Title: On dynamics of asymptotically AdS spaces

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Abstract: <p>The anti-de Sitter (AdS) space is of great interest in contemporary
theoretical physics due to the AdS/CFT correspondence. However, the
question of stability of AdS space is unanswered till now. After
giving the motivation for studies of asymptotically AdS spaces, I will
review dynamics of such spacetimes in the context of AdS instability
problem. This survey will include: evidence for instability of AdS
space, existence and properties of time-periodic solutions, and
finally an application of analytical technique called multiscale or
resonant approximation approach. If time permits, I will comment on
other asymptotically AdS solutions. Along with the results, I will
highlight some details of methods relevant to the topic.</p>

On dynamics of asymptotically AdS spaces

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Model equation

Consider

$$\partial_t^2 u - \partial_x^2 u + m^2 u = g(u), \quad (t,x) \in \mathbb{R} \times [0,\pi],$$

with boundary conditions $u(t,0) = 0 = u(t,\pi)$. For $g = 0$ analysis is simple

$$u(t,x) = \sum_{j \geq 0} \hat{u}_j(t) \sin(\omega_j x), \quad \omega_j = \sqrt{j^2 + m^2},$$

and mode energies

$$E_j(t) = \frac{1}{2}\omega_j^2 \hat{u}_j(t)^2 + \frac{1}{2}\hat{u}_j'(t)^2,$$

are constant.

- What is the long-time behaviour of solutions of the nonlinearly perturbed equation (e.g. $g(u) = \pm u^3$)?
- Is $u = 0$ stable? An open problem for $m^2 = 0$. The transfer of energy to higher frequencies.
- Existence and properties of time-periodic solutions.

The AdS space

Maximally symmetric solution of: $R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = 0$, with $\Lambda = -\frac{d(d-1)}{2\ell^2}$, $(a,\, b = 0, 1, \ldots, d = D-1)$.

- Hyperboloid of radius $\ell > 0$

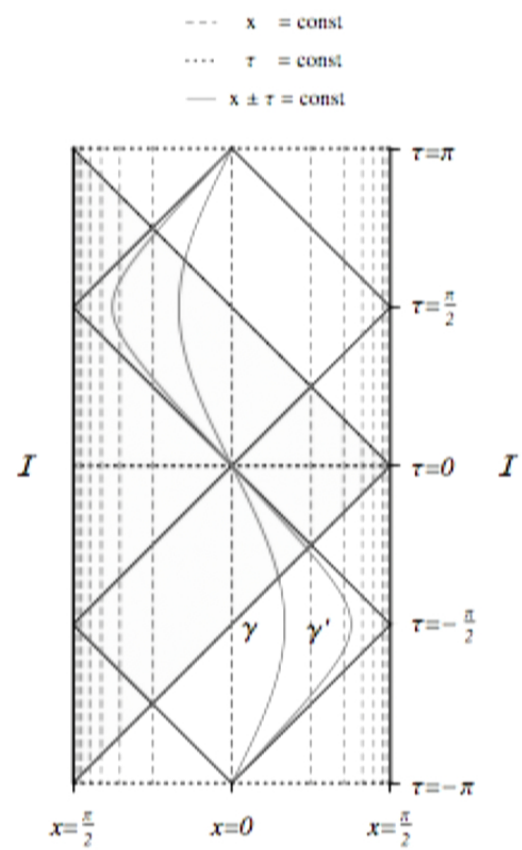
$$-X_0^2 + \sum_{k=1}^{d} X_k^2 - X_{d+1}^2 = -\ell^2,$$

- Global coordinates with $-\pi \leq \tau < \pi$, $0 \leq \rho < \infty$, and $\sum_{k=1}^{d} n_k^2 = 1$ (coordinates on $\mathbb{S}^{d-1}$)

$$X_0 = \ell \cosh\rho \cos\tau, \quad X_k = \ell \sinh\rho \, n_k,$$
$$X_{d+1} = \ell \cosh\rho \sin\tau, \qquad k = 1, \ldots, d,$$

Compactification: $\tan x = \sinh\rho$, $x \in [0, \pi/2)$ yields

$$\mathrm{d}s^2 = \frac{\ell^2}{\cos^2 x}\left(-\,\mathrm{d}\tau^2 + \mathrm{d}x^2 + \sin^2 x \;\mathrm{d}\Omega_{d-1}^2\right).$$



The AdS space

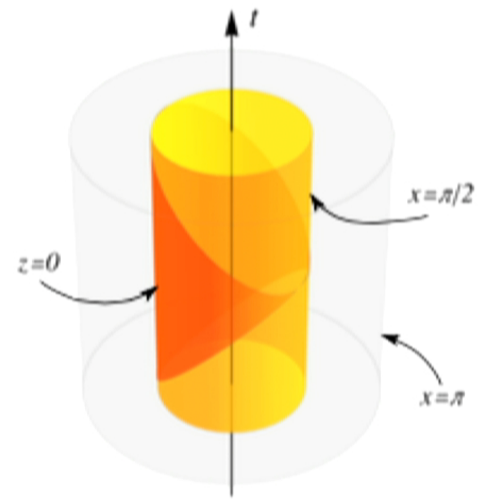
- Poincaré coordinates

$$X_0 = \frac{1}{2z}\left(z^2 + \ell^2 + \bar{x}^2 - t^2\right),$$
$$X_i = \frac{x_i}{z}\ell, \quad i = 1, \ldots, d-1,$$
$$X_d = \frac{1}{2z}\left(z^2 - \ell^2 + \bar{x}^2 - t^2\right),$$
$$X_{d+1} = \frac{t}{z}\ell,$$

where $\bar{x}^2 \equiv \sum_{i=1}^{d-1} x_i^2$, $z > 0$ or $z < 0$ $(X_0 = X_d)$. Metric

$$\mathrm{d}s^2 = \frac{\ell^2}{z^2}\left(-\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}\bar{x}^2\right),$$

- Coordinate transformation and boundary mapping [Bayona&Braga, '07] .



Motivation

- The AdS/CFT correspondence with applications to QGP [Chesler, '14] , [Bhattacharyya, '09] and CMP [Horowitz, '14] .
- Classical problem in General Relativity. Do *all* small perturbations of AdS remain small for all future times? (de Sitter [Friedrich, '86] , and Minkowski [Christodoulou&Klainerman, '93])
- Long-time evolution of closed conservative systems. If dissipation of energy by dispersion is absent what happens with generic perturbations? Is there any universal behavior? Whether time-periodic solution exists? If yes, are such solutions stable?
- More complex structure of critical behavior ([Santos-Oliván&Sopuerta, '15, '16] , also AdS_3 [Jałmużna, '15]) than in asymptotically flat case [Choptuik, '93] .
- Challenging problems for numerical analysis.

