Title: Uniform Additivity

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Abstract: <p>Information theory establishes the fundamental limits on data transmission, storage, and processing. Quantum information theory unites information theoretic ideas with an accurate quantum-mechanical description of reality to give a more accurate and complete theory with new and more powerful possibilities for information processing. The goal of both classical and quantum information theory is to quantify the optimal rates of interconversion of different resources. These rates are usually characterized in terms of entropies. However, nonadditivity of many entropic formulas often makes finding answers to information theoretic questions intractable. In a few auspicious cases, such as the classical capacity of a classical channel, the capacity region of a multiple access channel and the entanglement assisted capacity of a quantum channel, additivity allows a full characterization of optimal rates. Here we present a new mathematical property of entropic formulas, uniform additivity, that is both easily evaluated and rich enough to capture all known quantum additive formulas. We give a complete characterization of uniformly additive functions using the linear programming approach to entropy inequalities. In addition to all known quantum formulas, we find a new and intriguing additive quantity: the completely coherent information. We also uncover a remarkable coincidence---the classical and quantum uniformly additive functions are identical; the tractable answers in classical and quantum information theory are formally equivalent.</p>

Uniformly Additive Entropic Formulas

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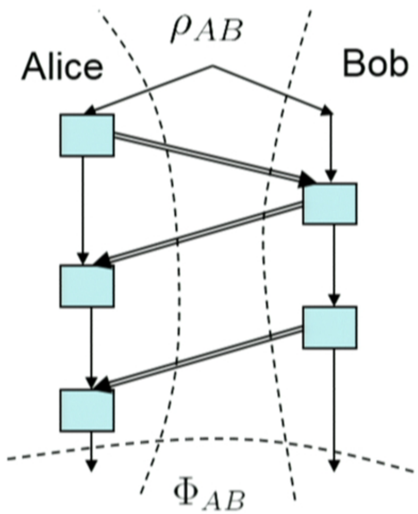
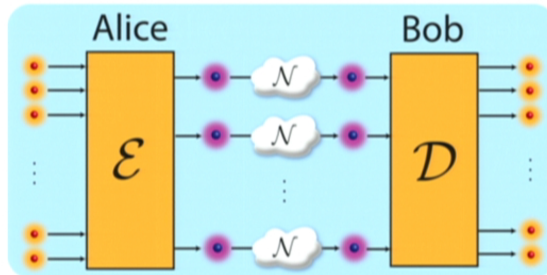
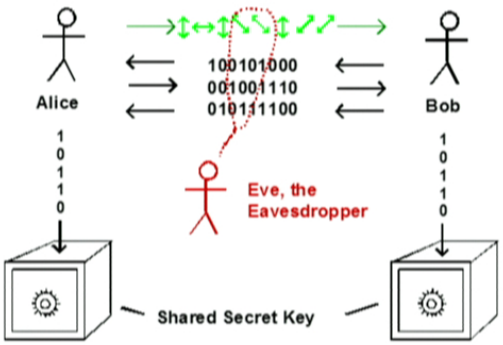
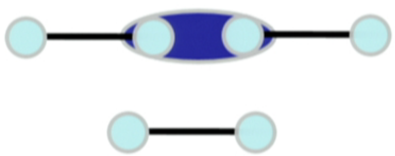
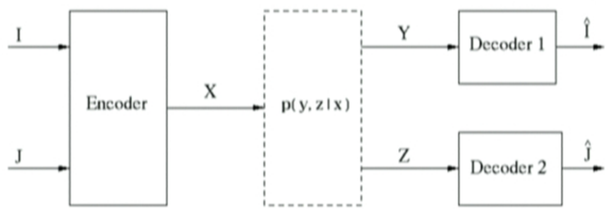
Information theory: optimal rates in sending, storing, processing data



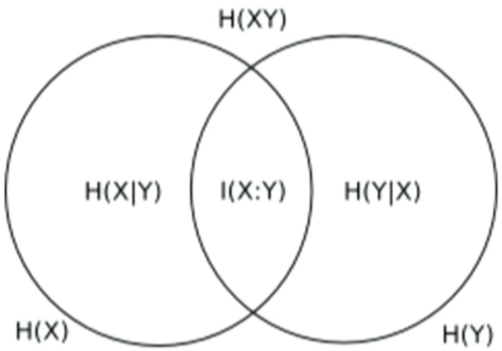








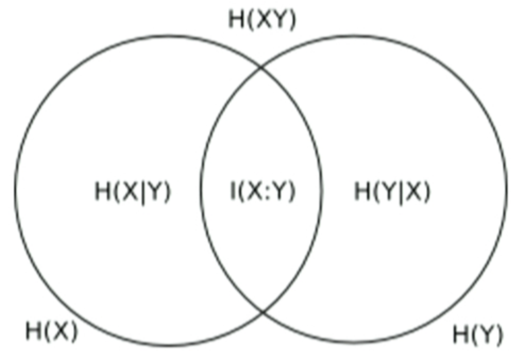
Entropy formulas quantify the answers



- $H(X) = -\sum_x p_x \log p_x$
- $H(\rho) = -\mathrm{Tr}\, \rho \log \rho$

- Optimal Compression: H(X)
- Schumacher Compression: $H(\rho)$
- Classical Channel capacity: max I(X;Y)
  $I(X;Y) = H(X) + H(Y) - H(XY)$
- Quantum Communication: max {H(B) – H(E)}
- Private capacity: max {I(V;B)-I(V;E)}

