Title: Non-Relativistic Scale Anomalies and Geometry

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Abstract: <p>I will discuss the coupling of non-relativistic field theories to curved spacetime, and develop a framework for analyzing the possible structure of non-relativistic (Lifshitz) scale anomalies using a cohomological formulation of the Wess-Zumino consistency condition. I will compare between cases with or without Galilean boost symmetry, and between cases with or without an equal time foliation of spacetime. In 2+1 dimensions with a dynamical critical exponent of z=2, the absence of a foliation structure allows for an A-type anomaly in the Galilean case, but also introduces the possibility of an infinite set of B-type anomalies.</p>

<p>I will also derive Ward identities for flat space correlation functions in Lifshitz field theories, and develop a method for calculating Lifshitz anomaly coefficients from these correlation functions using split dimensional regularization.</p>

Non-Relativistic Scale Anomalies and Geometry

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November 22, 2016

Talk at Perimeter Institute
Based On:
arXiv: 1410.5831, 1601.06795 with Shira Chapman, Yaron Oz
arXiv: To appear soon, with Yaron Oz, Avia Raviv-Moshe

Outline

Lifshitz Scaling Symmetry

I will consider **non-relativistic** field theories in $d+1$ spacetime dimensions, which are invariant under:

- Lifshitz scaling D:

$$t \to \lambda^z t, \qquad x^i \to \lambda x^i, \qquad i = 1, \dots, d,$$

  where z - the dynamical critical exponent,
- Time translations $H = i\partial_t$,
- Space translations $P_i = -i\partial_i$,
- Space rotations $L_{ij} = -i[x_i\partial_j - x_j\partial_i]$,

With the usual commutation relations, as well as:

$$[D, H] = izH, \qquad [D, P_i] = iP_i, \qquad [D, L_{ij}] = 0.$$

Lifshitz Scaling Symmetry

Occurs in **quantum critical points** of condensed matter systems:

- Critical points of zero temperature phase transitions induced by tuning an external parameter,
- Described by an effective quantum field theory with Lifshitz scaling symmetry,
- Believed to be the cause of ‘strange metal’ phases in certain high T_c superconductors and heavy fermion compounds.

Galilean Field Theories

I will also consider Lifshitz field theories which are symmetric under the full Galilean group, that contains in addition:

- Galilean boosts K_i:

$$x^i \to x^i + v^i t, \qquad t \to t,$$

- Global $U(1)$ symmetry M corresponding to conserved particle number.

Along with the commutation relations:

$$[K_i, K_j] = 0, \qquad [K_i, H] = iP_i, \qquad [K_i, P_j] = iM\delta_{ij},$$
$$[D, K_i] = i(1-z)K_i, \qquad [D, M] = i(2-z)M.$$

Note that:

- M is a central extension of the Galilean algebra,
- M has no Lifshitz dimension only for $z = 2$.

Examples of Lifshitz Field Theories

- Free real scalar for general d and even z (no Galilean invariance):

$$S = \int dt d^d x \left[\frac{1}{2}(\partial_t \phi)^2 - \frac{\kappa}{2}((\nabla^2)^{\frac{z}{2}} \phi)^2 \right],$$

  - κ is some parameter (with Lifshitz dimension 0).
  - Invariant under Lifshitz scaling:

$$t \to \lambda^z t, \qquad x^i \to \lambda x^i, \qquad \phi \to \lambda^{\frac{z-d}{2}} \phi.$$

Conformal / Weyl Anomalies - Review

- In relativistic conformal theories, the Ward identity corresponding to scale (or Weyl) symmetry is for the stress-energy tensor to be traceless:

$$T^{\mu}_{\mu} = 0$$

- In even spacetime dimension D, this identity is violated by the quantum theory when defined on a curved manifold. This is known as the **conformal / Weyl / trace anomaly**.
- In general:

$$< T^{\mu}_{\mu} >= \mathcal{A} = -(-1)^{D/2} a E_D + \sum_i c_i I_i,$$

  - E_D is the Euler density of the manifold (A-type anomaly),
  - I_i are Weyl invariant densities (B-type anomalies),
  - a, c_i are coefficients that depend on the content of the theory.

Conformal / Weyl Anomalies - Review

- These terms appear when the stress-tensor is required to be **conserved** and **symmetric**:

$$\nabla_\mu T^{\mu\nu} = 0, \qquad T_{[\mu\nu]} = 0.$$

- The curved space effective action $W[g_{\mu\nu}]$ is not Weyl-invariant:

$$\delta_\sigma^W W = \int \sqrt{-g}\sigma\mathcal{A}$$

- The anomalous terms also appear as contact terms in **flat space correlation functions** involving T_μ^μ.
- The possible structure of terms in the anomaly can be determined from the **Wess-Zumino consistency condition**. This can be formulated as a cohomological problem. [Bonora et al 1983]

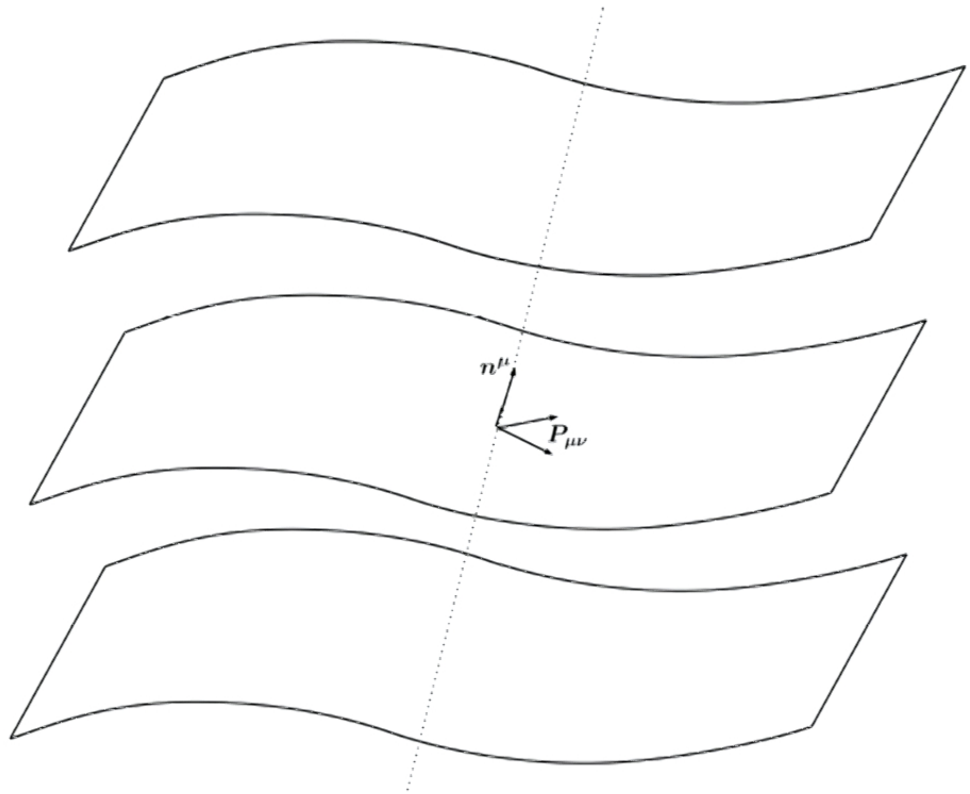
Non-Relativistic Scale Anomalies

- For a non-relativistic theory, the Ward identity corresponding to Lifshitz scale symmetry is:

$$zT^0_0 + T^i_i = 0,$$

- As in the relativistic case, this identity can be violated due to an anomaly.
- **Goal:** Find the general form for such scale anomalies in non-relativistic theories, and calculate their coefficients.