Self-gravitating massless scalar field

Simple model

$$R_{ab}-\frac{1}{2}g_{ab}R+\Lambda g_{ab}=8\pi G\left(\nabla_a\phi\,\nabla_b\phi-\frac{1}{2}g_{ab}\nabla_c\phi\,\nabla^c\phi\right),\quad \nabla_a\nabla^a\phi=0,$$

$$\Lambda=-d(d-1)/(2\ell^2)<0\,,\quad 8\pi G=d-1\,.$$

- Spherically symmetric parametrization of asymptotically AdS spacetimes

$$\mathrm{d}s^2=\frac{\ell^2}{\cos^2 x}\left(-Ae^{-2\delta}\,\mathrm{d}t^2+\frac{\mathrm{d}x^2}{A}+\sin^2 x\;\mathrm{d}\Omega^2_{d-1}\right),\quad (t,\,x)\in\mathbb{R}\times[0,\pi/2).$$

- Field equations

$$\partial_t\phi=Ae^{-\delta}\Pi,\quad \partial_t\Pi=\frac{1}{\tan^{d-1}x}\partial_x\left(\tan^{d-1}x\,Ae^{-\delta}\partial_x\phi\right),$$

$$\partial_x A=2\frac{d-2+2\sin^2 x}{\sin 2x}(1-A)+A\partial_x\delta\,,\quad \partial_x\delta=-\frac{1}{2}\sin 2x\left((\partial_x\phi)^2+\Pi^2\right).$$

Linear perturbations of AdS

- Linear equation on an AdS background [Ishibashi&Wald, '04]

$$\partial_t^2 \phi + \hat{L}\phi = 0\,, \quad \hat{L} = -\frac{1}{\tan^{d-1} x}\,\partial_x\left(\tan^{d-1} x\,\partial_x\right).$$

This operator is essentially self-adjoint on $\mathcal{H} = L^2\left([0,\pi/2);\tan^{d-1}x\ \mathrm{d}x\right)$.

- Eigenvalues and eigenvectors of operator $\hat{L}$ on Hilbert space $\mathcal{H}$ are

$$\omega_j^2 = (d+2j)^2, \quad e_j(x) = \mathcal{N}_j\ \cos^d x\, P_j^{(d/2-1,d/2)}(\cos 2x)\,,$$

$j = 0, 1, \ldots\,, \mathcal{N}_j \in \mathbb{R}$ normalization.

- AdS is linearly stable and any linear perturbation can be written as

$$\phi(t,x) = \textstyle\sum_{j\geq 0}\left(\alpha_j e^{-i\omega_j t} + \bar{\alpha}_j e^{i\omega_j t}\right) e_j(x)\,,$$

with constants α_j determined by the initial data $\phi(0,x)$ and $\partial_t\phi(0,x)$.

- Nondispersive spectrum + nonlinear coupling of linear modes = resonances.

Resonances and secular terms

$$\hat{L}e_j(x) = \omega_j^2 e_j(x), \quad j = 0, 1, \ldots .$$

- For *nondispersive* spectrum ($\mathrm{d}\omega_j / \mathrm{d}j = \mathrm{const}$) we expect resonances caused by the (nonlinear) coupling of linear modes.
- How resonances produce secular terms

$$u''(t) + \omega^2 u(t) = f(t), \quad f(t) = f_0 \cos(\chi t) \ ,$$

$$u(t) = \frac{u'(0)}{\omega} \sin(\omega t) + u(0)\cos(\omega t) + f_0 \cdot \begin{cases} \frac{\cos(\chi t) - \cos(\omega t)}{\omega^2 - \chi^2}, & \chi \neq \omega \,, \\ \frac{1}{2\omega} \mathbf{t} \sin(\omega t) \,, & \chi = \omega \,. \end{cases}$$

- Here the spectrum is *fully resonant* ($\exists \, \{k_i | \, k_i \in \mathbb{N}\}, \, \sum_i k_i \omega_i = 0$) which suggests *a large number of secular terms in non-linear perturbation theory.*
- Resonances are equally comon in dispersive cases, e.g. Yang-Mills propagating on the Einstein Universe

$$\omega_j = \sqrt{(j+2)^2 - 3}, \quad j = 0, 1, \ldots .$$

Resonances should be attributed to the structure of equations not only to the frequency spectrum.

Nonlinear evolution—boundary conditions

- We require smooth evolution and finiteness of the total mass

$$m(t,x) = \frac{\sin^{d-2} x}{\cos^d x}\left(1 - A(t,x)\right),$$

$$M = \lim_{x \to \pi/2} m(t,x) = \int_0^{\pi/2} A\left((\partial_x \phi)^2 + \Pi^2\right) \tan^{d-1} x \, \mathrm{d}x \, .$$

There is no freedom in prescribing boundary data at $x = \pi/2$. Reflecting (no-flux) BC.

- Expansion at $x = \pi/2$ ($y \to 0$) for odd d

$$\phi\left(t, \pi/2 - y\right) = \breve{\phi}_0 + \breve{\phi}_d(t) y^d + \left(\breve{\phi}_{d+2}(t) y^{d+2} + \cdots\right) + M\left(\breve{\phi}_{2d}(t) y^{2d} + \breve{\phi}_{2d+2}(t) y^{2d+2} + \cdots\right),$$

and for even d

$$\phi\left(t, \pi/2 - y\right) = \breve{\phi}_0 + \breve{\phi}(t)_d y^d + \left(\breve{\phi}_{d+2}(t) y^{d+2} + \cdots\right),$$

where $\partial_x^d A \mid_{x=\pi/2} = -d! M$.

- Local well-posedness has been proved [Friedrich, '95] , [Holzegel&Smulevici, '11] .

Nonlinear evolution—numerical approach

- Constrained evolution scheme. Method of lines with FD discretization in space. Runge-Kutta for time stepping (and to integrate constraints). High resolution with large number of grid points [Bizoń&Rostworowski, '11] or by using adaptive algorithm [Buchel *et al.*, '12, '13] . Parallelization.
- Pseudospectral approach using either eigenbasis (even d)

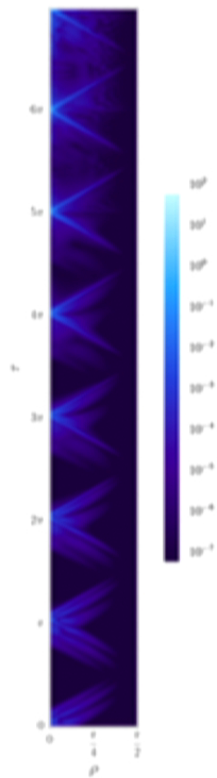
$$\phi(t,x) = \sum_{j=0}^{N-1} \hat{\phi}_j(t)\, e_j(x), \quad \Pi(t,x) = \sum_{j=0}^{N-1} \hat{\Pi}_j(t)\, e_j(x),$$

or Chebyshev polynomials (particularly in odd d)

$$\frac{\phi'}{\cos^{d-2} x} = \sum_{j=0}^{N-1} \hat{\Phi}_j(t)\, T_{2j}(2/\pi x), \; \frac{\Pi}{\cos^{d-1} x} = \sum_{j=0}^{N-1} \hat{\Pi}_j(t)\, T_{2j}(2/\pi x)\,.$$

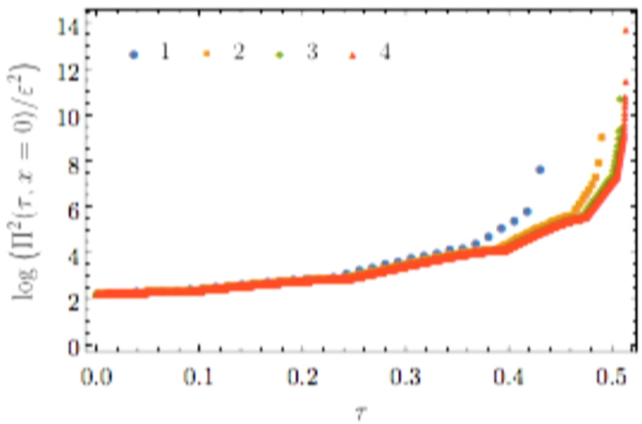
Coupled ODEs for $\hat{\phi}_j$ and $\hat{\Pi}_j$. Use symplectic method to integrate in time— preserve structure of equations and conserve energy

$$M = \sum_{j\geq 0} E_j, \quad E_j := \left(\sqrt{A}\Pi \,|\, e_j\right)^2 + \omega_j^{-2} \left(\sqrt{A}\phi' \,|\, e_j'\right)^2 .$$



The conjecture

Conjecture [Bizoń&Rostworowski, '11]
1. AdS$_{d+1}$ (for $d \geq 3$) is unstable against black hole formation under arbitrarily small perturbations.
2. There are perturbations for which turbulent energy transfer is not active (time-periodic and quasi-periodic solutions).



