Title: Tensor networks and the renormalization group

Date: Nov 09, 2016 02:00 PM

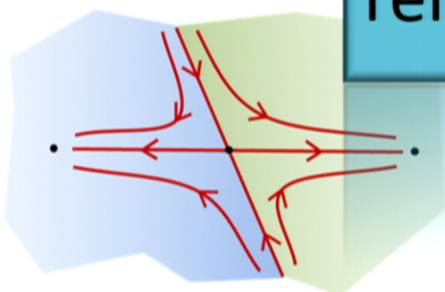
URL: http://pirsa.org/16110045

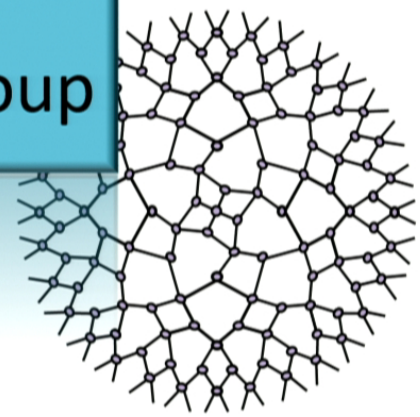
Abstract: <p>Tensor networks offer an efficient representation of many-body wave-functions in an exponentially large Hilbert space by exploiting the area law of ground state quantum entanglement. I will start with a gentle introduction to the tensor network formalism. Then I will describe its application to realizing Wilson's renormalization group directly on quantum lattice models (e.g. quantum spin chains), with emphasis on the RG fixed points corresponding to conformal field theories. We will see how to define both global and local scale transformations on the lattice in such a way that, intriguingly, conformal invariance can be tested (and conformal data extracted) right at the UV cut-off scale.</p>

Perimeter Nov 9th 2016

COLLOQUIUM

Tensor networks and the renormalization group





Guifre Vidal

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TENSOR NETWORKS INITIATIVE

SIMONS FOUNDATION

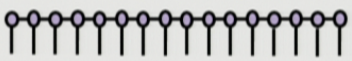
NSERC

Outline

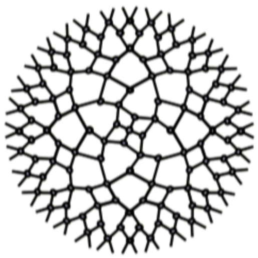
1 - Entanglement and tensor networks

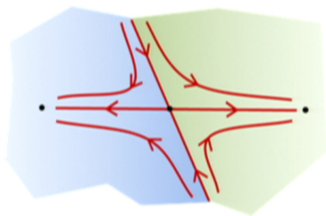
$S(A) \sim |\partial A|$

area law



2 - Application: The renormalization group

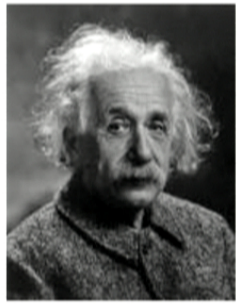




Quantum Mechanics

1920-1930


Niels Bohr


Albert Einstein


Wolfgang Pauli


Paul Dirac

$$i\hbar\frac{\partial}{\partial t}\Psi = \hat{H}\Psi$$

Schrodinger equation


Erwin Schrodinger


Enrico Fermi


Werner Heisenberg


Richard Feynman

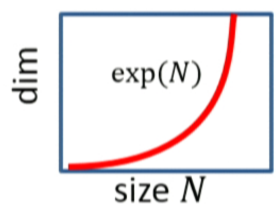
There is a problem…

Paul Dirac

"The fundamental laws necessary for the mathematical treatment of a large part of physics and the whole of chemistry are thus completely known,

and the difficulty lies only in the fact that application of these laws leads to equations that are too complex to be solved."

number of spins		Hilbert space dimension	
$N=1$	●	$2^1 = 2$	$\lvert\Psi\rangle = c_1\lvert\uparrow\rangle + c_2\lvert\downarrow\rangle$
$N=2$	● ●	$2^2 = 4$	$\lvert\Psi\rangle = c_1\lvert\uparrow\uparrow\rangle + c_2\lvert\uparrow\downarrow\rangle + c_3\lvert\downarrow\uparrow\rangle + c_4\lvert\downarrow\downarrow\rangle$
$N=3$	● ● ●	$2^3 = 8$	$\lvert\Psi\rangle = c_1\lvert\uparrow\uparrow\uparrow\rangle + c_2\lvert\uparrow\uparrow\downarrow\rangle + c_3\lvert\uparrow\downarrow\uparrow\rangle + c_4\lvert\uparrow\downarrow\downarrow\rangle + c_5\lvert\downarrow\uparrow\uparrow\rangle + c_6\lvert\downarrow\uparrow\downarrow\rangle + c_7\lvert\downarrow\downarrow\uparrow\rangle + c_8\lvert\downarrow\downarrow\downarrow\rangle$
$N=4$	● ● ● ●	$2^4 = 16$	$\lvert\Psi\rangle = c_1\lvert\uparrow\uparrow\uparrow\uparrow\rangle + c_2\lvert\uparrow\uparrow\uparrow\downarrow\rangle + c_3\lvert\uparrow\uparrow\downarrow\uparrow\rangle + c_4\lvert\uparrow\uparrow\downarrow\downarrow\rangle + c_5\lvert\uparrow\downarrow\uparrow\uparrow\rangle + c_6\lvert\uparrow\downarrow\uparrow\downarrow\rangle + c_7\lvert\uparrow\downarrow\downarrow\uparrow\rangle + c_8\lvert\uparrow\downarrow\downarrow\downarrow\rangle + c_9\lvert\downarrow\uparrow\uparrow\uparrow\rangle + c_{10}\lvert\downarrow\uparrow\uparrow\downarrow\rangle + c_{11}\lvert\downarrow\uparrow\downarrow\uparrow\rangle + c_{12}\lvert\downarrow\uparrow\downarrow\downarrow\rangle + c_{13}\lvert\downarrow\downarrow\uparrow\uparrow\rangle + c_{14}\lvert\downarrow\downarrow\uparrow\downarrow\rangle + c_{15}\lvert\downarrow\downarrow\downarrow\uparrow\rangle + c_{16}\lvert\downarrow\downarrow\downarrow\downarrow\rangle$
N	● ● ● ··· ●	$dim = 2^N$	



example: $N = 16 \times 16 = 256$ spins

$\dim = 2^{256} \approx 10^{77}$ complex numbers

10GB laptop

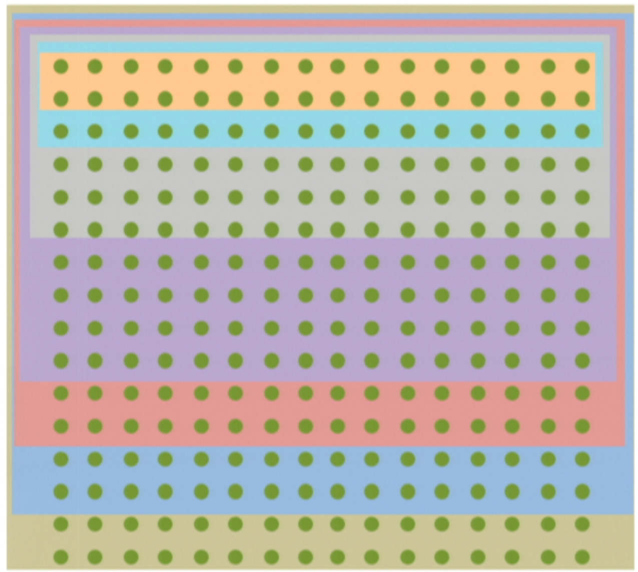
10^9 complex numbers

$N = 32$ spins

petabyte (10^{15} bytes) supercomputer

10^{14} complex numbers

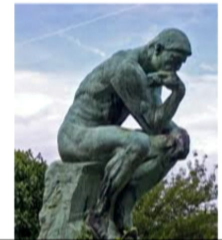
$N = 48$ spins



# of atoms in

human being

10^{28}



earth

10^{50}

solar system

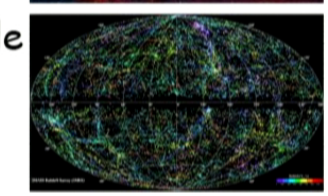
10^{57}

milky way

10^{69}

observable universe

10^{80}



Entanglement

quantum spin chain

$\mathcal{H}^A \otimes \mathcal{H}^B$

$$|\Psi^{AB}\rangle \stackrel{?}{=} |\psi^A\rangle \otimes |\varphi^B\rangle$$

Def.: product state $\quad |\Psi^{AB}\rangle = |\psi^A\rangle \otimes |\varphi^B\rangle$

example: $|\uparrow\uparrow \cdots \uparrow\uparrow^A\rangle \otimes |\uparrow\uparrow \cdots \uparrow\uparrow^B\rangle$

Def.: entangled state $\quad |\Psi^{AB}\rangle \neq |\psi^A\rangle \otimes |\varphi^B\rangle$

example:

$$\frac{1}{\sqrt{2}}|\uparrow\uparrow \cdots \uparrow\uparrow^A\rangle \otimes |\uparrow\uparrow \cdots \uparrow\uparrow^B\rangle + \frac{1}{\sqrt{2}}|\downarrow\downarrow \cdots \downarrow\downarrow^A\rangle \otimes |\downarrow\downarrow \cdots \downarrow\downarrow^B\rangle$$

How much entanglement does $|\Psi^{AB}\rangle$ a state have?