Entropy formulas quantify the answers



- $H(X) = -\sum_x p_x \log p_x$
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- Optimal Compression: $H(X)$
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- Classical Channel capacity: max $I(X;Y)$
  $I(X;Y) = H(X)+ H(Y) - H(XY)$
- Quantum Communication: max $\{H(B) - H(E)\}$
- Private capacity: max $\{I(V;B)-I(V;E)\}$

Additivity lets us calculate answers

C( x )

= C()+C()

Classical Capacity of Classical Channel

Nonadditivity is the rule
Especially quantumly

- Good: Better rates, e.g., for classical and quantum communication.

- Bad:
  Mostly don't know capacities, distillable entanglement, etc.
  Have upper and lower bounds that are far apart.

Outline

1. Entropy formulas and their additivity proofs
2. All the uniformly additive formulas under standard decouplings
3. Standard decoupling is typical
4. Completely coherent information: a new additive quantity
5. Observation:classical-quantum correspondence

Entropy formulas

- Quantum channel: unitary interaction with a inaccessible environment

- Entropy formula : linear combination of entropies

$$f_\alpha(U_\mathcal{N}, \phi_{V_1 \ldots V_n A}) = \sum_{s \in \mathcal{P}(V_1 \ldots V_n BE)} \alpha_s H(s)_\rho$$

with $\rho_{V_1 \ldots V_n BE} = (I \otimes U_\mathcal{N}) \phi_{V_1 \ldots V_n A} (I \otimes U_\mathcal{N}^\dagger)$

- Maximized version: $f_\alpha(U_\mathcal{N}) = \max_{\phi_{V_1 \ldots V_n A}} f_\alpha(U_\mathcal{N}, \phi_{V_1 \ldots V_n A})$

- Additivity: $f_\alpha(U_{\mathcal{N}_1} \otimes U_{\mathcal{N}_2}) = f_\alpha(U_{\mathcal{N}_1}) + f_\alpha(U_{\mathcal{N}_2})$

Additivity Proofs

$$f_\alpha(U_\mathcal{N}) = \max_{\phi_{V_1\ldots V_n A}} f_\alpha(U_\mathcal{N}, \phi_{V_1\ldots V_n A})$$

$$f_\alpha(U_\mathcal{N}, \phi_{V_1\ldots V_n A}) = \sum_{s\in\mathcal{P}(V_1\ldots V_n BE)} \alpha_s H(s)_\rho$$

- Enough to show subadditive:

additive:

$$f_\alpha(U_{\mathcal{N}_1} \otimes U_{\mathcal{N}_2}) = f_\alpha(U_{\mathcal{N}_1}) + f_\alpha(U_{\mathcal{N}_2})$$

"$\geq$" is obvious ⇕

subadditive:

$$f_\alpha(U_{\mathcal{N}_1} \otimes U_{\mathcal{N}_2}) \leq f_\alpha(U_{\mathcal{N}_1}) + f_\alpha(U_{\mathcal{N}_2})$$

⇕

$$f_\alpha(U_{\mathcal{N}_1} \otimes U_{\mathcal{N}_2}, \phi_{12}^*) \leq f_\alpha(U_{\mathcal{N}_1}, \phi_1^*) + f_\alpha(U_{\mathcal{N}_2}, \phi_2^*)$$