- Extrapolation (here $d = 4$)

$$\tau_H := \lim_{\varepsilon \to 0} \frac{t_H}{\varepsilon^2} \approx 0.514\,.$$

- Weak turbulence

$$E_j \sim j^{-2}, \quad j \gg 1\,.$$

- Secular terms.

$$\phi(0,x) = \varepsilon\left(\tfrac{1}{4}e_0(x) + \tfrac{1}{6}e_1(x)\right), \quad \partial_t\phi(0,x) = 0, \quad \varepsilon \sim 2^{-p}\,.$$

Independent confirmation [Buchel *et al.*, '12, '13] (complex SF), [Deppe&Frey, '15] (massive SF).

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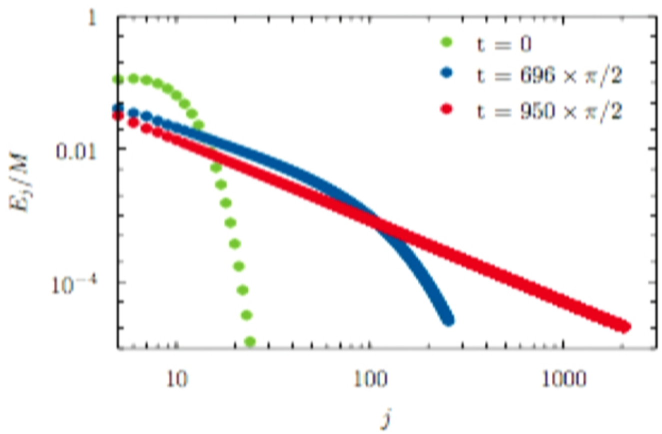

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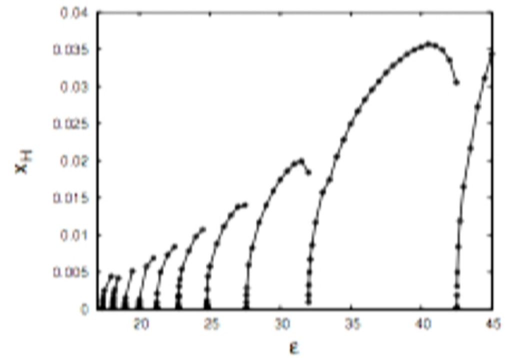
Critical collapse in AdS

- Studies in $D = 4$ indicate that critical exponent independent of value of Λ [Husain *et al.*, '03] (double-null characteristic scheme). Collapse in AdS_3 [Pretorius&Choptuik, '00] , [Garfinkle, '01] , [Jałmużna *et al.*, '15] .
- Mixed Cauchy-characteristic (like [Goldwirth&Piran, '87]) evolution scheme [Santos-Oliván&Sopuerta, '15, '16] . New feature of gravitational collapse in asymptotically AdS

$$M_{AH}(p) - M_g^{(n+1)} \propto (p_n - p)^{\xi} ,$$

$\xi \approx 0.7$, where $(n+1)$ enumerates bounces off the AdS boundary (evidence for $n = 0, 1, 2$).

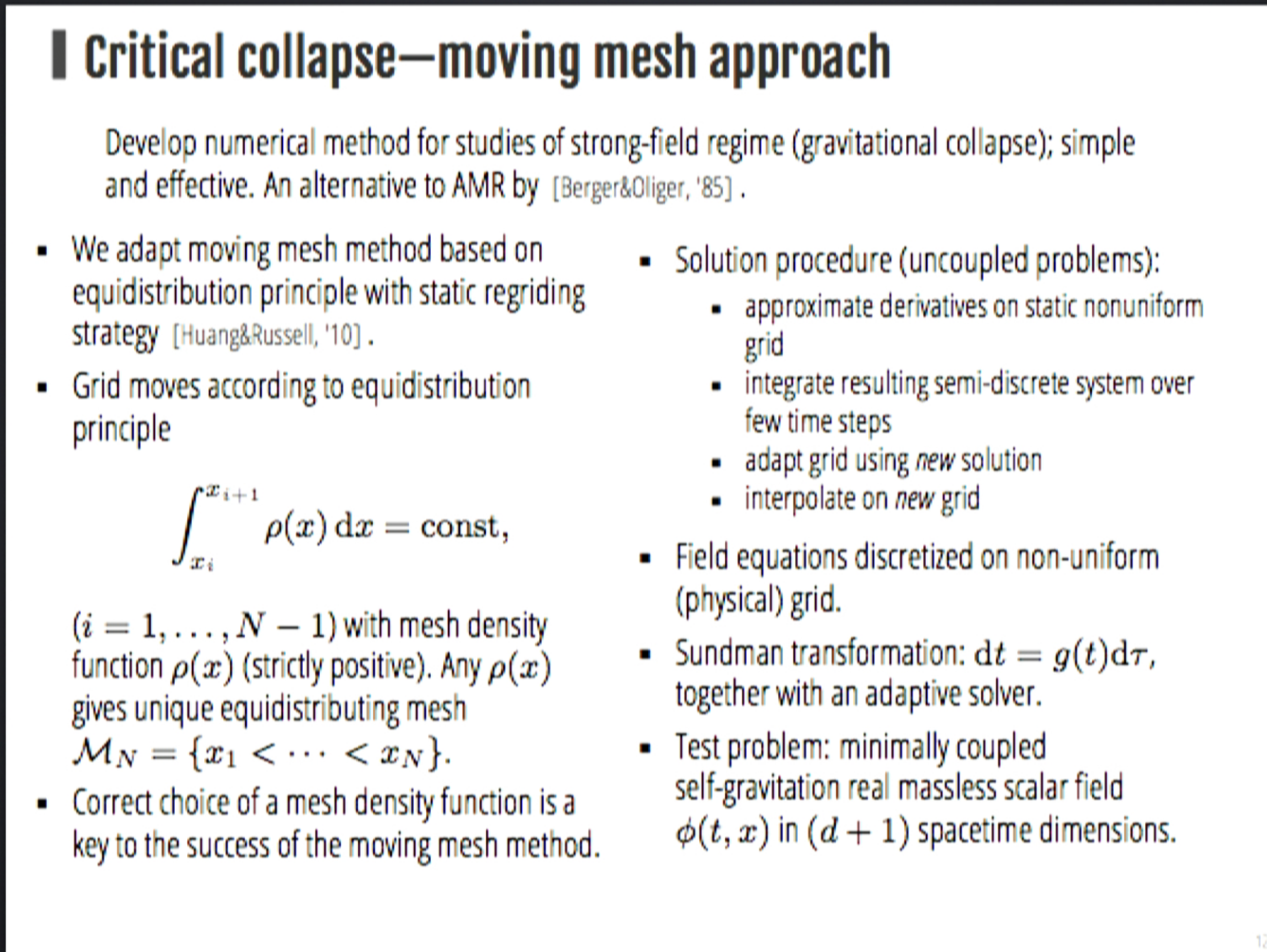
- Is the global geometry important? *Minkowski in a box* (Einstein-Maxwell-KG) and double-null coordinates [Cai&Yang, '16] . Scaling exponent $\xi \approx 0.36$.



