Title: Multi-Boundary Entanglement in Chern-Simons theory & Link Invariants

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Abstract: <p>We will study the entanglement structure of states in Chern-Simons (CS) theory defined on n-copies of a torus. We will focus on states created by performing the path-integral of CS theory on special 3-manifolds, namely link complements of n-component links in S^3. The corresponding entanglement entropies provide new framing independent link-invariants. In U(1)_k CS theory, we will give a general formula for the entanglement entropy across a bi-partition of a generic n-link into sub-links. In the non-Abelian case, we study various interesting 2 & 3-links including the Whitehead link & Borromean rings, both of which have non-trivial entanglement entropies. </p>

Multi-Boundary Entanglement in Chern-Simons theory and Link invariants

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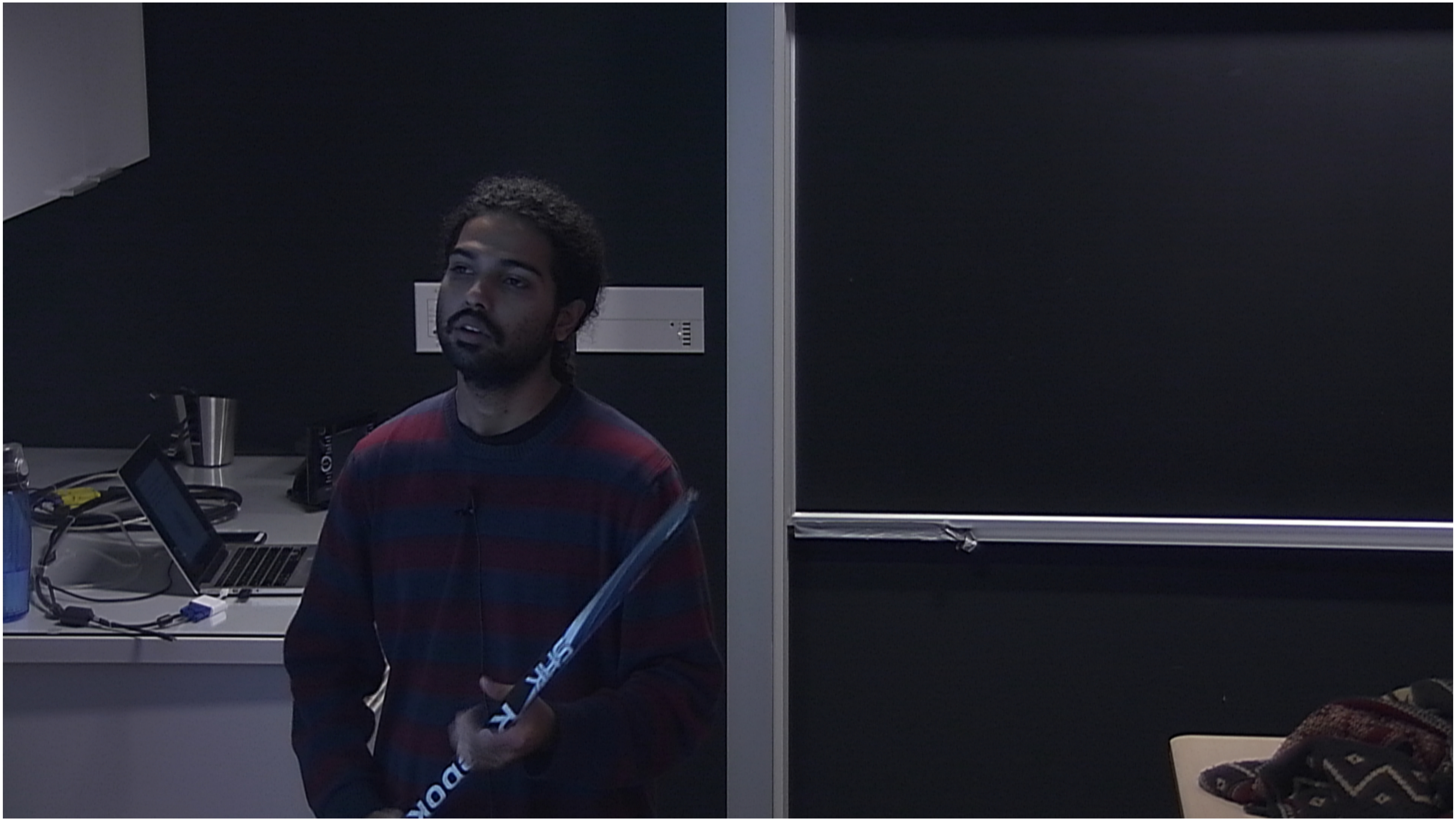
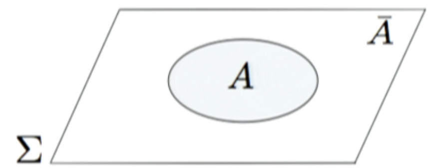


