Title: The twisted superpotential of 3d N=2 theories

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Abstract: <p>We study the effective twisted superpotential of 3d N=2 gauge theories compactified on a circle. This is a rich object which encodes much of the protected information in these theories. We review its properties, and survey some applications, including the algebra of Wilson loops, computation of supersymmetric partition functions on S^1 bundles, and the reduction of 3d dualities to two dimensions.</p>

The twisted superpotential of $3d$ $\mathcal{N} = 2$ theories

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based on work w/ C. Closset and H. Kim,
and w/ O. Aharony, N. Seiberg, and S. Razamat

Background

- $3d\ \mathcal{N} = 2$ gauge theories provide a rich class of SQFTs with many interesting physical and mathematical applications.

- They are asymptotically free, but typically flow to strongly coupled IR SCFTS, which exhibit many non-trivial dualities.
  - Mirror symmetry
  - Seiberg-like dualities
  - $3d - 3d$ correspondence
  - AdS/CFT duality

- To study them, we need probes of the theory which are RG invariant.
  - An important example is the computation of supersymmetric partition functions by localization.

Background

- [Nekrasov,Shatashvili] showed that we can learn about higher dimensional theories by compactifying them to effective 2*d* $\mathcal{N} = (2, 2)$ theories and studying their twisted chiral ring data.
- This turns out to lead to an interesting connection of supersymmetric gauge theories to integrable systems, known as the **Bethe/gauge correspondence**.
- The central object of study is the **effective twisted superpotential** on the Coulomb branch of the gauge theory.
- In this talk, we will see how this protected object teaches us about many interesting aspects of 3*d* gauge theories.

Outline

- Effective twisted superpotential
    - Algebra of Wilson loops
- Partition function on $U(1)$ bundles
- Applications to $3d$ dualities
- Reductions of $3d$ dualities to $2d$.
- Conclusion and Outlook

(Twisted) chiral operators in two dimensions

- A $2d$ $\mathcal{N} = (2,2)$ may have chiral and twisted chiral operators:

$$\bar{\mathcal{Q}}_+ \Phi = \bar{\mathcal{Q}}_- \Phi = 0 \quad \rightarrow \quad \text{chiral}$$

$$\bar{\mathcal{Q}}_+ \tilde{\Phi} = \mathcal{Q}_- \tilde{\Phi} = 0 \quad \rightarrow \quad \text{twisted chiral}$$

- The action may include a (twisted) superpotential, which may only depend on (twisted) chiral superfields.
- These define the (twisted) chiral ring:

$$\Phi_i \Phi_j = \mathcal{O}_{ij}^k \Phi_k + \mathcal{Q}(...)$$

  whose relations are generated by derivatives of the (twisted) superpotential
- We can define a topological A-model by twisting by the $U(1)_V$ R-symmetry, which computes correlators of twisted chiral operators (and similarly for B-model).

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Twisted chiral operators in a GLSM

- Many interesting $\mathcal{N} = (2,2)$ theories have a UV description as a gauged linear sigma model (GLSM).
- Here a natural basis of the twisted chiral ring comes from the twisted chiral field strength multiplet:

$$\Sigma = \mathcal{D}_+ \bar{\mathcal{D}}_- V = \tilde{\sigma} - i\bar{\Lambda}_+\theta^+ - i\Lambda_-\theta^- + \theta^+\bar{\theta}^-(D - if_{12}) + ...$$

- Here Σ runs over a basis of the dynamical and background vector multiplets. We will denote these:

$$\Sigma_\alpha \to \quad u_a, \quad a = 1, ..., r_G, \qquad m_i, \quad i = 1, ..., r_H$$

- The effective twisted superpotential is computed by integrating out charged matter on the Coulomb branch.
- E.g., for a charge one chiral multiplet in two dimensions, one finds:

$$\tilde{W}(\Sigma) = \frac{1}{2\pi i}\Sigma(\log(\Sigma/\mu) - 1)$$

- It may also depend on twisted chiral moduli, such as Fayet-Iliopolous (FI) parameters.

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Vacuum equations and twisted chiral ring

- Because of quantization of the flux, $\tilde{W}$ has branch cut ambiguities where $\tilde{W} \to \tilde{W} + n\Sigma$.
- Because of this ambiguity, the vacuum equations become:

$$\frac{\partial \tilde{W}}{\partial u_a} = 0 \mod \mathbb{Z} \quad \Leftrightarrow \quad \mathcal{P}_a \equiv \exp\left(2\pi i \frac{\partial \tilde{W}}{\partial u_a}\right) = 1$$

- **E.g.**, for a $U(1)$ gauge theory with charge Q_i chirals:

$$\tilde{W}(u, m_i) = \frac{1}{2\pi i} \sum_i (Q_i u + m_i)(\log(Q_i u + m_i) - 1) + tu$$

$$\Rightarrow \exp\left(2\pi i \frac{\partial \tilde{W}}{\partial u}\right) = e^t \prod_i (Q_i u + m_i)^{Q_i}$$

- Similarly, the twisted chiral ring relations in $\mathbb{C}[\Sigma_a]$ are generated by:

$$\exp\left(2\pi i \frac{\partial \tilde{W}}{\partial u_a}\right) - 1$$

  which are given by rational functions in the u_a and m_i.
- The twisted chiral ring is a quotient of $\mathbb{C}[\Sigma]$ by this polynomial.