Coupling to Curved Spacetime



Background Structure

- Given a $d+1$ dimensional manifold, the following background structures are required:
  1. A vector $v^\mu = \partial_t$ that contains information about the direction and units of time at each point,
  2. A 1-form t_μ such that $u^\mu t_\mu = 0 \Leftrightarrow u^\mu$ is in a spatial direction. Obviously we must require $t_\mu v^\mu \neq 0$ everywhere.
     - Alternatively normalized n_μ such that $n_\mu v^\mu = 1$.
  3. A spatial metric $P_{\mu\nu}$ defined such that $P_{\mu\nu} v^\mu = 0$.
- These 3 structures are equivalent to requiring the **1-form** t_μ and a **metric** defined as: $g_{\mu\nu} = P_{\mu\nu} - n_\mu n_\nu$.
- In this notation, $n^\mu = g^{\mu\nu} n_\nu = -v^\mu$.
- Alternatively, we can use ${e^a}_\mu$ and t^a in vielbein formalism.

Foliation Structure

- Globally on the manifold, the integral curves of n^μ define a global notion of time.
- However, the 1-form t_μ **doesn't** necessarily define global equal-time slices over the manifold.
- **The Frobenius theorem:** t_μ induces a foliation on the manifold ($\equiv$ hypersurface orthogonal) if and only if:

$$t \wedge dt = 0 \qquad (t_{[\alpha}\partial_\beta t_{\gamma]} = 0)$$

- When the condition is satisfied: $t_\mu = f\partial_\mu h$, where $h = const$ defines equal-time slices.
- Note that using ADM decomposition for the background metric implies the Frobenius condition!
- Should the Frobenius condition be **assumed**?

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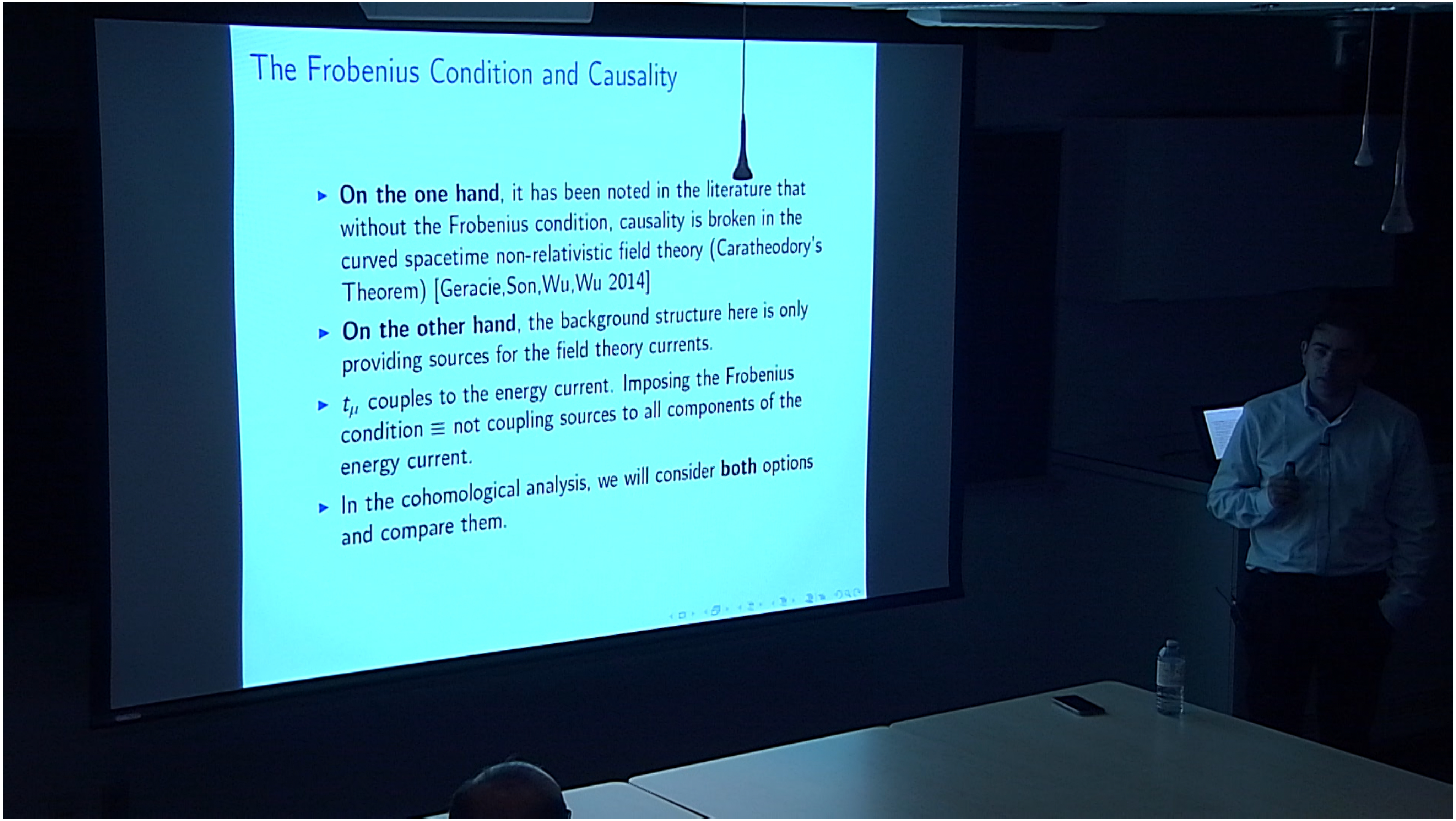
The Frobenius Condition and Causality
On the one hand, it has been noted in the literature that without the Frobenius condition, causality is broken in the curved spacetime non-relativistic field theory (Caratheodory's Theorem) [Geracie,Son,Wu,Wu 2014]
On the other hand, the background structure here is only providing sources for the field theory currents.
t_μ couples to the energy current. Imposing the Frobenius condition ≡ not coupling sources to all components of the energy current.
In the cohomological analysis, we will consider both options and compare them.