Entanglement Structure

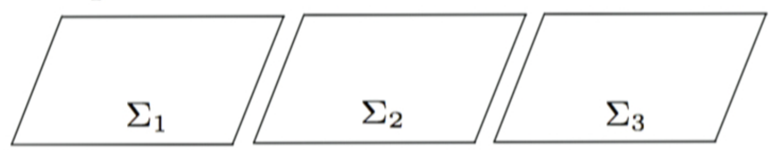
- Quantum information theory ideas are becoming central in quantum field theory.
- In AdS/CFT for instance, entanglement is widely expected to be crucial in understanding bulk emergence. [Van Ramsdonk '10, Lashkari et al '13, Faulkner et al '13.]
- Further, information inequalities, such as positivity & monotonicity of relative entropy, constrain field theories in interesting ways. [Casini & Huerta '04, Casini '08, Faulkner et al '16]
- An important question is to understand the possible patterns of entanglement in quantum field theory.
- However, classifying entanglement patterns in qubit systems is hard enough, let alone quantum field theories.

Multi-boundary Entanglement?

- One interesting case where we might be able to make some progress is multi-boundary entanglement.
- The usual setup for studying entanglement in field theories is to consider a QFT state on a connected spatial slice Σ, and then partition it into two or more regions.



- Instead, here we will consider a spatial slice which consists of n disconnected components.



- The Hilbert space is the tensor product

$$\mathcal{H}(\Sigma_1) \otimes \mathcal{H}(\Sigma_2) \otimes \mathcal{H}(\Sigma_3) \cdots$$

and we can ask for the entanglement between various components.

Multi-boundary Entanglement in AdS/CFT

- Holographic multi-boundary entanglement was studied recently in AdS_3/CFT_2 by [Balasubramanian et al '14, Marolf et al '15].

- The spatial slice in their case was n copies of a circle, and the states they considered were prepared by performing the Euclidean path integral on a Riemann surface with n circle boundaries, and lived in

$$\mathcal{H}(S^1) \otimes \mathcal{H}(S^1) \otimes \mathcal{H}(S^1) \cdots \otimes \mathcal{H}(S^1)$$

- The bulk duals are multiboundary wormhole geometries, where entanglement can be studied using the Ryu-Takayanagi formula.

Multi-boundary Entanglement in Quantum Field Theory

- The motivation for the present work was to study multiboundary entanglement within field theory.
- In general, this appears to be an involved calculation.
- However, one case where it can be done simply is a Topological Quantum Field Theory (TQFT). [Witten '88, Atiyah '88...]
- A TQFT is a quantum field theory which is sensitive only to topology, and not geometry (i.e. metric).
- We will focus on a particular TQFT namely Chern-Simons theory for groups $G = U(1)$ and $SU(2)$.

Chern-Simons theory

- The action for $d = 3$ Chern-Simons gauge theory at level k is given by

$$S_{CS}[A] = \frac{k}{4\pi} \int_M \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

- Bi-partite entanglement of connected spatial sections (i.e. single boundary component) in such theories was studied in [Kitaev & Preskill '05, Levin & Wen '05, Dong et al '08].

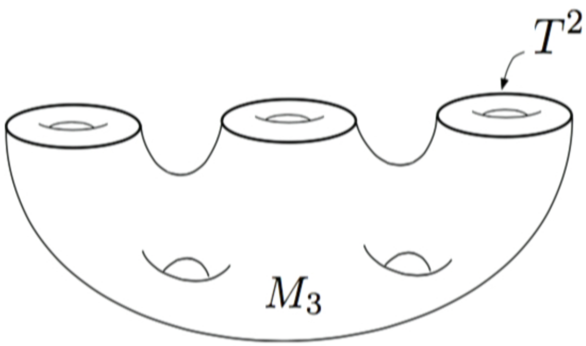
  By contrast, we are interested in multiboundary entanglement, which corresponds to states on a disconnected spatial slice:

$$\Sigma_1 \cup \Sigma_2 \cdots \cup \Sigma_n$$

- For simplicity, we will take all $\Sigma_i = T^2$.