Critical behavior [Bizoń&Rostworowski, '11]

$$M_{AH}(p) \propto (p_n - p)^{\gamma} ,$$

with $\gamma \approx 0.374$ [Choptuik, '93] , [Gundlach, '97] .


Critical collapse—moving mesh approach
Develop numerical method for studies of strong-field regime (gravitational collapse); simple and effective. An alternative to AMR by [Berger&Oliger, '85] .
We adapt moving mesh method based on equidistribution principle with static regriding strategy [Huang&Russell, '10] .
Grid moves according to equidistribution principle
$$\int_{x_i}^{x_{i+1}} \rho(x)\,\mathrm{d}x = \mathrm{const},$$
$(i = 1, \ldots, N-1)$ with mesh density function $\rho(x)$ (strictly positive). Any $\rho(x)$ gives unique equidistributing mesh $\mathcal{M}_N = \{x_1 < \cdots < x_N\}$.
Correct choice of a mesh density function is a key to the success of the moving mesh method.
Solution procedure (uncoupled problems):
approximate derivatives on static nonuniform grid
integrate resulting semi-discrete system over few time steps
adapt grid using new solution
interpolate on new grid
Field equations discretized on non-uniform (physical) grid.
Sundman transformation: $\mathrm{d}t = g(t)\mathrm{d}\tau$, together with an adaptive solver.
Test problem: minimally coupled self-gravitation real massless scalar field $\phi(t,x)$ in $(d+1)$ spacetime dimensions.
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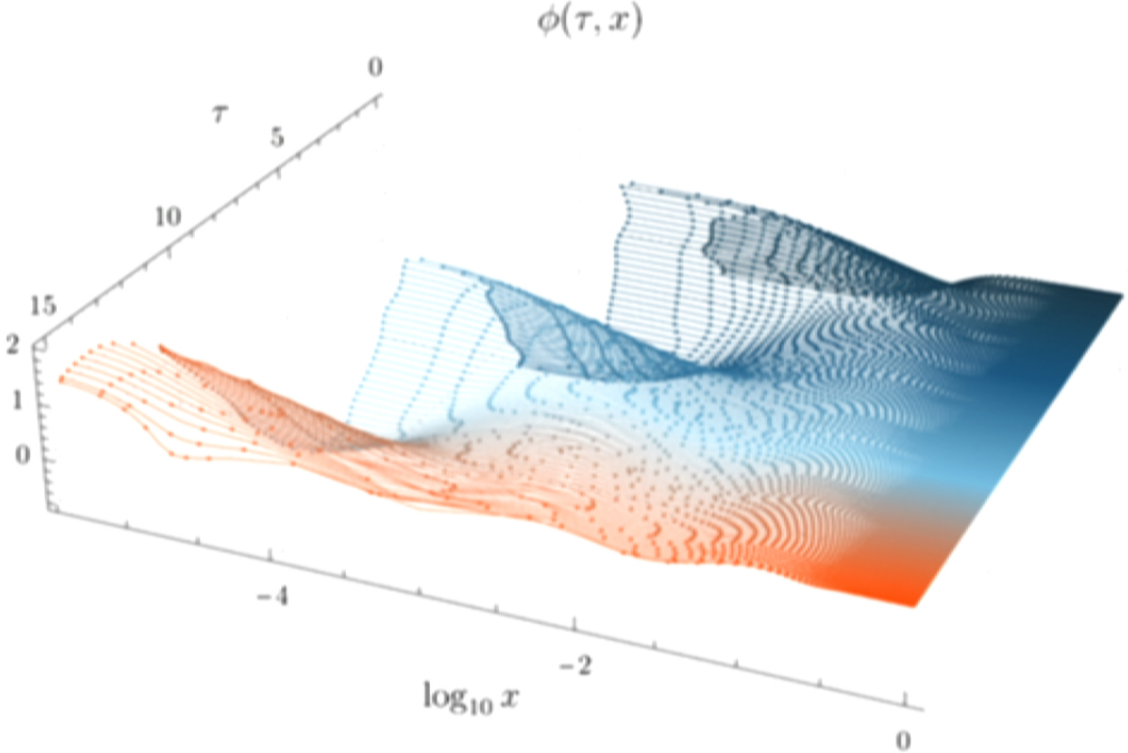

Resonant approximation

- Failure of naïve perturbative approach: $\phi(t,x) = \varepsilon\phi_1(t,x) + \varepsilon^3\phi_3(t,x) + \cdots$, and

$$\phi_1(t,x) = \sum_{j\geq 0} \left(\alpha_j e^{-i\omega_j t} + \bar{\alpha}_j e^{i\omega_j t}\right) e_j(x)\,,$$

  in general gives secular terms: $\phi(t,x) = \varepsilon\phi_1(t,x) + \varepsilon^3 t(\cdots) + \cdots$.
- Resummation (with *slow time* $\tau = \varepsilon^2 t$ dependence)

$$\phi_1(t,x) = \sum_{j\geq 0} \left(\alpha_j(\tau) e^{-i\omega_j t} + \bar{\alpha}_j(\tau) e^{i\omega_j t}\right) e_j(x).$$

- Resonant system ($\omega_i \pm \omega_j \pm \omega_k = \pm\omega_m$)

$$\alpha_m' = -\frac{i}{2\omega_m}\left(\sum_{ijk}^{-++} C_{ijkm}^{-++}\alpha_i\alpha_j\bar{\alpha}_k + \sum_{ijk}^{+--} C_{ijkm}^{+--}\bar{\alpha}_i\bar{\alpha}_j\alpha_k + \sum_{ijk}^{+++} C_{ijkm}^{+++}\alpha_i\alpha_j\alpha_k\right).$$

- Derived with multiscale [Balasubramanian *et al.*, '14] , renormalization group [Craps *et al.*, '14] (for EKG only $-++$ type resonances) and averaging [Craps *et al.*, '15] approaches.
- Invariant under: $\alpha_m(\tau) \to \varepsilon\,\alpha_m\left(\varepsilon^2\tau\right)$. Slow long-time energy flow between the modes. Symmetries—three constants of motion.

This infinite system has a solution that becomes singular in finite time. Singular solution governs generic blowup [Bizoń,M&Rostworowski, '15] .

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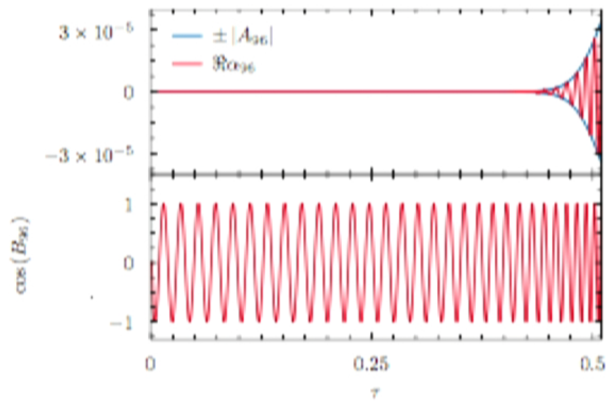
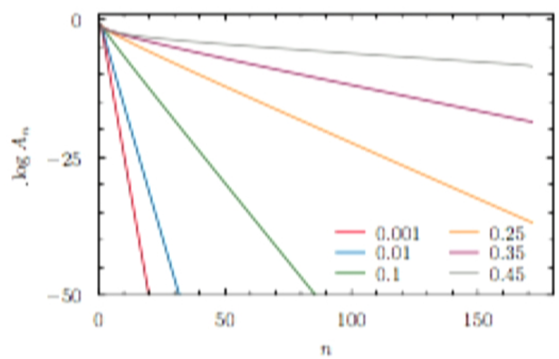
Resonant approximation—blowup