Def.: entanglement entropy $S(\Psi^{AB}) \equiv -Tr\,(\rho^A \log_2 \rho^A)$

$$\rho^A \equiv tr_B\,(|\Psi^{AB}\rangle\langle\Psi^{AB}|)$$

example: $\frac{1}{\sqrt{2}}|\uparrow\uparrow\cdots\uparrow\uparrow^A\rangle \otimes |\uparrow\uparrow\cdots\uparrow\uparrow^B\rangle + \frac{1}{\sqrt{2}}|\downarrow\downarrow\cdots\downarrow\downarrow^A\rangle \otimes |\downarrow\downarrow\cdots\downarrow\downarrow^B\rangle$

$$\rho^A = \frac{1}{2}|\uparrow\uparrow\cdots\uparrow\uparrow^A\rangle\langle\uparrow\uparrow\cdots\uparrow\uparrow^A| + \frac{1}{2}|\downarrow\downarrow\cdots\downarrow\downarrow^A\rangle\langle\downarrow\downarrow\cdots\downarrow\downarrow^A|$$

$S(\Psi^{AB}) = 1$ — 1 unit of entanglement

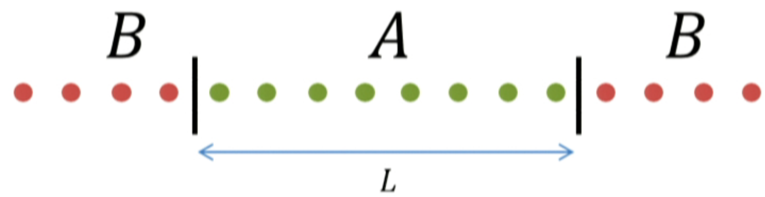
example: $|\uparrow\uparrow\cdots\uparrow\uparrow^A\rangle \otimes |\uparrow\uparrow\cdots\uparrow\uparrow^B\rangle$

$$\rho^A = |\uparrow\uparrow\cdots\uparrow\uparrow^A\rangle\langle\uparrow\uparrow\cdots\uparrow\uparrow^A|$$

$S(\Psi^{AB}) = 0$ — 0 units of entanglement

Scaling of entanglement

quantum spin chain



generic state $S(L) \approx L$

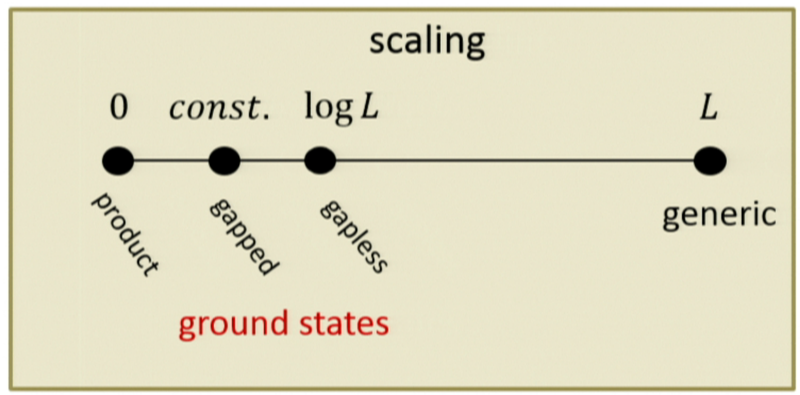
product state $S(L) = 0$

$|\uparrow\uparrow \cdots \uparrow\uparrow\rangle$

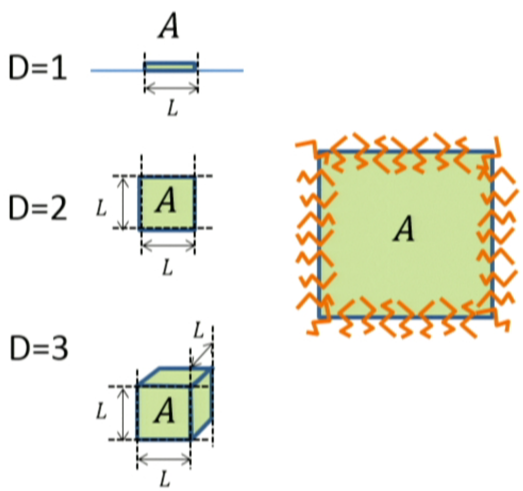
ground state of
local Hamiltonians

massive/gapped $S(L) = const.$

critical/gapless $S(L) = \log(L)$



Ground states of local Hamiltonians obey an entanglement area law



area law
for entanglement entropy

$$S(A) \sim |\partial A| \sim L^{D-1}$$

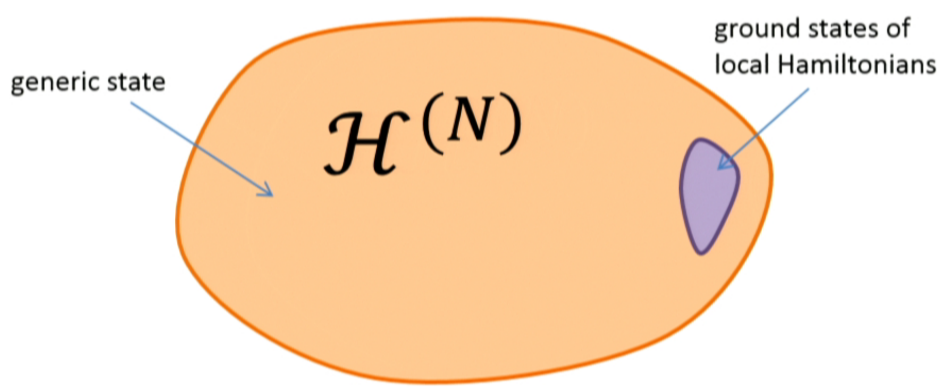
sometimes,
logarithmic corrections

$$S(A) \sim L^{D-1} \log(L)$$

instead of
volume law

$$S(A) \sim |A| \sim L^{D}$$

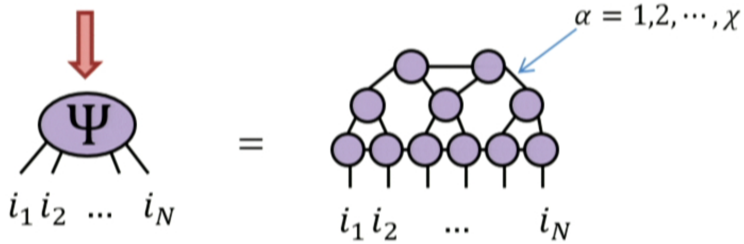
Ground states of local Hamiltonians are special/non-generic states



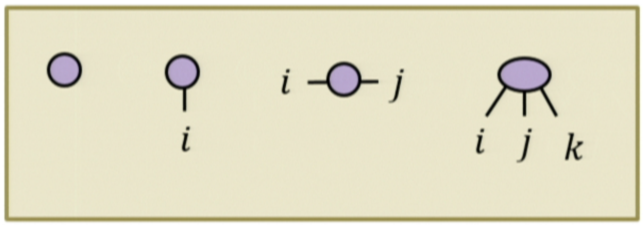
Tensor network state: variational many-body wavefunction

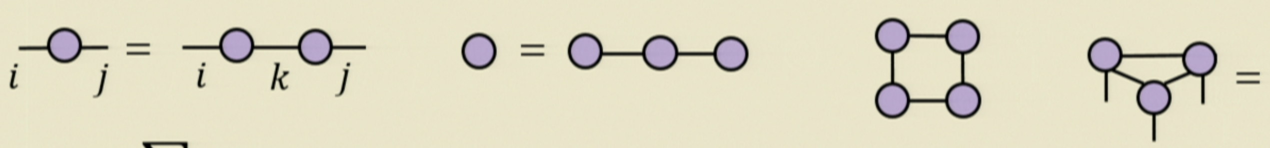
many-body wave-function

$$|\Psi\rangle = \sum_{i_1, i_2, \cdots, i_N} \Psi_{i_1 i_2 \cdots i_N} |i_1 i_2 \cdots i_N\rangle$$



graphical notation

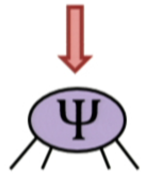




$T_{ij} = \sum_k R_{ik} S_{kj}$ $\quad a = \vec{y}^\dagger \cdot M \cdot \vec{x}$ $\quad tr(ABCD)$

$$|\Psi\rangle = \sum_{i_1, i_2, \cdots, i_N} \Psi_{i_1 i_2 \cdots i_N} |i_1 i_2 \cdots i_N\rangle$$