Some Geometric Implications

- For solving the cohomological problem, it is convenient to decompose any tensor to components which are either tangent or normal to the spatial directions.
- We can decompose $\nabla_\alpha n_\beta$ as:

$$\nabla_\alpha n_\beta = (K_S)_{\alpha\beta} + (K_A)_{\alpha\beta} - a_\beta n_\alpha,$$

  where $(K_S)_{\mu\nu}$, $(K_A)_{\mu\nu}$ and a_α are space tangent:
  - $(K_S)_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu}$ is symmetric,
  - $(K_A)_{\mu\nu} = P_\mu^{\mu'} P_\nu^{\nu'} \nabla_{[\mu'} n_{\nu']}$ is anti-symmetric,
  - $a_\alpha = \mathcal{L}_n n_\alpha$ is the acceleration vector.
- When the **Frobenius condition** is satisfied:
  - $(K_S)_{\mu\nu}$ is the extrinsic curvature of the induced foliation,
  - $(K_A)_{\mu\nu} = 0$.

Some Geometric Implications

- If we define the space tangent derivative by:

$$\widetilde{\nabla}_\mu \widetilde{T}_{\alpha\beta\ldots} \equiv P_\mu^{\mu'} P_\alpha^{\alpha'} P_\beta^{\beta'} \ldots \nabla_{\mu'} \widetilde{T}_{\alpha'\beta'\ldots},$$

- The commutation of two space tangent derivatives:

$$\left[\widetilde{\nabla}_\mu, \widetilde{\nabla}_\nu\right] \widetilde{V}_\alpha = \widetilde{R}_{\alpha\rho\mu\nu} \widetilde{V}^\rho + 2K^A_{\mu\nu} \mathcal{L}_n \widetilde{V}_\alpha,$$

- In the **Frobenius case**, $\widetilde{R}_{\alpha\rho\mu\nu}$ is the intrinsic curvature of the foliation.
- Generally it **does not** have all of the regular symmetries of the Riemann tensor.
- We can define another tensor $\widehat{R}_{\alpha\rho\mu\nu}$ that has all of them except for the second Bianchi identity.

Adding Galilean Symmetry

- In the case of a field theory invariant under the full Galilean group, there is an added $U(1)$ symmetry and a corresponding conserved particle number current.
- Therefore when coupling to curved spacetime we add a background gauge field A_μ that couples to the conserved particle number current.
- The gauge invariant data is encoded in the field-strength tensor $F_{\mu\nu}$, or alternatively in the electric and magnetic fields (which are space tangent):

$$E_\mu \equiv F_{\mu\nu} n^\nu, \qquad B_{\mu\nu} \equiv P_\mu^{\mu'} P_\nu^{\nu'} F_{\mu'\nu'},$$

Adding Galilean Symmetry

- The full Galilean symmetry means that we have to consider two additional symmetries in curved spacetime:
  - Flat space global $U(1) \to U(1)$ **gauge** symmetry in curved spacetime,
  - Flat space Galilean boosts $\to$ **Milne boost** symmetry in curved spacetime.
- We have to restrict the various terms to ones which are gauge and Milne boost invariant.
- Note: For our purposes, this structure is equivalent to the Newton-Cartan geometry.

Symmetries

The flat space symmetries translate to local symmetries in curved spacetime.

1. TPD Invariance

Rotation invariance $\rightarrow$ **time-direction-preserving diffeomorphisms** (TPD) invariance:

- Diffeomorphisms with parameter ξ^μ such that $\mathcal{L}_\xi\, t_\alpha \propto t_\alpha$.
- When the Frobenius condition is satisfied, amounts to **foliation-preserving** diffeomorphisms of the form:

$$t \to f(t), \qquad x \to g(x,t)$$

- Can be extended to any ξ^μ by having t_α transform appropriately:

$$\delta^D_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \quad \delta^D_\xi t_\alpha = \mathcal{L}_\xi\, t_\alpha = \xi^\beta \nabla_\beta t_\alpha + \nabla_\alpha \xi^\beta t_\beta,$$
$$\delta^D_\xi A_\alpha = \mathcal{L}_\xi\, A_\alpha = \xi^\beta \nabla_\beta A_\alpha + \nabla_\alpha \xi^\beta A_\beta.$$

Symmetries

2. Anisotropic Weyl Invariance

Lifshitz scaling invariance → **anisotropic Weyl** invariance:

$$\delta_\sigma^W t_\alpha = 0,$$
$$\delta_\sigma^W (g^{\alpha\beta} t_\alpha t_\beta) = -2\sigma z (g^{\alpha\beta} t_\alpha t_\beta),$$
$$\delta_\sigma^W P_{\alpha\beta} = 2\sigma P_{\alpha\beta},$$
$$\delta_\sigma^W n_\alpha = z\sigma n_\alpha, \qquad \delta_\sigma^W n^\alpha = -z\sigma n^\alpha,$$
$$\delta_\sigma^W A_\mu = (2-z)\sigma A_\mu.$$

(The weight of the gauge field is determined from the Galilean algebra.)

Symmetries

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Lifshitz scaling invariance $\rightarrow$ **anisotropic Weyl** invariance:

$$
\begin{aligned}
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&\delta_\sigma^W n_\alpha = z\sigma n_\alpha, \qquad \delta_\sigma^W n^\alpha = -z\sigma n^\alpha, \\
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\end{aligned}
$$

(The weight of the gauge field is determined from the Galilean algebra.)

Symmetries (The Galilean Case)

3. Milne Boost Invariance

Galilean boost invariance in flat space:

$$\partial_i \to \partial_i, \qquad \partial_t \to \partial_t - v^i \partial_i,$$

translates to local **Milne boost** invariance in curved spacetime:

$$\delta^B_W n^\mu = W^\mu, \qquad \delta^B_W n_\mu = 0,$$
$$\delta^B_W A_\mu = -W_\mu, \qquad \delta^B_W g_{\mu\nu} = W_\mu n_\nu + W_\nu n_\mu,$$

where W^μ is a space tangent ($W^\mu n_\mu = 0$) parameter of the transformation.

4. Gauge Invariance

Global $U(1)$ Invariance (particle number) $\to$ local gauge invariance:

$$\delta^G_\Lambda A_\mu = \partial_\mu \Lambda, \qquad \delta^G_\Lambda g_{\mu\nu} = \delta^G_\Lambda t_\mu = 0.$$

Currents and Ward Identities

Given the action $S(g_{\mu\nu}, t_\alpha, A_\alpha, \{\phi\})$, define the currents:

- Stress-energy tensor: $T^{\mu\nu}_{(g)} \equiv \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}$, $\quad T^{\mu\nu}_{(e)} \equiv \frac{1}{e} e^{a\nu} \frac{\delta S}{\delta e^a{}_\mu}$,
- $J^\alpha \equiv \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta t_\alpha}$,
- Mass current: $J^\alpha_m \equiv \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta A_\alpha}$.