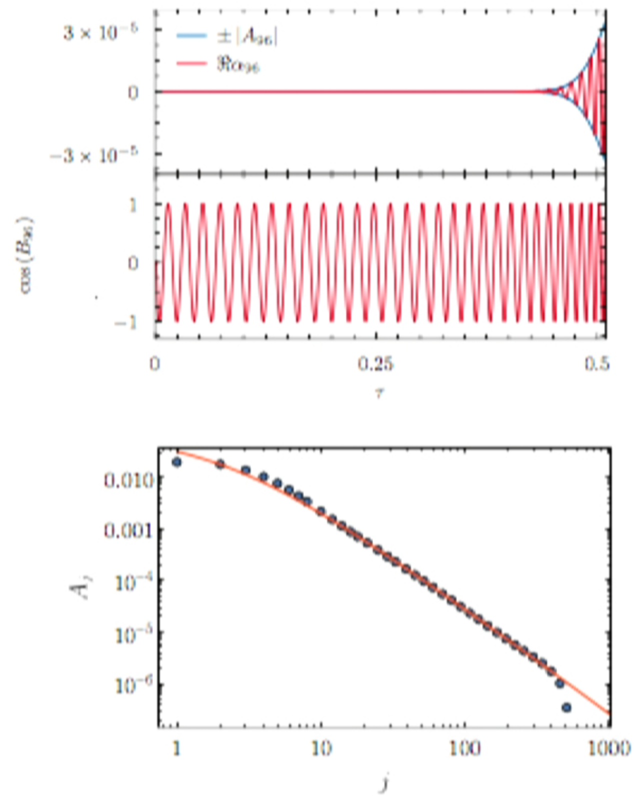


- Universal behavior in terms of $\alpha_m(\tau) = A_m(\tau)e^{iB_m(\tau)}$, i.e.: $A_m(\tau) \sim \tau^{m-1}$, and $B_m(\tau) \nearrow$.
- Analyticity strip method [Sulem *et al.*, '83], [Bizoń&Jałmużna, '13] (instability of AdS_3) with asymptotic ansatz ($m \gg 1$)

$$A_m(\tau) \sim m^{-\gamma(\tau)} e^{-\rho(\tau)m}.$$

- Numerical data indicate: $\gamma(\tau) \approx 2$, $\rho(\tau) \approx \rho_0(\tau_* - \tau)$, as $\tau \to \tau_*$ ($\approx 0.513 \approx \tau_H$) and synchronization of phases $B_j \sim j$ during evolution.

Resonant approximation—blowup



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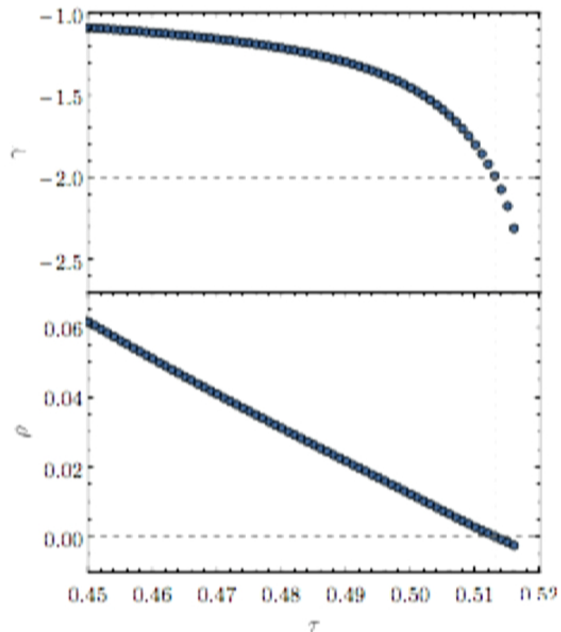

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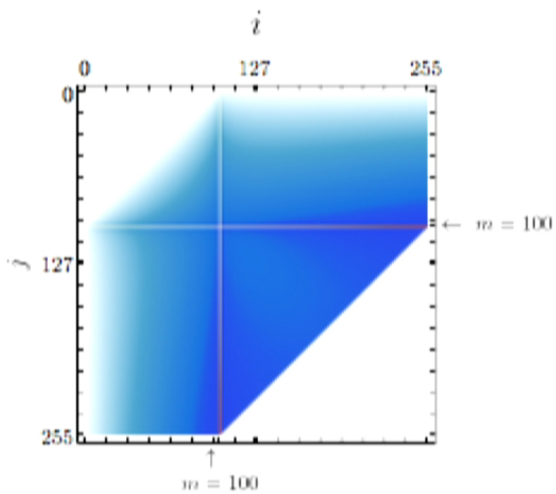
Resonant approximation—blowup

$$2\omega_m A'_m = \sum_{ijk}^{-++}{}' C^{-++}_{ijkm} A_i A_j A_k \Im\left(e^{i(B_i+B_j-B_k-B_m)}\right),$$

$$2\omega_m B'_m A_m = C^{-++}_{mmmm} A^3_m + A_m \sum_{j\neq m} \left(C^{-++}_{mjjm} + C^{-++}_{jmjm}\right) A^2_j + \cdots .$$



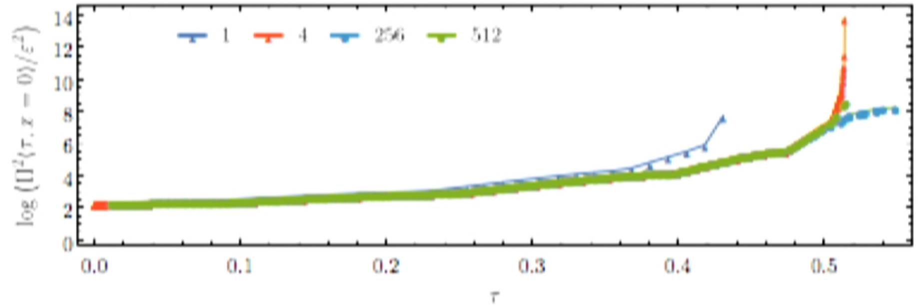
- Dominant contribution

$$2\omega_m B'_m \sim$$
$$\sim \sum_{j\neq m} \left(C^{-++}_{mjjm} + C^{-++}_{jmjm}\right) A^2_j$$
$$\sim m^2 \sum_{j\neq m} \frac{1}{j} e^{-2\rho_0(\tau_* - \tau)j} .$$

- Finite time ($\tau_* < \infty$) logarithmic blowup

$$B'_m(\tau) = a_m \log(\tau_* - \tau) + b_m .$$

Resonant approximation—blowup and collapse



- Method intended to provide uniformly bounded solution gives hints for instability. Note $|\alpha_j| < \infty \nRightarrow |\alpha_j'| < \infty$; here $(d = 4)$

$$|\alpha_j| < \infty \quad \text{but} \quad |\alpha_j^{(k)}| \sim \left(\frac{1}{\tau_* - \tau}\right)^{k-1}, \; k \geq 1,$$

- Generalization of asymptotic $\tau \to \tau_*$ solution to $d \geq 4$

$$A_j \sim j^{\gamma} e^{-\rho\tau}, \quad \gamma \to -d/2, \quad \rho \to 0,$$

which blows up in finite time $\tau_* < \infty$. The character of blowup is oscillatory, i.e. phases behave as $B_m'(\tau) \sim \log(\tau_* - \tau)$ (in the interior gauge).