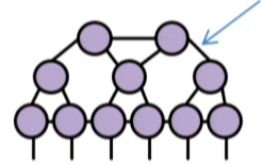
1 tensor
of size 2^N



$i_1 \, i_2 \; \dots \; i_N$

=

$\alpha = 1, 2, \cdots, \chi$



$i_1 \, i_2 \; \dots \; i_N$

$O(N)$ tensors
of size χ^p independent of N

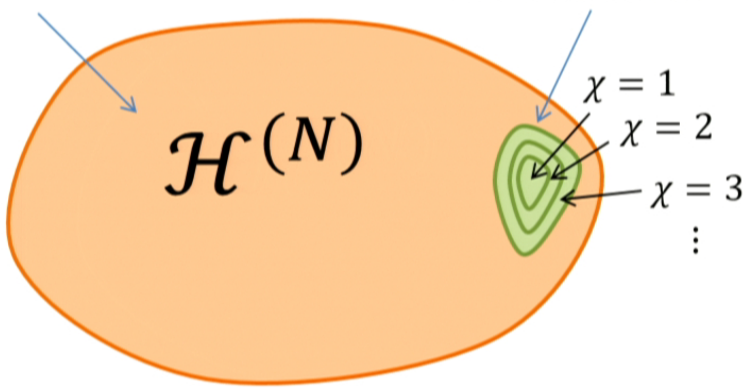
2^N
parameters
inefficient

→

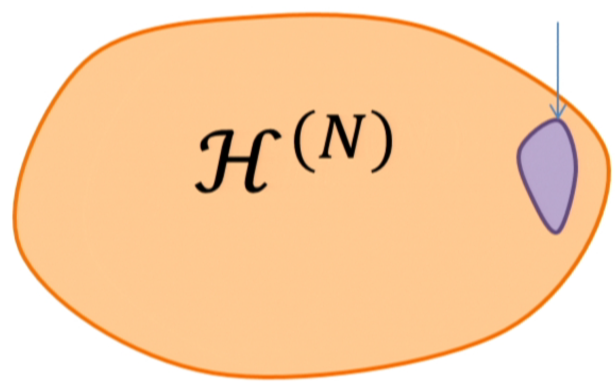
$O(N)$
parameters
efficient

generic states

tensor network states



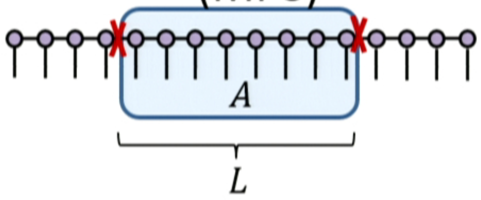
ground states of
local Hamiltonians



Scaling of entanglement

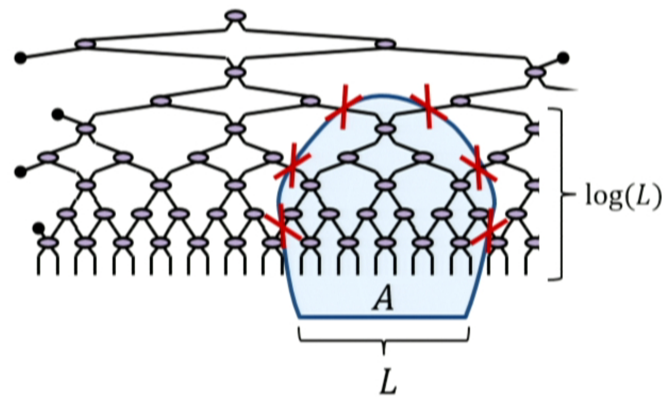
matrix product state

(MPS)



multi-scale entanglement renormalization ansatz

(MERA)



network connectivity:

$S(A) \approx const$

area law!

$S(A) \approx \log L$

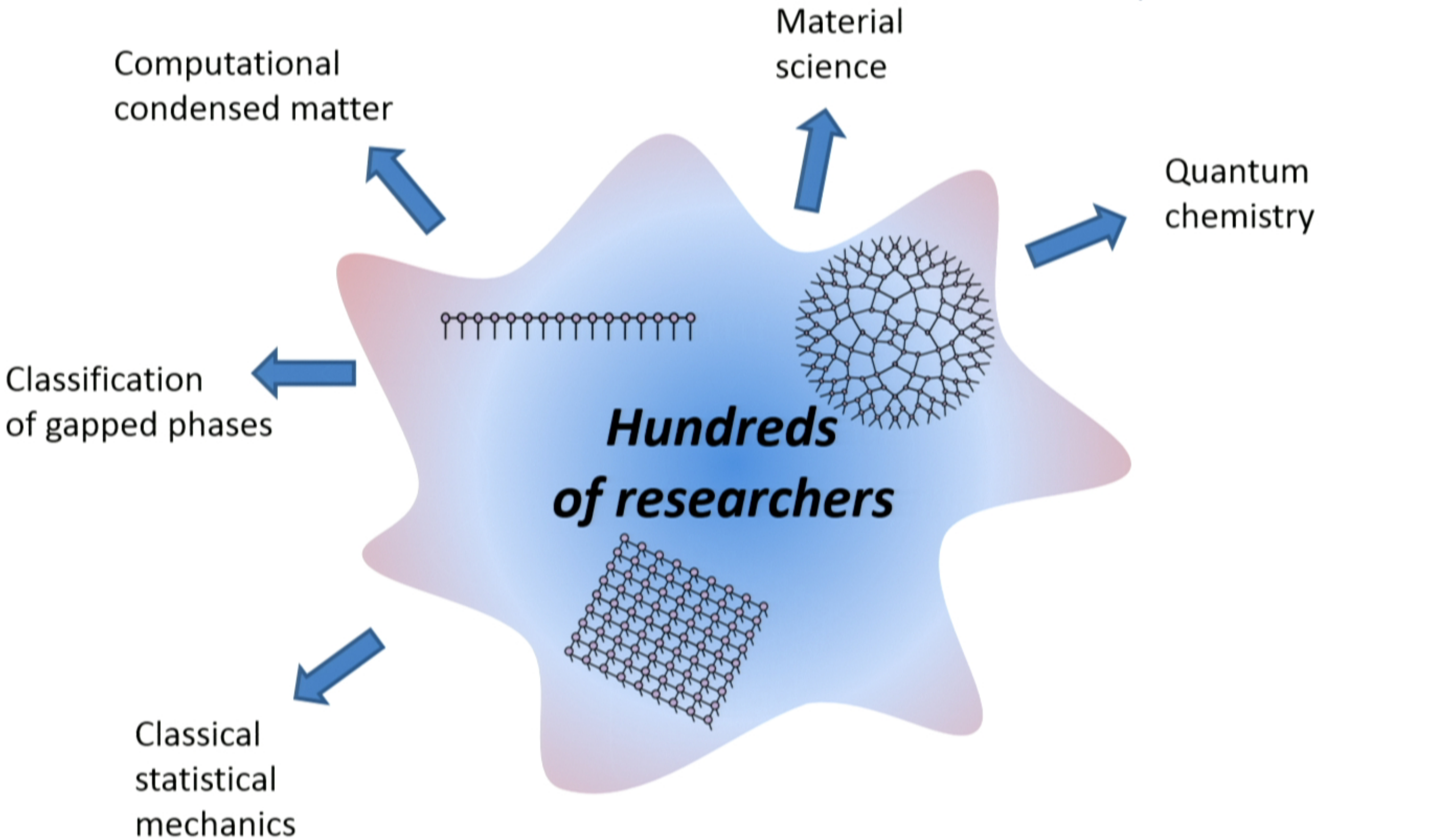
logarithmic correction!

just as ground state of
massive/gapped
1d system

just as ground state of
critical/gapless
1d system

CURRENT APPLICATIONS

tensor network =
sparse data structure



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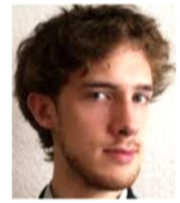
  - FROM DISCRETUUM TO CONTINUUM

TENSOR NETWORKS INITIATIVE

FROM ENTANGLED QUANTUM MATTER TO EMERGENT SPACE TIME

Tensor networks have in recent years emerged as a powerful tool to describe strongly entangled quantum many-body systems. Applications range from the study of collective phenomena in condensed matter to providing a discrete realization of the holographic principle in quantum gravity.



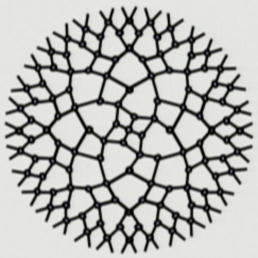
Ash Milsted
PI postdoc

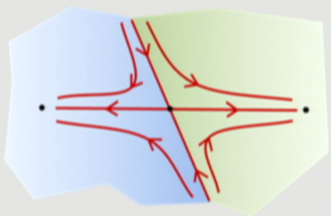
1 – Entanglement and tensor networks

$S(A) \sim |\partial A|$

area law

2 – Application: The renormalization group

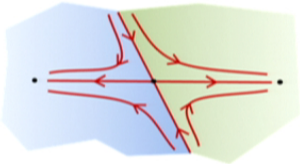




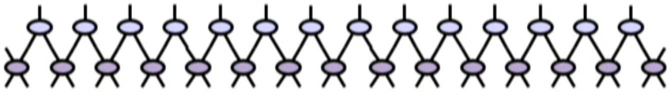
Part II

Application: the Renormalization Group

A – The renormalization group and its fixed points

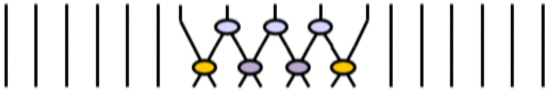


B – Global scale transformation/invariance



Glen Evenbly
University of Sherbrooke

C – Local scale transformation/invariance



The Renormalization Group

Hamiltonian

$$H[\vec{k}] \longrightarrow H[\vec{k}(s)]$$

scale

coupling constants

$$\vec{k} = (k_1, k_2, k_3, \cdots)$$

Ising model

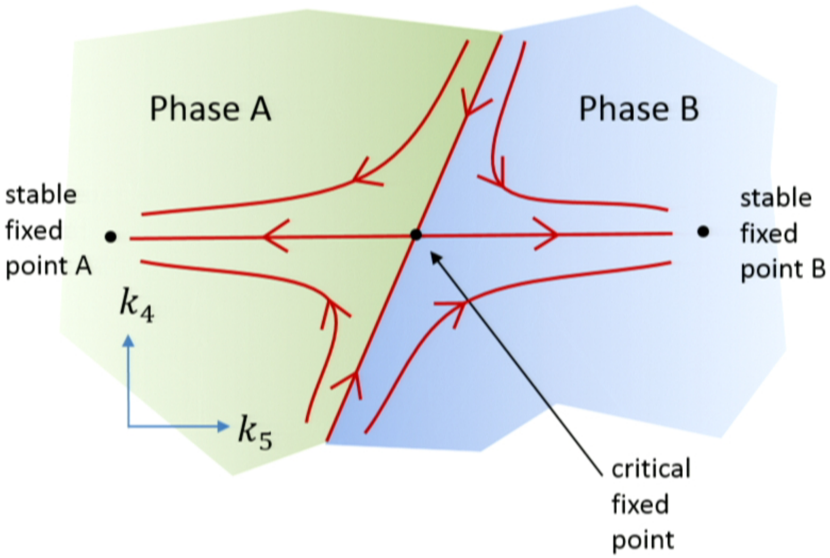
$$H[J,\lambda] = J\sum_i \sigma_i^x \sigma_j^x + \lambda \sum_i \sigma_i^z$$