Note that $T^{\mu\nu}_{(g)}$ and $T^{\mu\nu}_{(e)}$ are related by: $T^{\mu\nu}_{(e)} = T^{\mu\nu}_{(g)} + J^\mu t^\nu$.

These currents satisfy the following Ward identities:

- From TPD invariance:

$$\nabla_\mu T^\mu_{(g)\nu} = J^\mu \nabla_\nu t_\mu - \nabla_\mu (J^\mu t_\nu) + J^\mu_m F_{\nu\mu},$$

$$\nabla_\mu T_{(e)}{}^\mu{}_\nu = J^\mu \nabla_\nu t_\mu + J^\mu_m F_{\nu\mu}, \qquad T_{(e)[\mu\nu]} = J_{[\mu} t_{\nu]}.$$

Currents and Ward Identities

- From anisotropic Weyl invariance:

$$D \equiv T^{\mu\nu}_{(g)} P_{\mu\nu} - z T^{\mu\nu}_{(g)} n_\mu n_\nu + \frac{2-z}{2} J^\mu_m A_\mu = 0$$

- From Milne boost invariance:

$$P_{\nu\alpha} T^{\mu\nu}_{(e)} n_\mu = P_{\alpha\beta} J^\beta_m,$$

($\Rightarrow$ momentum density = particle number current).

- From gauge invariance:

$$\nabla_\mu J^\mu_m = 0.$$

- Our goal is to find the possible anomalous corrections to D, assuming the other Ward identities are not anomalous.

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- Our goal is to find the possible anomalous corrections to D, assuming the other Ward identities are not anomalous.

Structure of Non-Relativistic Scale Anomalies

Based On: arXiv: 1410.5831, 1601.06795
with Shira Chapman, Yaron Oz

Goal:

- Find the general form for scale anomalies in non-relativistic theories **with and without** Galilean boost invariance as allowed by the Wess-Zumino consistency condition.
- Compare the cases with and without a foliation structure.

The Cohomological Problem

- Given the quantum effective action in curved spacetime $W(g_{\mu\nu}, t_\alpha, A_\alpha)$, the anomalous Ward identity corresponding to the anisotropic Weyl symmetry is:

$$\delta_\sigma^W W = \mathcal{A}_\sigma,$$

where $\mathcal{A}_\sigma$ is a local functional of the background fields and σ.
- The Wess-Zumino consistency condition takes the form:

$$\delta_{\sigma_1}^W \mathcal{A}_{\sigma_2} - \delta_{\sigma_2}^W \mathcal{A}_{\sigma_1} = 0.$$

- A **trivial solution** of the form $\mathcal{A}_\sigma = \delta_\sigma^W \mathcal{G}$ where $\mathcal{G}$ is a local functional of the background fields can be cancelled by appropriate counterterms.
- We are looking for non-trivial solutions to the consistency condition.

The Cohomological Problem

. An equivalent cohomological BRST-like description:

- Replace the parameter σ by a Grassmannian ghost.
- Define δ_σ^W such that it's **nilpotent**: $(\delta_\sigma^W)^2 = 0$.
- The Wess-Zumino condition takes the form: $\delta_\sigma^W A_\sigma = 0$.
- Solutions are **cocycles** of δ_σ^W (δ_σ^W-closed).
- Trivial solutions are **coboundaries** of δ_σ^W (δ_σ^W-exact).
- Possible anomalies are local terms which are cocycles but not coboundaries:

$$\delta_\sigma^W A_\sigma = 0, \qquad A_\sigma \neq \delta_\sigma^W G$$

where A_σ, G are local and invariant under the other symmetries.

$\Rightarrow$ We are looking for the **relative cohomology** of δ_σ^W with respect to TPD, Milne boost and gauge transformations.

Basic Tangent Tensors

Basic Tangent Tensor		(n_T, n_S, n_ϵ)
Acceleration	$a_\mu \equiv \mathcal{L}_n n_\mu = n^\nu \nabla_\nu n_\mu$	$(0,1,0)$
"Extrinsic curvature"	$K^S_{\mu\nu}$	$(1,0,0)$
	$K^A_{\mu\nu}$	$(-\mathbf{1},2,0)$
"Intrinsic curvature"	$\widehat{R}_{\mu\nu\rho\sigma}$	$(0,2,0)$
Spatial Levi-Civita tensor	$\tilde{\epsilon}^{\mu\nu\rho\ldots} = n_\alpha \epsilon^{\alpha\mu\nu\rho\ldots}$	$(0,0,1)$
Temporal derivative	$\mathcal{L}_n$	$(1,0,0)$
Space tangent derivative	$\widetilde{\nabla}_\mu$	$(0,1,0)$
The electric field	E_μ	$(2,-1,0)$
The magnetic field	$B_{\mu\nu}$	$(1,0,0)$

- TPD invariant terms in the cohomology can be built from a set of **basic space tangent tensors**.
- Each has a Lifshitz dimension $-(zn_T + n_S)$ and parity $(-1)^{n_\epsilon}$.
- In the Frobenius case: $n_T, n_S =$ Total number of time / space derivatives.
- Generally $n_D = n_T + n_S =$ Total number of derivatives.

Classification by Sectors

- The cohomological problem can be solved for each (n_T, n_S, n_ϵ) sector separately (the sectors don't "mix").
- The possible sectors are given by the conditions:

$$z n_T + n_S = d + z,$$
$$n_S + d n_\epsilon \quad \text{is even.}$$

- When the Frobenius condition is not satisfied $(K^A_{\mu\nu} \neq 0)$ and $z \geq 2$, there's an infinite number of sectors! $\rightarrow$ Possible to have an infinite set of independent anomalies.

Scale Anomalies for $2+1$ Dimensions $z=2$

We consider 4 cases:

1. With Frobenius and Galilean boost invariance,
2. With Frobenius and no Galilean boost invariance,
3. Without Frobenius and with Galilean boost invariance,
4. Without Frobenius and no Galilean boost invariance.