Non-abelian theories

- In the case of a non-abelian gauge theory, the twisted superpotential is a Weyl-invariant function on the Cartan.
- In addition, solutions with enhanced gauge symmetry (e.g., $u_a = u_b$ for some $a \neq b$) have an unbroken nonabelian gauge symmetry.
- Pure $SU(2)$ theory is equivalent to the theory of a free twisted chiral [Aharony,Razamat,Seiberg,BW]:

$$\Phi = \mathrm{Tr}\Sigma^2$$

Generically there is a term in $\tilde{W}$ linear in Φ, which breaks SUSY. Then:

$$\mathcal{S}_{vac} = \{u_a^* \ \mid \ \mathcal{P}_a = 1, \quad w \cdot u_a^* \neq u_a^*\}/\mathcal{W}$$

- **E.g.,** for $U(N_c)$ with N_f flavors, the vacuum equations are:

$$e^t \prod_{i=1}^{N_f} (u_a + m_i) = \prod_{i=1}^{N_f} (u_a + \tilde{m}_i)$$

this has N_f solutions for each a, and imposing the Weyl symmetry, this leads to $\binom{N_f}{N_c}$ vacua.

Bethe/gauge correspondence

- Consider a mass-deformed $2d$ $\mathcal{N} = (2,2)$ gauge theory, or more generally, a higher dimensional gauge theory compactified to an effective $2d$ $\mathcal{N} = (2,2)$ theory, eg:
  - **E.g., 3d $\mathcal{N} = 2$ on $\mathbb{R}^2 \times \mathbf{S}^1$**
  - E.g., $4d$ $\mathcal{N} = 1$ on $\mathbb{R}^2 \times T^2$
  - E.g., $4d$ $\mathcal{N} = 2$ with Ω-deformation
- Then an important insight of [Nekrasov,Shatashvili] was that, for many $d \geq 2$ gauge theories of physical interest, these equations coincide with the **Bethe equations** for certain integrable systems.
- In addition, these equations generate the relations which define the twisted chiral ring of the gauge theory.
- In the case of higher dimensional gauge theories compactified on a circle, the twisted chiral operators are line and surface operators.

$3d\ \mathcal{N}=2$ theories

- $3d\ \mathcal{N}=2$ gauge theories are specified by a gauge group G, with matter content:,

$$\mathcal{V}=(A_\mu,\sigma,D,\Lambda)\in Ad(G),\qquad \Phi=(\phi,\psi,F)\in\mathcal{R}=\bigoplus_\alpha \mathcal{R}_\alpha^{(G)}\otimes\mathcal{R}_\alpha^{(H)}$$

where H is the flavor symmetry group.

- The action takes the form:

$$S=S_{YM}+S_{chiral}+S_{CS}+S_W$$

where:

$$S_{CS}=\frac{1}{4\pi}\int Tr_{CS}\Big(A\wedge dA+\frac{2}{3}A\wedge A\wedge A+\sqrt{g}d^3x(2D\sigma-\bar{\Lambda}\Lambda)\Big)$$

- A natural observable is a supersymmetric Wilson loop:

$$\mathcal{W}=\mathrm{Tr}_R\mathcal{P}\exp\Big(\oint(iA+\sigma d|x|)\Big)$$

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$$\mathcal{W} = \mathrm{Tr}_R \mathcal{P} \exp\left(\oint (iA + \sigma d|x|)\right)$$

$3d$ theories on $\mathbb{R}^2 \times S^1_r$

- We can treat a $3d$ $\mathcal{N}=2$ theory on S^1_r as we treated $2d$ $\mathcal{N}=(2,2)$ theories earlier.
- Now Σ is periodic:

$$\Sigma_\alpha \sim \Sigma_\alpha + \frac{1}{r}$$

- Then the effective twisted superpotential of a $3d$ gauge theory is given by:

$$\tilde{W} = \sum_\ell \tilde{W}_\chi(Q^\alpha_\ell \Sigma_\alpha) + \frac{r}{2} k^{\alpha\beta}(\Sigma_\alpha \Sigma_\beta + \delta_{\alpha\beta}\frac{1}{r}\Sigma_\alpha) + \frac{\kappa_g}{r}$$

where $k^{\alpha\beta}$ are the Chern-Simons levels, κ_g is the gravitational Chern-Simons level, and:

$$\tilde{W}_\chi(\Sigma) = \sum_{n\in\mathbb{Z}} \Sigma(\log(\Sigma + \frac{in}{r}) - 1) = \frac{1}{4\pi^2 r}\mathrm{Li}_2(e^{-2\pi i r\Sigma}) - \frac{r}{4}\Sigma(\Sigma + \frac{1}{r})$$

$$\Rightarrow 2\pi i\, \partial_\Sigma \tilde{W}_{chi}(\Sigma) = -\log(2i \sin \pi r \Sigma)$$

Vacuum equations and algebra of line operators

- The equations for the supersymmetric vacua are given by:

$$\exp\left(2\pi i \frac{\partial \tilde{W}}{\partial u_a}\right) = 1, \quad a = 1, ..., r_G$$

- Provided the theory is free of anomalies, the LHS will be a rational function of the gauge-invariant variables:

$$x_a = e^{2\pi i r u_a}, \qquad \mu_i = e^{2\pi i r m_i}$$

- We can also define twisted chiral operators. These are line operators, which lie at a point on $\mathbb{R}^2$ and wrap the S^1. They are generated by Weyl-invariant polynomials in x_a, subject to the relations:

$$\exp\left(2\pi i \frac{\partial \tilde{W}}{\partial u_a}\right) - 1$$

- Thus we can define an algebra of Wilson lines by a certain quotient of the representation ring of G by polynomial relations [Kapustin,BW],[Closset,Kim] of dimension $|\mathcal{S}_{vac}|$.

