Title: GEOMETRY OF QUANTUM ENTANGLEMENT

Date: Dec 01, 2016 03:30 PM

URL: http://pirsa.org/16120010

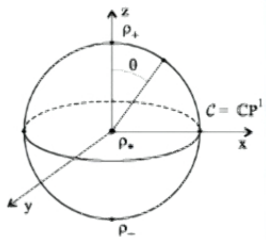
Abstract: <p>A geometric approach to investigation of quantum entanglement is advocated.
We discuss first the geometry of the (N^2-1)--dimensional convex body
of mixed quantum states acting on an N--dimensional Hilbert space
and study projections of this set into 2- and 3-dimensional spaces.
For composed dimensions, N=K^2, one consideres the subset
of separable states and shows that it has a positive measure.
Analyzing its properties contributes to our understanding of
quantum entanglement and its time evolution.</p>

Pure states in a finite dimensional Hilbert space $\mathcal{H}_N$

Qubit = quantum bit; $N = 2$, $\langle\psi|\psi\rangle = 1$, $|\psi\rangle \sim e^{i\alpha}|\psi\rangle$

$$|\psi\rangle = \cos\frac{\vartheta}{2}|1\rangle + e^{i\phi}\sin\frac{\vartheta}{2}|0\rangle$$

Bloch sphere of $N = 2$ pure states



Space of pure states for an arbitrary N:

a complex projective space $\mathbb{C}P^{N-1}$ of $2N-2$ real dimensions.

Unitary evolution

Fubini-Study distance in $\mathbb{C}P^{N-1}$

$$D_{FS}(|\psi\rangle, |\varphi\rangle) := \arccos |\langle\psi|\varphi\rangle|$$

Unitary evolution

Let $U = \exp(iHt)$. Then $|\psi'\rangle = U|\psi\rangle$.
Since $|\langle\psi|\varphi\rangle|^2 = |\langle\psi|U^\dagger U|\varphi\rangle|^2$ any unitary evolution is an **isometry**
(with respect to any standard distance !)

Quantum Chaos: what happends for large N?

How an **isometry** may lead to a classically chaotic dynamics?
The limits $t \to \infty$ and $N \to \infty$ do not commute.

Otton Nikodym & Stefan Banach,

talking at a bench in Planty Garden, **Cracow, summer 1916**

Mixed quantum states

Set $\mathcal{M}_N$ of all mixed states of size N

$$\mathcal{M}_N := \{\rho : \mathcal{H}_N \to \mathcal{H}_N ; \rho = \rho^\dagger, \rho \geq 0, \mathrm{Tr}\rho = 1\}$$

Example: $N = 2$, **One–qubit** states: Bloch sphere + its interior,
$\mathcal{M}_2 = B_3 \subset \mathbb{R}^3$ - **Bloch ball** with all pure states at the boundary

The set $\mathcal{M}_N$ is compact and convex:

$\rho = \sum_i a_i |\psi_i\rangle\langle\psi_i|$ where $a_i \geq 0$ and $\sum_i a_i = 1$.
It has $N^2 - 1$ real dimensions, $\mathcal{M}_N \subset \mathbb{R}^{N^2-1}$.

What the set of all $N = 3$ mixed states looks like?

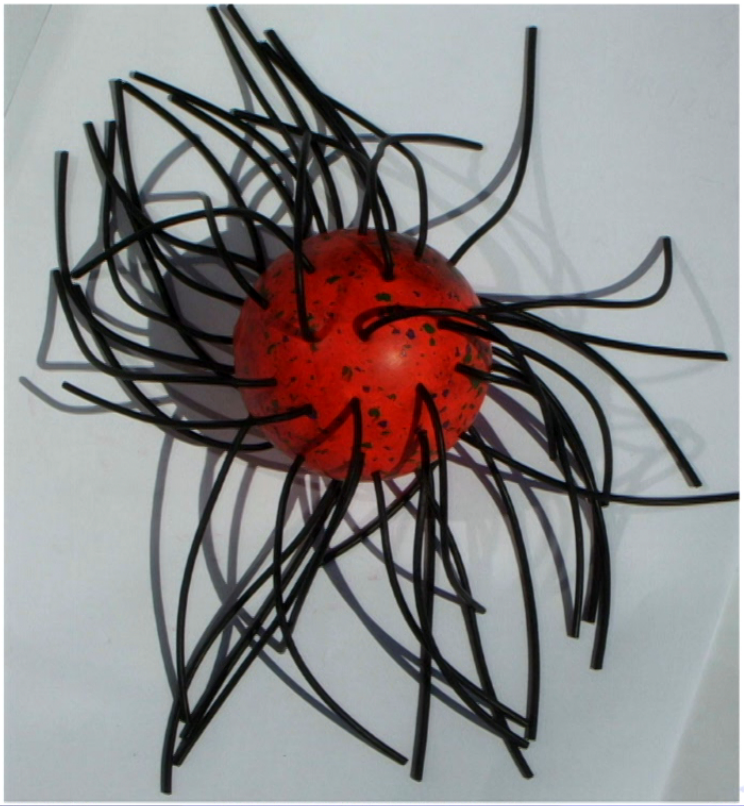
An 8 dimensional convex set with only 4 dimensional subset of pure (extremal) states, which belong to its 7 dim boundary

The set $\mathcal{M}_N$ of quantum mixed states:

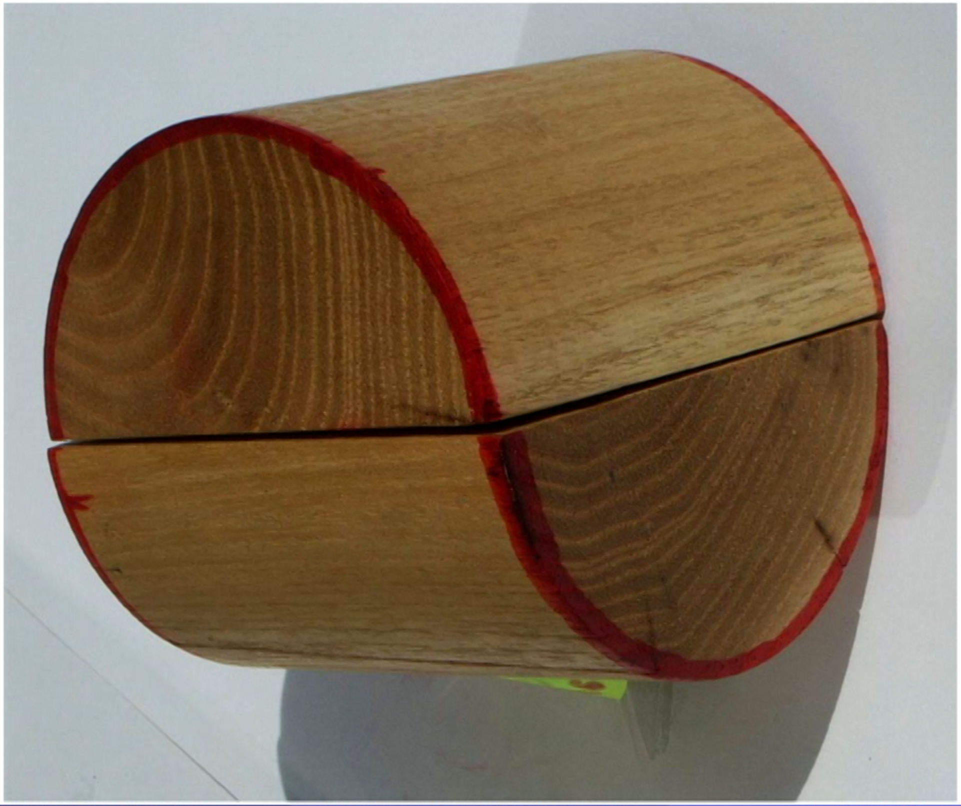
What it looks like for (**for** $N \geq 3$)

?

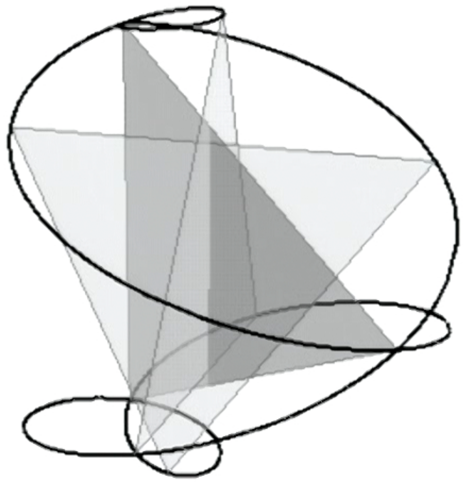
An apophatic approach :

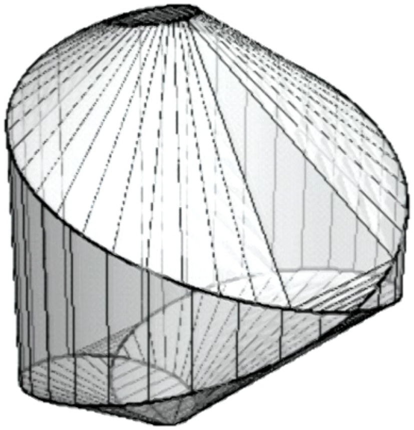



5
APPEL
THEE
20 Tea Bags
APPEL THEE




B)





rotated edges of an **equilateral triangle** and its **convex hull**

Vistula river and **Wawel castle** in **Cracow**

The set $\mathcal{M}_N$ of quantum mixed states for $N \geq 3$

A constructive approach:

Analysis of its structure with aid of notions of operator theory like

Numerical Range

The same tools are useful to investigate the structure of the subsets of $\mathcal{M}_N$, namely sets of

a) **separable** states

and

b) **maximally entangled** states.

Operator theory: **Numerical Range** (Field of Values)

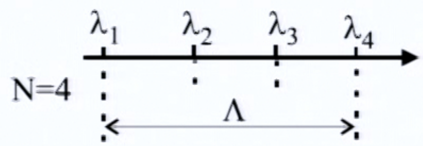
Definition

For any operator A acting on $\mathcal{H}_N$ one defines its **NUMERICAL RANGE** (**Wertevorrat**) as a subset of the complex plane defined by:

$$\Lambda(A) = \{\langle x|A|x\rangle : |x\rangle \in \mathcal{H}^N, \ \langle x|x\rangle = 1\}. \qquad (1)$$

Hermitian case

For any hermitian operator $A = A^\dagger$ with spectrum $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ its **numerical range** forms an interval: the set of all possible expectation values of the observable A among arbitrary pure states, $\Lambda(A) = [\lambda_1, \lambda_N]$.



Numerical range and its properties

Compactness

$\Lambda(A)$ is a **compact** subset of $\mathbb{C}$.

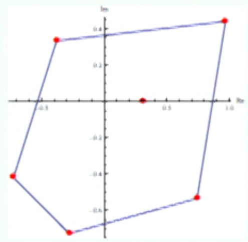
Convexity: Hausdorf-Toeplitz theorem

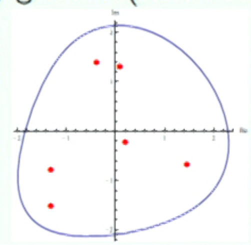
- $\Lambda(A)$ is a **convex** subset of $\mathbb{C}$.