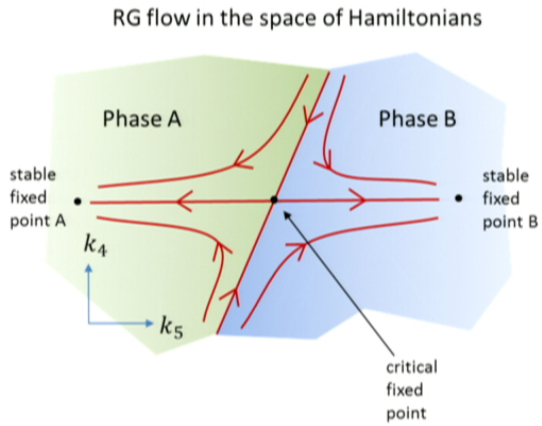
$$H[k_1, k_2, 0, 0, \cdots] = k_1 \sum_i \sigma_i^x \sigma_j^x + k_2 \sum_i \sigma_i^z$$

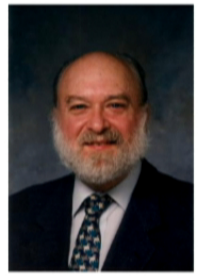
Leo Kadanoff

Kenneth Wilson

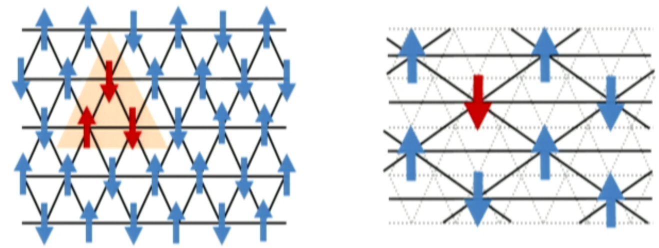
RG flow in the space of Hamiltonians







Leo Kadanoff



block spin

+ some rule: majority vote, etc

Change of scale?

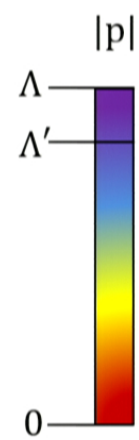
coarse-graining transformation

Kenneth Wilson

$$Z = \int_{|p|\leq\Lambda} \mathcal{D}\phi e^{-H[\phi,\vec{k}]}$$

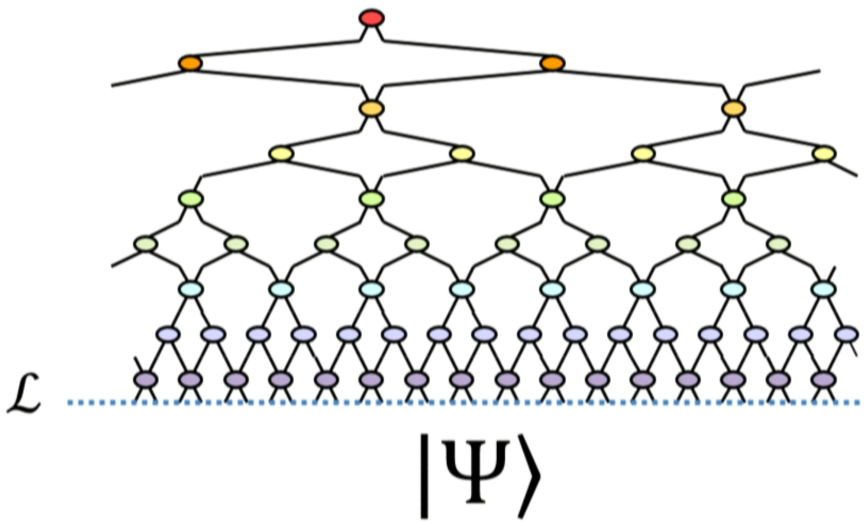
$$e^{-H[\phi,\vec{k}']} = \int_{\Lambda' \leq |p| \leq \Lambda} \mathcal{D}\phi e^{-H[\phi,\vec{k}]}$$

$$Z = \int_{|p|\leq\Lambda'} \mathcal{D}\phi e^{-H[\phi,\vec{k}']}$$

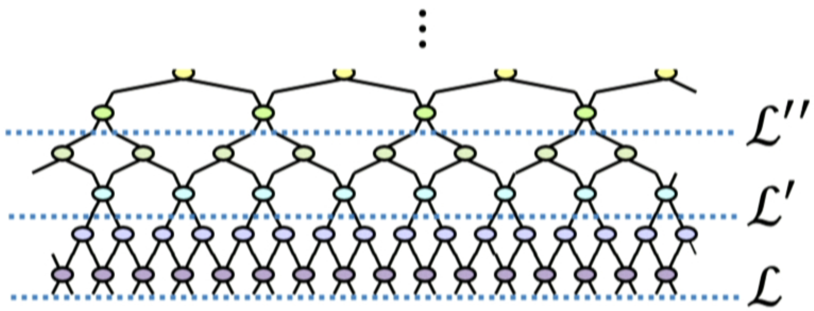


exact renormalization group equation (ERGE)

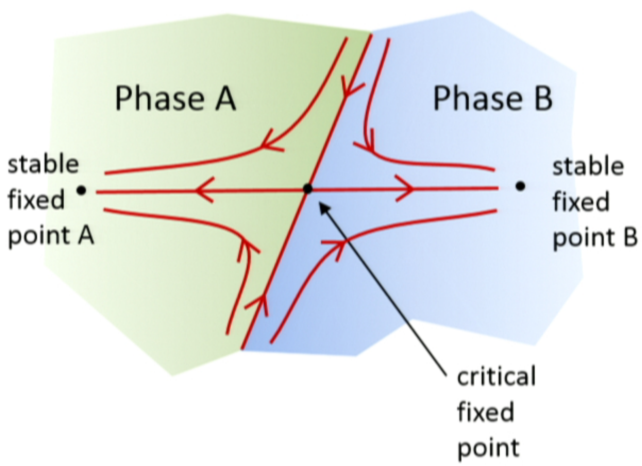
MERA as a Renormalization Group transformation

