{G}}, \overline{\mathcal{G}} * \overline{\mathcal{F}}) \longleftarrow$ multiply by some power of $(z_1 - z_2)$ to get nonzero extension



- $\mathcal{F}, \mathcal{G}$ on Gr_G

  $\rightsquigarrow$ extend to $\overline{\mathcal{F}}, \overline{\mathcal{G}}$ on $Gr^{\mathbb{C}}$

- study $\mathrm{Hom}_{\mathbb{C}[z_1, z_2]}\left((\overline{\mathcal{F}} \boxtimes \mathcal{O}) * (\mathcal{O} \boxtimes \overline{\mathcal{G}}), (\mathcal{O} \boxtimes \overline{\mathcal{G}}) * (\overline{\mathcal{F}} \boxtimes \mathcal{O}) \right)$

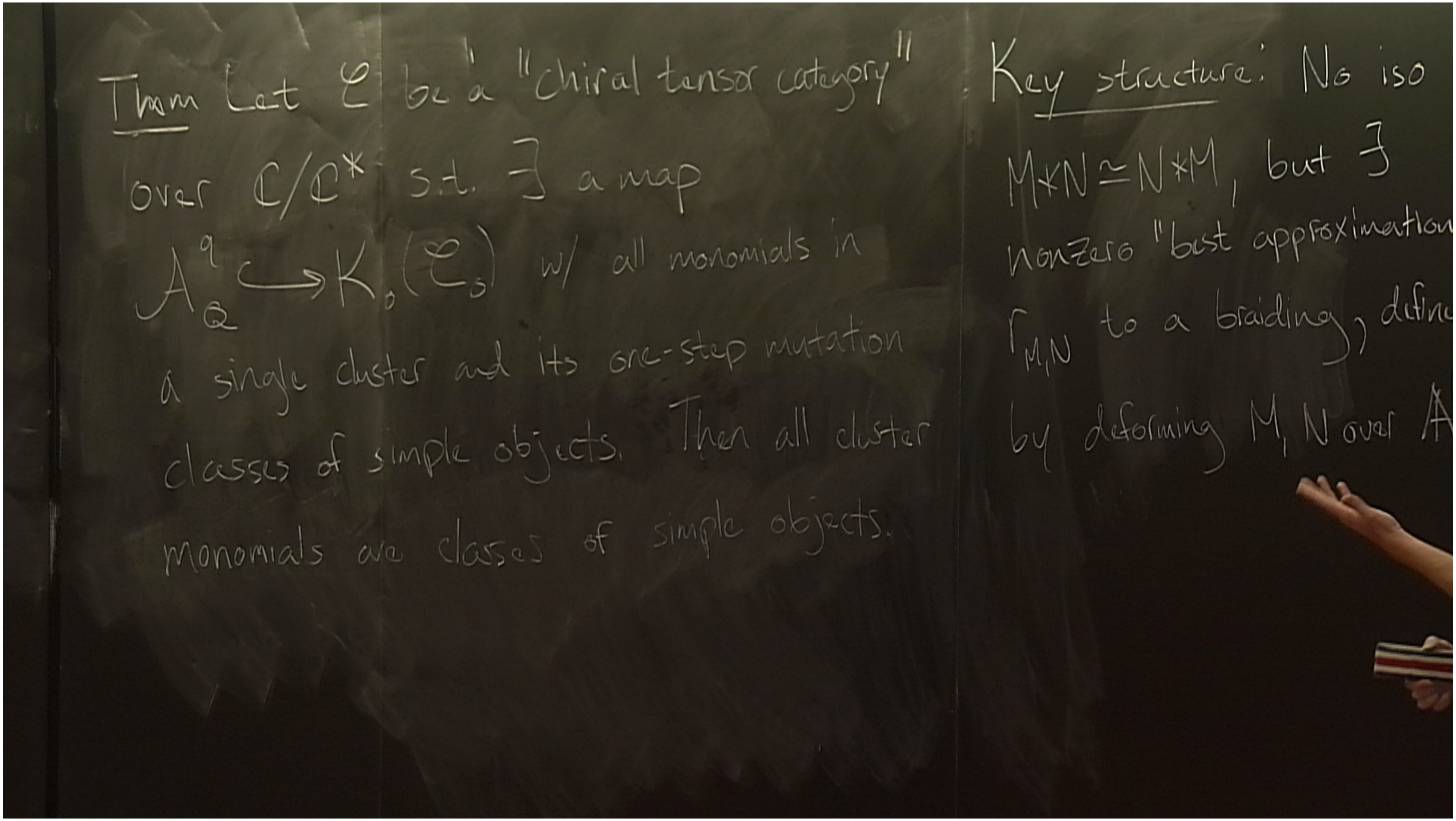
  - on $\mathbb{C}^2 \setminus \Delta$: $\mathrm{Hom}(\overline{\mathcal{F}} \boxtimes \overline{\mathcal{G}}, \overline{\mathcal{F}} \boxtimes \overline{\mathcal{G}}) \leftarrow$ id is nonvanishing section