The conditions for the possible sectors here are:

$$\begin{aligned} 2n_T + n_S &= 4, \\ n_S \quad &\text{is even.} \end{aligned}$$

Some Definitions

For calculations in $2+1$ dimensions we define:

$$B_{\mu\nu} \equiv B\tilde{\epsilon}_{\mu\nu}, \qquad K^A_{\mu\nu} \equiv K_A\tilde{\epsilon}_{\mu\nu}, \qquad \tilde{K}^S_{\alpha\beta} \equiv \tilde{\epsilon}_{(\alpha}{}^{\gamma} K^S_{\beta)\gamma}.$$

1. With Frobenius and Galilean boost invariance

- This case contains 6 sectors: $(2,0,0)$, $(2,0,1)$, $(1,2,0)$, $(1,2,1)$, $(0,4,0)$, $(0,4,1)$.
- In the sectors with $n_T > 0$ there are no boost and gauge invariant expressions.
- In the purely spatial sectors ($n_T = 0$) all TPD invariants are also boost invariant. $\Rightarrow$ Identical to the same sectors in the non-Galilean case.
- This leaves only 1 possible anomaly, which is B-type:

$$\mathcal{A}^{(0,4,0)} = \left(\widehat{R} + \widetilde{\nabla}_\alpha a^\alpha\right)^2 .$$

2. With Frobenius and no Galilean boost invariance

- This case also contains the 6 sectors: (2, 0, 0), (2, 0, 1), (1, 2, 0), (1, 2, 1), (0, 4, 0), (0, 4, 1).
- There are 2 possible anomalies, in the (2, 0, 0) and (0, 4, 0) sectors:

$$\mathcal{A}_1^{(2,0,0)} = \mathrm{Tr}(K_S^2) - \frac{1}{2}K_S^2,$$
$$\mathcal{A}_2^{(0,4,0)} = \left(\widehat{R} + \widetilde{\nabla}_\alpha a^\alpha\right)^2.$$

(where $K_S \equiv (K_S)^\mu_\mu$, $Tr(K_S^2) \equiv (K_S)^{\mu\nu}(K_S)_{\mu\nu}$.)

- Both anomalies are B-type.

3. Without Frobenius and with Galilean boost invariance

- Since $K_A \neq 0$, there's an **infinite** number of sectors.
- Full analysis was performed for those with $n_D < 4$ and the parity even sector with $n_D = 4$:
  $(2,0,0)$, $(2,0,1)$, $(1,2,0)$, $(1,2,1)$, $(0,4,0)$.
- This case can also be derived from **null reduction** of a $3+1$ Lorentzian manifold with a null isometry. [Jensen 2014]
  We didn't use it for the analysis here.
- There are no boost invariant expressions in sectors with $n_D < 4$ $(n_T > 0)$.
  - Expected from the null reduction as there are no scalars of dimension 4 with $n_D < 4$.
- The cohomology in $(0,4,0)$ mirrors the relativistic Weyl cohomology in $3+1$ dimensions.
  - Expected from the null reduction, as this sector corresponds to scalars in $3+1$ which involve only the curvature.

3. Without Frobenius and with Galilean boost invariance

- There are 2 anomalies in this sector:

$$\mathcal{A}_{E_4}^{(0,4,0)} =(\widetilde{\nabla}_\mu + a_\mu)\Big(4K_A B a^\mu + 8E^\mu K_A^2 + 8K_A \tilde{K}_S^{\mu\nu} a_\nu + 4K_A K_S \tilde{\epsilon}^{\mu\nu} a_\nu + 2(a_\nu \widetilde{\nabla}^\nu a^\mu - a^\mu \widetilde{\nabla}_\nu a^\nu) - 8\tilde{\epsilon}^{\mu\nu} K_A \mathcal{L}_n a_\nu\Big) + (\mathcal{L}_n + K_S)\left(16 K_A \mathcal{L}_n K_A + 8K_S K_A^2\right),$$

$$\mathcal{A}_{W^2}^{(0,4,0)} =(\widehat{R} + \widetilde{\nabla}_\mu a^\mu - 8K_A B)^2 + 12\widetilde{\nabla}^\mu K_A(\widetilde{\nabla}_\mu + a_\mu)B + 12\tilde{\epsilon}^{\mu\nu}\widetilde{\nabla}_\nu K_A(\widetilde{\nabla}_\mu + a_\mu)K_S + 12\tilde{\epsilon}^{\mu\nu}\widetilde{\nabla}_\mu K_A \mathcal{L}_n a_\nu + 24\widetilde{\nabla}_\alpha K_A \widetilde{\nabla}_\beta \tilde{K}_S^{\alpha\beta} - 72K_A E^\mu \widetilde{\nabla}_\mu K_A - 36(\mathcal{L}_n K_A)^2.$$

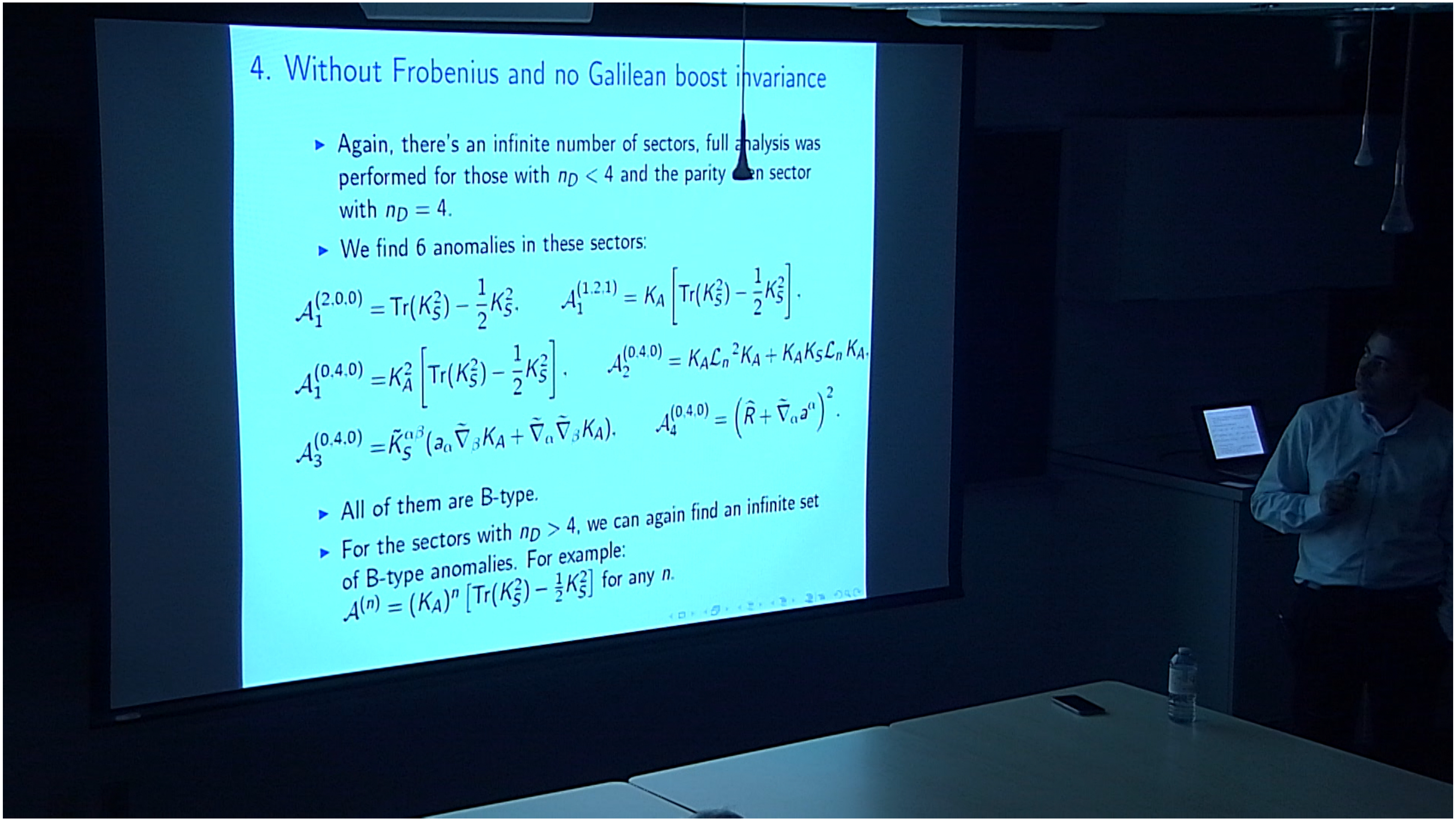
- $\mathcal{A}_{E_4}^{(0,4,0)}$ is A-type and corresponds to the $3+1$ dimensional Euler density.
- $\mathcal{A}_{W^2}^{(0,4,0)}$ is B-type and corresponds to the $3+1$ dimensional Weyl tensor squared.