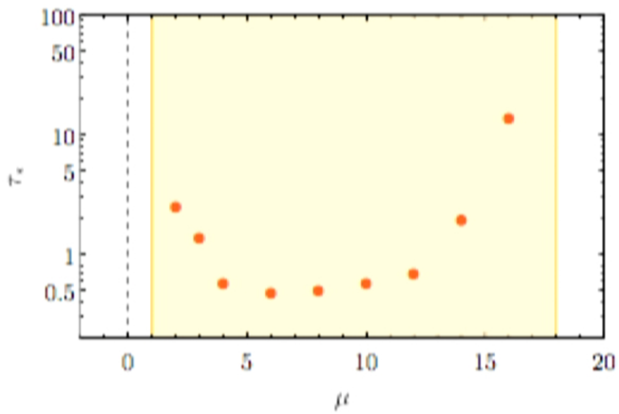
- Energy spectra: $|\alpha_j| \sim j^{-d/2} \Rightarrow E_j \sim j^{2-d}$. Dimensional argument [Bizoń&Rostworowski, '12?] , also [Freivogel&Yang, '15] .

Resonant approximation—two-mode initial data

But not all data leads to unbounded growth of higher Sobolev type norms.

$$\phi(0,x) = \varepsilon(\mu)\left(\frac{1}{\omega_0}e_0(x) + \frac{1}{\mu}e_1(x)\right), \quad \partial_t\phi(0,x) = 0\,.$$

Stability islands (of time-periodic solutions [M&Rostworowski, '13] , [Kim, '15] , [Fodor *et al.*, '15] ?)



- $\mu \to \infty, \alpha_0(\tau) = \varepsilon e^{-i\frac{T_0}{2\omega_0}\tau}$,
- $\mu \to 0, \alpha_1(\tau) = \varepsilon e^{-i\frac{T_1}{2\omega_1}\tau}$.
- Stationary solutions [Balasubramanian *et al.*, '14]

$$\alpha_j(\tau) = A_j e^{iB_j\tau},\; B_j = aj + b,$$

Stability [Green *et al.*, '15] , asymptotics [Craps *et al.*, '15] . Role in dynamics of generic initial conditions?

The same picture for *narrow/wide gaussians* [Buchel *et al.*, '13] , [M&Rostworowski, '13] .

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Resonant approxmiation—models

- Szegő system

$$i\alpha_n' = \sum_{j=0}^{\infty}\sum_{k=0}^{j+n} \bar{\alpha}_j \alpha_k \alpha_{n+j-k},$$

which is an exact system for cubic Szegő equation

$$i\partial_t u = \Pi\left(|u|^2 u\right), \quad u(t, e^{i\theta}) = \sum_{n=0}^{\infty} \alpha_n(t) e^{in\theta},$$

$\sum_{n=0}^{\infty} |\alpha_n|^2 < \infty$, $\Pi(\sum_{n=-\infty}^{\infty} \alpha_n(t)e^{in\theta}) := \sum_{n=0}^{\infty} \alpha_n(t)e^{in\theta}$. Has a Lax pair, finite-dimensional invariant subspaces and weakly turbulent solutions [Gérard&Grellier, '10,'12,'15] .

- Conformally invariant wave equation on $\mathbb{R} \times \mathbb{S}^3$ (a geometric PDE)

$$\Box_g \phi - \frac{1}{6} R(g)\phi - \phi^3 = 0, \quad \Rightarrow \quad \partial_t^2 v - \partial_x^2 v + \frac{v^3}{\sin^2 x} = 0,$$

for $g = -\mathrm{d}t^2 + \mathrm{d}x^2 + \sin^2 x \, \mathrm{d}\Omega^2$, and $v = \sin x \, \phi$. Its resonant approximation yields

$$i(n+1)\alpha_n' = \sum_{j=0}^{\infty}\sum_{k=0}^{j+n} \left[\min(n, j, k, n+j-k) + 1\right] \bar{\alpha}_j \alpha_k \alpha_{n+j-k},$$

which displays a number of dynamical parallels [Bizoń *et al.*, '16] .

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Time-periodic solutions—perturbative approach

- We search for solutions of the form ($|\varepsilon| \ll 1$)

$$\phi(t,x) = \varepsilon \cos(\omega_\gamma t) e_\gamma(x) + \mathcal{O}(\varepsilon^3)\,,$$

solution bifurcating from a single eigenmode γ.

- We make an ansatz for the ε-expansion

$$\phi(\tau,x;\varepsilon) = \sum_{\substack{\lambda\geq 1\\ \text{odd}}} \varepsilon^\lambda \phi_\lambda(\tau,x),$$

$$\delta(\tau,x;\varepsilon) = \sum_{\substack{\lambda\geq 2\\ \text{even}}} \varepsilon^\lambda \delta_\lambda(\tau,x), \qquad A(\tau,x;\varepsilon) = 1 - \sum_{\substack{\lambda\geq 2\\ \text{even}}} \varepsilon^\lambda A_\lambda(\tau,x),$$

where we rescaled the time variable

$$\tau = \Omega\, t, \qquad \Omega(\varepsilon) = \omega_\gamma + \sum_{\lambda\geq 1} \varepsilon^\lambda \xi_\lambda.$$

- Crucial part in the construction—solution to: $\left(\omega_\gamma^2\, \partial_\tau^2 - \hat{L}\right)\phi_\lambda = S_\lambda.$

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Time-periodic solutions—perturbative approach

- Assuming

$$\phi_\lambda(\tau,x) = \sum_{j\geq 0} \hat{\phi}_{\lambda,j}(\tau) e_j(x),$$

we get a set of forced harmonic oscillator equations

$$\left(\omega_\gamma^2\, \partial_\tau^2 - \omega_k^2\right) \hat{\phi}_{\lambda,k}(\tau) = \int_0^{\pi/2} S_\lambda(\tau,x) e_k(x) \tan^{d-1} x \,\mathrm{d}x,$$

with initial conditions $\hat{\phi}_{\lambda,k}(0) = c_{\lambda,k}$, and $\partial_\tau \hat{\phi}_{\lambda,k}(0) = \tilde{c}_{\lambda,k}$.

- We use the integration constants $\{c_{\lambda,k}, \tilde{c}_{\lambda,k}\}$ and frequency shift parameters ξ_λ to remove all of the resonant terms: $\cos\left(\left(\omega_k/\omega_\gamma\right)\tau\right), \sin\left(\left(\omega_k/\omega_\gamma\right)\tau\right)$.

- Regular structure for each dominant mode (d even)

$$\phi_\lambda(\tau,x) = \sum_{j=0}^{[(\lambda-1)(d+1)/2+\lambda\gamma]} \sum_{k=0}^{(\lambda-1)/2} \hat{\phi}_{\lambda,j,2k+1} \cos\left((2k+1)\,\tau\right) e_j(x),$$

(exceptional cancellation of resonant terms [Craps *et al.*, '14, '15]).

- Extension to massive case [Kim, '15] .