Standard Additivity Proof

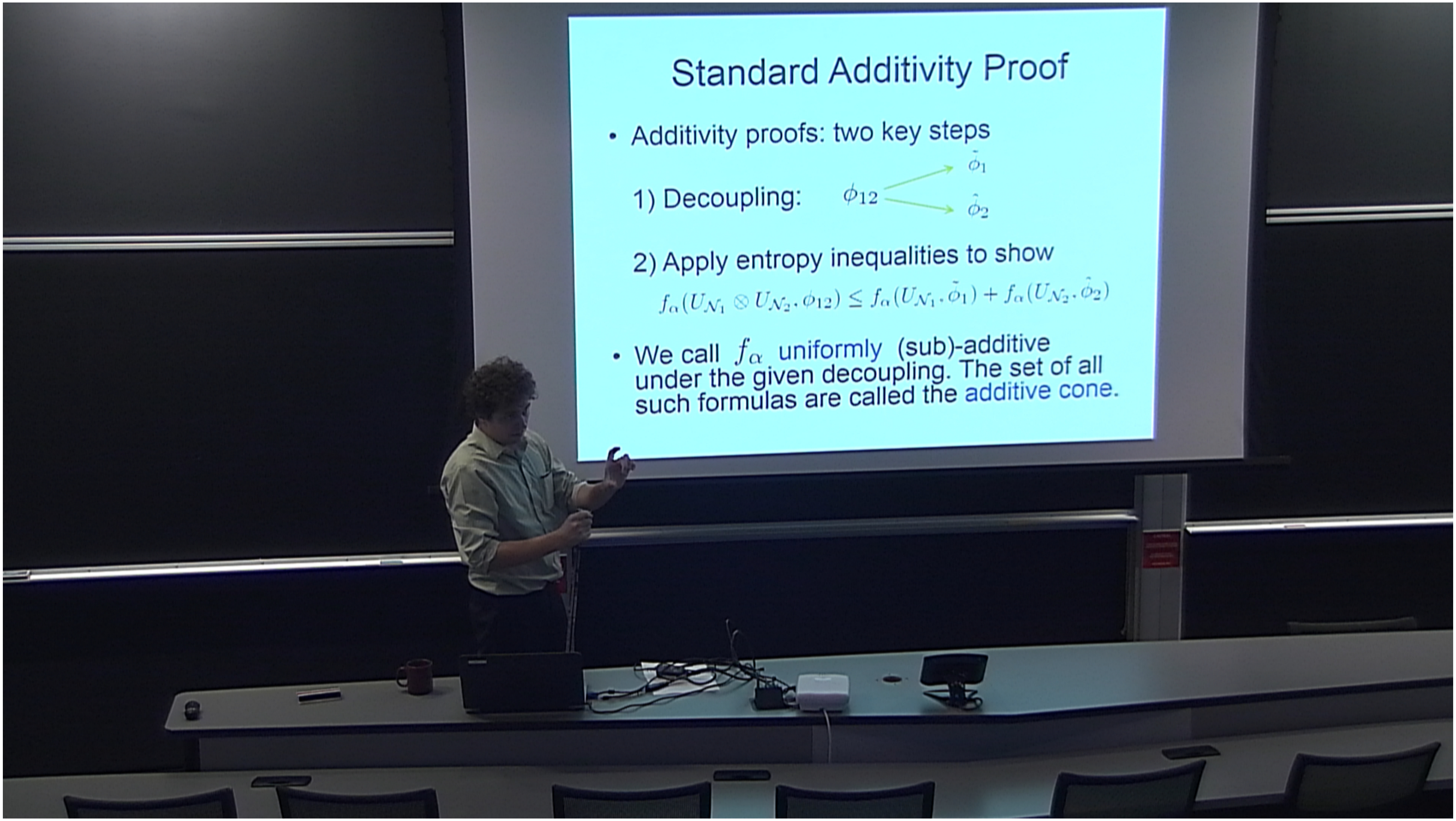
- Additivity proofs: two key steps

  1) Decoupling: $\phi_{12} \rightarrow \tilde{\phi}_1, \; \phi_{12} \rightarrow \hat{\phi}_2$

  2) Apply entropy inequalities to show

$$f_\alpha(U_{\mathcal{N}_1} \otimes U_{\mathcal{N}_2}, \phi_{12}) \leq f_\alpha(U_{\mathcal{N}_1}, \tilde{\phi}_1) + f_\alpha(U_{\mathcal{N}_2}, \hat{\phi}_2)$$

- We call f_α uniformly (sub)-additive under the given decoupling. The set of all such formulas are called the additive cone.


Standard Additivity Proof
• Additivity proofs: two key steps
1) Decoupling: ϕ_{12} → $\tilde{\phi}_1$, $\tilde{\phi}_2$
2) Apply entropy inequalities to show
$f_\alpha(U_{\mathcal{N}_1} \otimes U_{\mathcal{N}_2}, \phi_{12}) \leq f_\alpha(U_{\mathcal{N}_1}, \tilde{\phi}_1) + f_\alpha(U_{\mathcal{N}_2}, \tilde{\phi}_2)$
• We call f_α uniformly (sub)-additive under the given decoupling. The set of all such formulas are called the additive cone.

A canonical example

- Entanglement assisted capacity:

$$C_{ea}(\mathcal{N}) = \mathbf{max}_{\phi_{VA}} I(V;B)$$

1) Decoupling

$$\phi_{VA_1A_2} \rightarrow \hat{\phi}_{\hat{V}A_2} = \phi_{VB_1|A_2}$$

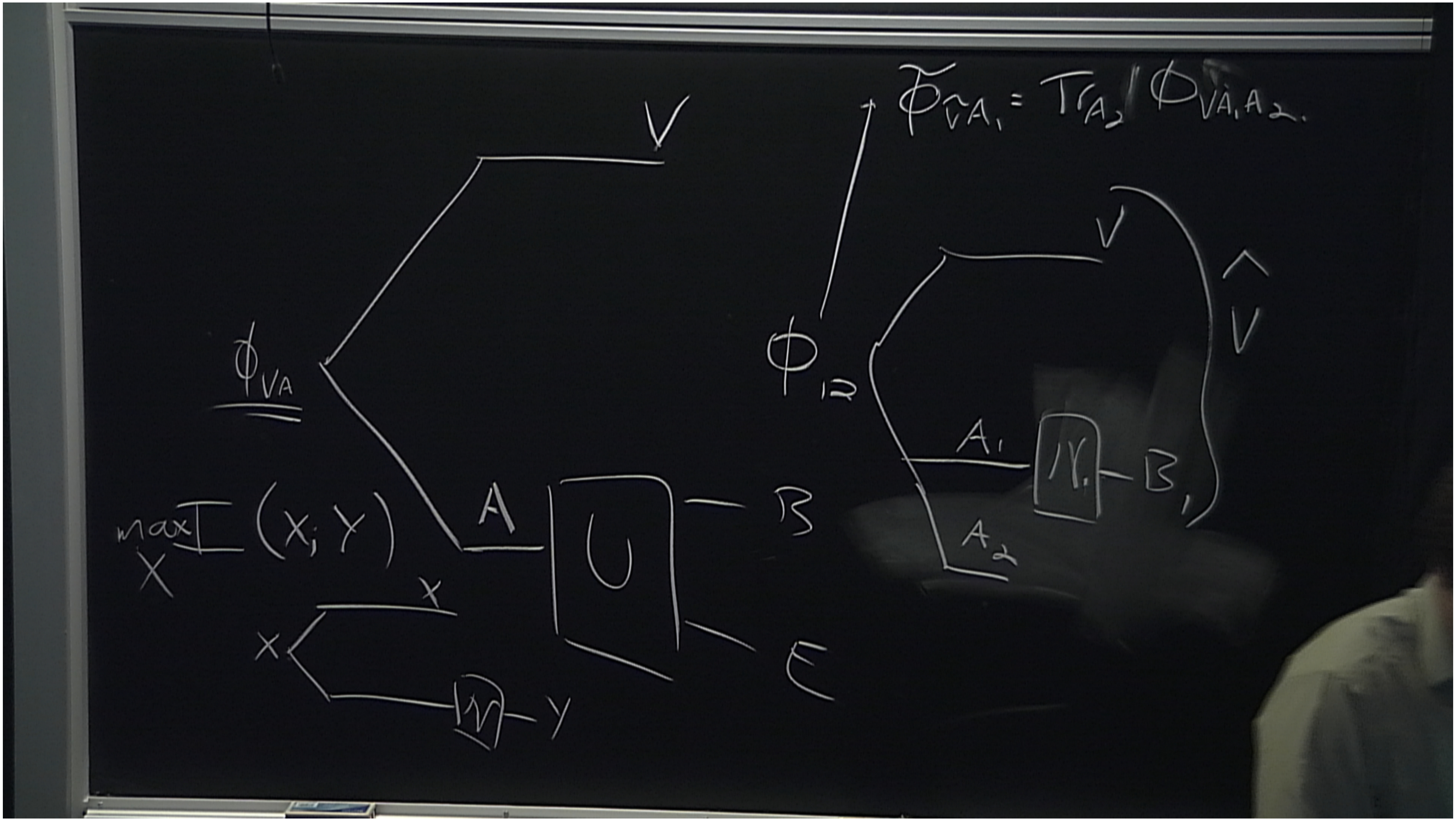
$$\phi_{VA_1A_2} \rightarrow \tilde{\phi}_{\tilde{V}A_1} = \phi_{VA_1}$$

2) Entropy inequality

$$\begin{aligned} I(V;B_1B_2) &= I(V;B_1) + I(V;B_2|B_1) \\ &= I(V;B_1) + I(VB_1;B_2) - I(B_1;B_2) \\ &\leq I(V;B_1) + I(VB_1;B_2) \leq C_{ea}(\mathcal{N}_1) + C_{ea}(\mathcal{N}_2) \end{aligned}$$

Outline

1. Entropy formulas and their additivity proofs
2. All the uniformly additive formulas under standard decouplings
3. standard decoupling is typical
4. Completely coherent information: a new additive quantity
5. Observation:classical-quantum correspondence


V
ϕ_{VA}
$\max_X I(X;Y)$
A
U
B
E
X
N
Y
$\tilde{\phi}_{VA_1} = Tr_{A_2} \phi_{VA_1A_2}$
ϕ_{12}
A_1
N_1
B_1
A_2
$\hat{V}$

Decoupling

- We focus on "standard decoupling".

$$\phi_{V_1\ldots V_n A_1 A_2} \to \phi_{\tilde{V}_1\ldots\tilde{V}_n A_1}$$

$$\phi_{V_1\ldots V_n A_1 A_2} \to \phi_{\hat{V}_1\ldots\hat{V}_n A_2}$$

$\mathcal{N}_1$, $\mathcal{N}_2$

$$\left(\phi_{V_1\ldots V_n B_1 E_1 A_2},\ \phi_{V_1\ldots V_n B_2 E_2 A_1}\right)$$

$$\tilde{V}_i = \tilde{M}_2 V_i\ , \quad \tilde{M}_2 \in \mathcal{P}(B_2E_2); \quad \hat{V}_i = \hat{M}_1 V_i\ , \quad \hat{M}_1 \in \mathcal{P}(B_1E_1)$$

- Example

$$\tilde{V}_1 = V_1 B_2,\ \tilde{V}_2 = V_2 E_2,\ \tilde{V}_3 = V_3$$

$$\hat{V}_1 = V_1,\ \hat{V}_2 = V_2 B_1 E_1,\ \hat{V}_3 = V_3$$

Entropy Inequalities

- Strong subadditivity:

  I(A;B|C) = H(AC)+ H(BC)-H(ABC)-H(C)>=0

  [H(A)>=0, H(A)+H(B)-H(AB)>=0, H(AB)+H(A)-H(B)>=0, H(AB)+H(AC)-H(B)-H(C)>=0]

- There may be more, but we don't know them! (Classically, there is more: H(A|B)>=0, Non-Shannon inequalities.)
- Luckily, we don't need them!