3. Without Frobenius and with Galilean boost invariance

- When $K_A = 0$:
  - $\mathcal{A}^{(0,4,0)}_{W^2}$ reduced to the B-type anomaly of the Frobenius case.
  - $\mathcal{A}^{(0,4,0)}_{E_4}$ reduced to an expression that becomes **trivial** and can be cancelled by the counterterm:

$$W_{c.t.} = \int \sqrt{-g} \left(\frac{1}{2} a^{\alpha} \widetilde{\nabla}_{\alpha}(a^2) + \frac{3}{8} a^4 \right) .$$

- For the sectors with $n_D > 4$ we find an **infinite** set of independent anomalies:
  - K_A is invariant under Weyl, gauge and Milne boost transformations.
    $\Rightarrow$ For any n, $\mathcal{A}^{(n)} = (K_A)^n \mathcal{A}^{(0,4,0)}_{W^2}$ is a possible B-type anomaly with $n_D = 4 + n$.
  - There may be other anomalies in these sectors.


4. Without Frobenius and no Galilean boost invariance
Again, there's an infinite number of sectors, full analysis was performed for those with $n_D < 4$ and the parity en sector with $n_D = 4$.
We find 6 anomalies in these sectors:
$\mathcal{A}_1^{(2,0,0)} = \mathrm{Tr}(K_S^2) - \frac{1}{2}K_S^2.$
$\mathcal{A}_1^{(1,2,1)} = K_A\left[\mathrm{Tr}(K_S^2) - \frac{1}{2}K_S^2\right].$
$\mathcal{A}_1^{(0,4,0)} = K_A^2\left[\mathrm{Tr}(K_S^2) - \frac{1}{2}K_S^2\right].$
$\mathcal{A}_2^{(0,4,0)} = K_A\mathcal{L}_n{}^2K_A + K_AK_S\mathcal{L}_nK_A.$
$\mathcal{A}_3^{(0,4,0)} = \tilde{K}_S^{\alpha\beta}(a_\alpha\tilde{\nabla}_\beta K_A + \tilde{\nabla}_\alpha\tilde{\nabla}_\beta K_A).$
$\mathcal{A}_4^{(0,4,0)} = \left(\hat{R} + \tilde{\nabla}_\alpha a^\alpha\right)^2.$
All of them are B-type.
For the sectors with $n_D > 4$, we can again find an infinite set of B-type anomalies. For example:
$\mathcal{A}^{(n)} = (K_A)^n\left[\mathrm{Tr}(K_S^2) - \frac{1}{2}K_S^2\right]$ for any n.

Summary

- When coupling non-relativistic theories to curved spacetime, the existence of a **foliation structure** (Frobenius condition) has important consequences, even for calculation of flat space quantities.
- For $2+1$ dimensions with $z=2$, it changes the possible forms of Lifshitz scale anomalies considerably, both in the Galilean and the non-Galilean cases.
- An **A-type anomaly** is possible in this case only if we both impose **Galilean boost invariance** and give up the **foliation structure** of spacetime.
- However, when giving up the foliation structure we introduce the possibility of having an **infinite** set of independent B-type anomalies.

Lifshitz Anomalies, Ward Identities and Split Dimensional Regularization

Based on work with Yaron Oz, Avia Raviv-Moshe (to appear soon)

Goal:

- Understand the structure of Ward identities for flat space correlation functions in Lifshitz field theories,
- Develop a method for calculating Lifshitz anomaly coefficients from correlation functions using split dimensional regularization.

Lifshitz Anomaly From Correlation Functions

- Lifshitz anomaly coefficients have been calculated in the past using heat kernel and zeta function regularization. [e.g. Baggio, de Boer, Holsheimer 2012]
- However, they can also be computed directly from **flat space** field theory correlation functions:

$$\langle T^{\mu_1}{}_{a_1}(x_1)\dots T^{\mu_n}{}_{a_n}(x_n)\rangle \equiv (-i)^{n-1}\frac{\delta^n W}{\delta e^{a_1}{}_{\mu_1}(x_1)\dots\delta e^{a_n}{}_{\mu_n}(x_n)}$$

- From the curved spacetime anomalous Ward identity:

$$\langle D\rangle \equiv D^{\mu\nu}\langle T_{\mu\nu}\rangle = \mathcal{A},$$

where $D^{\mu\nu} \equiv P^{\mu\nu} - z\, n^\mu n^\nu$, we derive identities for flat space correlation functions.

