Title: Symmetries of Flat Quantum Spacetime

Date: Nov 24, 2016 02:30 PM

URL: http://pirsa.org/16110089

Abstract: <p>By applying loop quantum gravity techniques to 2+1 gravity with a positive cosmological constant Î›, we show how the local gauge symmetry of the theory encoded in the constraint algebra acquires the quantum group structure of SOq(4). By means of an Inonu-Wigner contraction of the quantum group bi-algebra we obtain the kappa-PoincarÃ© algebra of the flat quantum space-time symmetries.</p>

κ-Poincare as a symmetry of flat quantum spacetime

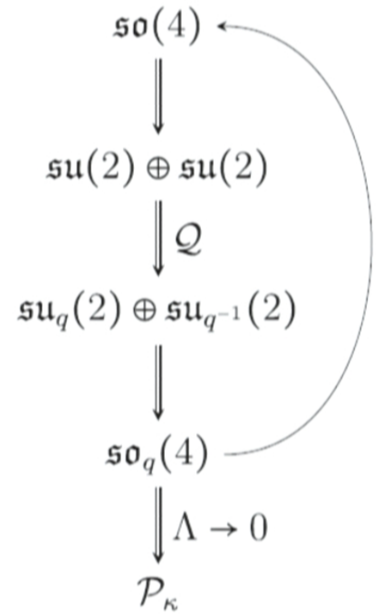
Based on PRD 94, 084044 [arXiv 1606.03085]
with F. Cianfrani, D. Pranzetti & G. Rosati

κ-Poincare

- It was already 25 years ago when κ-Poincare algebra (a quantum deformation of Poincare) was first derived.
- Since the deformation parameter of this algebra has dimension of mass, and is identified with Planck mass, it was speculated from the very beginning that it must have something to do with quantum gravity, and be relevant for quantum gravity phenomenology.
- It took 25 years to prove that *in 3D, kappa-Poincare is the algebra of symmetries of flat spacetime of quantum gravity*.
- Here is how it comes about.

The steps

1. Classical algebra of constraints of 3D gravity and algebra of spacetime symmetries.
2. Algebra of quantum constraint operators.
3. An argument for co-algebra.
4. Contraction.



Constraints and symmetries

- In metric formulation canonical gravity is characterized by (smeared) diffeomorphism and Hamiltonian constraints

$$D[f] = \int d^2x\, f^a(x) D_a(x), \quad H[g] = \int d^2x\, g(x) H(x)$$

- The (Poisson bracket) algebra of these constraints is not a Lie algebra as it contains metric components on the RHS.

$$\big[D[f_1], D[f_2]\big] = D\big[[f_1, f_2]\big]$$

$$\big[D[f], H[g]\big] = H[f^a \partial_a g]$$

$$\big[H[g_1], H[g_2]\big] = D[f(g_1, g_2)]$$

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Constraints and symmetries

$$D[f] = \int d^2x\, f^a(x) D_a(x), \quad H[g] = \int d^2x\, g(x) H(x)$$

- However, in the case of a maximally symmetric spacetime, if we chose $f^a(x)$ and $g(x)$ to be components of the Killing vectors and the metric on the RHS to be the one of the spacetime in question, the resulting algebra becomes the algebra of symmetries of spacetime (Poincare or (Anti) de Sitter).
- **We assume that an analogous statement holds in the quantum case.**

3D gravity

- The basic variables of (Euclidean) 3D gravity are triad e and connection ω. The Chern-Simons action is

$$S[e,\omega] = \kappa \int_M tr[e \wedge F(\omega) + \frac{\Lambda}{3} e \wedge e \wedge e]$$

- In this action Λ is the cosmological constant, while κ (an inverse of Newton's constant G) is the 3D Planck mass.