Zero Auxiliary Variable

$$f_\alpha(\mathcal{N}, \phi_A) = \alpha_B H(B) + \alpha_E H(E) + \alpha_{BE} H(BE)$$

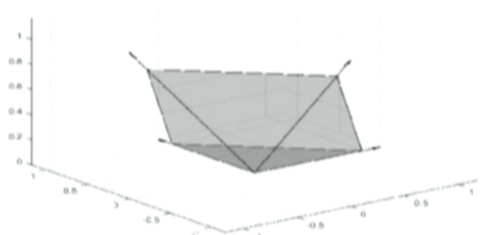
$$\text{Decoupling: } \phi_{A_1A_2} \to (\phi_{A_1}, \phi_{A_2})$$

$$\Pi^{\varnothing} := \{f_\alpha \,|\, f_\alpha(U_{\mathcal{N}_1} \otimes U_{\mathcal{N}_2}, \phi_{A_1A_2}) \leq f_\alpha(U_{\mathcal{N}_1}, \phi_{A_1}) + f_\alpha(U_{\mathcal{N}_2}, \phi_{A_2})\}$$

Result: full characterization of $\Pi^{\varnothing}$

$\Pi^{\varnothing}$ Rays

$$f_\alpha = \lambda_1 H(B) + \lambda_2 H(E) + \lambda_3 H(B|E) + \lambda_4 H(E|B)$$



$\Pi^{\varnothing}$ Faces

$$\alpha_B + \alpha_{BE} \geq 0$$
$$\alpha_E + \alpha_{BE} \geq 0$$
$$\alpha_B + \alpha_E + \alpha_{BE} \geq 0$$
$$\alpha_{BE} \geq 0.$$

Anything inside the cone is uniformly additive. Outside the cone, there is A state that makes f_α not subadditive.

One Auxiliary Variable

At first, consider

$$f_{\alpha^V}(\mathcal{N},\phi_{VA}) = \alpha_V H(V) + \alpha_{BV} H(BV) + \alpha_{EV} H(EV) + \alpha_{BEV} H(BEV)$$

Fix a standard decoupling:

$\tilde{V} \in \{V, B_2V, E_2V, B_2E_2V\}$ and

$\hat{V} \in \{V, B_1V, E_1V, B_1E_1V\}$

These are labeled by (a,b) $a,b = 0...3$

for each decoupling (a,b), define the additive cone:

$$\Pi^{V,(a,b)} :=$$
$$\{f_{\alpha^V} \mid f_{\alpha^V}(U_{\mathcal{N}_1} \otimes U_{\mathcal{N}_2}, \phi_{VA_1A_2}) \leq f_{\alpha^V}(U_{\mathcal{N}_1}, \phi_{\tilde{V}A_1}) + f_{\alpha^V}(U_{\mathcal{N}_2}, \phi_{\hat{V}A_2})\}$$

We give a full characterization of $\Pi^{V,(a,b)}$.

One Auxiliary Variable

The additive cone $\Pi^{V,(a,b)}$

case	(a,b)	$\hat{M}_1$	$\tilde{M}_2$	equivalents	Additive Cone	Extreme Rays
1.	(3,3)	B_1E_1	B_2E_2	(0,0)	$\alpha_V + \alpha_{BV} + \alpha_{EV} \geq 0$ $\alpha_V + \alpha_{BV} \geq 0$ $\alpha_V + \alpha_{EV} \geq 0$ $\alpha_V \geq 0$	$-H(E\|BV)$ $-H(E\|V)$ $-H(B\|EV)$ $-H(B\|V)$
2.	(3,2)	B_1E_1	E_2	$(2,3),(3,1)$ $(1,3),(1,0),(0,1)$ $(2,0),(0,2)$	$\alpha_{BV} \leq 0$ $\alpha_V + \alpha_{BV} \geq 0$	$-H(BE\|V)$ $\pm H(B\|EV)$ $-H(B\|V)$
3.	(3,0)	B_1E_1	ϕ	(0,3)	$\alpha_{EV} \leq 0$ $\alpha_{BV} \leq 0$	$H(E\|BV)$ $-H(E\|V)$ $\pm H(BE\|V)$
4.	(1,1)	B_1	B_2	(2,2)	$\alpha_{EV} = 0$ $\alpha_V \geq 0$ $\alpha_{BEV} \geq 0$	$-H(B\|V)$ $H(E\|BV)$
5.	(1,2)	B_1	E_2	(2,1)	$\alpha_{BEV} \geq 0$ $\alpha_V \geq 0$	$\pm[H(EV) - H(BV)]$ $H(E\|BV)$ $-H(E\|V)$