.g.

  - on Δ : $\mathrm{Hom}(\overline{\mathcal{F}} * \overline{\mathcal{G}}, \overline{\mathcal{G}} * \overline{\mathcal{F}}) \leftarrow$ multiply by some power of $(z_1 - z_2)$ to get nonzero extension

- many good properties, e.g. $\mathcal{F}, \mathcal{G}, \mathcal{F}^{*2}$ all simple

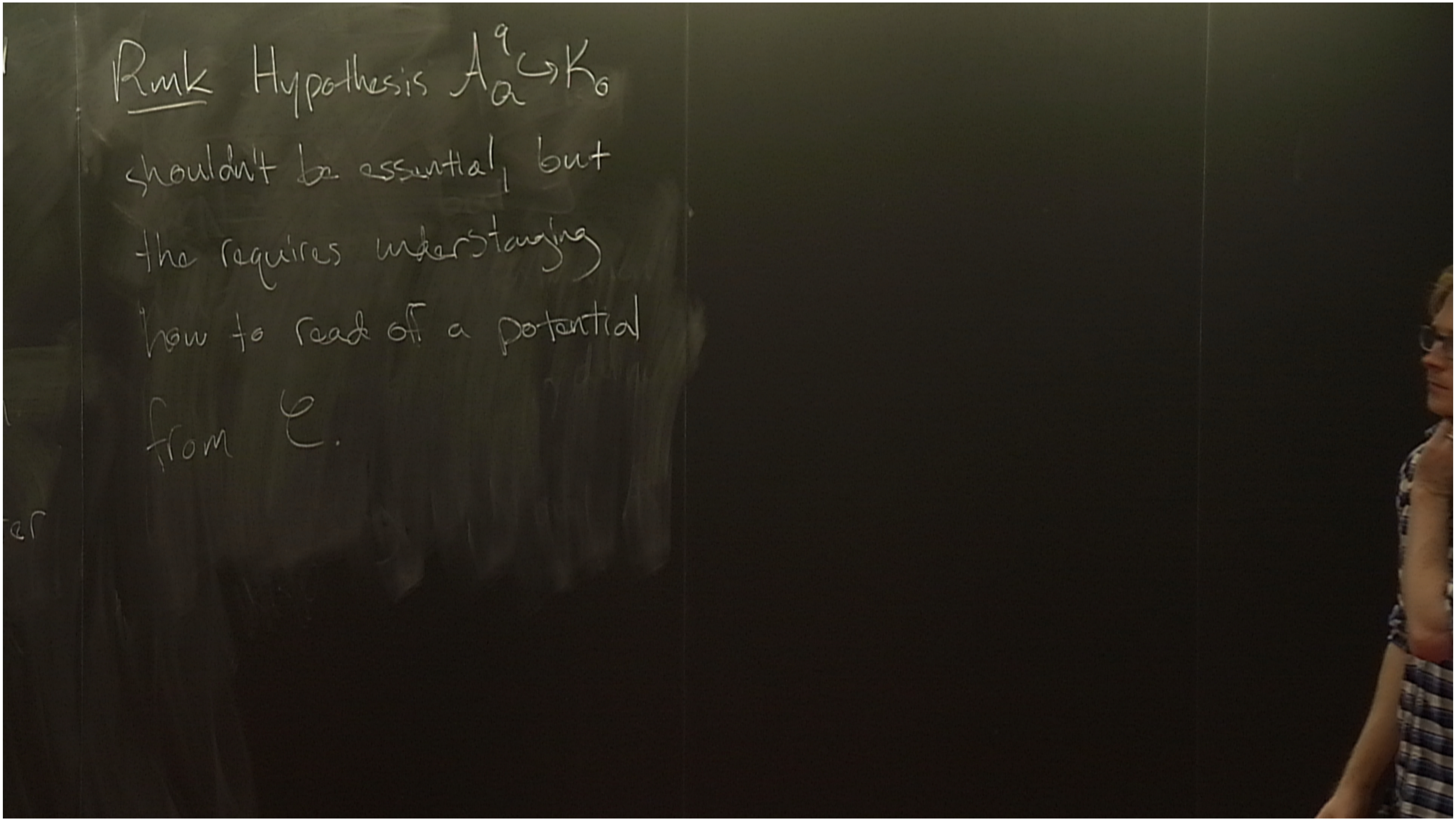
  $\Longrightarrow$ im $\Gamma_{\mathcal{F},\mathcal{G}}$ is simple