3D gravity

$$S[e,\omega] = \kappa\int_M tr[e \wedge F(\omega) + \frac{\Lambda}{3} e \wedge e \wedge e]$$

- The constraints (3 Gauss & 3 curvature ones) can be cast to the form

$$H^{\pm}[N] \equiv \kappa\int_\Sigma N_i F^i(A^{\pm}),$$

$$A_a^{\pm i} = A_a^i \pm \sqrt{\Lambda} e_a^i = A_a^i \pm \frac{\sqrt{\Lambda}}{\kappa} \epsilon_{ba} E_i^b$$

- Notice that connection $A^{\pm}$ is non-commutative, because

$$\{A_a^i(x), E_j^b(y)\} = \delta_a^b \delta_j^i \delta^{(2)}(x,y)$$

3D gravity

- The Poisson algebra of constraints $H^{\pm}$ reads

$$\{H^{\pm}[N], H^{\pm}[M]\} = \pm 2\sqrt{\Lambda}\, H^{\pm}[[N, M]]$$
$$\{H^{+}[N], H^{-}[M]\} = 0$$

$$[N, M]^{i} = \epsilon^{i}{}_{jk} N^{j} M^{k}$$

Comments

- The relation of Gauss and curvature constraints with the ones of metric gravity is not trivial in general. However, the situation simplifies considerably in the case of the maximally symmetric spacetime.
- There, there exists a convenient choice of the Killing vectors, which makes it easy to associate the algebra of symmetry generators of metric formalism in the case of (Euclidean) de Sitter space-time with the algebra of $H^{\pm}$, in which the smearing functions are proportional to delta functions.

Comments

- Then the algebra of $H^{\pm}$ constraints becomes two copies of the *su(2)* algebra (forming together the *so(4)* algebra).

$$\{H_i^{\pm}, H_j^{\pm}\} = \pm 2\sqrt{\Lambda}\,\epsilon^k{}_{ij} H_k^{\pm}$$

$$\{H_i^{+}, H_j^{-}\} = 0$$

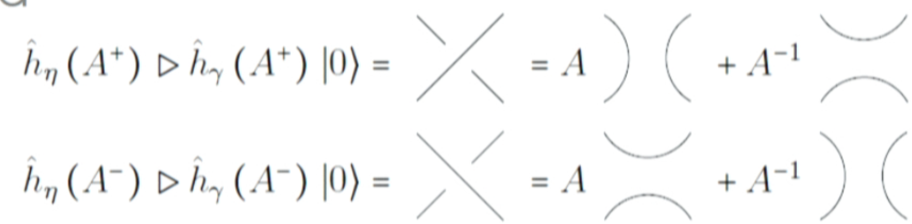
- There is one to one correspondence between these generators and the ones corresponding to energy E linear momenta P_i (time and space translations), rotations M, and boosts N_i of spacetime symmetries.
- This is the system that we are going to quantize and then express as an algebra of physical symmetries.

LQG quantization

- Consider now the action of a holonomy on the vacuum

$$\hat{h}_\gamma[A^\pm]|0\rangle = |h_\gamma[A^\pm]\rangle$$

- If we apply two holonomy operators in succession we find



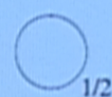
- with

$$A = e^{h/2} = \exp\left(i\hbar \frac{\sqrt{\Lambda}}{4\kappa} \right)$$

- Noui, Perez, Pranzetti, JHEP 10 (2011) 036

The quantum algebra

- Using these one can compute the form of the algebra of commutators of quantum constraints.
- It turns out that this algebra is anomalous in general, but the anomaly can be cancelled by assuming that

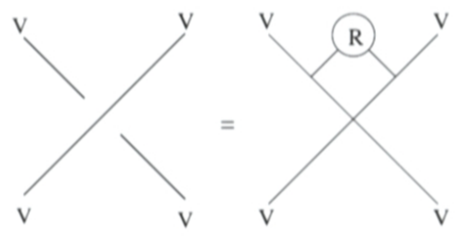
$\mathrm{tr}[W_p] =$  $= -(A^2 + A^{-2})$

- Then the algebra becomes deformed and can be cast into the form of $su_q(2)$ algebra for H^+ and $su_{q^{-1}}(2)$ for H^- with

$$q = A^2 = \exp\left(i\hbar \frac{\sqrt{\Lambda}}{2\kappa} \right)$$

- Pranzetti, Phys. Rev. D 89, 084058 (2014)