Lifshitz Anomaly from Correlation Functions

- For the 2-point function:

$$D^a_\mu \langle T^\mu{}_a(x) T^\rho{}_b(y) \rangle = -i \left. \frac{\delta \mathcal{A}(x)}{\delta e^b{}_\rho(y)} \right|_{\text{flat}}$$

- For the 3-point function:

$$\begin{aligned} & D^a_\mu \langle T^\mu{}_a(x) T^\rho{}_b(y) T^\alpha{}_c(z) \rangle \\ & - i(\delta^\rho_\mu D^a_b - \delta^\rho_b D^a_\mu)\delta(x-y) \langle T^\mu{}_a(x) T^\alpha{}_c(z) \rangle \\ & - i(\delta^\alpha_\mu D^a_c - \delta^\alpha_c D^a_\mu)\delta(x-z) \langle T^\mu{}_a(x) T^\rho{}_b(y) \rangle = - \left. \frac{\delta^2 \mathcal{A}(x)}{\delta e^b{}_\rho(y) \delta e^c{}_\alpha(z)} \right|_{\text{flat}} \end{aligned}$$

- By calculating the renormalized correlation functions and using these identities we can extract the anomaly coefficients.
- Unlike the relativistic case, we may need to calculate all n-point functions to find all the anomaly coefficients.

Split Dimensional Regularization

- Like in the relativistic case, the correlation functions of the stress-energy tensor need to be renormalized.
- We use a **split dimensional regularization** scheme: [Leibbrandt, Williams 1995]
  - Define the theory in d_t time dimensions and d_s space dimensions (invariant under "time rotations" and space rotations).
  - Calculate a correlation function $I(d_t, d_s)$ and analytically continue to: $d_t = 1 - \varepsilon_t$, $d_s = d - \varepsilon_s$.
- To one-loop order I has the form:

$$I(\varepsilon_t, \varepsilon_s) = \frac{1}{\varepsilon_{\text{lif}}} f(\varepsilon_t, \varepsilon_s)$$

where $\varepsilon_{\text{lif}} \equiv z\varepsilon_t + \varepsilon_s$, and $f(\varepsilon_t, \varepsilon_s)$ is a regular function.
- In order to renormalize, we have to choose a parameter $\tilde{\varepsilon}(\varepsilon_t, \varepsilon_s)$ to keep fixed as we take the limit $(\varepsilon_t, \varepsilon_s) \to 0$.

Split Dimensional Regularization

- Then:

$$I(\varepsilon_{\text{lif}}, \tilde{\varepsilon}) = \frac{1}{\varepsilon_{\text{lif}}} f(\varepsilon_{\text{lif}}, \tilde{\varepsilon}) = \frac{1}{\varepsilon_{\text{lif}}} I^{(\text{res})}(\tilde{\varepsilon}) + I^{(\text{ren})} + O(\varepsilon_{\text{lif}})$$

  where:
  - $I^{(\text{ren})} = \left.\frac{\partial f}{\partial \varepsilon_{\text{lif}}}\right|_{\tilde{\varepsilon}}$ is the renormalized correlation function,
  - $I^{(\text{res})} = f(0, \tilde{\varepsilon})$ is the pole residue, and represents a **local** counterterm (polynomial in external momenta). [Anselmi, Halat 2007]

- The renormalization **depends on the choice of** $\tilde{\varepsilon}$. Changing $\tilde{\varepsilon}' \to \tilde{\varepsilon}$ will change the renormalized expression by a **local** term:

$$I^{(\text{ren})} \to \left(I^{(\text{ren})}\right)' - \alpha \left.\frac{\partial f}{\partial \tilde{\varepsilon}'}\right|_{\varepsilon_{\text{lif}}},$$

  where $\alpha \equiv -\left.\frac{\partial \tilde{\varepsilon}'}{\partial \varepsilon_{\text{lif}}}\right|_{\tilde{\varepsilon}}$.

Anomaly from the Pole

- Feynman diagrams are more difficult to evaluate in the Lifshitz case since the propagator denominators are polynomials of degree $2z$.
- Luckily, in some cases, the anomalous Ward identities can be computed from the ε_{lif} pole (the divergent part) alone!
- Suppose the Lifshitz Ward identity is:

$$T(\varepsilon_{\text{lif}}, \tilde{\varepsilon})[I_k] = 0,$$

  where $\{I_k\}$ is a set of correlation functions and T is some linear operator.
- As long as the split dimensional regularization scheme doesn't explicitly break Lifshitz symmetry, the unrenormalized functions satisfy this exactly.

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Anomaly from the Pole

- The anomaly then comes purely from the counterterm:

$$A = T(0,0)\left[I_k^{(\mathrm{ren})}\right] = -\lim_{(\varepsilon_{\mathrm{lif}},\tilde{\varepsilon})\to 0}\left(\frac{1}{\varepsilon_{\mathrm{lif}}}T(\varepsilon_{\mathrm{lif}},\tilde{\varepsilon})\left[I_k^{(\mathrm{res})}(\tilde{\varepsilon})\right]\right)$$

- If we change the choice of $\tilde{\varepsilon}$, A changes by a **trivial term** proportional to α:

$$A = A' - \alpha T(0,0)\left[\left.\frac{\partial f}{\partial \tilde{\varepsilon}'}\right|_{\varepsilon_{\mathrm{lif}}}\right]$$

$\Rightarrow$ As expected, only coefficients of trivial terms can depend on α.

Expansion in External Momenta

- The ε_{lif} pole residue ($\equiv$ divergent part of the Feynman diagram) can be computed by expanding the integrand in the external momenta.
- The result is a **polynomial** in the external momenta.
- The coefficients are solvable integrals of a well-known form that depend only on the loop momentum.
- Extract the ε_{lif} pole, plug it back to the Ward identity and take the limit $(\varepsilon_{\text{lif}}, \tilde{\varepsilon}) \to 0$ to get the anomalous contribution (both anomalies and trivial terms!).

Example: Free $z = 2$ Scalar in $2 + 1$ Dimensions

- Consider a free Lifshitz scalar in $2 + 1$ dimensions, with $z = 2$,with the action:

$$S = \int dtd^2x \left[\frac{1}{2}(\partial_t\phi)^2 - \frac{\kappa}{2}(\nabla^2\phi)^2\right]$$

- In order to perform split dimensional regularization, we have to couple the theory to curved spacetime with d_t time dimensions and d_s space dimensions.
- The coupling has to be **anisotropic Weyl invariant** and **non-singular** as $d_t \to 1$ and $d_s \to 2$!
- There is more than one way to do this.