Example 1: $U(1)$ with one flavor

- The twisted superpotential for $U(1)$ with one flavors is (setting $r = 1$):

$$\tilde{W} = \tilde{W}_{chi}(u+m) + \tilde{W}_{chi}(-u+m) + \zeta u$$

  where u is the gauge parameter, and m and ζ are flavor parameters.
- Then the vacuum equation is:

$$1 = \exp\left(2\pi i \frac{\partial \tilde{W}}{\partial u}\right) = \frac{\sin \pi(-u+m)}{\sin \pi(u+m)} e^{2\pi i \zeta}$$

- Defining $x = e^{2\pi i u}, \mu = e^{2\pi i m}, z = e^{2\pi i \zeta}$, this can be rearranged to:

$$\frac{\mu - x}{x\mu - 1} z = 1 \qquad \Rightarrow \mathcal{S}_{vac} = \{x^* = \frac{1+\mu z}{\mu + z}\}$$

  giving a single vacuum.
- A charge q Wilson loop is represented by $x^q \in \mathbb{C}[x]$, but the relation $x(\mu + z) = \mu z + 1$ relates these to c-numbers, so the algebra is trivial.

Example 2: $U(N_c)_k$ with N_f flavors

- The twisted superpoential for $U(N_c)_k$ with N_f flavors is:

$$\tilde{W} = \sum_{a=1}^{N_c} \left(\sum_{i=1}^{N_f} \left(\tilde{W}_{chi}(u_a + m_i) + \tilde{W}_{chi}(-u_a + \tilde{m}_i) \right) + \frac{k}{2} u_a(u_a + 1) + \zeta u_i \right.$$

where u_a are the gauge parameters for the Cartan of $U(N_c)$, and $(m_i, \tilde{m}_i, \zeta)$ are the parameters for the $SU(N_f) \times SU(N_f) \times U(1)_J$ flavor symmetry.

- The vacuum equations are:

$$1 = \exp\left(2\pi i \frac{\partial \tilde{W}}{\partial u_a}\right) = e^{2\pi i (k u_a + \zeta)} \prod_{j=1}^{N_f} \frac{\sin \pi(-u_a + m_i)}{\sin \pi(u_a + \tilde{m}_i)}$$

$$\Leftrightarrow \quad p(x_a) \equiv \prod_{i=1}^{N_f} (x_a \mu - 1) - x_a{}^k z \prod_{i=1}^{N_f} (\tilde{\mu}_i - x_a) = 0$$

- This has $N_f + k$ solutions for each a, and taking Weyl symmetry into account, this leaves:

$$\mathcal{S}_{vac} = \{\text{choice of } N_c \text{ roots of } p(x)\}, \quad |\mathcal{S}_{vac}| = \binom{k + N_f}{N_c}$$

Example 2: $U(N_c)_k$ with N_f flavors (cont'd)

- The algebra of Wilson lines is given by a quotient of the representation ring:

$$\mathcal{A} = \mathbb{C}[x_a]^{\mathcal{W}}/\mathcal{I}$$

where:

$$\mathcal{I} = \langle p(x_a) \rangle$$

- In the case $N_f = 0$ this reproduces the Verlinde algebra.
- More generally, the space of Wilson loop irreps is truncated by quantum relations. E.g., for $N_c = k = N_f = 2$, we have:

$$\mathbf{W}_{\square\square\square} = -(\mu + \mu^{-1})\mathbf{W}_{\square\square} + (z-1)\mathbf{W}_{\square} + z(\mu+\mu^{-1})\mathbf{1}$$

where $\mu = e^{2\pi i m}$, $z = e^{2\pi i \zeta}$

- The Young diagram can always be truncated to fit inside a $(k + N_f - N_c) \times N_c$ box, leading to:

$$\dim \mathcal{A} = \binom{k+N_f}{N_c} = |S_{vac}|$$

Partition functions on

$U(1)$ bundles

A $3d$ uplift of the A-twist

- [Nekrasov-Shatashvili],[Closset, Kim], [Benini,Zaffaroni], argued that the partially topologically twisted partition function on $\Sigma_g \times S^1$ can be considered as a $3d$ uplift of the A-twisted partition function.
- In particular, the partition function is independent of the metric on Σ_g, and is computed by a $2d$ TQFT. This implies it takes the general form:

$$\langle \mathcal{O}_1 ... \mathcal{O}_s \rangle_{\Sigma_g \times S^1} = \mathrm{Tr}_{\mathcal{V}}(\mathcal{H}^{g-1} \mathcal{O}_1 ... \mathcal{O}_s)$$

- Here $\mathcal{O}_i$ are local operators on Σ_g, corresponding to loop operators which wrap the S^1 factor.
- We will argue that this statement can be generalized to spaces which are $U(1)$ bundles over a Riemann surface:

$$S^1 \to \mathcal{M}_{g,p} \stackrel{\pi}{\to} \Sigma_g$$

- They are labeled by two integers, the genus g and Chern-degree p of the bundle.