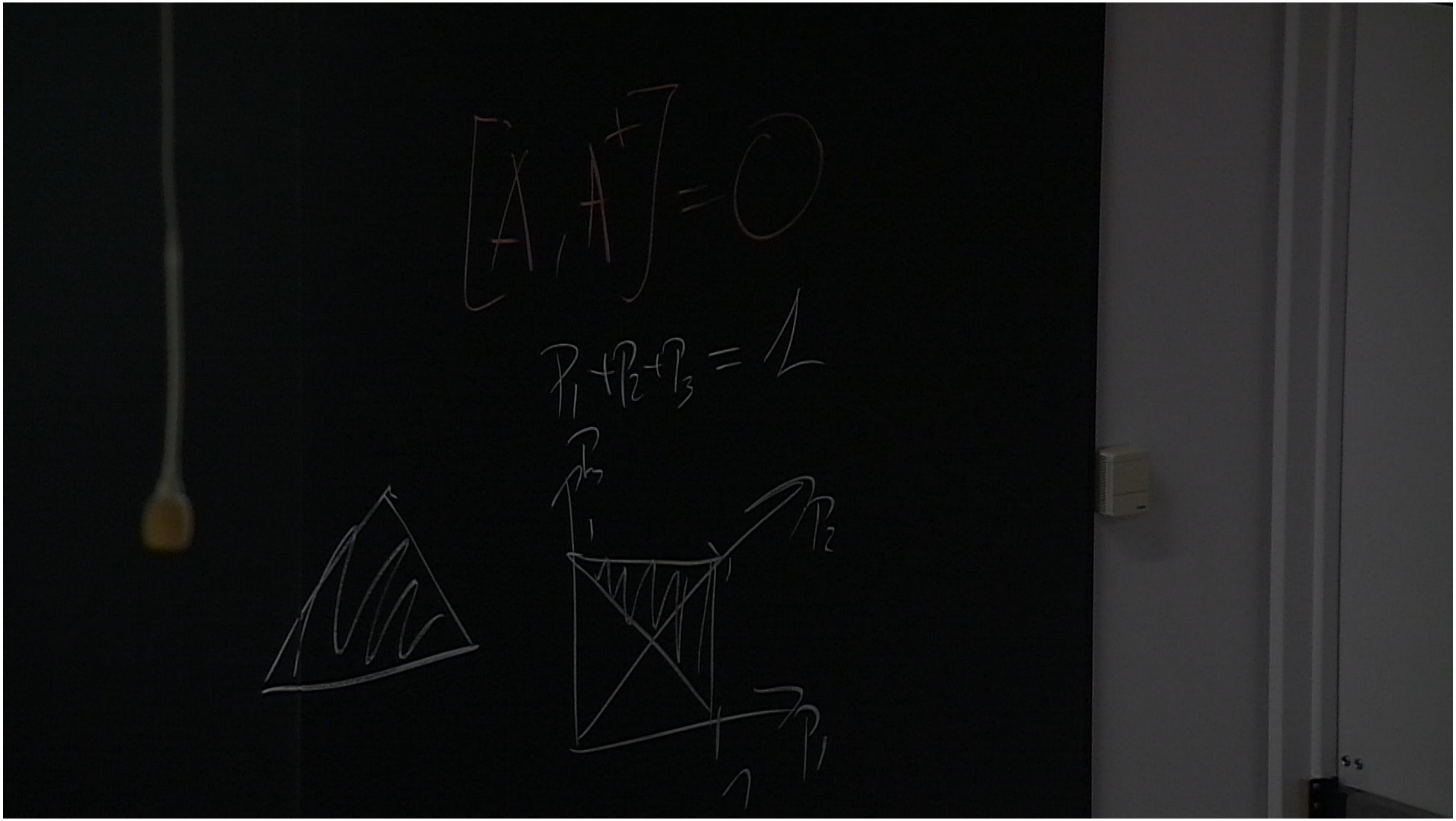
Example

Numerical range for random matrices of order $N = 6$

a) normal, b) generic (non-normal)





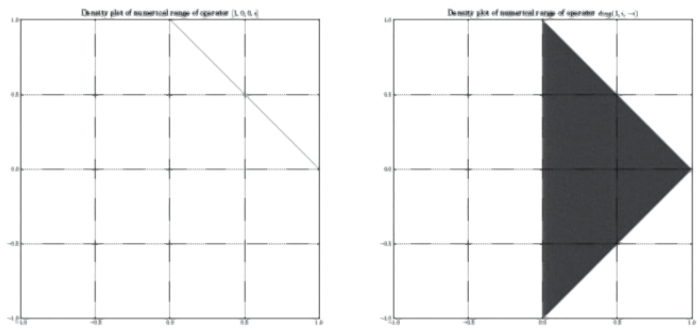

[A, A+] = 0
P1+P2+P3 = 1
P3
P2
P1

Normal case: a projection of the classical simplex...

Normal matrix, ($[A, A^*] = 0$), of size $N = 2$ with spectrum $\{\lambda_1, \lambda_2\}$

Numerical range $\Lambda(A)$ forms the **interval** $[\lambda_1, \lambda_2]$ on the complex plane,

Examples for diagonal matrices A of size **two** and **three**



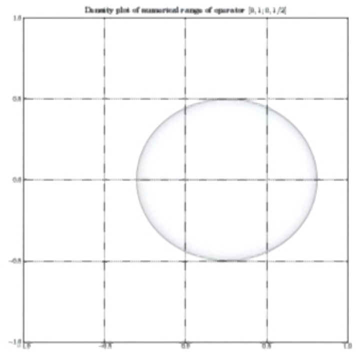
Normal matrices of order $N = 3$ with spectrum $\{\lambda_1, \lambda_2, \lambda_3\}$

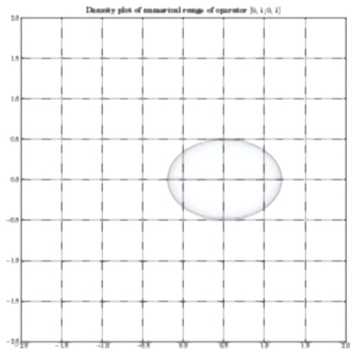
Numerical range $\Lambda(A)$ forms the **triangle** $\Delta(\lambda_1, \lambda_2, \lambda_3)$ on the complex plane.

Numerical range for $N = 2$

Non–normal matrices of size $N = 2$

Numerical range $\Lambda(A)$ forms an (eliptical) disk on the complex plane: **projection** of (empty!) **Bloch sphere**, $S^2 = \mathbb{C}P^1$ on the complex plane.





Wawel castle in Cracow

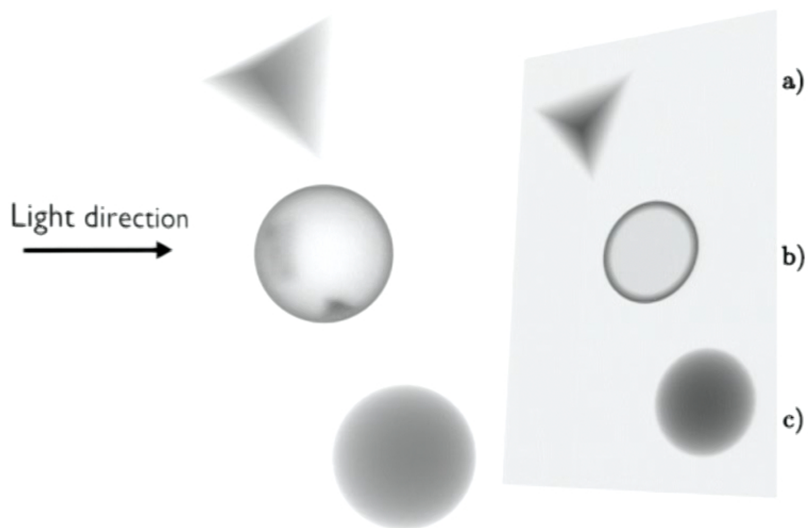
Ciesielski theorem

Ciesielski theorem: With probability $1-\epsilon$ the bench **Banach** talked to **Nikodym** in 1916 was localized in η-neighbourhood of the **red arrow**.