Free Scalar: Coupling to Curved Spacetime

- We use the following action:

$$S = \int d^{d_t+d_s}x\sqrt{-g}\left\{\frac{1}{2}\left[\mathcal{L}_{n^{(i)}}\phi + \xi_1 K_S^{(i)}\phi\right]^2 - \frac{\kappa}{2}\left[\widetilde{\nabla}^2\phi + \xi_2\, a^\mu\widetilde{\nabla}_\mu\phi + \xi_3\, a^2\phi + \xi_4\,\widetilde{\nabla}_\mu a^\mu\phi\right]^2\right\}$$

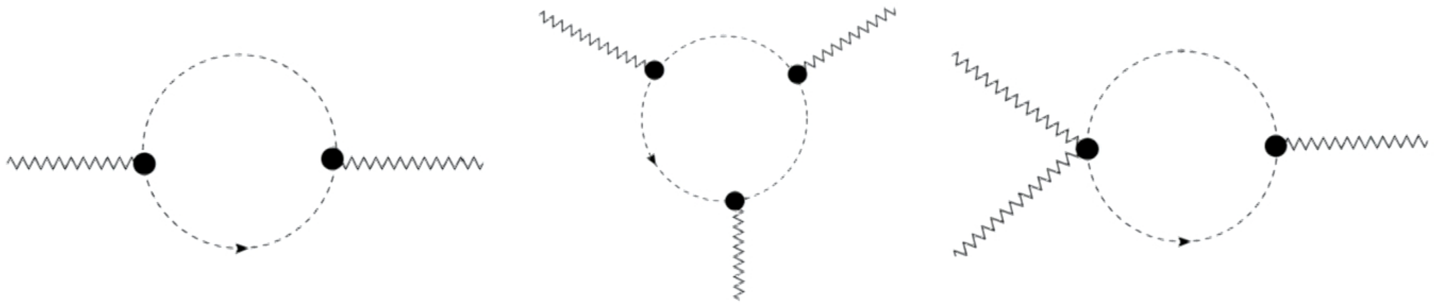
$$\xi_1 \equiv \frac{1}{d_s}\left(\frac{1}{2}d_{\text{lif}} - 2\right), \quad \xi_2 \equiv \frac{d_t - 1}{d_t}, \qquad (d_{\text{lif}} \equiv zd_t + d_s)$$

$$\xi_3 \equiv \frac{1}{4d_t^2}\left(\frac{1}{2}d_{\text{lif}} - 2\right)\left(d_t - \frac{1}{2}d_s\right), \quad \xi_4 \equiv \frac{1}{2d_t}\left(\frac{1}{2}d_{\text{lif}} - 2\right)$$

- $n_\mu^{(i)}$, $i = 1, \dots, d_t$ are orthonormal 1-forms corresponding to the time directions,
- The background expressions are defined similarly to the 1 time dimension case.
- This action is invariant under TPD, anisotropic Weyl transformations and time rotations: $n^{(i)} = U^{ij} n^{(j)}$.

Free Scalar: Correlation Function ε_{lif} Poles

- From the curved spacetime action, the flat space stress-energy tensor in $d_s + d_t$ dimensions is derived.
- The following diagrams contribute to the two-point function and three-point function of the stress-energy tensor:



- By computing the divergent parts of these diagrams and plugging them into the Ward identities, we obtain the anomalous contributions to these identities.
- Finally, comparing with the variations of the possible anomaly (and trivial terms) densities we can extract the anomaly coefficients.

Free Scalar: Main Results

- Computation was performed for the two-point and three-point functions.
  $\Rightarrow$ Only the coefficients of terms of order ≤ 2 in the background fields can be extracted.
- Since the computation involves thousands of terms, we used a Mathematica script to perform it.
- The results are consistent with previous calculations using the heat kernel method.
- As expected, only coefficients of trivial terms depend on α.
- For the $(2,0,0)$ $(n_D = 2)$ sector:
  - There is one anomaly $\mathrm{Tr}(K_S^2) - \frac{1}{2}K_S^2$ with a coefficient: $\frac{1}{32\sqrt{\kappa\pi}}$.
  - There is one trivial term: $\mathcal{L}_n K_S + K_S^2$ with a coefficient: $\frac{3-2\alpha}{96\sqrt{\kappa\pi}}$.

Free Scalar: Main Results

For the $(0,4,0)$ $(n_D = 4)$ sector:

- There are 3 independent anomalies up to second order, all of them with **vanishing coefficients**.
- There are 9 independent trivial terms up to second order. All of them have coefficients proportional to α.
- Some trivial terms with non-vanishing coefficients are ones that vanish in the Frobenius case.
- For example, the term: $(\widetilde{\nabla}_\alpha + a_\alpha)\left(K_A \widetilde{\nabla}_\beta \tilde{K}_S^{\alpha\beta}\right)$ has a coefficient: $-\frac{\sqrt{\kappa}\alpha}{12\pi}$.
- $\Rightarrow$ For $\alpha \neq 0$, one needs to violate the Frobenius condition to cancel these terms!

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Anomaly Ambiguity?

- For theories that include a Lifshitz scalar with $z = d$, an interesting ambiguity seems to occur.
- In these cases, the scalar ϕ is dimensionless.
- If $\mathcal{A}$ is a **B-type** anomaly density of the theory which is **second order** in the background fields, we can add to the curved spacetime action a term:

$$S_0 = \beta \int d^{d+1}x \sqrt{-g}\, \mathcal{A}\phi^n$$

  (an example was suggested by [Griffin, Hořava, Melby-Thompson 2012])
- Neither the flat space action nor the flat space currents change as a result of this addition.
- However, the anomaly coefficient of $\mathcal{A}$ will change by βc, where c is defined by the anomalous Ward identity:

$$\langle D(x)\phi^n(y)\rangle = ic\delta(x-y)$$

Anomaly Ambiguity?

- Unlike the relativistic conformal case, specifying the flat space action and currents is **not enough** to determine the consistent anomaly coefficients - one has to specify the curved spacetime coupling.
- This type of ambiguity doesn't occur in the relativistic case since in the $d = 2$ dimensionless scalar case there are no B-type anomalies.
- **Note:** $\langle \phi^n(x) \phi^n(y) \rangle$ diverges logarithmically in the infinite volume limit $\Rightarrow$ No ambiguity in theories with no built-in IR cutoff.
- Our results in this work are for the minimal coupling case.

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Summary

- A split dimensional regularization method was used to compute Lifshitz anomaly coefficients in the free scalar case.
- Some of the trivial terms found in this case require a curved spacetime description that **violates the Frobenius condition**.
- In the $z = d$ scalar case, the relation between flat space correlation functions of the stress-energy tensor and the consistent anomaly coefficients seems to be **ambiguous**.

Outlook and Open Questions

- The Frobenius condition:
  - Can the non-relativistic quantum theory be consistently defined on a curved background without a foliation structure?
  - What are the implications on the calculation of flat space quantities?
  - Are there theories with non-vanishing coefficients for anomalies that violate it?
- Is there a general structure for Lifshitz scale anomalies for general d and z, with or without Galilean boost invariance?
- Can the split dimensional regularization method be used to compute anomalies in other Lifshitz or Galilean theories?
- Does the coefficient of the A-type anomaly in the boost invariant case have an interesting behaviour along RG flows? Can it lead to an a-theorem for Galilean theories?
- What are the implications of the ambiguity in the coefficients of B-type anomalies for the $z = d$ scalar case?
- Lifshitz scale invariance vs. full Schrödinger invariance in Galilean theories.