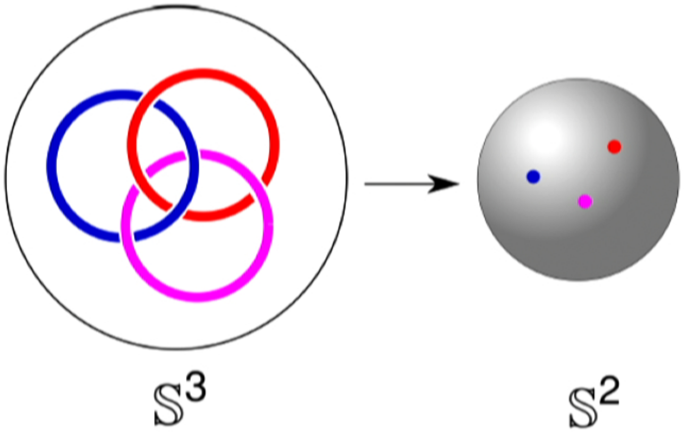
$U(1)$ bundles over Riemann surfaces

- E.g.:

$$\mathcal{M}_{g,0} \equiv \Sigma_g \times S^1,$$

$$\mathcal{M}_{0,1} = \mathbb{S}^3,$$

$$\mathcal{M}_{0,p} \equiv \mathbb{S}^3/\mathbb{Z}_p \text{ (lens space)}$$



- We take local coordinates, $(z, \bar{z}, \psi)$, on $\mathcal{M}_{g,p}$, such that the metric takes the form:

$$ds^2 = (d\psi + a)^2 + c(z, \bar{z})dzd\bar{z}$$

- Here $a = a(z, \bar{z})dz + \bar{a}(z, \bar{z})d\bar{z}$ is a $U(1)$ connection on Σ_g with flux p:

$$\frac{1}{2\pi}\int_{\Sigma_g} da = p$$

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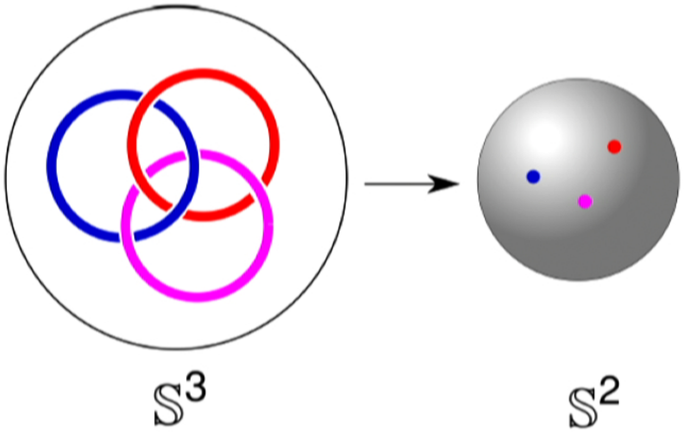

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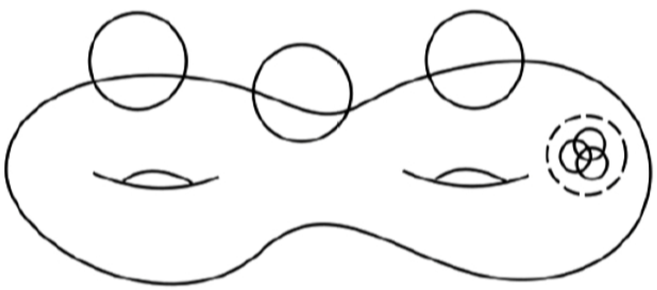
$$\frac{1}{2\pi} \int_{\Sigma_g} da = p$$

A $3d$ uplift of the A-twist (cont'd)

- For non-zero p, the independence of the partition function on the metric of Σ_g continues to hold.
- Moreover, the graviphoton background $a(z, \bar{z})$ enters only through its total flux p.
- By concentrating the flux at a single point, we can think of it as a local insertion in Σ_g, implying:

$$\langle \mathcal{O}_1 ... \mathcal{O}_s \rangle_{\mathcal{M}_{g,p}} = \langle \mathcal{F}^p \mathcal{O}_1 ... \mathcal{O}_s \rangle_{\Sigma_g \times S^1} = \text{Tr}_{\mathcal{V}}(\mathcal{H}^{g-1} \mathcal{F}^p \mathcal{O}_1 ... \mathcal{O}_s)$$

where $\mathcal{F}$, which we denote the "fibering operator" inserts one unit of graviphoton flux.



Comparisons to other work

- **$\mathbf{g} = \mathbf{0}, \mathbf{p} = \mathbf{1}$ - $\mathbb{S}^3$**
  - Taking the round metric on S^2, this gives the round $\mathbb{S}^3$ background. [Kapustin, BW, Yaakov],[Jafferis]
  - The "squashed sphere" backgrounds [Hama,Hosomichi,Lee] correspond to taking K to include a components along an S^2 isometry. These cannot be extended to higher genus.
- **$\mathbf{g} = \mathbf{0}, \mathbf{p} > \mathbf{1}$ - $\mathbb{S}^3/\mathbb{Z}_p$**
  - Here we compute the lens space $L(p, -1)$. This differs from the lens space partition function computed [Benini,Nishioka,Yamazaki], corresponding to $L(p, 1)$.
- **General $\mathbf{g}, \mathbf{p}$**
  - Localization of Chern-Simons theory on $\mathcal{M}_{g,p}$ was considered by [Blau,Thompson] and [Källén], and we reproduce their results.
  - Localization of 3d $\mathcal{N} = 2$ theories on $\mathcal{M}_{g,p}$ was previously considered by [Ohta,Yoshida]. Our backgrounds and supersymmetry transformations are similar to theirs, however our final answers differ.

Bethe/gauge correspondence on curved space

- [Nekrasov,Shatashvili] argued the $\Sigma_g \times S^1$ partition function can now be interpreted as an A-twisted partition function for this low energy theory.
- The low energy action on this curved space is given by:

$$S = \int d^2x \left(\frac{\partial \tilde{W}}{\partial \Sigma_\alpha}(D + if_{12}) + \frac{\partial^2 \tilde{W}}{\partial \Sigma_\alpha \partial \Sigma_\beta} \Lambda_\alpha \Lambda_\beta + \mathcal{U}\mathcal{R}^{(2)} \right) + \mathcal{Q}(...)$$

where $\mathcal{U}$ is the "effective dilaton interaction." It depends on the choice of R-charge, and couples to the curvature of Σ_g.