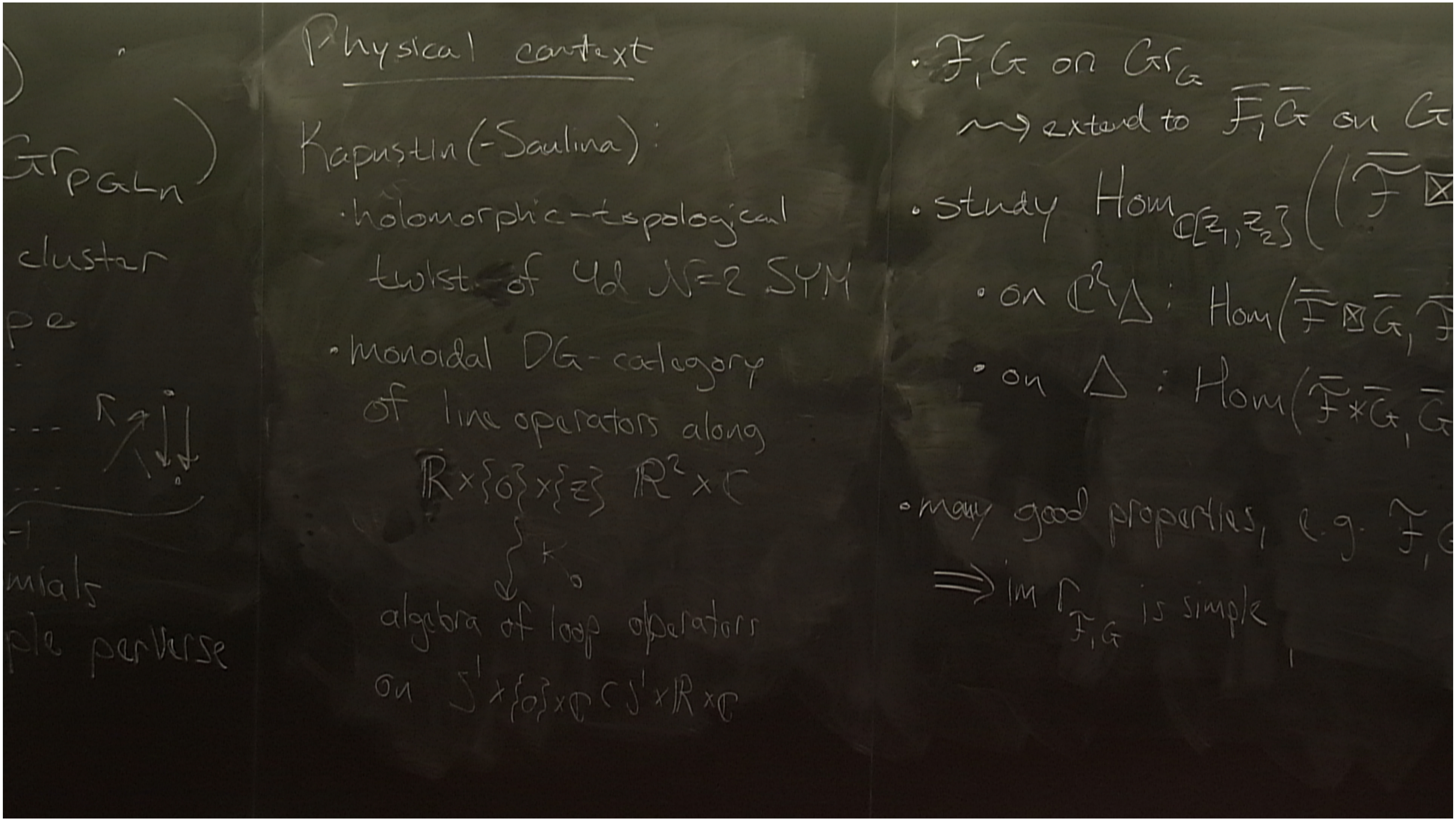
Thm Let $\mathcal{C}$ be a "chiral tensor category" over $\mathbb{C}/\mathbb{C}^*$ s.t. ∃ a map $\mathcal{A}^q_Q \hookrightarrow K_0(\mathcal{C}_0)$ w/ all monomials in a single cluster and its one-step mutation classes of simple objects. Then all cluster monomials are classes of simple objects.