The R-matrix



- We show that indeed the action of crossing operators H^+ is exactly the same as the action of $su_q(2)$ R-matrix (and $su_{q^{-1}}(2)$ R-matrix for H^- operators.)
- Thus the algebra of quantum operators is a direct sum

$$su_q(2) \oplus su_{q^{-1}}(2)$$

LQG quantization

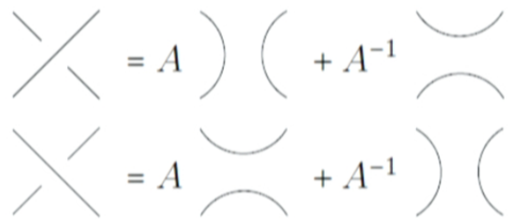
- Consider now the action of a holonomy on the vacuum

$$\hat{h}_\gamma[A^\pm]|0\rangle = |h_\gamma[A^\pm]\rangle$$

- If we apply two holonomy operators in succession we find

$$\hat{h}_\eta(A^+) \triangleright \hat{h}_\gamma(A^+)\,|0\rangle =$$

$$\hat{h}_\eta(A^-) \triangleright \hat{h}_\gamma(A^-)\,|0\rangle =$$



- with

$$A = e^{h/2} = \exp\left(i\hbar\frac{\sqrt{\Lambda}}{4\kappa}\right)$$

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The su$_q$(2)

$$X_+X_- - X_-X_+ = \frac{e^{2hH} - e^{-2hH}}{q - q^{-1}}, \quad \Delta X_+ = X_+ \otimes e^{hH} + e^{-hH} \otimes X_+ ,$$

$$e^{hH}X_+ = qX_+e^{hH}, \quad \Delta X_- = X_- \otimes e^{hH} + e^{-hH} \otimes X_- ,$$

$$e^{hH}X_- = q^{-1}X_-e^{hH} \quad \Delta e^{hH} = e^{hH} \otimes e^{hH}$$

$$S(H) = -H, \qquad S(X_\pm) = -e^{\pm h}X_\pm$$

From $su_q(2) \oplus su_{q^{-1}}(2)$ to $so_q(4)$

- By changing the generators we get the algebraic part of $so_q(4)$ expressed in terms of physical generators ($z = \sqrt{\Lambda}/2\kappa$ and $\hbar = 1$) .

$$[E,P_a] = \Lambda N_a, \qquad [N_a,E] = P_a,$$

$$[P_1,P_2] = \Lambda \frac{\sinh(zM)}{\sin(z)} \cosh\left(zE/\sqrt{\Lambda}\right),$$

$$[N_a,P_b] = -\delta_{ab}\sqrt{\Lambda}\frac{\sinh\left(zE/\sqrt{\Lambda}\right)}{\sin(z)}\cosh(zM),$$

$$[N_1,N_2] = \frac{\sinh(zM)}{\sin(z)}\cosh\left(zE/\sqrt{\Lambda}\right),$$

$$[M,N_a] = \epsilon_a{}^b N_b, \quad [M,P_a] = \epsilon_a{}^b P_b, \qquad [M,E] = 0$$

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From $su_q(2) \oplus su_{q^{-1}}(2)$ to $so_q(4)$

- Analogously, we get the co-algebraic part of $so_q(4)$

$$\Delta E = E \otimes 1 + 1 \otimes E\,, \qquad \Delta M = M \otimes 1 + 1 \otimes M\,,$$

$$\Delta P_a = P_a \otimes e^{\frac{1}{2}zE/\sqrt{\Lambda}} \cosh\left(\frac{1}{2}zM\right) + e^{-\frac{1}{2}zE/\sqrt{\Lambda}} \cosh\left(\frac{1}{2}zM\right) \otimes P_a$$
$$+\epsilon_{ab}\left(\sqrt{\Lambda}N_b \otimes e^{\frac{1}{2}zE/\sqrt{\Lambda}} \sinh\left(\frac{1}{2}zM\right) - \sqrt{\Lambda}e^{-\frac{1}{2}zE/\sqrt{\Lambda}} \sinh\left(\frac{1}{2}zM\right) \otimes N_b\right)$$

$$\Delta N_a = N_a \otimes e^{\frac{1}{2}zE/\sqrt{\Lambda}} \cosh\left(\frac{1}{2}zM\right) + e^{-\frac{1}{2}zE/\sqrt{\Lambda}} \cosh\left(\frac{1}{2}zM\right) \otimes N_a$$
$$-\epsilon_{ab}\left(\frac{1}{\sqrt{\Lambda}}P_b \otimes e^{\frac{1}{2}zE/\sqrt{\Lambda}} \sinh\left(\frac{1}{2}zM\right) - \frac{1}{\sqrt{\Lambda}}e^{-\frac{1}{2}zE/\sqrt{\Lambda}} \sinh\left(\frac{1}{2}zM\right) \otimes P_b\right)$$