Which states?

- The states we will consider are created by performing the path integral of Chern-Simons theory on 3-manifolds M_n with boundary consisting of n copies of T^2.



- For a given M_n of this form, the path-integral of Chern-Simons theory on M_n defines a state

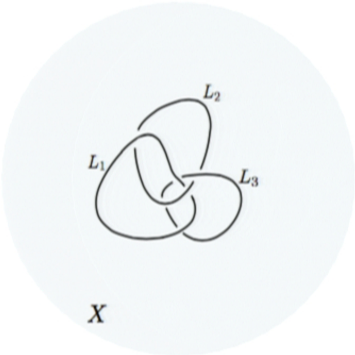
$$\Psi \in \mathcal{H}(T^2) \otimes \mathcal{H}(T^2) \otimes \ldots \otimes \mathcal{H}(T^2)$$

$$\Psi[A_{(0)}] = \int_{A|_\Sigma = A_{(0)}} [DA]\; e^{iS_{CS}[A]}$$

Link-Complements

- Clearly, the choice of M_n is far from unique. But there is a simple way to construct such manifolds.
- We start with a closed 3-manifold (i.e., a compact 3-manifold without boundary) X, and an n-component link in X

$$\mathcal{L}^n = L_1 \cup L_2 \cup \cdots L_n$$



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- It has the desired property, namely that

$$\partial M_n = T^2 \cup T^2 \cup \cdots \cup T^2.$$

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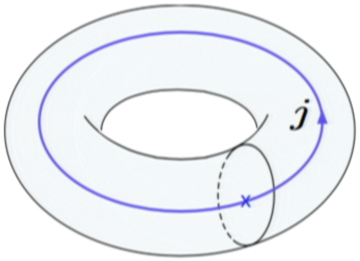
$$\partial M_n = T^2 \cup T^2 \cup \cdots \cup T^2.$$

- The path-integral of Chern-Simons theory on the link-complement assigns to a link $\mathcal{L}^n$ in S^3 a state $|\mathcal{L}^n\rangle \in \mathcal{H}(T^2)^{\otimes n}$.

The Hilbert space on a Torus

- Let us recall some details about the Hilbert space of CS theory on a torus [Witten '88].
- To construct a basis, we perform the path-integral on the "interior" solid torus, with a Wilson line in the representation R_j placed along the non-contractible cycle in the bulk. We call this state $|j\rangle$.



- The conjugate state $\langle j|$ is the path integral on the solid torus with a Wilson line in the conjugate representation R_j^*.

The Hilbert space on a Torus...

- By letting j run over all *integrable* representations, we obtain an orthonormal basis for $\mathcal{H}(T^2)$

$$\langle j|j'\rangle = \delta_{j,j'}$$

- For example if we take $G = SU(2)_k$, the integrable representations are labelled by their spin j for $j = 0, \frac{1}{2}, \cdots, \frac{k}{2}$.

Back to Link complements

- Now we can write the state prepared by path integration on the link complement $S^3 - \mathcal{L}^n$ as:

$$|\mathcal{L}^n\rangle = \sum_{j_1,\cdots,j_n} C_{\mathcal{L}^n}(j_1, j_2, \cdots j_n)|j_1\rangle \otimes |j_2\rangle \cdots \otimes |j_n\rangle$$

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- A little bit of thought shows that

$$C_{\mathcal{L}^n}(j_1, \cdots, j_n) = \left\langle W_{R^*_{j_1}}(L_1) \cdots W_{R^*_{j_n}}(L_n) \right\rangle_{S^3}$$

Entanglement Entropy

- We wish to study the entanglement structure of these states.
- So we partition the n-component link into an m-component sub-link $\mathcal{L}_A^m$ and the rest $\mathcal{L}_{\bar{A}}^{n-m}$.