Time-periodic solutions—numerical construction

- Search for solution in a finite-dimensional subspace of some Hilbert space [trial functions $\cos(k\tau)\, e_j(x)$]

$$\mathsf{B}_{K,N} = \mathrm{span}\left\{\cos(k\tau)\, e_j(x) \mid k = 0, 1, \ldots, K,\, j = 0, 1, \ldots, N\right\}.$$

- Assuming ($\tau = \Omega\, t$)

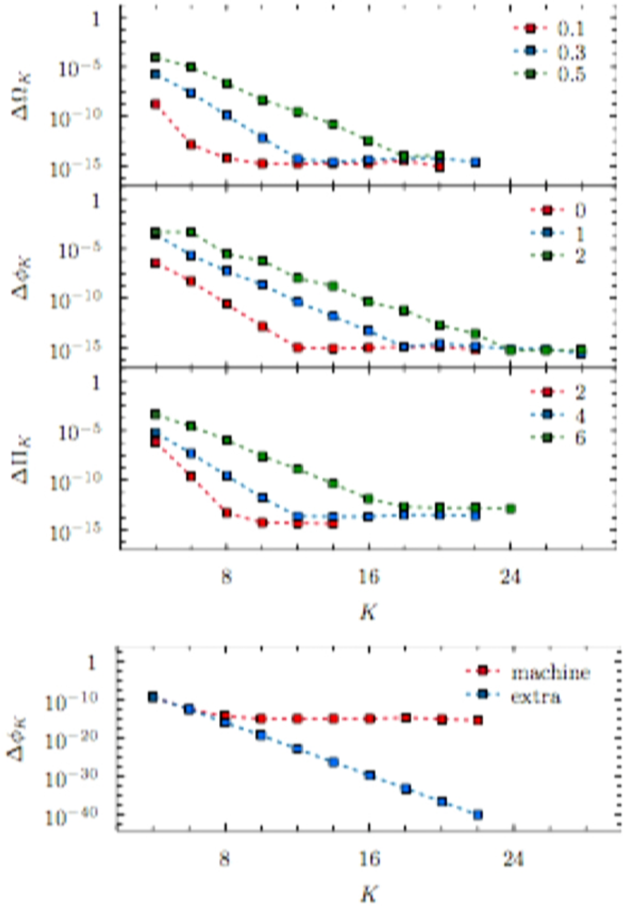
$$\mathcal{I}_{K,N}\phi(\tau, x) = \sum_{k=0}^{K-1} \sum_{j=0}^{N-1} \hat{\phi}_{k,j} \cos\left((2k+1)\,\tau\right)\, e_j(x),$$

$$\mathcal{I}_{K,N}\Pi(\tau, x) = \sum_{k=0}^{K-1} \sum_{j=0}^{N-1} \hat{\Pi}_{k,j} \sin\left((2k+1)\,\tau\right)\, e_j(x).$$

solution is represented by the set of $2KN + 1$ coefficients. Collocation approach [test functions $\delta(x - x_j)\delta(\tau - \tau_i)$], two equations on each grid point—use of time evolution code.

- One extra equation—the normalization condition e.g. $\left.(\phi \mid e_\gamma)\right|_{\tau=0} = \varepsilon$.
- Alternative approaches: [Boyd, '90], [Ambrose&Wilkening, '10], [Fodor *et al.*, '14].

Time-periodic solutions—structure



- Fast (spectral) convergence.
- For each family $\gamma = 0, 1, \ldots$, there is a finite range of ε for which solutions do exist [M&Rostworowski, '13] , [Kim, '15] .
- With perturbative series one can find an estimate for that limits with Padé resumation.
- Normalization condition problem (*central density*, $\phi(0,0) = \varepsilon$) [Fodor *et al.*, '14] .
- Upper bound on total mass of the solutions.
- Similar structure of standing waves (complex ϕ)

$$\phi(t,x) = e^{i\Omega t} f(x).$$

[Buchel *et al.*, '13] , [M&Rostworowski, '14] .

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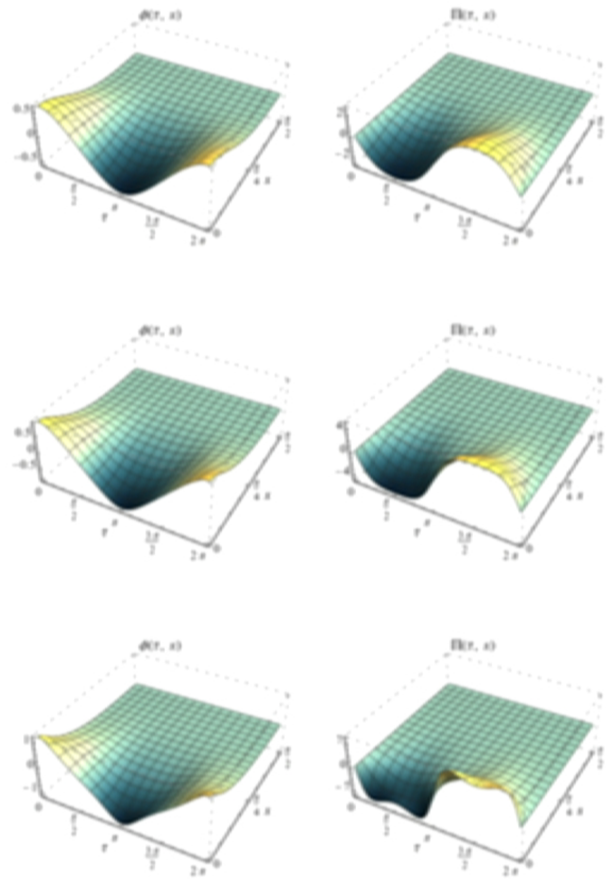

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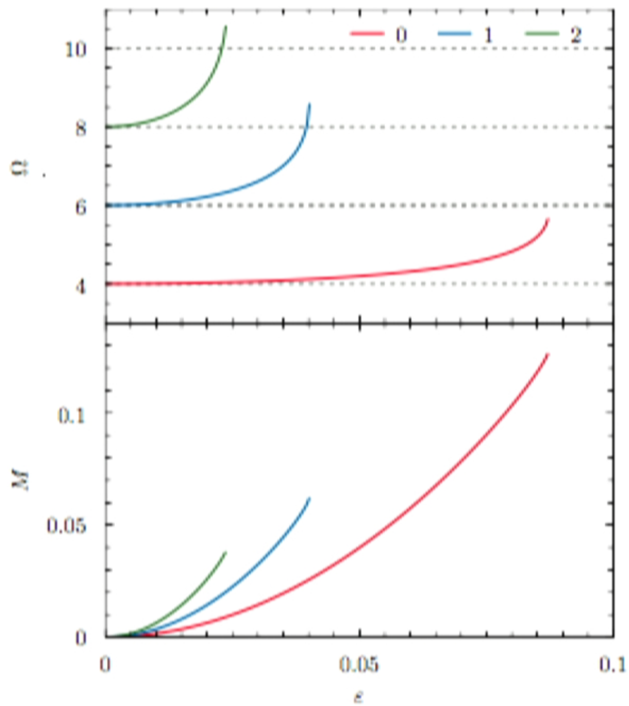