Plate commemorating the discussion between

Stefan Banach and **Otton Nikodym** (**Kraków, summer 1916**)

Shadows of three dimensional objects...



Quantum States and Numerical Range/Shadow

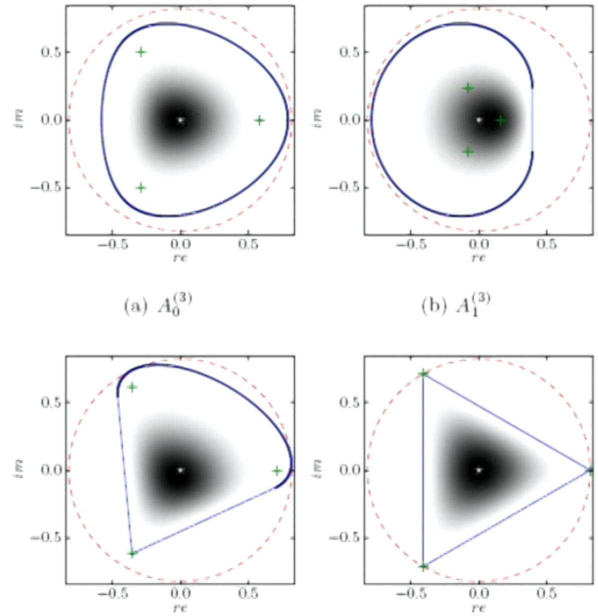
Classical States & normal matrices

Proposition 1. *Let $\mathcal{C}_N$ denote the set of* **classical states** *of size N, which forms the regular simplex Δ_{N-1} in $\mathbb{R}^{N-1}$. Then the set of similar images of orthogonal projections of $\mathcal{C}_N$ on a 2–plane is equivalent to the set of all possible numerical ranges $\Lambda(A)$ of all* **normal matrices** *A of order N (such that $AA^* = A^*A$).*

Quantum States & non–normal matrices

Proposition 2. *Let $\mathcal{M}_N$ denotes the set of* **quantum states** *size N embedded in $\mathbb{R}^{N^2-1}$ with respect to Euclidean geometry induced by Hilbert-Schmidt distance. Then the set of similar images of orthogonal projections $\mathcal{M}_N$ on a 2-plane is equivalent to the set of all possible numerical ranges $\Lambda(A)$ of* **all matrices** *A of order N.*

Numerical range of matrices of size $N = 3$



(a) $A_0^{(3)}$ (b) $A_1^{(3)}$

belong to one of **four** different classes specified e.g. by the number s of **flat segments** of the boundary, $s = 0, 1, 2, 3$.

Numerical range for matrices of order $N = 3$.

Classification by **Keeler, Rodman, Spitkovsky 1997**

Numerical range Λ of a 3×3 matrix A forms:

a) $\Lambda(A)$ is a compact set of an 'ovular' shape
(which contains three eigenvalues!) – the **generic** case, $s = 0$

b) a compact set with **one** flat part (e.g. convex hull of a **cardioid**), $s = 1$

c) a compact set with **two** flat parts
(e.g. convex hull of an ellipse and a point outside it), $s = 2$

d) **triangle** of eigenvalues, $\Lambda(A) = \Delta(\lambda_1, \lambda_2, \lambda_3)$
for any **normal matrix** A one has $s = 3$

These four cases describe the shape of possible projections of the $8D$ set $\mathcal{M}_3$ of mixed quantum states of size $N = 3$ onto a 2–**plane**.

Joint Numerical Range & Quantum States

Joint Numerical Range (JNR) of a set of m operators

$$\Lambda(A_1, A_2, \ldots, A_m) = (\langle\psi|A_1|\psi\rangle, \langle\psi|A_2|\psi\rangle, \ldots, \langle\psi|A_m|\psi\rangle) \subset \mathbb{R}^m.$$

For $m \geq N^2 - 1$ JNR is (in general) **not** a **convex set**!

Set $m = 2$, decompose $A = A_H + iA_A$ into its **Hermitian** and **anti–Hermitian** part. Then $\Lambda(A) = \Lambda(A_H, A_A)$

Proposition 3. *Take a set $\{A_1, \ldots, A_{N^2-1}\}$ of matrices of size N forming an* **orthonormal basis** *in the space of Hermitian, traceless matrices. Then*

- $\Lambda(A_1, A_2, \ldots, A_{N^2-1})$ *is affine isomorphic to the set $\Omega_N = \mathbb{C}P^{N-1}$ of* **pure quantum states** *of size N (embedded in $\mathbb{R}^{N^2-1}$),*
- *The* **convex hull** *of $\Lambda(A_1, A_2, \ldots, A_{N^2-1})$ is isomorphic to the set $\mathcal{M}_N$ of* **mixed quantum states** *of size N.*
- $\Lambda(A_1, A_2, \ldots, A_m)$ *with $m \leq N^2 - 1$ forms a projection of Ω_N into $\mathbb{R}^m$.*

Joint Numerical Range: some examples

$N = 2$: one qubit states

Let $\sigma_1, \sigma_2, \sigma_3$ denote three trace-less **Pauli matrices** of size $N = 2$. Then

- $\Lambda(\sigma_1, \sigma_2, \sigma_3) = \Omega_2 = \mathbb{C}P^1$ forms the **Bloch sphere** S^2 of all one–qubit pure states.
- The **convex hull** of $\Lambda(\sigma_1, \sigma_2, \sigma_3)$ forms the **Bloch ball**, $\mathcal{M}_2 = B_3 \subset \mathbb{R}^3$ of all one–qubit mixed states.