- The reduced density matrix is obtained by tracing out $\bar{A}$:

$$\rho_A = \frac{1}{\langle \mathcal{L}_n | \mathcal{L}_n \rangle} \mathrm{Tr}_{\mathcal{L}_{\bar{A}}} |\mathcal{L}^n\rangle\langle\mathcal{L}^n|$$

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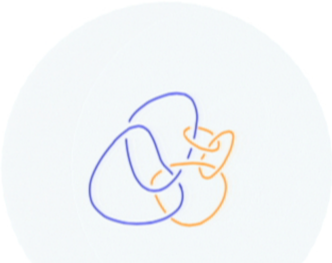

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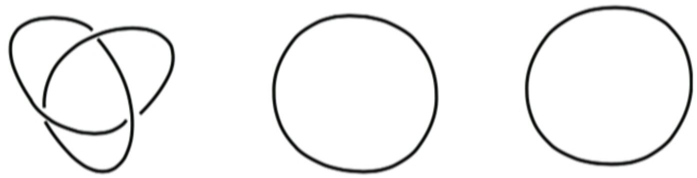
- The entanglement entropy is then defined as the Von Neumann entropy of this density matrix:

$$S_{EE} = -\mathrm{Tr}_{\mathcal{L}_A}(\rho_A \ln \rho_A)$$

The Unlink

- To see why these entropies are potentially interesting, we consider the simple but illuminating example of the unlink.
- So take $\mathcal{L}^n$ to be n un-linked knots.



- It is well-known that in this case the colored link-invariant factorizes [Witten '88]

$$|\mathcal{L}^n\rangle \propto |L_1\rangle \otimes |L_2\rangle \cdots \otimes |L_n\rangle$$

  and consequently all the entanglement entropies vanish. This is our first hint that quantum entanglement can detect topological linking.
- **Remark**: The entanglement entropies are all framing independent.

The Abelian case: $G = U(1)_k$

- For $U(1)$, we can compute the entropies in fair generality.

- Given a generic n-component link $\mathcal{L}^n$, the corresponding link state is given by [Witten '88]

$$|\mathcal{L}^n\rangle = \sum_{q_1, \cdots, q_n} \exp\left(\frac{2\pi i}{k} \sum_{i<j} q_i q_j \ell_{i,j}\right) |q_1\rangle \otimes |q_2\rangle \cdots \otimes |q_n\rangle$$

where $q_i \in \mathbb{Z}_k$ are charges and $\ell_{i,j} \in \mathbb{Z}_k$ is the Gauss linking number between the circles L_i and L_j.

Warm-up: 2-component links

- For a two component link $\mathcal{L}^2$, the state is given by

$$|\mathcal{L}^2\rangle = \frac{1}{k}\sum_{q_1,q_2} e^{\frac{2\pi i q_1 q_2}{k}\ell_{1,2}}|q_1\rangle \otimes |q_2\rangle$$

- If we trace out the second factor, then the reduced density matrix is given by

$$\rho_1 = \mathrm{Tr}_{L_2}|\mathcal{L}^2\rangle\langle\mathcal{L}^2| = \frac{1}{k}\sum_{q_1,q_1'} \eta_{q_1,q_1'}(k,\ell_{1,2})\,|q_1\rangle\langle q_1'|$$

where

$$\eta_{q_1,q_1'}(k,\ell_{12}) \equiv \begin{cases} 1 & \cdots \quad \ell_{12}(q_1 - q_1') = 0 \,(\mathrm{mod}\, k) \\ 0 & \cdots \quad \ell_{12}(q_1 - q_1') \neq 0 \,(\mathrm{mod}\, k) \end{cases}$$

Warm-up: 2-component links...

- From here, it is straightforward to compute the Von Neumann entropy, either by finding the spectrum of ρ_1 or by using the replica trick. We find

$$S_{EE} = \ln\left(\frac{k}{\gcd(k, \ell_{1,2})}\right)$$

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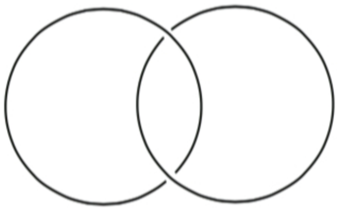

- As an example, note that the Hopf link is maximally entangled. We will encounter this fact more generally in the non-Abelian case.

n-component links

- Let us now consider a general n-link $\mathcal{L}^n$, and bi-partition into sublinks

$$\mathcal{L}_A^m = L_1 \cup L_2 \cup \cdots \cup L_m, \quad \mathcal{L}_{\bar{A}}^{n-m} = L_{m+1} \cup L_{m+2} \cup \cdots \cup L_n$$