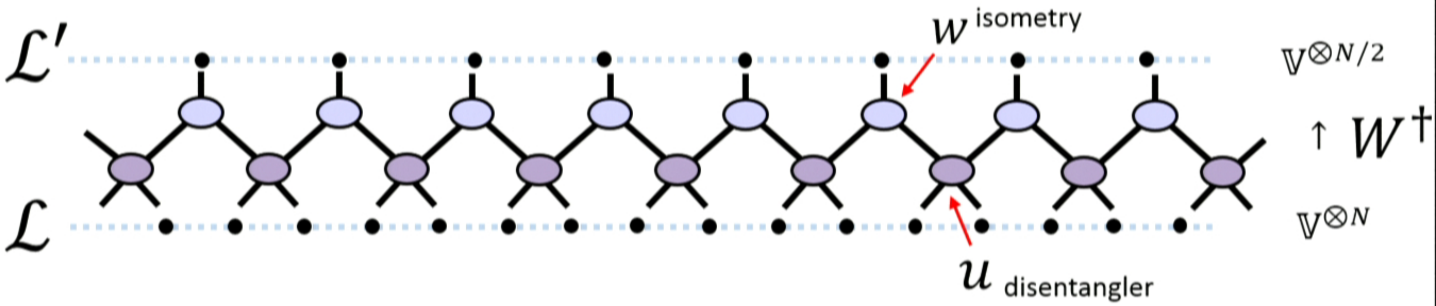


MERA defines an RG flow
in the space of wave-functions

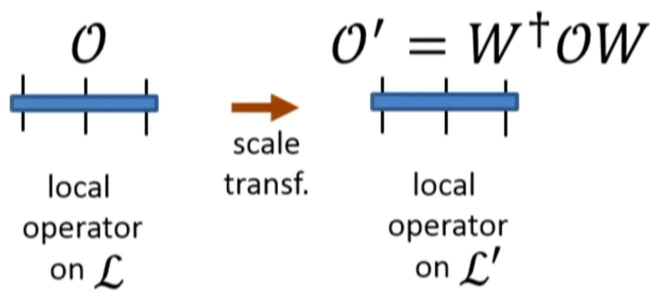


$$|\Psi\rangle \to |\Psi'\rangle \to |\Psi''\rangle \to \cdots$$

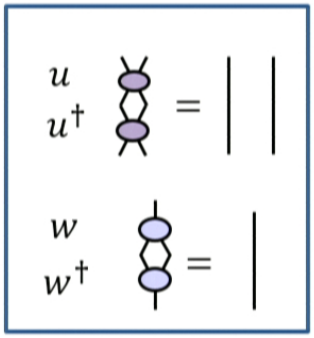




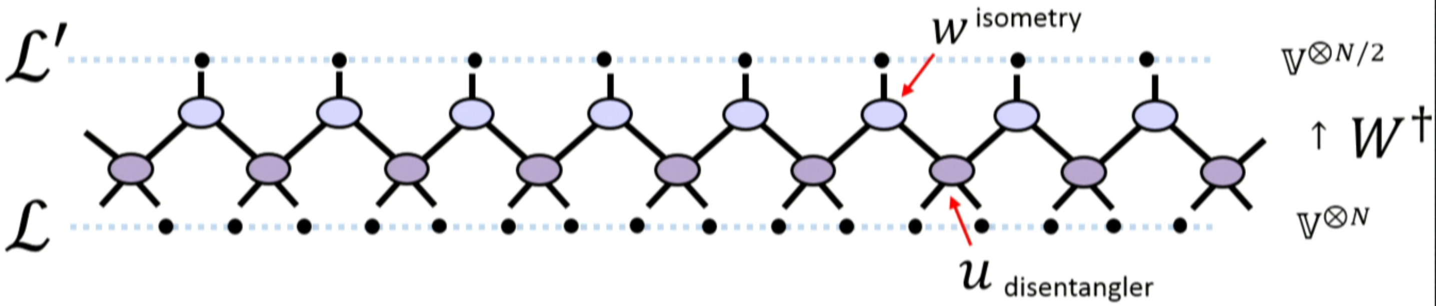
$\mathcal{L}'$
w isometry
$\mathbb{V}^{\otimes N/2}$
$\uparrow W^\dagger$
$\mathcal{L}$
$\mathbb{V}^{\otimes N}$
u disentangler



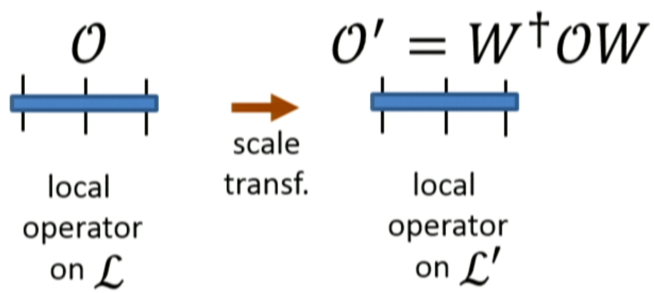
$\mathcal{O}$
$\mathcal{O}' = W^\dagger \mathcal{O} W$
scale transf.
local operator on $\mathcal{L}$
local operator on $\mathcal{L}'$



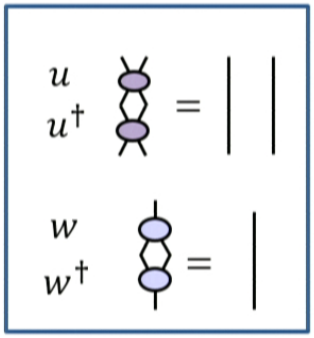
u
$u^\dagger$
=
w
$w^\dagger$
=



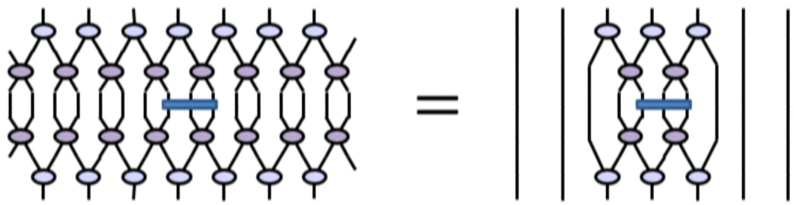
$\mathcal{L}'$
w isometry
$\mathbb{V}^{\otimes N/2}$
$\uparrow W^\dagger$
$\mathcal{L}$
$\mathbb{V}^{\otimes N}$
u disentangler



$\mathcal{O}$
local operator on $\mathcal{L}$
scale transf.
$\mathcal{O}' = W^\dagger \mathcal{O} W$
local operator on $\mathcal{L}'$



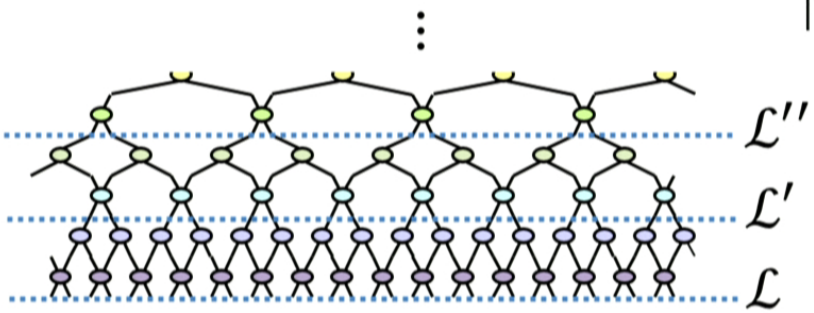
u
$u^\dagger$
$=$
w
$w^\dagger$
$=$

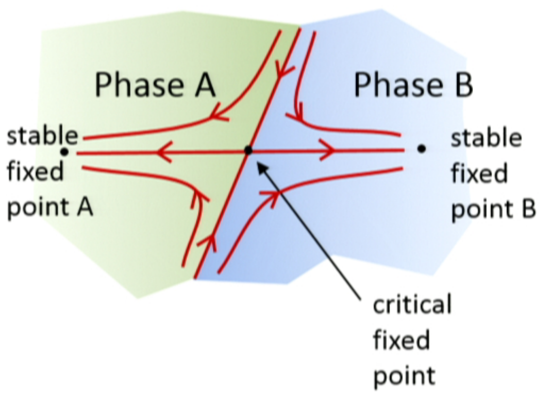


$=$

MERA defines an RG flow in the space of wave-functions:

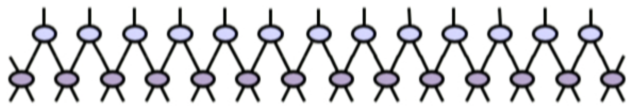
$$|\Psi\rangle \to |\Psi'\rangle \to |\Psi''\rangle \to \cdots$$



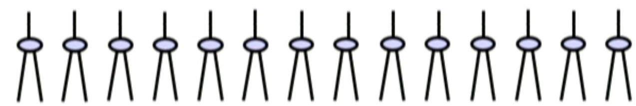


What is a proper RG transformation on the lattice?

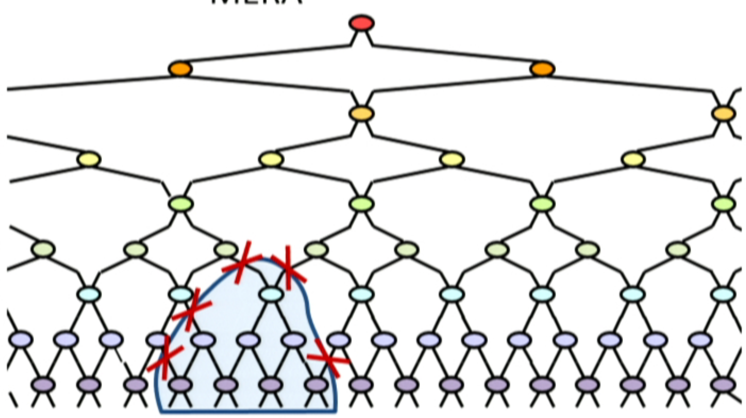
why use this?



and not use this? (closer to Kadanoff spin-blocking)

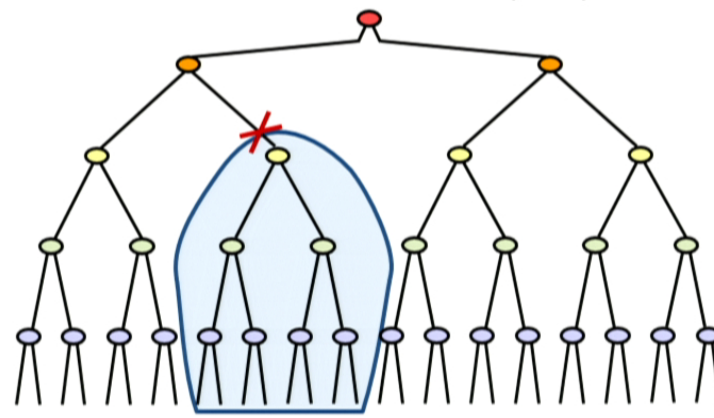


MERA



$S(A) \approx \log L$

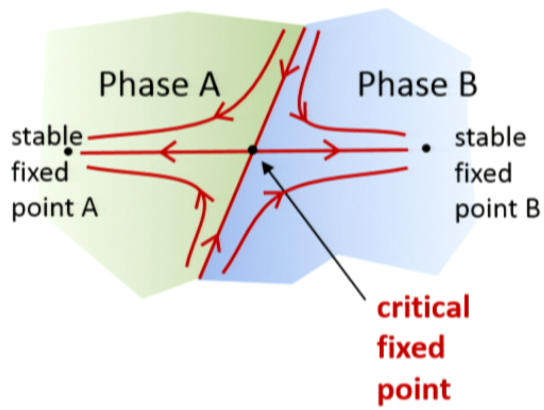
tree tensor network (TTN)



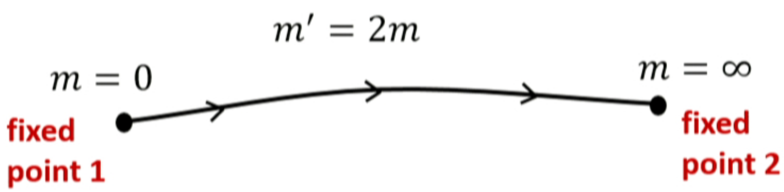
$S(A) \approx const.$

What is a proper RG transformation on the lattice?