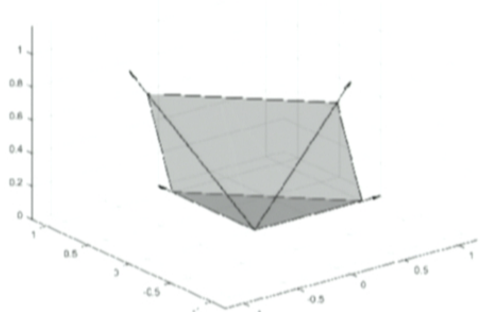
One Auxiliary Variable

$$f_\alpha = f_{\alpha^\varnothing} + f_{\alpha^V}$$

$$f_{\alpha^\varnothing} := \alpha_B H(B) + \alpha_E H(E) + \alpha_{BE} H(BE)$$
$$f_{\alpha^V} := \alpha_V H(V) + \alpha_{BV} H(BV) + \alpha_{EV} H(EV) + \alpha_{BEV} H(BEV)$$

Result: $f_\alpha(U_\mathcal{N})$ U.A. w.r.t. (a,b) iff $f_{\alpha^\varnothing} \in \Pi^\varnothing$ & $f_{\alpha^V} \in \Pi^{V,(a,b)}$

$\Pi^\varnothing$



$f_{\alpha^\varnothing}$

$\Pi^{V,(a,b)}$

case	(a,b)	$\tilde{M}_1$	$\tilde{M}_2$	equivalents	Additive Cone	Extreme Rays
1.	(3,3)	B_1E_1	B_2E_2	(0,0)	$\alpha_V + \alpha_{BV} + \alpha_{EV} \geq 0$ $\alpha_V + \alpha_{BV} \geq 0$ $\alpha_V + \alpha_{EV} \geq 0$ $\alpha_V \geq 0$	$-H(E\|BV)$ $-H(E\|V)$ $-H(B\|EV)$ $-H(B\|V)$
2.	(3,2)	B_1E_1	E_2	$(2,3), (3,1)$ $(1,3), (1,0), (0,1)$ $(2,0), (0,2)$	$\alpha_{BV} \leq 0$ $\alpha_V + \alpha_{BV} \geq 0$	$-H(BE\|V)$ $\pm H(B\|EV)$ $-H(B\|V)$
3.	(3,0)	B_1E_1	ϕ	(0,3)	$\alpha_{EV} \leq 0$ $\alpha_{BV} \leq 0$	$H(E\|BV)$ $-H(E\|V)$ $\pm H(BE\|V)$
4.	(1,1)	B_1	B_2	(2,2)	$\alpha_{EV} = 0$ $\alpha_V \geq 0$ $\alpha_{BEV} \geq 0$	$-H(B\|V)$ $H(E\|BV)$
5.	(1,2)	B_1	E_2	(2,1)	$\alpha_{BEV} \geq 0$ $\alpha_V \geq 0$	$\pm[H(EV) - H(BV)]$ $H(E\|BV)$ $-H(E\|V)$

f_{α^V}

Non-standard Decouplings

$$\phi_{V_1 \ldots V_n A_1 A_2} \rightarrow \phi_{\tilde{V}_1 \ldots \tilde{V}_n A_1}, \quad \phi_{\hat{V}_1 \ldots \hat{V}_n A_2}$$

$$\phi_{V_1 \ldots V_n A_1 A_2} \xrightarrow{\mathcal{N}_1,\ \mathcal{N}_2} (\phi_{V_1 \ldots V_n B_1 E_1 A_2},\ \phi_{V_1 \ldots V_n B_2 E_2 A_1})$$

- Standard decoupling (a special relabeling)

$$\tilde{V}_i = \tilde{M}_2 V_i\,, \quad \tilde{M}_2 \in \mathcal{P}(B_2 E_2); \quad \hat{V}_i = \hat{M}_1 V_i\,, \quad \hat{M}_1 \in \mathcal{P}(B_1 E_1)$$

- Consistent Decoupling (general relabeling)

$$\tilde{V}_i \in \mathcal{P}(V_1 ... V_n B_2 E_2) \text{ with } \tilde{V}_i \cap \tilde{V}_j = \varnothing;$$

$$\hat{V}_i \in \mathcal{P}(V_1 ... V_n B_1 E_1) \text{ with } \hat{V}_i \cap \hat{V}_j = \varnothing.$$

example:

$$\tilde{V}_1 = V_2 B_2, \tilde{V}_2 = V_3 E_2, \tilde{V}_3 = V_1$$

$$\hat{V}_1 = V_2 V_3, \hat{V}_2 = B_1, \hat{V}_3 = \varnothing$$

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Completely Coherent Information