- To state the answer for the entropy, we first define the linking matrix between the two sublinks

$$\boldsymbol{G} = \begin{pmatrix} \ell_{1,m+1} & \ell_{2,m+1} & \cdots & \ell_{m,m+1} \\ \ell_{1,m+2} & \ell_{2,m+2} & \cdots & \ell_{m,m+2} \\ \vdots & \vdots & & \vdots \\ \ell_{1,n} & \ell_{2,n} & \cdots & \ell_{m,n} \end{pmatrix}$$

- Further, let us define $|\ker \boldsymbol{G}|$ as the number of solutions $\vec{x} \in \mathbb{Z}_k^m$ to the system of congruences (or equivalently Diophantine equations)

$$\boldsymbol{G} \cdot \vec{x} = 0 \,(\mathrm{mod}\, k)$$

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n-component links...

- The entangement entropy is given by

$$S_{EE} = \ln\left(\frac{k^m}{|\ker \boldsymbol{G}|}\right)$$

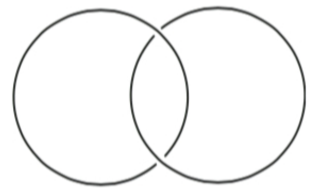
- **Corollary**: S_{EE} for a bi-partition vanishes if and only if the linking matrix $\boldsymbol{G}$ between the two sub-links vanishes $(\bmod\, k)$.
- In this sense, Abelian quantum entanglement detects Gauss linking between the sublinks $(\bmod\, k)$.

Non-Abelian case: $G = SU(2)_k$

- In contrast to $U(1)_k$, the calculation of non-Abelian entropies cannot be carried out in complete generality.
- So we will work out the entropies for several interesting two- and three-component links for $G = SU(2)_k$.

The Hopf Link

- The simplest non-trivial two-component link is the Hopf link.



- The corresponding link state is given by

$$|\text{Hopf}\rangle = \sum_{j_1, j_2} \mathcal{S}_{j_1, j_2} \, |j_1\rangle \otimes |j_2\rangle$$

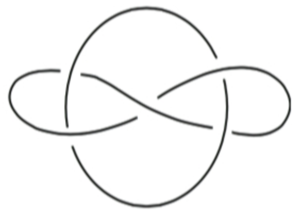
  where $\mathcal{S}$ is a unitary matrix which generates the modular transformation $\tau \to -\frac{1}{\tau}$ on the torus.
- Since $\mathcal{S}$ is unitary, the entanglement entropy is maximal:

$$S_{EE} = \ln(k+1)$$

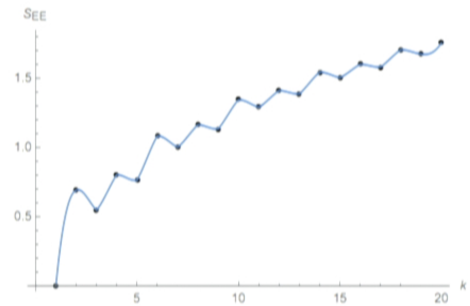
- In this sense, the Hopf link is the analog of the Bell pair from quantum information theory.

The Whitehead Link

- The Whitehead link is an example of a non-trivial link which has zero Gauss linking number.

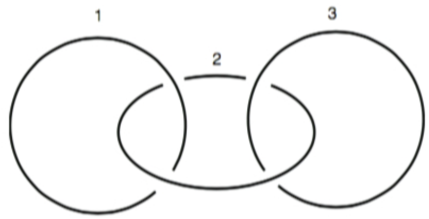


- The entropy can be computed using properties of chiral $SU(2)_k$ WZW conformal blocks, or by using a formula due to K. Habiro.



The 3-chain

- Let us now consider the 3-chain



- In this case all the entropies are equal and given by

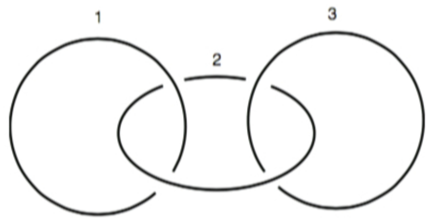
$$S_{EE} = -\sum_i p_i \ln p_i, \quad p_i = \frac{d_i^{-2}}{\sum_j d_j^{-2}}$$

where $d_j = \frac{S_{0j}}{S_{00}}$ is the quantum dimension of the representation j.

- Further, tracing out any of the links leaves a separable reduced density matrix on the other two links. In this sense, the above link has a "GHZ-like" entanglement structure.

Projected Entropies

- The entropy is a fairly coarse invariant.