MERA RG flow seems to be qualitatively correct
= consistent with expectation from QFT



e.g.: when adding a mass m in a free particle theory, mass grows with coarse-graining transformation as expected:



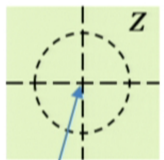
Is the MERA RG flow quantitatively correct?

Let us analyze critical fixed-points

In the continuum, conformal field theory (CFT) critical RG fixed-point = Conformal field theory (CFT)

1) scaling operators $\phi_\alpha(x)$

complex plane

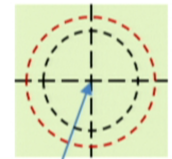


$\phi_\alpha(0)$

scale transformation

re-scaling

$z \to e^{\lambda} z$



$e^{\Delta_\alpha \lambda} \phi_\alpha(0)$

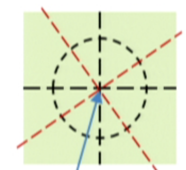
scaling dimension

Lorentz boost

(= rotation in Euclidean time)

rotation

$z \to e^{i\theta} z$



$e^{i s_\alpha \theta} \phi_\alpha(0)$

conformal spin

example: CFT for critical Ising model

$$H = \sum_i \sigma_i^x \sigma_j^x + \sum_i \sigma_i^z$$

primary fields

$$\{\mathbb{1}, \sigma, \varepsilon\}$$

spin

identity

energy density

scaling dimensions and conformal spins

$$(\Delta_{\mathbb{1}}, s_{\mathbb{1}}) = (0,0)$$

$$(\Delta_\sigma, s_\sigma) = (1/8,0)$$

$$(\Delta_\varepsilon, s_\varepsilon) = (1,0)$$

Ising CFT

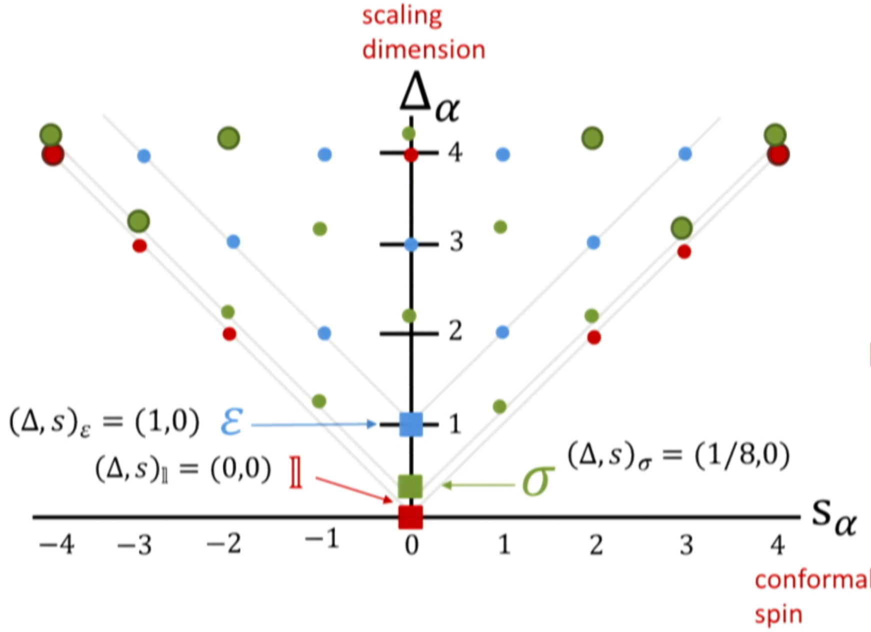
primary fields $\{\mathbb{1}, \sigma, \varepsilon\}$

spin

identity

energy density

global scale/rotation invariance + locality = local scale/rotation invariance (conformal group)



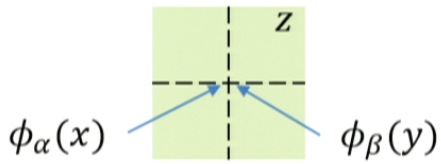
2) conformal towers

under conformal transformations, scaling operators mix

conformal towers = irreducible representations of the conformal group

primary field = highest weight state

3) Operator product expansion (OPE)



$$\phi_\alpha(x)\phi_\beta(y) \sim \sum_\gamma C_{\alpha\beta\gamma}(x-y)\phi_\gamma(x)$$

CFT completely specified by conformal data:

- central charge c
- list of primary fields $\{\phi_\alpha\}$ with their
  - scaling dimensions $\Delta_\alpha \equiv h_\alpha + \bar{h}_\alpha$
  - conformal spin $s_\alpha \equiv h_\alpha - \bar{h}_\alpha$
  - OPE coefficients $C_{\alpha\beta\gamma}$

example: Ising CFT

central charge $c = \frac{1}{2}$

primary fields

$\{\mathbb{1}, \sigma, \varepsilon\}$ ($\mathbb{1}$: identity, σ: spin, ε: energy density)

scaling dimensions and conformal spins

$$(\Delta_{\mathbb{1}}, s_{\mathbb{1}}) = (0,0)$$

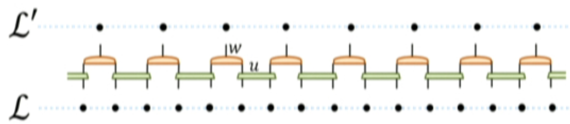
$$(\Delta_\sigma, s_\sigma) = (1/8,0)$$

$$(\Delta_\varepsilon, s_\varepsilon) = (1,0)$$

OPE

$$C_{\epsilon\sigma\sigma} = \frac{1}{2}$$

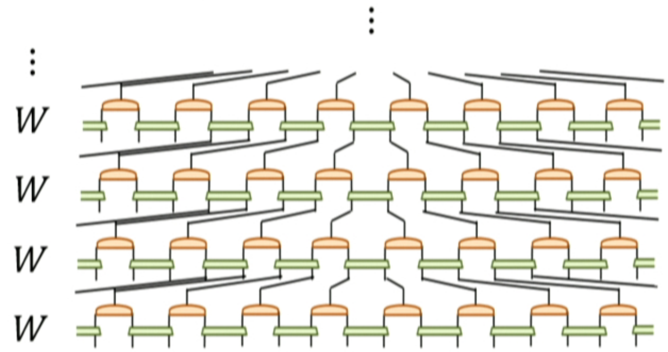
With notion of scale/RG transformation on the lattice,



$\equiv W$

define notion of (discrete) **scale invariance**

example: scale-invariant wavefunction



$|\Psi\rangle$

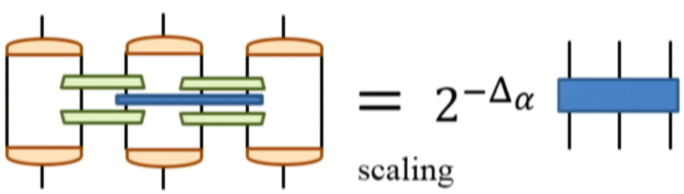
> Def.: $|\Psi\rangle$ is scale invariant
>
> iff $\quad W|\Psi\rangle = |\Psi\rangle$

Intriguing...

scale invariance with a length scale (UV cut-off)

Is this a useful notion of scale transformation / scale invariance ?

We can access the correct *conformal data* of underlying CFT from the lattice



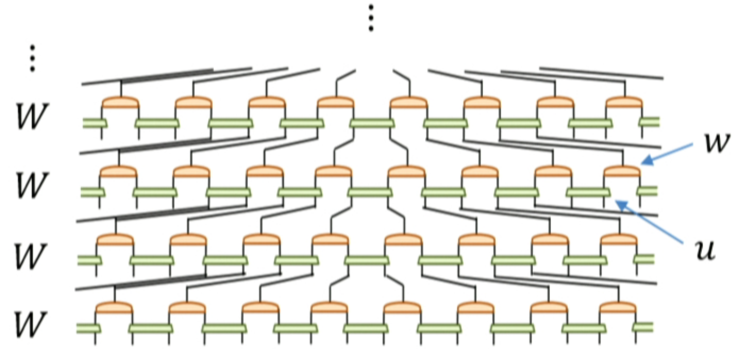
$= 2^{-\Delta_\alpha}$ [scaling operators]

- scaling dimensions $\quad \Delta_\alpha \equiv h_\alpha + \bar{h}_\alpha$
- OPE coefficients $C_{\alpha\beta\gamma}$

e.g. critical Ising model

critical Hamiltonian

$$H = \sum_i \sigma_i^x \sigma_j^x + \sum_i \sigma_i^z$$



ansatz $|\Psi\rangle \leftrightarrow (u, w)$

isometry

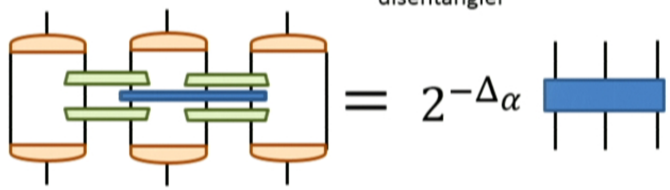
disentangler

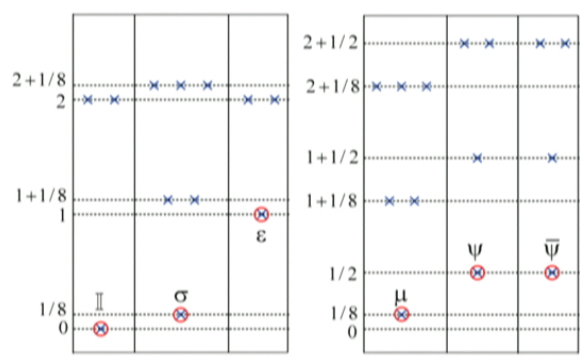
1) variational optimization

$$\min_{u,w} \langle \Psi | H | \Psi \rangle$$

(approx. an hour on your laptop)