$N = 3$: one qutrit states

Let $\lambda_1, \dots \lambda_8$ denote eight traceless **Gell–Man matrices** of size 3: the generators of $SU(3)$.
Then

- $\Lambda(\lambda_1, \dots \lambda_8) = \Omega_3 = \mathbb{C}P^2$ forms the set of all one–qutrit pure states.
- The **convex hull** of $\Lambda(\lambda_1, \dots, \lambda_8)$ forms the set of $N = 3$ mixed states – a convex body $\mathcal{M}_3$ embedded in $\mathbb{R}^8$

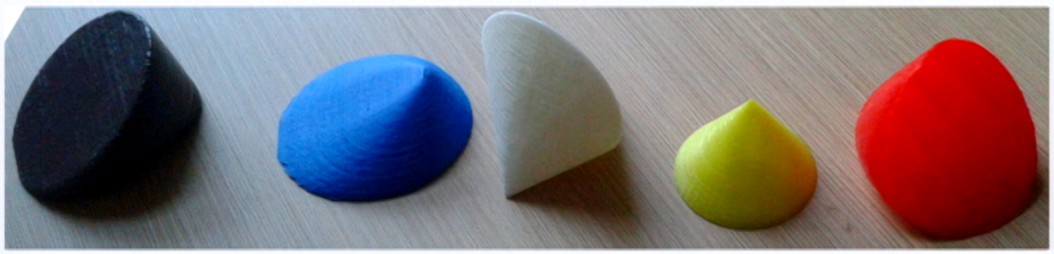
Joint Numerical Range: 3D examples for $m = 3$

$N = 3$: one qutrit

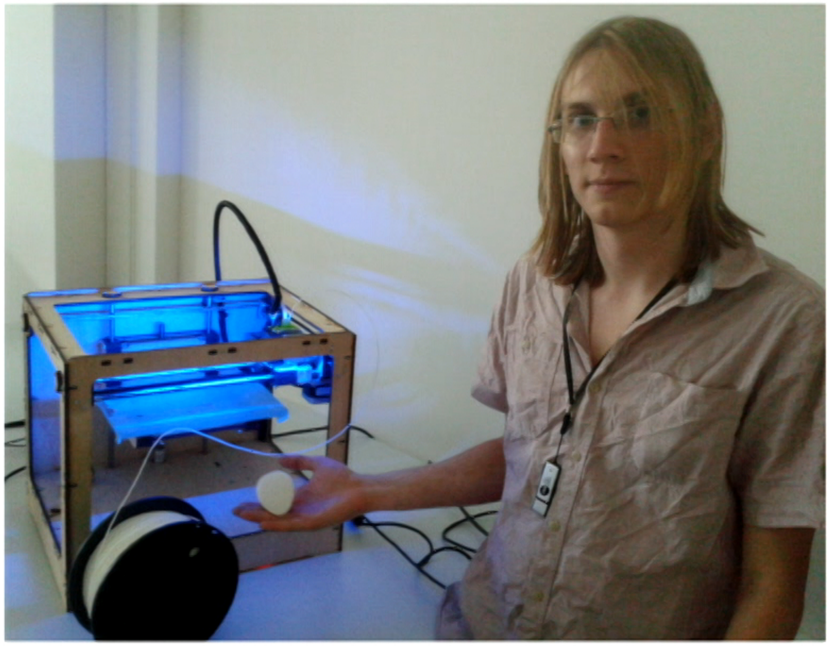
Take any triple of hermitian operators $\{A_1, A_2, A_3\}$ of size $N = 3$.

Then **joint numerical range** $\Lambda(A_1, A_2, A_3) \subset \mathbb{R}^3$ gives
a **projection** of the 8D set $\mathcal{M}_3$ of mixed states of a qutrit into **3D**.

Examples:



Different classes of 3D JNR: their further projections into 2D belong to one of **four** classes of **Keeler** *et al.* –
the possible shapes of the standard numerical range for $N = 3$.



Konrad Szymański producing a 3D joint numerical range

Recall the shadows on the wall of the cave of **Plato**:

we do not understand all details of the $8D$ set $\mathcal{M}_3$ of quantum states of size three, but at least we can study its 2D and 3D **projections**

How to classify possible shapes of JNR of three Hermitian matrices A_1, A_2, A_3 of size $N = 3$?

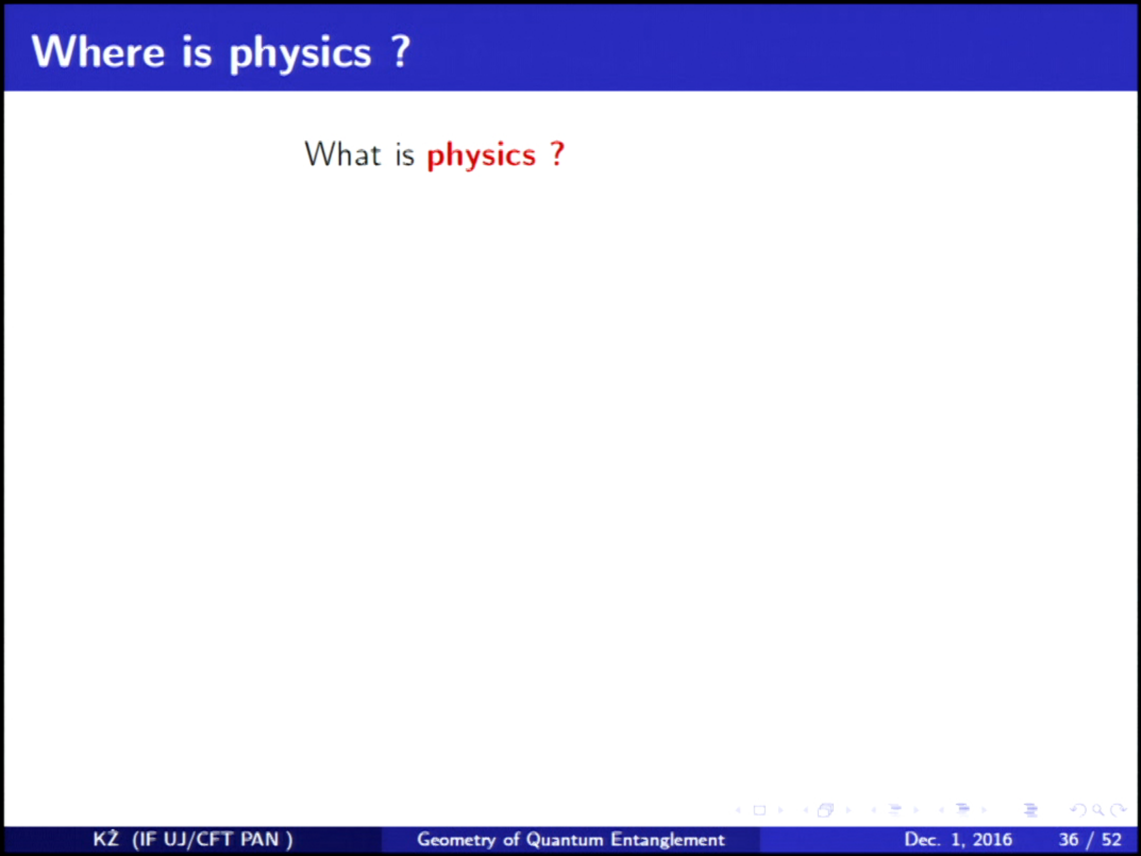
To classify the $3D$ numerical ranges for each body we count:

a) the number s of flat **segments** in the boundary
b) the number e of flat **faces** **(ellipses)** in the boundary

a) Ex. 6.2 $s = 0, e = 1$ b) Ex. 6.4 $s = 0, e = 3$ c) Ex. 6.5 $s = 0, e = 4$

d) Ex. 6.6 $s = 1, e = 0$ e) Ex. 6.7 $s = 1, e = 1$ f) Ex. 6.8 $s = 1, e = 2$

g) $s = e = 0$ h) $s = \infty, e = 0$ i) $s = \infty, e = 1$


Where is physics ?
What is physics ?
KŻ (IF UJ/CFT PAN)
Geometry of Quantum Entanglement
Dec. 1, 2016
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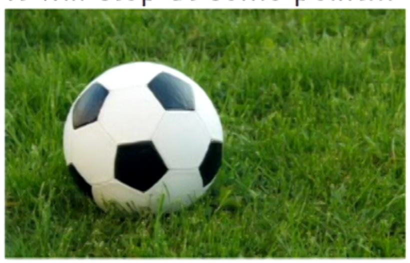
Where is physics ?

What is **physics ?**

Kick a ball !

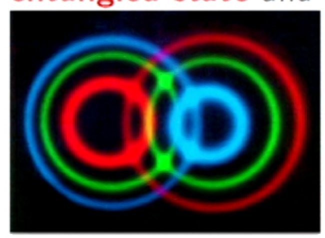
It will stop at some point...



Buy an **icecream** and wait a while..

It will melt !

Create an **entangled state** and do



nothing...

Composed systems & entangled states

bi-partite systems: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

- **separable pure states:** $|\psi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$
- **entangled pure states:** all states not of the above product form.

Two–qubit system: $N = 2 \times 2 = 4$

Maximally entangled **Bell state** $|\varphi^+\rangle := \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big)$

Entanglement measures

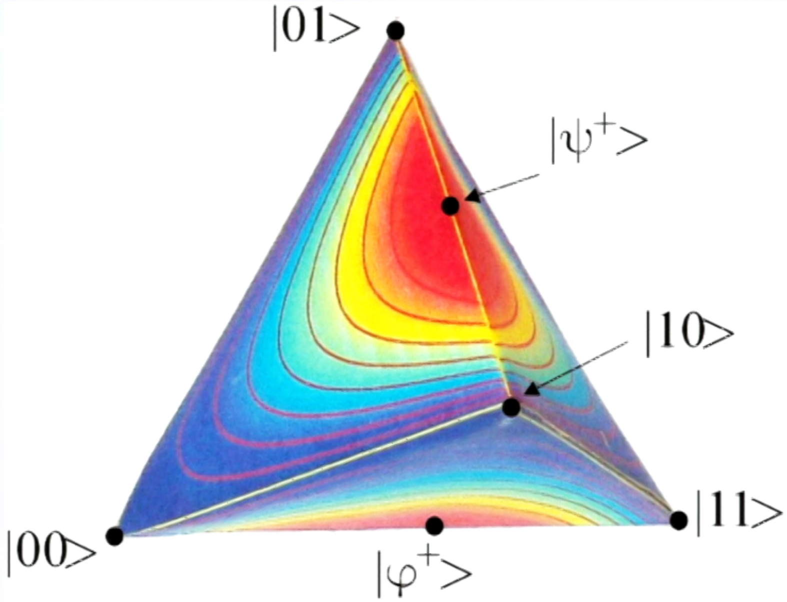
For any pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ define its partial trace $\sigma = \mathrm{Tr}_B|\psi\rangle\langle\psi|$.
Definition: Entanglement entropy of $|\psi\rangle$ is equal to von Neuman entropy of the partial trace

$$E(|\psi\rangle) := \ -\mathrm{Tr}\ \sigma \ln \sigma$$

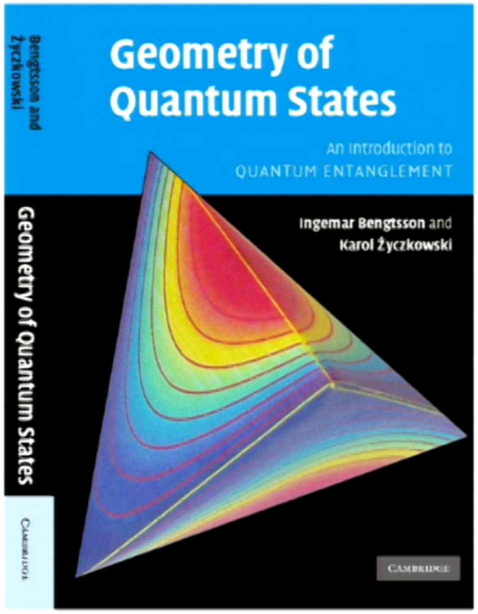
The more mixed partial trace, the more entangled initial pure state...

Entanglement of two real qubits

Entanglement entropy at the thetrahedron of $N = 4$ real pure states



Book completed in 2005 at **Perimeter Institute** !



II edition (with new chapters on MUBs & multipartite entanglement), Cambridge University Press, **2017**

Entanglement of mixed quantum states

Mixed states

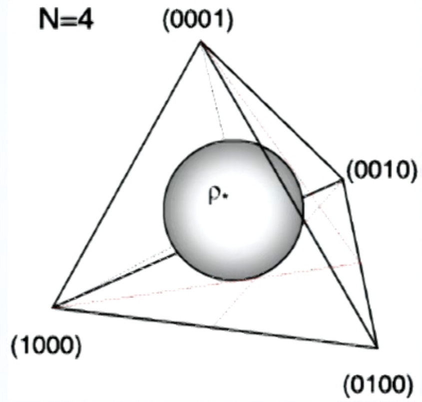
- **separable mixed states:** $\rho_{\rm sep} = \sum_j p_j \rho_j^A \otimes \rho_j^B \quad (**)$
- **entangled mixed states:** all states not of the above product form.