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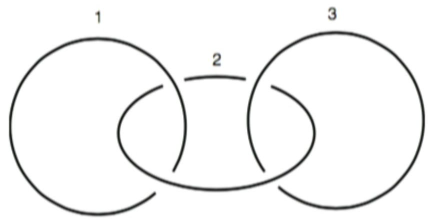
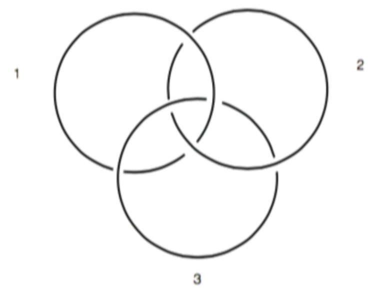


- For instance, it doesn't differentiate the topological linking between L_1 and L_2 or L_1 and L_3.
- Of course, the quantum state has much more fine-grained information. One simple-minded probe is the projector

$$\boldsymbol{P}(L_\alpha) = |0\rangle\langle 0|_{L_\alpha}$$

6^3_3

- The next 3-component link we look at is called 6^3_3 (in Rolfsen notation)



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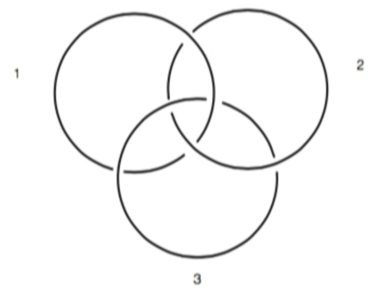

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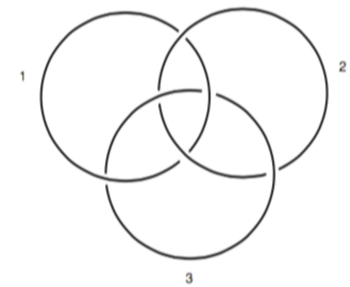
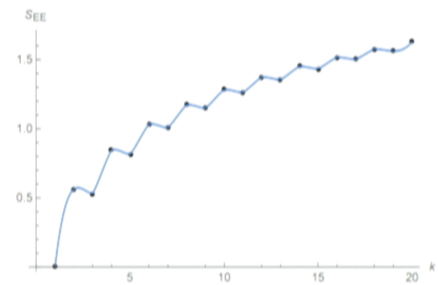
- The 6^3_3 state also has GHZ-like entanglement.
- We can use projected entropies or relative entropy to distinguish it from the 3-chain.

Borromean rings

- Finally we consider Borromean rings

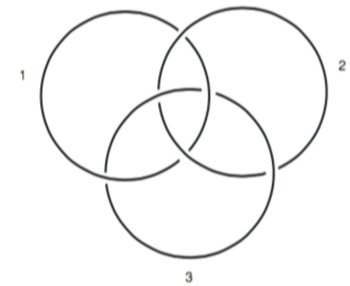


- In this case the entropy can be computed using properties of conformal blocks or from a formula due to Habiro:



Borromean rings...

- The Borromean rings have trivial Gauss linking, but they nevertheless have non-Abelian entanglement.



- Further, they have the special property that if we erase any circle from the link, the remaining two circles become unlinked; such links are called *Brunnian* links.
- This latter property can be cast as the statement that all the projected entropies are zero:

$$S_{EE}(\boldsymbol{P}(L_\alpha)|BR\rangle) = 0.$$

- Interestingly, Borromean rings have a W-like entanglement structure.

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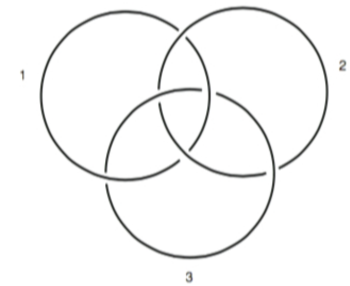

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Summary

- In summary, we have interpreted multi-boundary entanglement entropy of a class of states in Chern-Simons theory as a new link invariant.
- It is an old idea that quantum entanglement should be interpreted in terms of topological entanglement in links [Aravind '97, Kauffman et al '02]
- We have argued here that Chern-Simons theory is the right framework to realize this idea.
- A promising future direction is to study the large k limit of the entropy.
- In particular, it would be interesting if there is a geometric interpretation for the entropy as $k \to \infty$, in the spirit of the volume conjecture [Kashaev '97, Gukov '05].