2) diagonalize linear map





scaling dimensions

$(\Delta_{\mathbb{I}} = 0)$

$\Delta_\sigma \approx 0.124997$

$\Delta_\varepsilon \approx 0.99993$

$\Delta_\mu \approx 0.125002$

$\Delta_\psi \approx 0.500001$

$\Delta_{\bar\psi} \approx 0.500001$

OPE

$C_{\epsilon\sigma\sigma} = \frac{1}{2}$

$C_{\epsilon\mu\mu} = -\frac{1}{2}$

$C_{\epsilon\psi\bar\psi} = i$

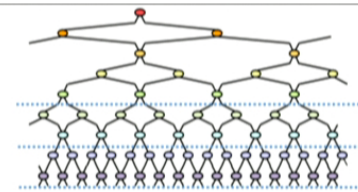
$C_{\epsilon\bar\psi\psi} = -i$

$C_{\psi\mu\sigma} = \frac{e^{-\frac{i\pi}{4}}}{\sqrt{2}}$

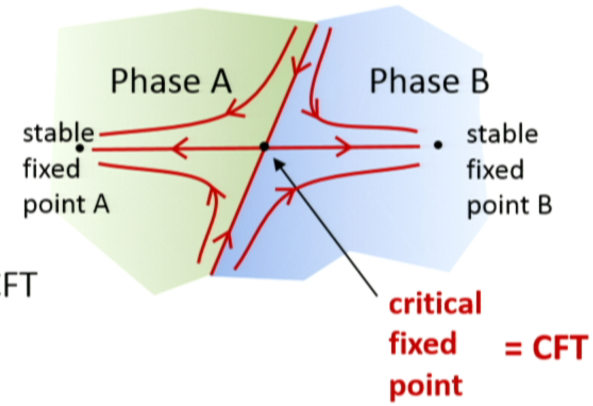
$C_{\bar\psi\mu\sigma} = \frac{e^{\frac{i\pi}{4}}}{\sqrt{2}}$

$(\pm 6 \times 10^{-4})$

What is a proper RG transformation on the lattice?

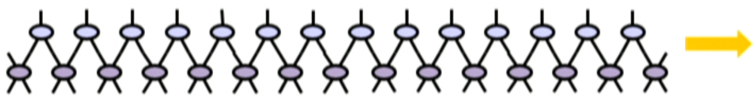


MERA RG flow seems to be qualitatively correct
= consistent with expectation from QFT



At a critical RG fixed-point, MERA accurately reproduces the universal data of the underlying CFT

RG transformation $\sim$ global scale transformation



- scaling dimensions Δ_α
- OPE coefficients $C_{\alpha\beta\gamma}$

Glen Evenbly

- conformal spins s_α?
- which scaling operators ϕ_α are primary operators?

answer will come from defining/studying local scale transformations on the lattice another **CFT fact** :

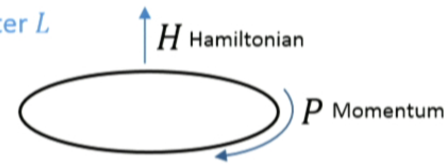
A CFT on a circle of perimeter L has
energy and momentum spectra given by

$$E_\alpha = \frac{2\pi}{L}\left(\Delta_\alpha - \frac{c}{12}\right)$$

$$p_\alpha = \frac{2\pi}{L}\, s_\alpha$$

continuum → discrete

circle of perimeter L



periodic chain made of L sites

$$H|\Psi_\alpha\rangle = E_\alpha|\Psi_\alpha\rangle$$

$$P|\Psi_\alpha\rangle = p_\alpha|\Psi_\alpha\rangle$$

$$H|\Psi_\alpha\rangle = E_\alpha|\Psi_\alpha\rangle$$ continuous time translations

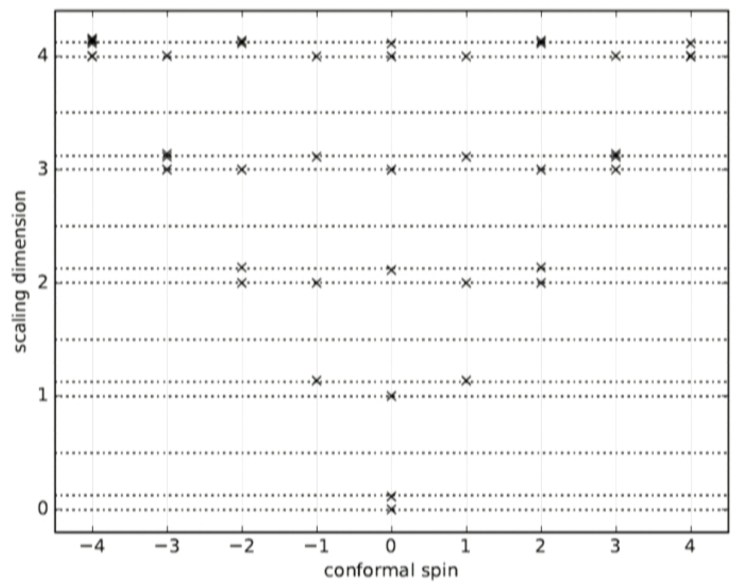
$$T|\Psi_\alpha\rangle = e^{ip_\alpha}|\Psi_\alpha\rangle$$ discrete space translations

DVRC

John Cardy 1989

$$E_\alpha = \frac{2\pi}{L}\left(\Delta_\alpha - \frac{c}{12}\right)$$

$$p_\alpha = \frac{2\pi}{L}\, s_\alpha$$



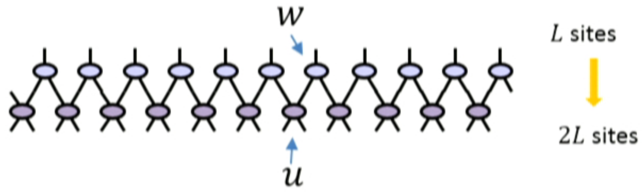
example:
(coarse-grained)
Ising model
L = 8 sites

→
- scaling dimensions Δ_α
- conformal spins s_α

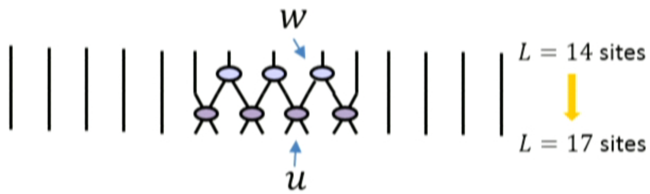
great! However,

- primary operators ϕ_α?
- conformal towers ?

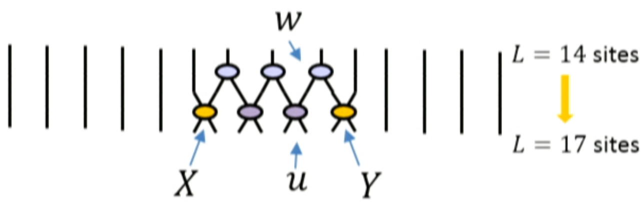
global scale transformation

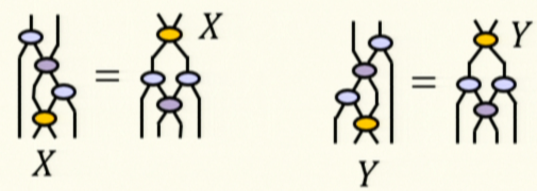


local scale transformation v1.0 (incorrect)