Key structure: No iso $M * N \simeq N * M$, but ∃ nonzero "best approximation $r_{M,N}$ to a braiding, defin

by deforming M, N over $\mathbb{A}$



Rmk Hypothesis $A^{q}_{a} \hookrightarrow K_G$ shouldn't be essential, but the requires understanding how to read off a potential from $\mathcal{C}$.



)

Gr_{PGL_n})

cluster

pe

mials

ple perverse

Physical context

Kapustin(-Saulina):

- holomorphic-topological twist of 4d $\mathcal{N}=2$ SYM
- monoidal DG-category of line operators along $\mathbb{R}\times\{0\}\times\{z\} \; \mathbb{R}^2\times\mathbb{C}$

$$\Big\downarrow{\scriptstyle K_0}$$

algebra of loop operators on $S^1\times\{0\}\times\mathbb{C} \subset S^1\times\mathbb{R}\times\mathbb{C}$

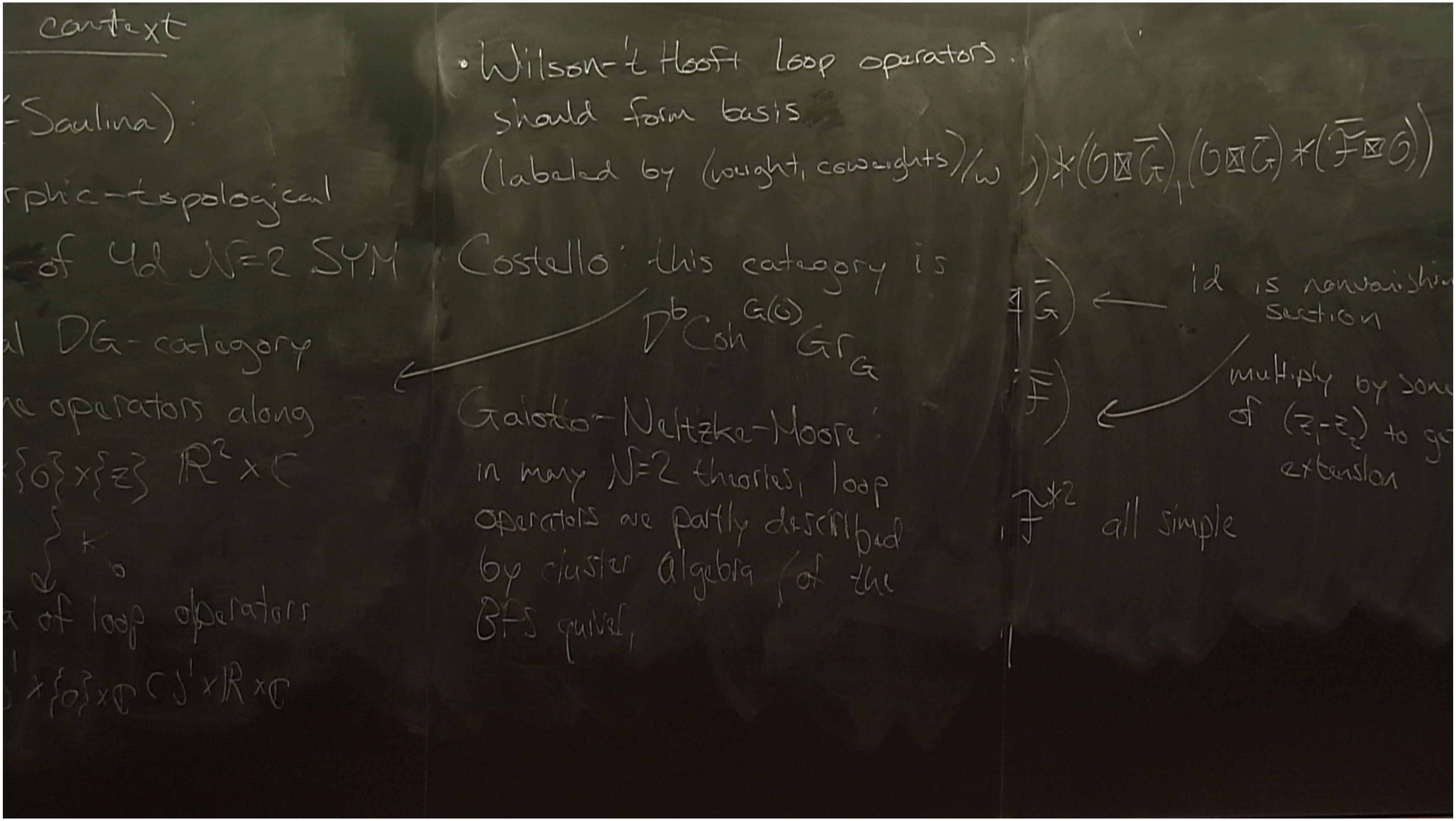
- $\mathcal{F}, \mathcal{G}$ on Gr_G

  $\rightsquigarrow$ extend to $\overline{\mathcal{F}}, \overline{\mathcal{G}}$ on G

- study $\mathrm{Hom}_{\mathbb{C}[z_1,z_2]}((\overline{\mathcal{F}}\boxtimes$
  - on $\mathbb{C}^2\setminus\Delta$: $\mathrm{Hom}(\overline{\mathcal{F}}\boxtimes\overline{\mathcal{G}}, \overline{\mathcal{F}}$
  - on Δ: $\mathrm{Hom}(\overline{\mathcal{F}}*\overline{\mathcal{G}}, \overline{\mathcal{G}}$

- many good properties, e.g. $\mathcal{F},$

  $\Longrightarrow$ im $\Gamma_{\mathcal{F},\mathcal{G}}$ is simple



context

-Saulina):

rphic-topological

of 4d $\mathcal{N}=2$ SYM

al DG-category

ne operators along

$\{0\}\times\{z\}$ $\mathbb{R}^2\times\mathbb{C}$

$\downarrow$ K_0

a of loop operators

$\times\{0\}\times\mathbb{C} \subset S^1\times\mathbb{R}\times\mathbb{C}$

- Wilson-'t Hooft loop operators should form basis (labeled by (weight, coweights)/W