- The partition function on $\Sigma_g \times S^1$ then recieves contributions from the SUSY vacua:

$$\mathcal{Z}_{\Sigma_g \times S^1}(m_i) = \sum_{u_{a*} \in \mathcal{S}_{BE}} \mathcal{H}^{g-1}(u_a^*, m_i)$$

where $\mathcal{H}$ is the "handle-gluing operator."

Handle-gluing operator

- This is given by:

$$\mathcal{H}(u_a, m_i) = e^{\mathcal{U}(u_a, m_i)} \det_{a,b} \frac{\partial^2 \tilde{W}}{\partial u_a \partial u_b}$$

- The effective dilaton interaction is:

$$\mathcal{U} = \sum_{\ell} (1 - r_\ell) \log 2 \sin(\pi Q_\ell^\alpha \Sigma_\alpha) + \sum_{\omega \in Ad(G)} \log(1 - e^{2\pi i \omega(u)}) + k^{R\alpha} \Sigma_\alpha + k^{RR}$$

where $r_\ell \in \mathbb{Z}$ is the R-charge of the chiral, and $k^{R\alpha}, k^{RR}$ are contact terms.

Flux operator

- We can generalize this formula by allowing flux through Σ_g for flavor symmetry gauge fields.
- For a background gauge multiplet m_i, if we define:

$$\exp\left(i\frac{\partial\tilde{W}}{\partial m_i}\int\sqrt{g}d^2xi\, f_{12}^{(i)}\right) = {\pi_i}^{s_i}$$

where $s_i = \frac{1}{2\pi}\int_{\Sigma_g} da^{(i)} \in \mathbb{Z}$ is the flux, and

$$\pi_i = \exp\left(2\pi i\frac{\partial\tilde{W}}{\partial m_i}\right)$$

- Then the formula in the presence of generic flavor symmetry fluxes is [Closset,Kim],[Benini,Zaffaroni]:

$$\mathcal{Z}_{\Sigma_g\times S^1}(m_i, s_i) = \sum_{u_a^*\in\mathcal{S}_{BE}} \mathcal{H}(u_a^*, m_a)^{g-1}\pi_i(u_a^*, m_i)^{s_i}$$

- By concentrating the flux near a point, such a configuration defines a supersymmetric vortex loop operator.

Fibering operator

- A $3d$ theory on $\mathbb{R}^2 \times S^1$ has a priveleged global symmety, which we denote $U(1)_{KK}$, which acts by translations along S^1. The charged fields are KK modes.
- **Then placing the theory on a non-trivial $U(1)$ bundle, $\mathcal{M}_{g,p}$, is the same as turning on a flux p for this symmetry.**
- On a circle of radius r, the KK modes have twisted mass $\frac{n}{r}$, and so we can define $\mathbf{m_{KK}} = \frac{\mathbf{1}}{\mathbf{r}}$. Then the flux operator is given by:

$$\exp\left(2\pi i \frac{\partial \tilde{W}}{\partial m_{KK}}\right) = \exp\left(-2\pi i r^2 \frac{\partial \tilde{W}}{\partial r}\right)$$

- The dependence on r is fixed by dimensional analysis, and one computes the *fibering operator* for a general theory as:

$$\mathcal{F} = \exp\left(2\pi i\left(\tilde{W} - \sum_a u_a \frac{\partial \tilde{W}}{\partial u_a} - \sum_i m_i \frac{\partial \tilde{W}}{\partial m_i}\right)\right)$$

Fibering operator (cont'd)

- Independent of branch cut ambiguities in $\tilde{W}$.
- It satisfies the difference equation:

$$\mathcal{F}(..., m_i + 1, ...) = \mathcal{F}(..., m_i, ...)\pi_i{}^{-1}$$

- This reflects the fact that, on $\mathcal{M}_{g,p}$, large gauge transformations identify:

$$(m, s) \sim (m+1, s+p) \Rightarrow \mathcal{F}(m+1)^p \pi(m)^{s+p} = \mathcal{F}(m)^p \pi(m)^s$$

so the fluxes m_i takes values in:

$$\mathbb{Z}_p \subset H^2(\mathcal{M}_{g,p})$$

- For a chiral multiplet, one has:

$$\mathcal{F}_{chi}(u) = \exp\left(2\pi i\left(\frac{1}{4\pi^2}\mathrm{Li}_2(e^{-2\pi i u}) - \frac{u}{2\pi i}\log(e^{2\pi i u} - 1)\right)\right) = s_{b=1}(u)$$

- For a Chern-Simons and gravitational Chern-Simons term, one finds:

$$\mathcal{F}_{CS} = e^{k\pi i u^2}, \quad \mathcal{F}_{grav} = e^{2\pi i \kappa_g}$$

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$$\exp\left(2\pi i \frac{\partial \tilde{W}}{\partial m_{KK}}\right) = \exp\left(-2\pi i r^2 \frac{\partial \tilde{W}}{\partial r}\right)$$

- The dependence on r is fixed by dimensional analysis, and one computes the *fibering operator* for a general theory as:

$$\mathcal{F} = \exp\left(2\pi i \left(\tilde{W} - \sum_a u_a \frac{\partial \tilde{W}}{\partial u_a} - \sum_i m_i \frac{\partial \tilde{W}}{\partial m_i}\right)\right)$$