case	(a,b)	$\hat{M}_1$	$\tilde{M}_2$	equivalents	Additive Cone	Extreme Rays
1.	(3,3)	B_1E_1	B_2E_2	(0,0)	$\alpha_V + \alpha_{BV} + \alpha_{EV} \geq 0$ $\alpha_V + \alpha_{BV} \geq 0$ $\alpha_V + \alpha_{EV} \geq 0$ $\alpha_V \geq 0$	$-H(E\|BV)$ $-H(E\|V)$ $-H(B\|EV)$ $-H(B\|V)$
2.	(3,2)	B_1E_1	E_2	$(2,3),(3,1)$ $(1,3),(1,0),(0,1)$ $(2,0),(0,2)$	$\alpha_{BV} \leq 0$ $\alpha_V + \alpha_{BV} \geq 0$	$-H(BE\|V)$ $\pm H(B\|EV)$ $-H(B\|V)$
3.	(3,0)	B_1E_1	ϕ	(0,3)	$\alpha_{EV} \leq 0$ $\alpha_{BV} \leq 0$	$H(E\|BV)$ $-H(E\|V)$ $\pm H(BE\|V)$
4.	(1,1)	B_1	B_2	(2,2)	$\alpha_{EV} = 0$ $\alpha_V \geq 0$ $\alpha_{BEV} \geq 0$	$-H(B\|V)$ $H(E\|BV)$
5.	(1,2)	B_1	E_2	(2,1)	$\alpha_{BEV} \geq 0$ $\alpha_V \geq 0$	$\pm[H(EV) - H(BV)]$ $H(E\|BV)$ $-H(E\|V)$

Completely Coherent Information

$$I^{cc}(\mathcal{N}) = \max_{\phi_{VA}}[H(VB) - H(VE)]$$

properties:

- Symmetric in B $\leftrightarrow$ E .
- Lower bound for cost of swapping B and E.
  [J. Oppenheim and A. Winter, arXiv:quant-ph/0511082]
- Upper bound for simultaneous quantum communication rate to B and E.
- For degradable channels, $I^{cc}(N) = Q(N) = Q^{(1)}(N)$.
- WANT: operational meaning.