How to find,
whether a given density matrix ρ can be written in the form (**)
and is **separable ?**

The **separability problem** is solved only for the simplest cases of 2×2 and 2×3 problems...

Two–qubit mixed states

The maximal ball inscribed into $\mathcal{M}^{(4)}$ of radius $r_4 = 1/\sqrt{12}$ centred at $\rho_* = \mathbb{1}/4$ is separable !

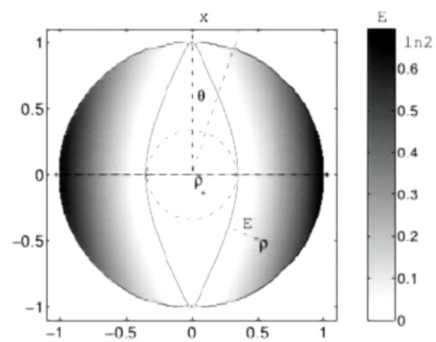


thetrahedron of eigenvalues

K.Ż, P.Horodecki, M.Lewenstein, A.Sanpera, 1998

Two–qubit mixed states

Degree of entanglement: a distance to the closest separable state



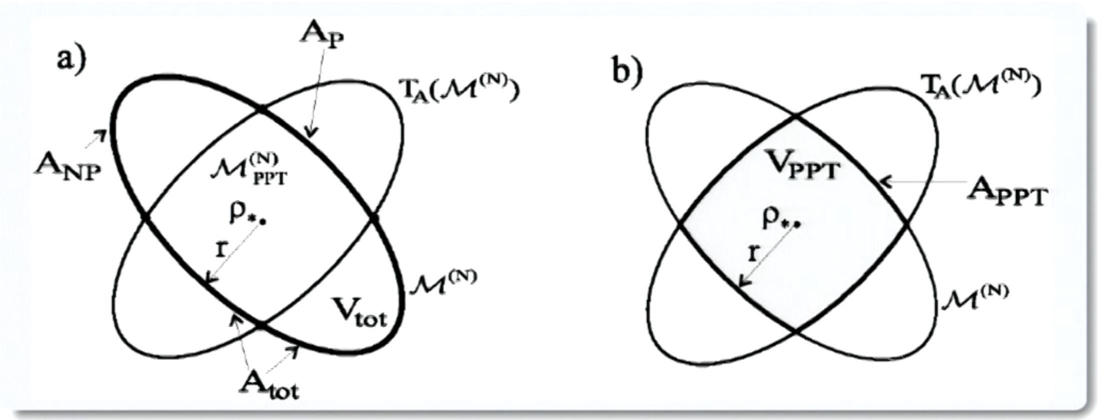
(entanglement of formation)

K.Ż, M. Kuś, 2001

Positive partial transpose criterion: Two–qubit mixed states

$$(\mathbb{I} \otimes T)\rho \;=\; \rho^{T_2} \geq 0 \;\Leftrightarrow\; \rho \text{ is separable}$$

The set of separable states of two–qubit system arises as an intersection of $\mathcal{M}^{(4)}$ and its mirror image with respect to partial transposition $T_A(\mathcal{M}^{(4)})$.



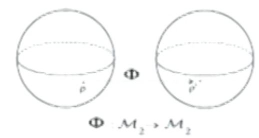
Stefan Banach sitting at his bench close to the Wawel Castle

Sculpture: Stefan Dousa

Fot. Andrzej Kobos

Quantum maps

Quantum operation: linear, completely positive trace preserving map



Enviromental form

$$\rho' = \Phi(\rho) = \mathrm{Tr}_E[U\,(\rho \otimes \omega_E)\,U^\dagger]\ .$$

where ω_E is an initial state of the environment while $UU^\dagger = \mathbb{1}$.

Kraus form

$$\rho' = \Phi(\rho) = \sum\nolimits_i A_i \rho A_i^\dagger\ ,$$

where the Kraus operators satisfy $\sum_i A_i^\dagger A_i = \mathbb{1}$.

A model discrete quantum dynamics

a) unitary dynamics (rotation), $\rho' = U\rho U^\dagger$
b) decoherence (contraction), $\rho'' = \sum_i^k A_i \rho' A_i^\dagger$

Two qubit model - $N = 2 \times 2 = 4$

a) free evolution: $U = \exp(itH)$ where $H = \sigma_x \otimes \sigma_y$
(non-local unitary dynamics !)

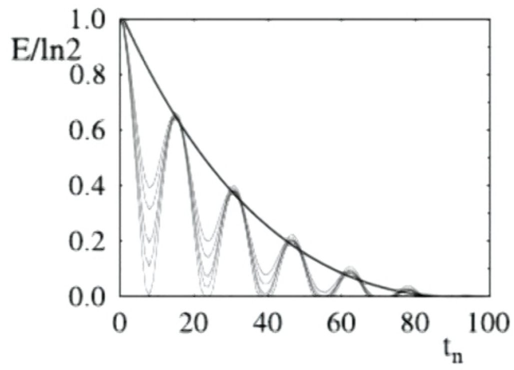
variant b1) bistochastic channel: $\Phi(\mathbb{1}/N) = \mathbb{1}/N$,
One–qubit **Pauli channel**: $k = 4$, $A_1 = \sqrt{1-\epsilon}\, \mathbb{1} \otimes \mathbb{1}$,
$A_2 = \sqrt{\epsilon/3}\, \mathbb{1} \otimes \sigma_x$, $A_3 = \sqrt{\epsilon/3}\, \mathbb{1} \otimes \sigma_y$, $A_4 = \sqrt{\epsilon/3}\, \mathbb{1} \otimes \sigma_z$.

variant b2) non bistochastic channel:
One qubit **amplitude damping channel**, *(decaying channel)*, $k = 2$,
where $A_1 = \mathbb{1} \otimes B_1$ and $A_2 = \mathbb{1} \otimes B_2$
with $B_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{p} \end{pmatrix}$ and $B_2 = \begin{pmatrix} 0 & \sqrt{1-p} \\ 0 & 0 \end{pmatrix}$

Dynamics of entanglement

Entanglement of formation E as a function of time t_n

for some initially pure states of a two–qubit system.