Final formula

- Putting the ingredients together, the partition function on $\mathcal{M}_{g,p}$ is given by:

$$\mathcal{Z}_{\mathcal{M}_{g,p}}(m_i, s_i) = \sum_{u_a^* \in \mathcal{S}_{vac}} \mathcal{H}^{g-1} {\pi_i}^{s_i} \mathcal{F}^p$$

- One can also include Wilson loop operators, which recall are defined by gauge-invariant polynomials $\mathcal{W}(x_a)$:

$$\langle \mathcal{W}(x_a) \rangle_{\mathcal{M}_{g,p}}(m_i, s_i) = \sum_{u_a^* \in \mathcal{S}_{vac}} \mathcal{H}^{g-1} {\pi_i}^{s_i} \mathcal{F}^p \mathcal{W}(x_a)$$

- Note that the Wilson loop algebra relations are automatically satisfied in this formula since we sum over the solutions to the vacuum equations.

UV localization on $\mathcal{M}_{g,p}$

- The partition function can also be computed by localization, and one finds:

$$\mathcal{Z}_{\mathcal{M}_{g,p}} = \frac{1}{|W|} \sum_{\mathfrak{m}_a \in \mathbb{Z}} \oint_{C_{JK}} du_a \, \mathcal{F}^p \mathcal{P}_a{}^{\mathfrak{m}_a} \pi_i{}^{s_i} e^{(g-1)\mathcal{U}} (\det H_{ab})^g$$

  where C_{JK} is an appropriate Jeffrey-Kirwan contour.
- This can be shown to agree with the TQFT expression derived earlier.
- Moreover, it reduces to the Coulomb-branch-integral formulation of the S^3 partition function.

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Applications to dualities

Applications to Dualities

- Given two dual theories, their partition functions on all manifolds should agree, as should expectation values of dual Wilson loop operators:

$$< \mathcal{W} >^{(A)}_{\mathcal{M}_{g,p}} = \sum_{u_a{}^* \in \mathcal{S}_{vac}} \mathcal{H}^{g-1}(u_a{}^*)\pi_i^{s_i}(u_a{}^*)\mathcal{F}^p(u_a{}^*)\mathcal{W}(u_a{}^*)$$

$$= < \hat{\mathcal{W}} >^{(B)}_{\mathcal{M}_{g,p}} = \sum_{\hat{u}_a^* \in \hat{\mathcal{S}}_{vac}} \hat{\mathcal{H}}^{g-1}(\hat{u}_a^*)\hat{\pi}_i^{s_i}(\hat{u}_a^*)\hat{\mathcal{F}}^p(\hat{u}_a^*)\hat{\mathcal{W}}(\hat{u}_a^*)$$

- For this to hold for all g, p, s_i, and arbitrary insertions $\mathcal{W}(u_a)$ is equivalent to the existence of a **duality map** between vacua:

$$D : \mathcal{S}_{vac} \to \hat{\mathcal{S}}_{vac}$$

  such that:

$$\mathcal{H}(u_a{}^*) = \hat{\mathcal{H}}(D(u_a{}^*)), \quad \pi_i(u_a{}^*) = \hat{\pi}_i(D(u_a{}^*)), \quad \mathcal{F}(u_a{}^*) = \hat{\mathcal{F}}(D(u_a{}^*))$$

- These equalities in turn follow provided:

$$\mathcal{W}(u_a{}^*) \cong \hat{\mathcal{W}}(D(u_a{}^*)), \quad \mathcal{U}(u_a{}^*) \equiv \hat{\mathcal{U}}(D(u_a{}^*)),$$

  where these hold modulo branch ambiguities.

$U(1)_{k=\frac{1}{2}} \leftrightarrow$ free chiral

- The simplest 3d duality relates a free chiral multiplet to a $U(1)_{k=1/2}$ theory with one charged chiral. The first theory has (writing $\hat{\tilde{W}} = 4\pi^2 \tilde{W}$)

$$\hat{\tilde{W}}_A(\mu) = -\mathrm{Li}_2(\mu)$$

- For the second, we have:

$$\hat{\tilde{W}}_B(x, \mu) = \mathrm{Li}_2(x) + \log x \log \mu$$

which leads to a vacuum equation:

$$1 = \exp\left(\frac{\partial \tilde{W}_B}{\partial u}\right) = \frac{\mu}{1-x} \quad \Rightarrow \quad x = 1 - \mu$$

$$\Rightarrow \hat{\tilde{W}}_B(\mu) = \mathrm{Li}_2(1-\mu) + \log(1-\mu)\log\mu$$

- By a basic dilogarithm identity, one checks:

$$\tilde{W}_A(\mu) = \tilde{W}_B(\mu) - \frac{\pi^2}{6}$$

$U(1)$ $N_f = 1 \leftrightarrow$ XYZ

- Another simple duality relates $U(1)$ with one flavor to the *XYZ* model. The *XYZ* model has:

$$\hat{\tilde{W}}_A(\mu, z) = \mathrm{Li}_2(\mu z) + \mathrm{Li}_2(\mu z^{-1}) + \mathrm{Li}_2(\mu^2)$$

- The $U(1)$ theory has:

$$\hat{\tilde{W}}_B(x, \mu, z) = \mathrm{Li}_2(\mu^{-1} x) + \mathrm{Li}_2(\mu^{-1} x^{-1}) + \log x \log z$$

$$\text{B.E.} \Rightarrow x_* = \frac{1 - \mu z}{\mu - z}$$

$$\Rightarrow \tilde{W}_B(\mu, z) = \mathrm{Li}_2(\mu^{-1} \frac{1 - \mu z}{\mu - z}) + \mathrm{Li}_2(\mu^{-1} \frac{1 - \mu z}{\mu - z}) + \log\left(\frac{1 - \mu z}{\mu - z}\right) \log z$$