Costello: this category is $D^b\mathrm{Coh}^{G(\mathcal{O})}\mathrm{Gr}_G$

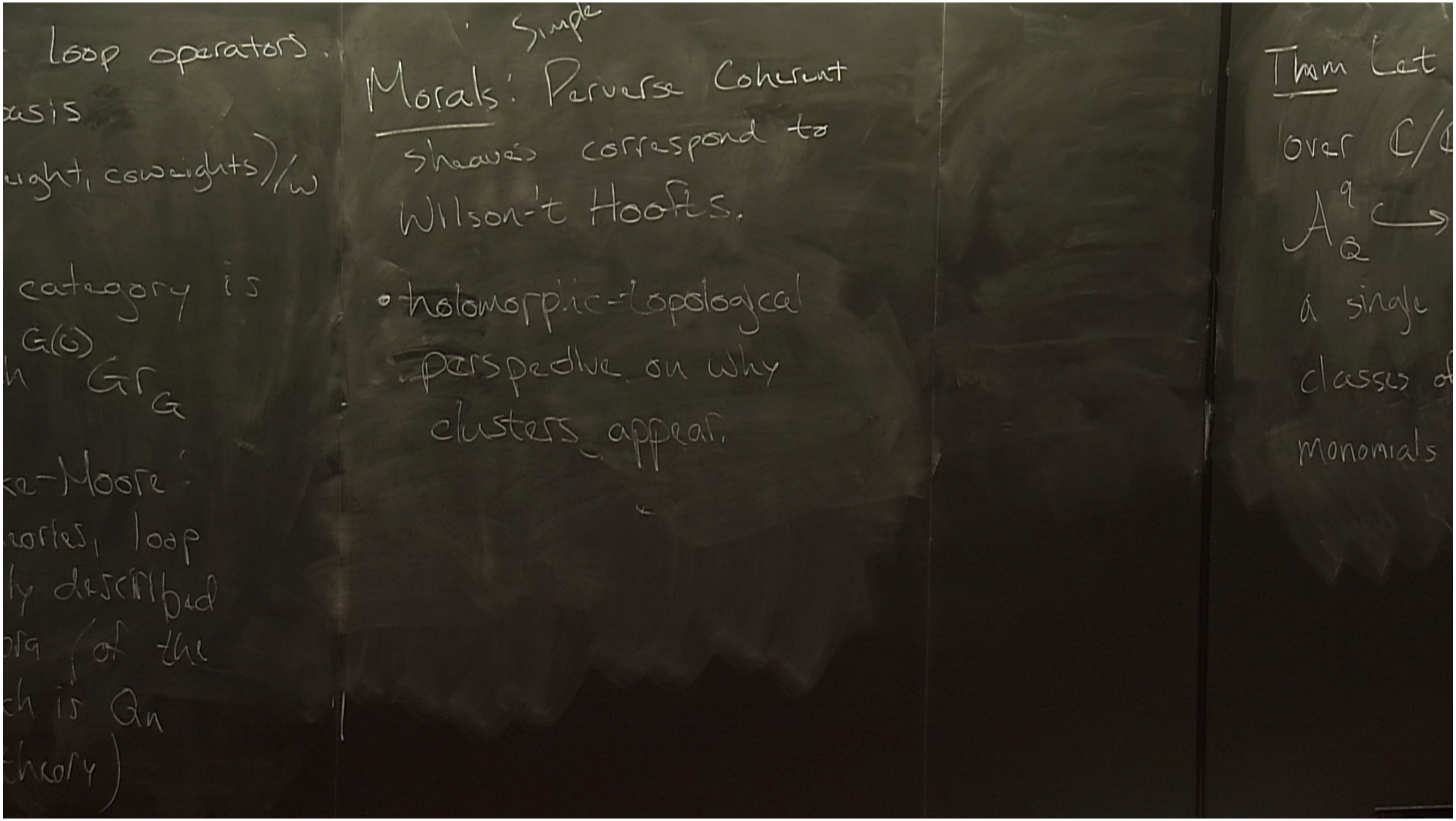
Gaiotto-Neitzke-Moore: in many $\mathcal{N}=2$ theories, loop operators are partly described by cluster algebra (of the BPS quiver,

$)) * (\mathcal{O}\boxtimes\overline{G})$, $(\mathcal{O}\boxtimes\overline{G}) * (\overline{\mathcal{F}}\boxtimes\mathcal{O}))$

$\boxtimes\overline{G})$ $\leftarrow$ id is nonvanishin section

$\overline{\mathcal{F}})$ $\leftarrow$ multiply by som of (z_1-z_2) to ge extension

$\overline{\mathcal{F}}^{*2}$ all simple



loop operators.

asis

ight, coweights)/W

category is

$G(\mathcal{O})$

h Gr_G

ke-Moore:

ories, loop

ly described

ra (of the

ch is Qn

theory)

Simple

Morals: Perverse Coherent sheaves correspond to Wilson-'t Hoofts.

- holomorphic-topological perspective on why clusters appear.

Thm Let

over $\mathbb{C}/$

$A^q_Q \hookrightarrow$

a single

classes of

monomials