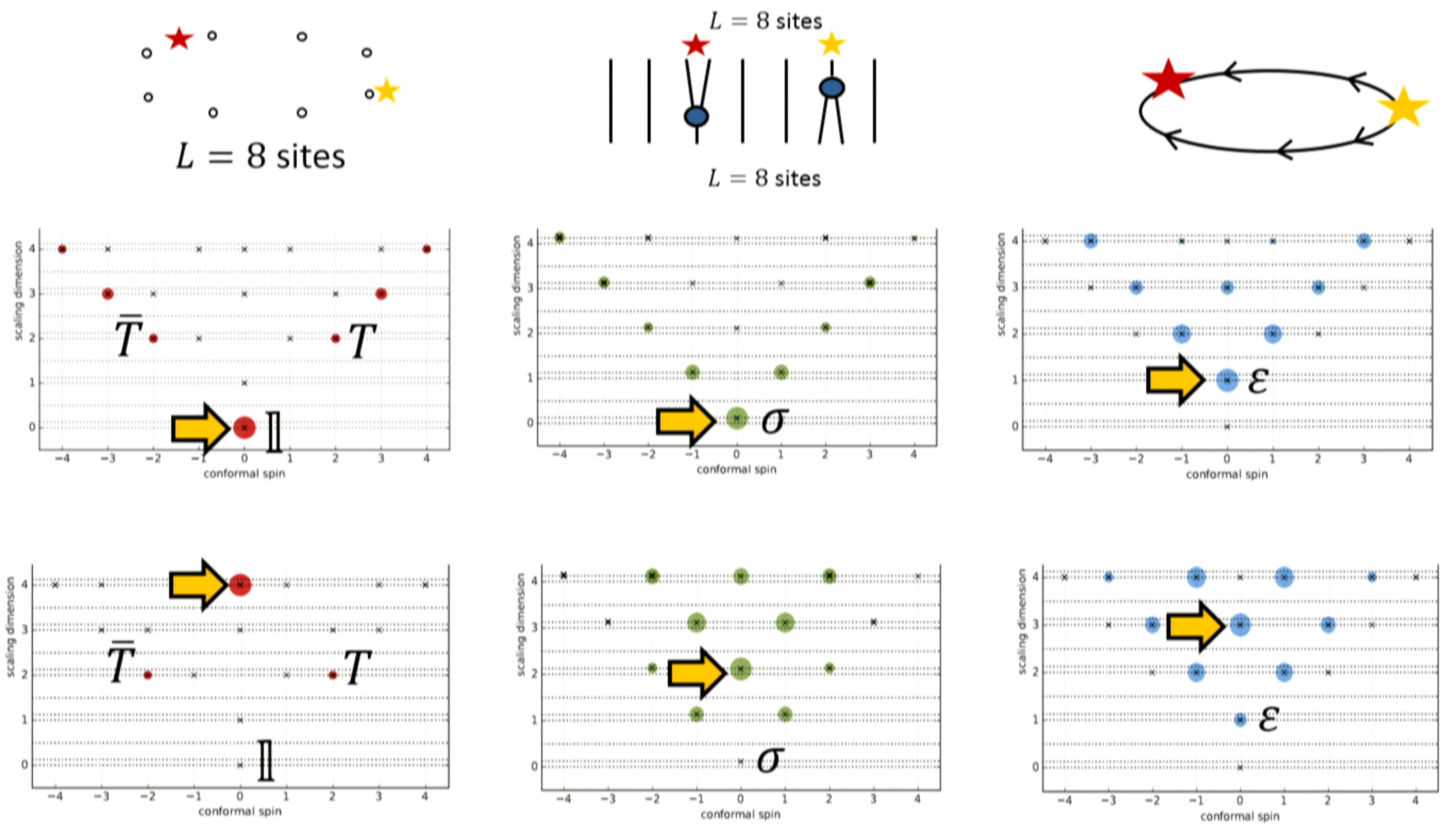
local scale transformation v2.0 (correct)





tensor network equalities

example 1: generic local scale transformation

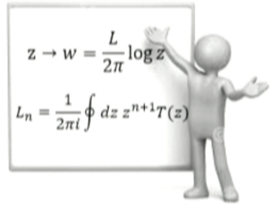


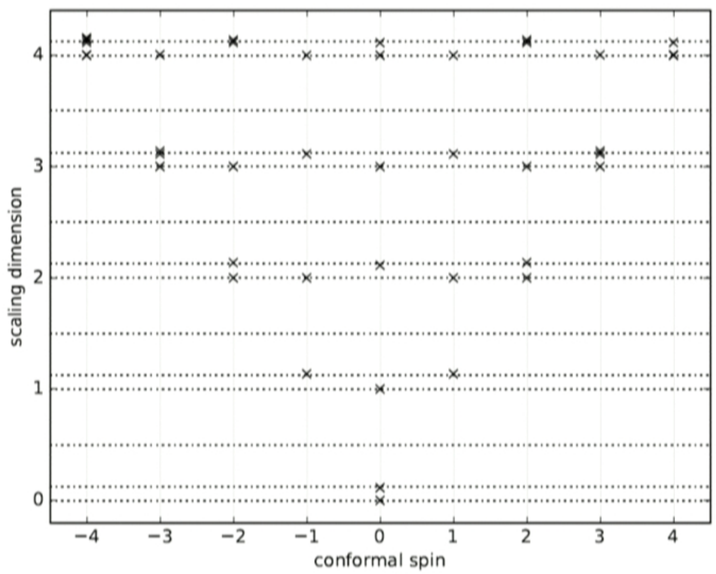
conformal tower of identity $\mathbb{I}$

conformal tower of spin σ

conformal tower of energy density ε

Cardy 1989

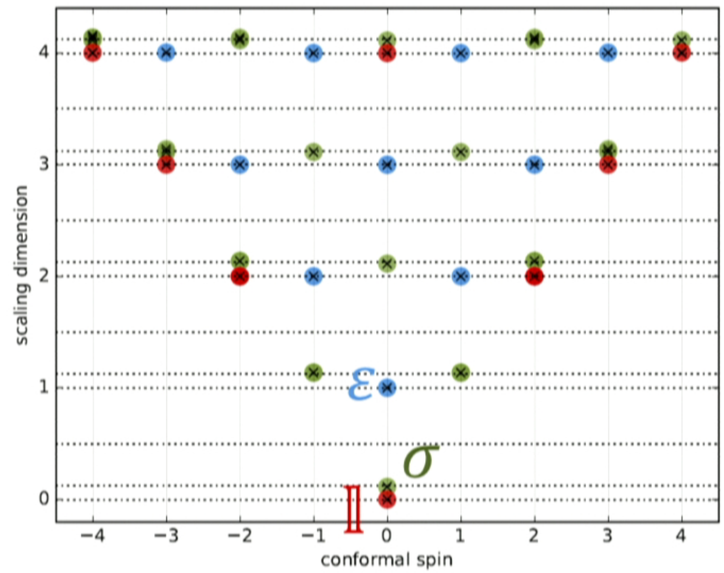




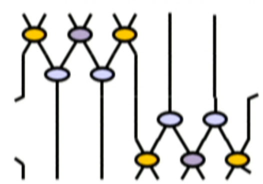
$$p_\alpha = \frac{2\pi}{L} s_\alpha$$

$$E_\alpha = \frac{2\pi}{L} \left(\Delta_\alpha - \frac{c}{12} \right)$$

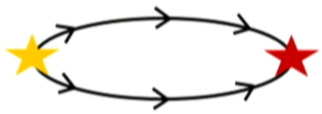
Ashley 2016

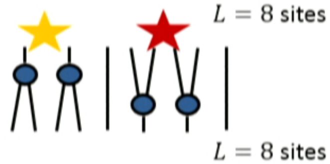


use tensor networks to build and apply local scale transformations on the lattice



example 2: local scale transformation corresponding to global conformal generators





Virasoro $L_0 \quad L_{\pm 1} \quad L_{\pm 2} \quad L_{\pm 3} \quad \cdots$

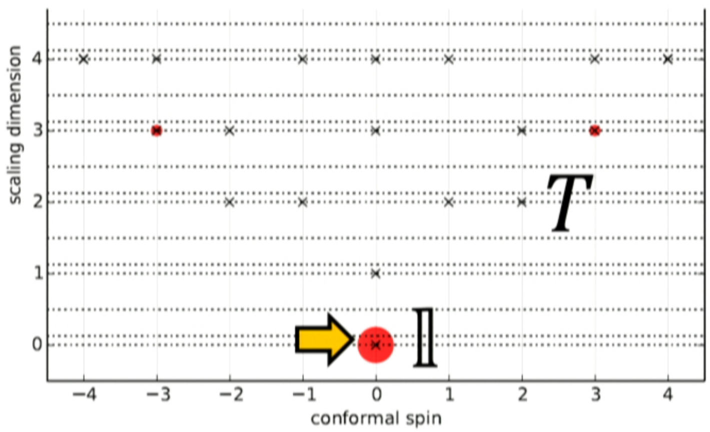
global conformal

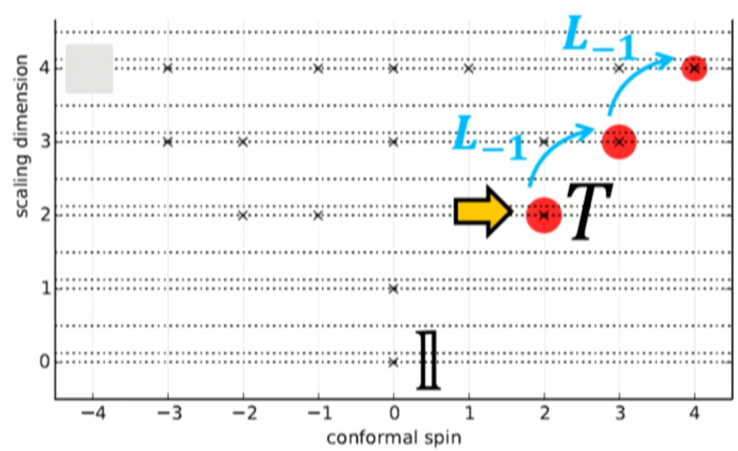
$$q_1 \equiv (l_1 - \bar{l}_{-1}) + (l_{-1} - \bar{l}_1)$$

$$\sim \cos\left(\frac{2\pi}{L}x\right)\partial_x$$

$$Q_1 \equiv (L_1 - \bar{L}_{-1}) + (L_{-1} - \bar{L}_1)$$

$e^{iQ_1} \sim$ mixing within global conformal towers





identify quasi-primaries and their global conformal towers

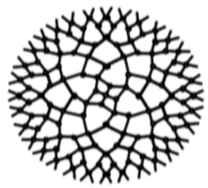
Summary

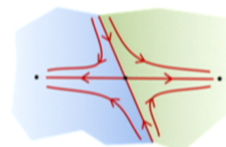
1 - Entanglement and tensor networks

$S(A) \sim |\partial A|$

area law

2 - Application: The renormalization group

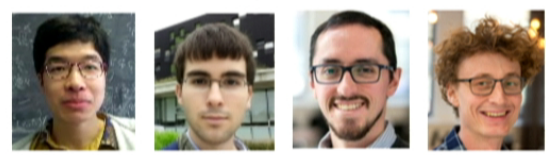




What else?

Euclidean path integrals
(Lagrangian formalism)

Continuous
tensor networks
for QFTs



Tensor network holography: AdS/CFT