- Then the five-term dilogarithm relations directly implies:

$$\tilde{W}_A(\mu, z) = \tilde{W}_B(\mu, z) + \frac{\pi^2}{2}$$

Aharony and Giveon-Kutasov dualities

- Next consider the duality of Giveon-Kutasov relating:
  - $U(N_c)_k$ with N_f flavors.
  - $U(k+N_f-N_c)_{-k}$ with N_f flavors and N_f^2 mesons, with $W = q^a M_{ab} \tilde{q}^b$.
- The vacuum equations of the two theories are the same, and can be written:

$$p(x_i) = x_i{}^k z \prod_i (1 - x_i \mu_i) - \prod_i (x_i - \tilde{\mu}_i) = 0$$

- Let x_α, $\alpha = 1, .., k + N_f$ be the roots of p. Then the duality map is:

$$D\Big(\{x_i^* = x_\alpha, \alpha \in A\} \Big) = \{\hat{x}_i^* = x_\alpha, \alpha \in A^c\}$$

- The matching of the twisted superpotential follows from a dilogarithm identity of [Ray,1991].

Reduction of

$3d\ \mathcal{N}=2$ dualities

Reduction of 3d dualities

- To understand the reduction of 3d theories on S^1_r, we can study the corresponding limit of the twisted superpotential:

$$\tilde{W}_{2d} = \lim_{r \to 0} \tilde{W}_{3d}(r)$$

- When taking this limit, we must also choose how to scale parameters in the action:
  - **Take m** $<< 1/r$ - the m is not important for the compactification, but appears as a relevant parameter (twisted mass) in the 2d UV description.
  - **Take** $\gamma \equiv mr \sim O(1)$ - Then γ appears as a (classically) marginal parameter (Kahler moduli) in 2d. Quantum effects may renormalize γ.
- We have such a choice for each relevant parameter in the theory, and different choices lead to different 2d reductions.
- Starting with a single 3d dual pair, we can then find many new 2d dual pairs.

Example 1

- *Example:* $U(1)_{k=1/2}$ with charge one chiral:

$$2\pi i \frac{d\tilde{W}_{3d}}{du} = \log 2i\sin(\pi r u) - \pi i r u - 2\pi i r\zeta$$

- **Limit 1**: Compare to $2d$ version of this theory:

$$2\pi i \frac{d\tilde{W}_{2d}}{du} = \log\ u + t$$

  One finds $\tilde{W}_{3d} \to \tilde{W}_{2d}$ as $r \to 0$ provided we set
  $\zeta = \frac{1}{2\pi i r}(t - \log(\pi r))$.
- Claim: In this limit, we obtain the $U(1)$ gauge theory with one chiral.

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- Claim: In this limit, we obtain the $U(1)$ gauge theory with one chiral.
- **Limit 2**: Now we take ζ finite. Then one computes $< u >\sim -\frac{\log(r\zeta)}{2\pi i r}$.
- This suggests we should define a renormalized field $X = 2\pi i r u + \log \; r$, and the twisted superpotential is regular in terms of X as $r \to 0$:

$$\tilde{W}_{3d} \to e^X + \zeta X + O(r)$$

- This time we find the $\mathcal{N} = 2$ Liouville theory.

Example 2

- **Example:** $U(1)_{k=1}$ with charge $(1,-1)$ chirals:

$$2\pi i \frac{d\tilde{W}_{3d}}{du} = \log 2i \sin(\pi r(u+m)) - \log 2\sinh(\pi r(-u+m)) + 2\pi i r u + 2\pi i r\zeta$$

- Compare to the $2d$ version of this theory:

$$2\pi i \frac{d\tilde{W}_{2d}}{du} = \log(u+m) - \log(-u+m) + t$$

- Now $\tilde{W}_{3d} \to \tilde{W}_{2d}$ as $r \to 0$ provided we hold m, Σ finite and $\zeta = \frac{t}{2\pi r}$.
- Note we treated the mass $m \sim O(1)$ and FI parameter $\zeta \sim O(r^{-1})$ asymetrically in the $r \to 0$ limit, although they appear on the same footing in $3d$, and are exchanged under mirror symmetry.

Example 2 - dual description

- $3d$ mirror dual description: $U(1)_{k=-1}$ with charge $(1,1)$ chirals:

$$2\pi i \frac{d\tilde{W}_{3d}}{du} = \log 2i \sin(\pi r(u+\zeta)) + \log 2i \sin(\pi r(u-\zeta)) - 2\pi iru + 2\pi irm$$

- Taking the same limit of parameters, $m, t = 2\pi ir\zeta$ fixed, one defineds a rescaled twisted chiral field $Y = 2\pi iru - \log(r)$, which has finite VEV, and we find:

$$\tilde{W}_{3d} \rightarrow_{r\rightarrow 0} = Ym + \frac{1}{2\pi}(e^{Y+t} + e^{Y-t}) + O(r)$$

- This is precisely the $2d$ Hori-Vafa dual of the previous $2d$ gauge theory, as shown first by [Aganagic, Hori, Karch, Tong].
- This can be repeated for an arbitrary $3d$ abelian mirror symmetry pair, and one recovers the general case of Hori-Vafa duality. Thus $3d$ mirror symmetry implies $2d$ mirror symmetry.