Contraction

- Now we want to contract down to flat quantum spacetime, by taking the limit $\Lambda \to 0$.
- Actually, it turns out that the limit (in the co-sector) will be divergent if instead of $su_q(2) \oplus su_{q^{-1}}(2)$ we take $su_q(2) \oplus su_q(2)$. This was known for years, but it is nice to see that quantum gravity takes care of the problem without any external input.
- Thus although the non-zero cosmological constant is necessary in the intermediate steps, the theory still has a non-trivial flat space limit.

Contraction down to κ-Poincare

- To contract down to the bicrossproduct basis of κ-Poincare, we must first change the basis of $so_q(4)$ as follows

$$E=\tilde{E}, \qquad M=\tilde{M},$$

$$P_a = e^{z\tilde{E}/\left(2\sqrt{\Lambda}\right)}\tilde{P}_a,$$

$$N_a = e^{z\tilde{E}/\left(2\sqrt{\Lambda}\right)}\left(\tilde{N}_a - \frac{z}{2\sqrt{\Lambda}}\epsilon_{ab}\tilde{M}\tilde{P}_b\right)$$

Contraction down to κ-Poincare

- Now we take the contraction limit $\Lambda \to 0$ with κ kept finite
- As a result we get the κ-Poincare algebra

$$[E,P_a]=[P_1,P_2]=0,$$

$$[N_a,E]=P_a,[N_a,P_b]=-\delta_{ab}\left(\frac{\kappa}{2}\left(1-e^{-2E/\kappa}\right)-\frac{1}{2\kappa}\vec{P}^2\right)-\frac{1}{\kappa}P_aP_b,$$

$$[N_1,N_2]=M,\quad [M,N_a]=\epsilon_{ab}N_b,\qquad [M,P_a]=\epsilon_{ab}P_b,\qquad [M,E]=0$$

- co-algebra

$$\Delta E=E\otimes 1+1\otimes E,\qquad \Delta M=M\otimes 1+1\otimes M,$$

$$\Delta P_a = P_a\otimes 1+e^{-E/\kappa}\otimes P_a$$

$$\Delta N_a=N_a\otimes 1+e^{-E/\kappa}\otimes N_a-\frac{1}{\kappa}\epsilon_{ab}P_b\otimes M$$

What happened here?

1. We started with algebra of constraints of gravity H^+ and H^- , in the case of maximally symmetric spacetime with positive cosmological constant, the (Euclidean) de Sitter space.
2. These constraints, classically, satisfy the algebra of symmetry of de Sitter.
3. We interpret the algebra of **quantum** constraints as an algebra of symmetries of quantum de Sitter space. This algebra happens to be $so_q(4)$.

What happened here?

1. We take the limit of vanishing cosmological constant, with Planck mass kept finite.
2. As a result we find the 3D κ-Poincare algebra.
3. Thus as claimed for years

κ-Poincare is the algebra of symmetries of flat quantum spacetime!

(in 3D)

Beyond 3D

- There is a simple argument indicating that what was said above may have some relevance for 4D physics
- Consider a planar system in 4D QG. Such system, after dimensional reduction and field truncations can be described by 3D QG. But we have shown that in the former case the symmetry of flat quantum spacetime is κ-Poincare. Thus it seems that the symmetry of flat spacetime in 4D should be κ-Poincare as well.
- But it seems to be rather hard to device a direct proof; most of the techniques used here cannot be really applied to 4D.

$\Lambda \sim \text{const}$

$z \to 0$

$SO_q(4) \to SO(4)$

$\lambda \to 0$

$\kappa = \sqrt{\Lambda}/z\tau - \text{const}$

$SO_q(4) \to \kappa\text{-Poincaré}$