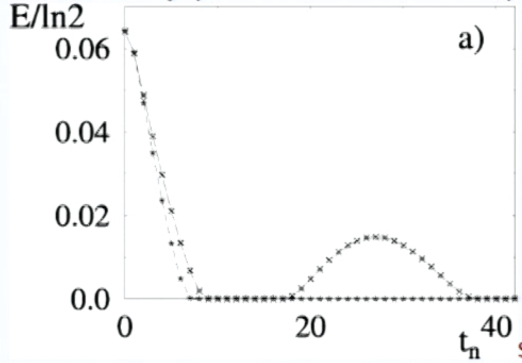


revivals of entanglement

Dynamics of entanglement

Entanglement of formation E as a function of time t_n

for some initially pure states of a two–qubit system.



sudden death of entanglement

K.Ż, P.Horodecki, M.Horodecki, R.Horodecki, PRA 2001

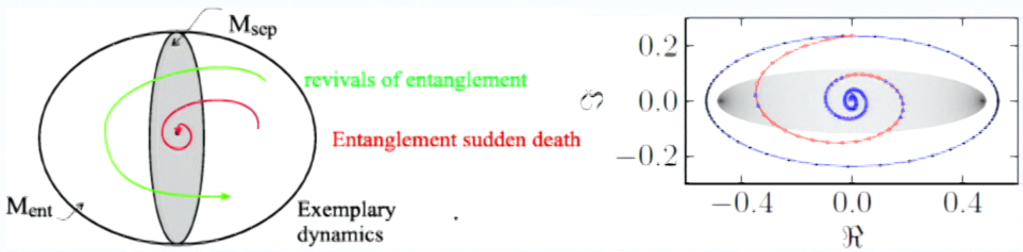
the name coined by Yau and Eberly,

who independently reported this effect in 2003.

Dynamics of Entanglement and separable shadow

Trajectories of quantum dynamics on the complex plane

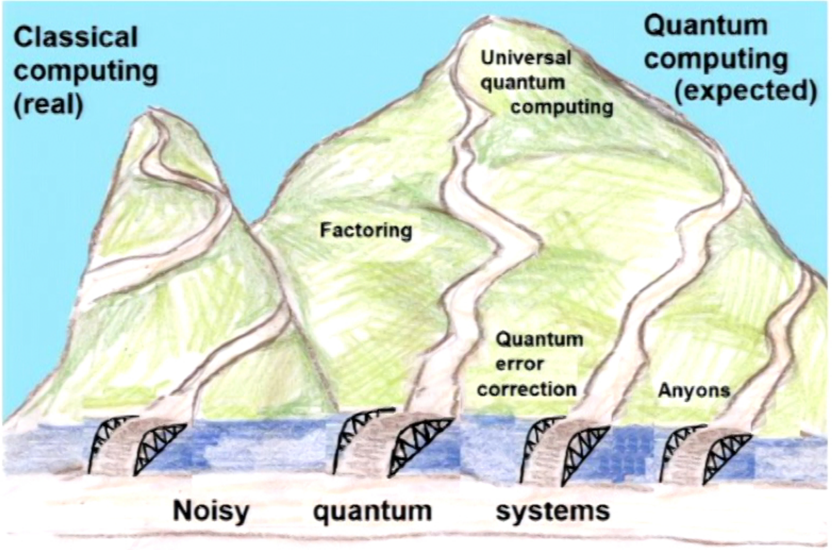
$$z(t) = \langle\psi(t)|A|\psi(t)\rangle$$



a) sketch of the problem; b) data for 2×2 system with initial separable pure state $|\psi(0)\rangle$ and suitably chosen (non–Hermitian !) operator A of size $N = 4$ **visualize** possible behaviour of quantum entanglement...

Quantum computing and coping with noise

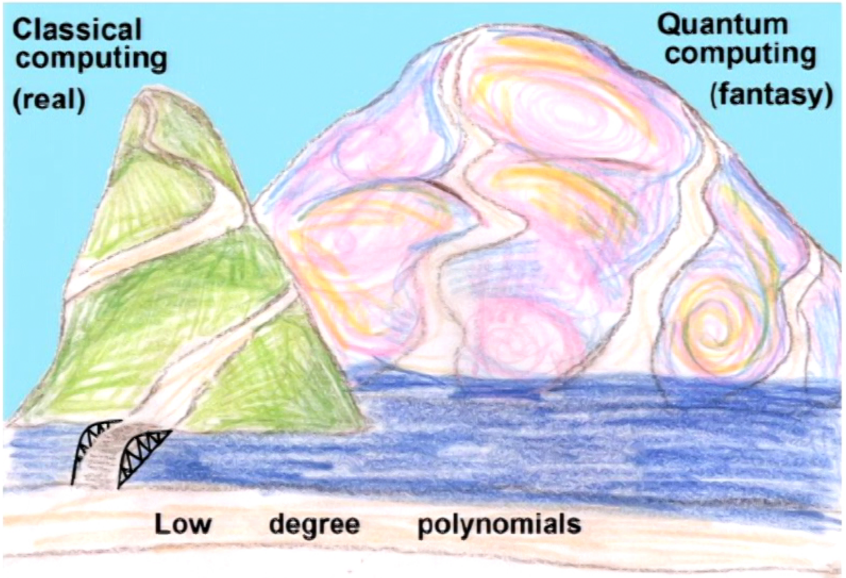
Alternative 1 (optimistic)



Gil Kalai (2016)

Quantum computing and coping with noise

Alternative 2 (pessimistic)



Gil Kalai (2016)

Concluding Remarks

- The set $\mathcal{M}_N$ of **mixed quantum states** of size N forms a scene for which the screenplays of **quantum information** processing are written.
  It is useful for any author to learn about the **structure & geometry** of the scene.
- As the set $\mathcal{M}_N$ has $N^2 - 1$ dimensions for $N \geq 3$ it is possible to investigate it by studying the **numerical range**:
  its projections onto a 2− or 3− planes.
- **Geometric approach** is usefull to study **quantum entanglement** and its dynamics. It allows one to explain the effects of entanglement revival and entanglement sudden death.
- More work is still required to understand the consequences of noise and decoherence for known schemes of **quantum computation**.

Bench commemorating the discussion between
Stefan Banach and **Otton Nikodym** (**Kraków, summer 1916**)

Sculpture: Stefan Dousa

Fot. Andrzej Kobos

opened in Planty Garden, Cracow, Oct. 14, 2016