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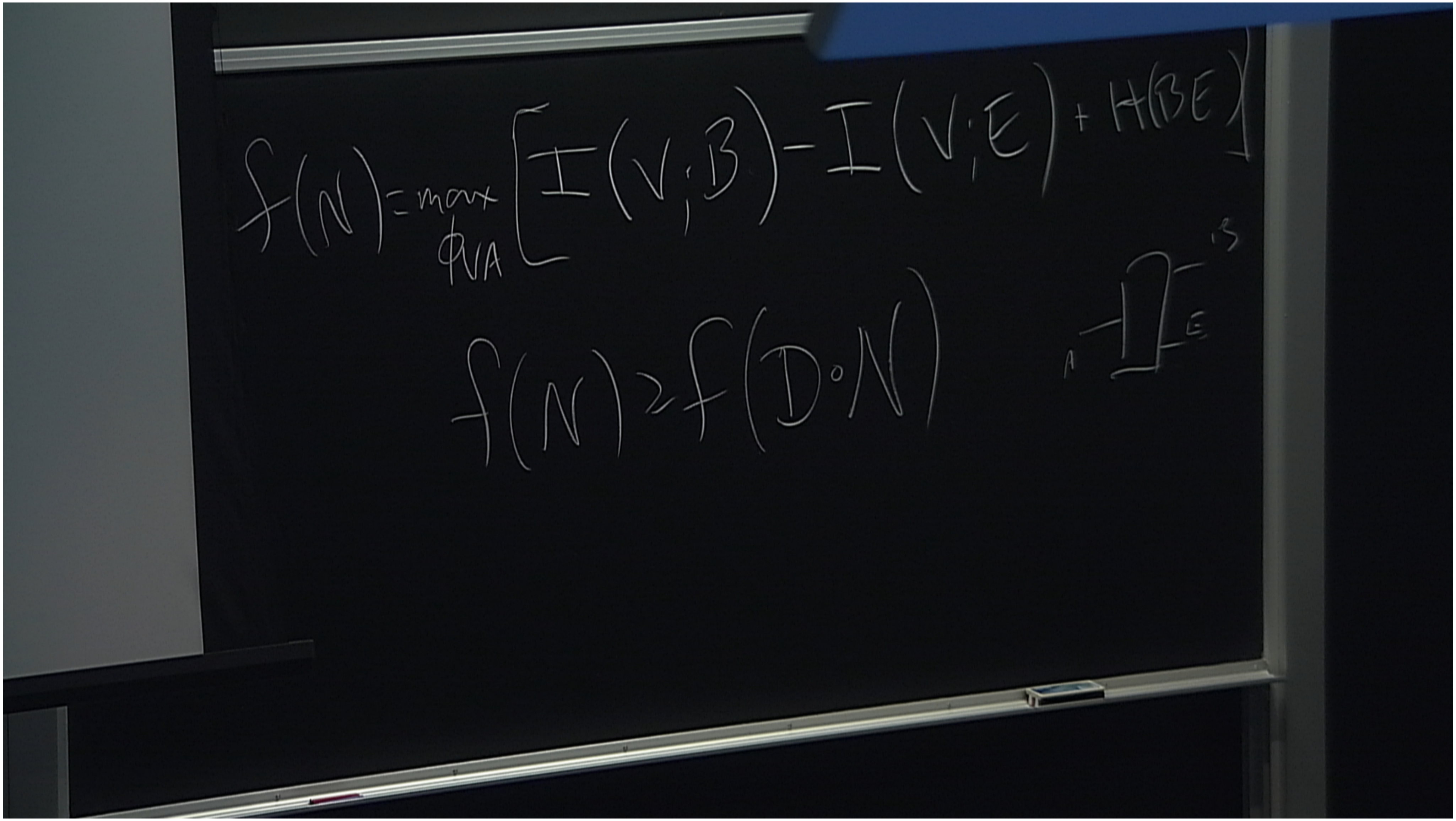
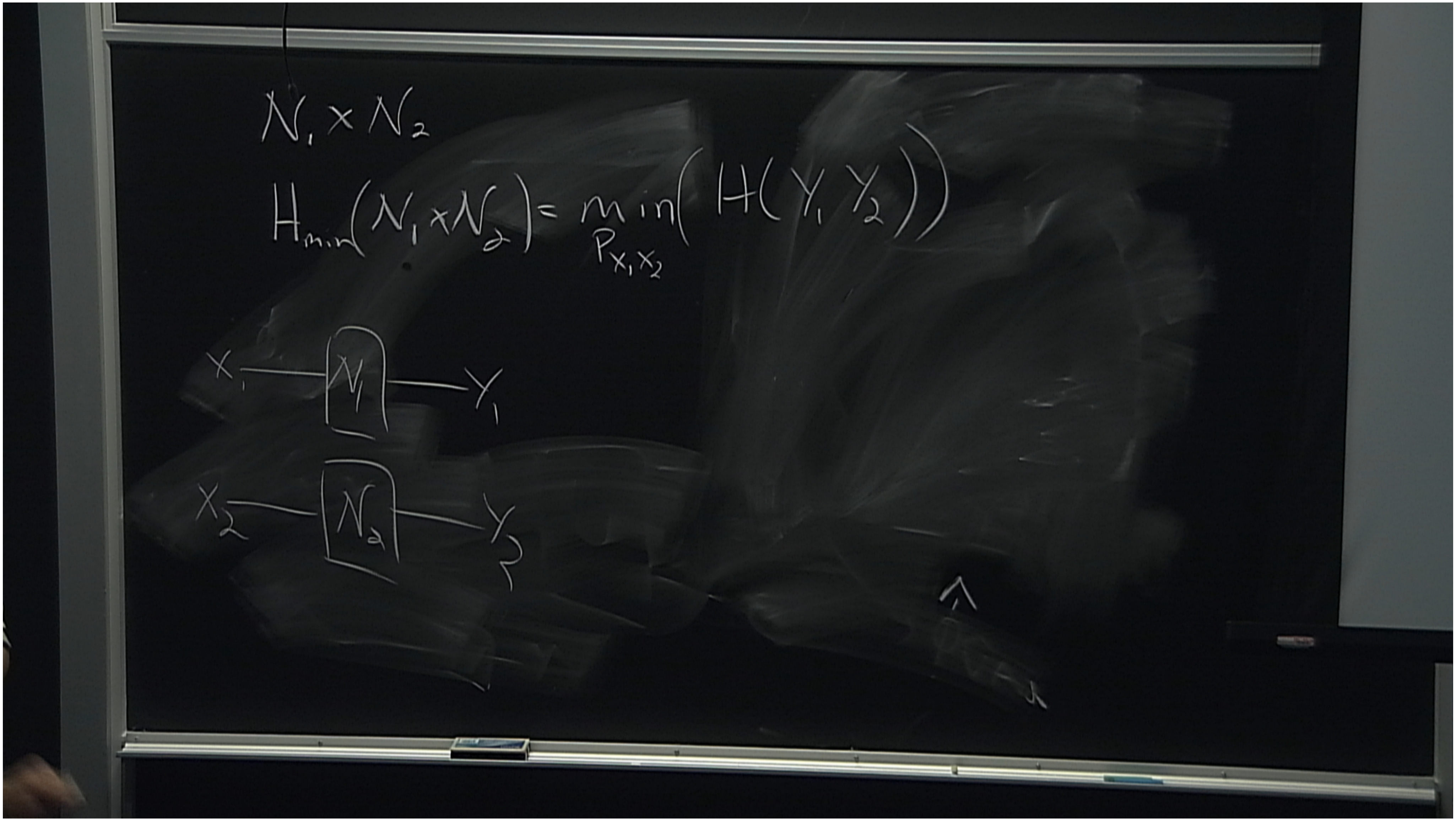

f(N) = max φVA [I(V;B) − I(V;E) + H(B|E)]
f(N) ≥ f(D∘N)
A
B
E


N1 × N2
Hmin(N1 × N2) = min Px1,x2 (H(Y1 Y2))
X1
N1
Y1
X2
N2
Y2

Open Questions

- Additivity other than uniform additivity?
- More general decouplings? $\phi_{12} \rightarrow \tilde{\phi}_1$, $\phi_{12} \rightarrow \hat{\phi}_2$
- Completely coherent information: operational meaning?
- Understand classical-quantum correspondence better.

Thank you!