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Example 3

- Next consider $U(N_c)_k$ with N_f flavors. Then we have:

$$\sum_{i=1}^{N_f}(\log 2i\sin(\pi r(u_a+m_i)) - \log 2i\sin(\pi r(-u_a+\tilde{m}_i))) + 2\pi irku_a + 2\pi ir\zeta$$

  Taking masses finite and $2\pi ir\zeta = t$, one finds the twisted superpotential of the $2d$ $U(N_c)$ theory:

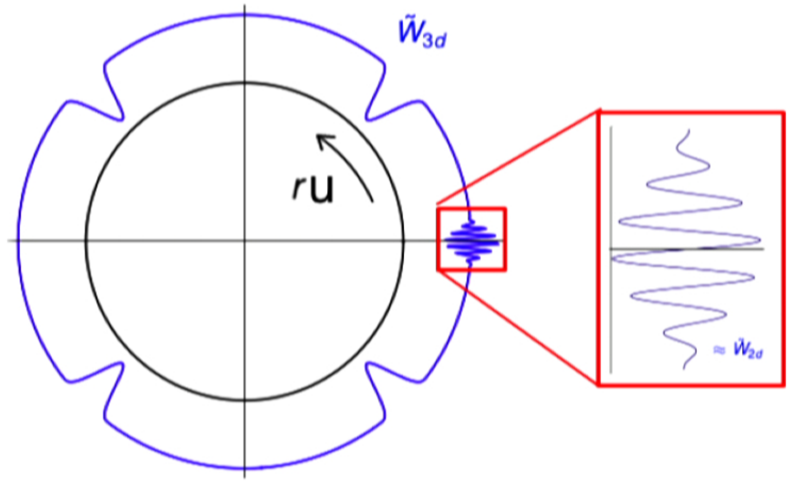
$$2\pi i\frac{d\tilde{W}_{2d}}{du_a} = \sum_{i=1}^{N_f}(\log(u_a+m_i) - \log(-u_a+m_i)) + t$$

- Suggests we find a $2d$ $U(N_c)$ theory with N_f flavors - however, *this is inconsistent with Giveon-Kutasov duality*, as it would imply this is dual to $U(N_f+k-N_c)$ with N_f flavors for any k, which is clearly false.
- Another problem: the theory above has only $\binom{N_f}{N_c}$ vacua - we must be missing some!

Example 3 (cont'd)

- The resolution is that there are k additional solutions with $u \sim r^{-1}$:

$$u \approx \frac{-t + in}{kr}, \quad n = 0, ..., k-1$$



- Thus when we pick vacua, we can let $\ell \leq k$ of the eigenvalues to be large, and the remaining $N - \ell$ will form a $2d$ $U(N - \ell)$ gauge theory. These sectors decouple from each other, and we find a direct sum:

$$U(N)_{N_f flavors} \oplus U(N-1)_{N_f flavors} \oplus ... \oplus U(N-k)_{N_f flavors}$$

- The $3d$ Giveon-Kutasov dual reduces similarly as (suppressing mesons):

$$U(k + N_f - N)_{N_f flavors} \oplus U(k + N_f - N - 1)_{N_f flavors} \oplus ... \oplus U(N_f - N)_{N_f flavors}$$

- These direct-sum theories are related termwise by the $2d$ Seiberg-like duality of [Benini,Park,Zhao] relating $U(N)_{N_f flavors}$ to $U(N_f - N)_{N_f flavors}$.

Some other examples

- 3d duality: "Duality appetizer" $SU(2)_{k=1}+$ adjoint dual to free chiral. [Jafferis,Yin]:

  $\Rightarrow$ 2d duality: $W = XY^2/\mathbb{Z}_2$ dual to free chiral [Hori]. Here one must scale $\Sigma \sim r^{-1/2}$.

- 3d duality: $SU(N_c)_k$ (N_f, N_a) (anti-)fundamental chirals dual to $SU(N_f - N_c)_{-k}$ with (N_f, N_a) for $N_f > N_a + 1,\ k < \frac{N_f - N_a}{2}$. [Aharony,Fleischer]

  $\Rightarrow$ 2d duality: $SU(N_c)$ (N_f, N_a) dual to $SU(N_f - N_c)$ (N_f, N_a) for $N_f > N_a + 1 \rightarrow$ generalization of [Hori,Tong].

- S^2 partition functions match in all these examples, as follows by reducing the 3d index identites.
- One can also check the elliptic genera (T^2 partition functions) also match $\rightarrow$ independent checks of the dualities.

Summary

- The twisted superpotential is a probe which explores many aspects of a 3*d* $\mathcal{N} = 2$ theory.
- It determines the supersymmetric vacua and the algebra of Wilson loops.
- We computed the partition function on wide class of manifolds, and showed it can be expressed in terms of data associated to the supersymmetric vacua.
- We described subtleties that arise in compactifying 3*d* dualites, and how the twisted superpotential gives control over these.

Outlook

- Study the reduction of more examples of 3*d* dualities, and of higher dimensional systems.
  - E.g., 4*d* theories on a Riemann surface.
- Many applications for $\mathcal{M}_{g,p}$ partition function to 3*d* $\mathcal{N} = 2$ theories;
  - Supersymmetric dualities
  - Large *N* calculations and AdS/CFT comparisons
  - $3d - 3d$ correspondence
- Connections to integrable systems through the gauge-Bethe correspondence
  - What is the meaning of the fibering operator in this context?
- Extend to more general Seifert manifolds and to 4*d* theories
  - Involves understanding the algebra of half-BPS surface operators.