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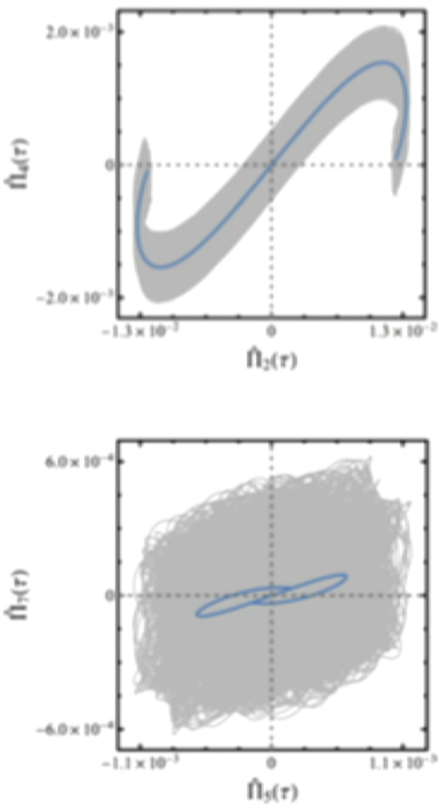
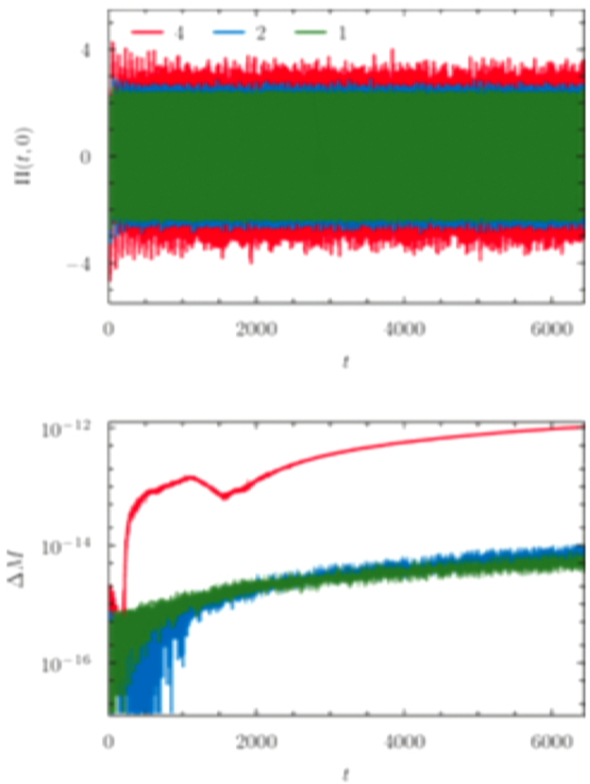
Time-periodic solutions—nonlinear stability



- MOL with pseudospectral discretization in space (dedicated schemes for even and odd d) and symplectic time stepping (Gauss-RK).
- Nonlinear stability for $|\varepsilon| < \varepsilon_{*}$.
- Long time evolution of generic perturbation imposed on time-periodic background—dispersive spectra [M&Rostworowski, '13, '14] .
- Unstable branch for $|\varepsilon| > \varepsilon_{*}$.
- Quality of numerical solution—convergence and conservation of mass.

Time-periodic solutions—nonlinear stability



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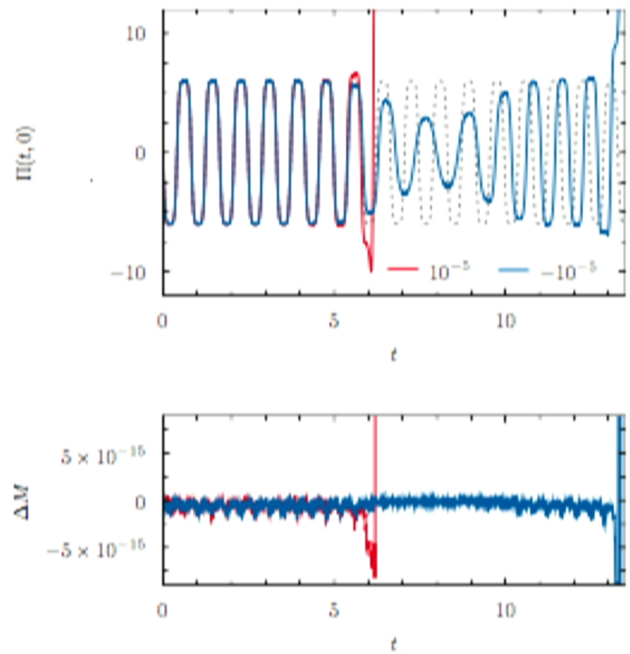

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Relaxing symmetry assumption

- Cohomogenity-two biaxial Bianchi IX ansatz [Bizoń *et al.*, '06] : turbulence [Bizoń&Rostworowski, '14] and time-periodic solutions [M, '14] .
- *Geons*—a time-periodic solutions solutions to $R_{ab} + \frac{3}{\ell^2} g_{ab} = 0$, with Killing vector $K = \partial_t + \Omega\, \partial_\varphi$ [Dias *et al.*, '12] , [Horowitz&Santos, '15] , [Dias&Santos, '16] . Using naïve Poincaré-Lindstedt method $g_{\mu\nu} = \bar{g}_{\mu\nu} + \sum_{k \geq 1} \varepsilon^k h_{\mu\nu}^{(k)}$

$$\Delta_{\hat{L}}(\bar{g}) h_{ab}^{(k)} = T_{ab}^{(k)} \left(h_{cd}^{(j \leq k-1)} \right),$$

(at third perturbative order) one will find *"(...) normal modes without a nonlinear extension and geons"*. Their numerical construction uses de Turck method [Headrick *et al.*, '10] , [Figueras *et al.*, '11] (based on harmonic formulation)

$$R_{ab} + \frac{3}{\ell^2} g_{ab} - \nabla_{(a}\, \xi_{b)} = 0,$$

where $\xi^a = g^{bc} \left(\Gamma^a{}_{bc} - \bar{\Gamma}^a{}_{bc} \right)$, with the Levi-Civita connection $\bar{\Gamma}$ of $\bar{g}$. Requires solution to nonlinear PDEs on compact domain.

Black holes in AdS

- The end state of instability? Schwarzschild-AdS candidate in spherical symmetry [Holzegel&Smulevici, '13] .
- Outside spherical symmetry Kerr-AdS [Cardoso&Dias, '04] ?
- Superradiant instability [Hawking&Reall, '00] , [Dias *et al.*, '15] , [Bosch *et al.*, '16] .
- Dynamics of asymptotically AdS solutions with black holes [Bantilan *et al.*, '12] , [Bantilan, '13] , [Bantilan&Romatschke, '15] .
- Stationary solutions with AdS asymptotics—higher dimensions and lumpy black holes, black rings, black belts, etc. Application of de Turck method [Dias *et al.*, '15] .
- Studies motivated by AdS/CFT, e.g. collisions of shocks [Chesler&Yaffe, '14] , and many more.

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Conclusions and questions

- So, is AdS stable?
- Extend studies of the resonant system. Study simple models—conformally invariant cubic wave equation on the Einstein cylinder [Bizoń *et al.*, '16] (low-dimensional invariant subspaces, a wealth of stationary states).
- How to transfer oscillatory blowup to the full system? How to interpret oscillatory singularity? Is it related to Choptuik's critical solution?
- Nontrivial (*complicated*) phase-space of solutions to the Einstein's equation with negative cosmological constant. How large the islands of stability are? Understand the role of stationary solutions in the dynamics [Green *et al.*, '15] . Explore the borderline between collapse and quasiperiodic motion.
- The resonant structure [Craps *et al.*, '14, '15] and its impact on nonlinear evolution. Is *Minkowski in a box* with reflecting BC a good model for EKG system with $\Lambda < 0$?
- Prove the existence of time-periodic solutions [Gentile *et al.*, '05] .
- Clash between different numerical approaches ([Balasubramanian *et al.*, '14] and [Bizoń&Rostworowski, '14] , see also [Deppe&Frey, '15]) shows that long-time evolution of asymptotically AdS solutions is particularly demanding.

Conclusions and questions

- Weak turbulence—common phenomena for nonlinear wave equations on bounded domains (NLS on torus [Colliander *et al.*, '10] , [Carles&Faou, '12]).
- Challenging mathematical problems, both for any attempts to rigorous proofs and numerical analysis. Meeting point of GR, theory of PDEs, turbulence, and HEP, makes it exciting field of research.