Title: A conceptual viewpoint on information decomposition

Date: Nov 08, 2016 03:30 PM

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Abstract: <p>Can we decompose the information of a composite system into terms arising from its parts and their interactions?
For a bipartite system (X,Y), the joint entropy can be written as an algebraic sum of three terms: the entropy of X alone, the entropy of Y alone, and the mutual information of X and Y, which comes with an opposite sign. This suggests a set-theoretical analogy: mutual information is a sort of "intersection", and joint entropy is a sort of "union".
The same picture cannot be generalized to three or more parts in a straightforward way, and the problem is still considered open. Is there a deep reason for why the set-theoretical analogy fails?
Category theory can give an alternative, conceptual point of view on the problem. As Shannon already noted, information appears to be related to symmetry. This suggests a natural lattice structure for information, which is compatible with a set-theoretical picture only for bipartite systems.
The categorical approach favors objects with a structure in place of just numbers to describe information quantities. We hope that this can clarify the mathematical structure underlying information theory, and leave it open to wider generalizations.</p>

A Conceptual Viewpoint on Information Decomposition

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Leipzig, Germany

Perimeter Institute
for Theoretical Physics
8th November 2016

Information Decomposition

Question:

How much do parts of a system contribute to the total information?

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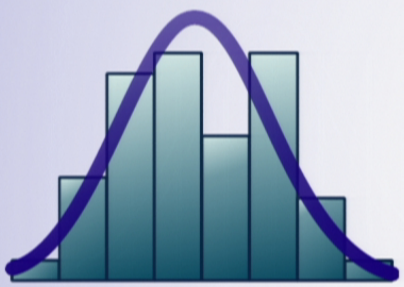

Statistics

Nonlinear higher
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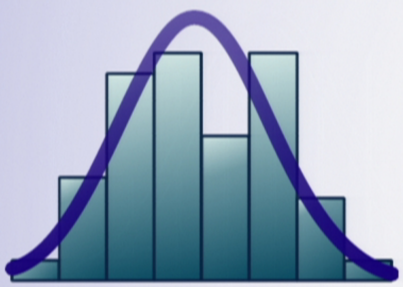

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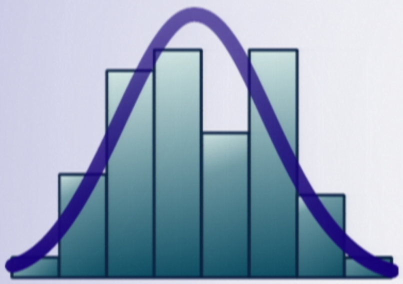

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Quantify complexity of neural networks.

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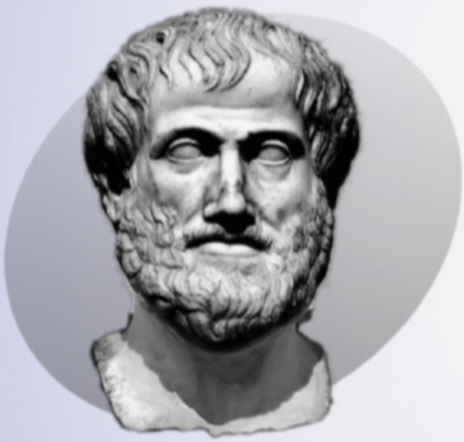

"The totality is not [...] a mere heap, but the whole is something besides the parts."

– Aristotle, Metaphysics VIII

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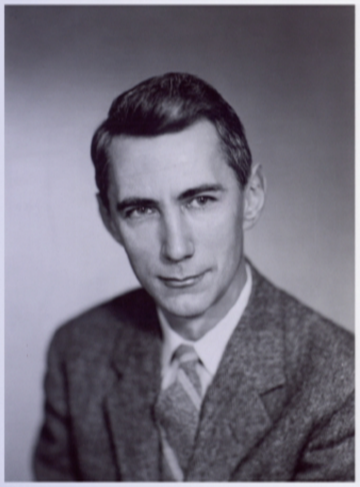

- Shannon entropy:

$$H(X) := -\sum_{x \in X} p(x) \log p(x)$$

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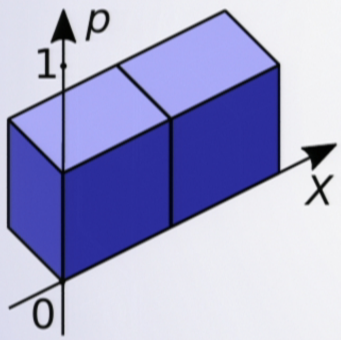

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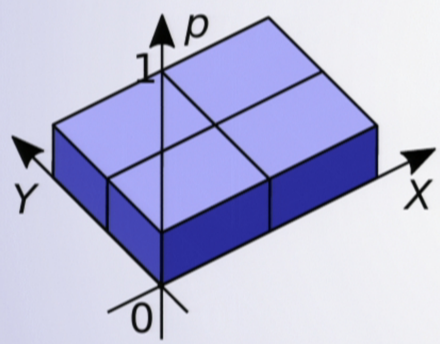

- Joint entropy:

$$H(X, Y) := -\sum_{x,y} p(x, y) \log p(x, y)$$

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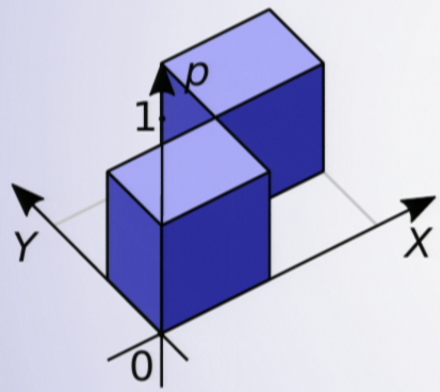

- Conditional entropy:

$$H(X|Y) := -\sum_{x,y} p(x,y) \log p(x|y)$$

In the example, $H(X|Y) = 0$ bit.

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"The whole is greater than the sum of its parts."

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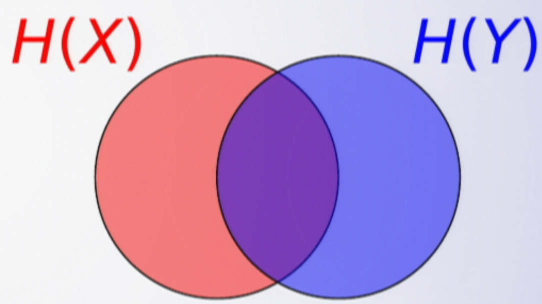

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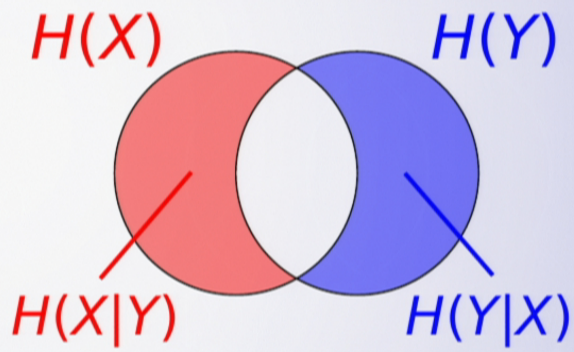

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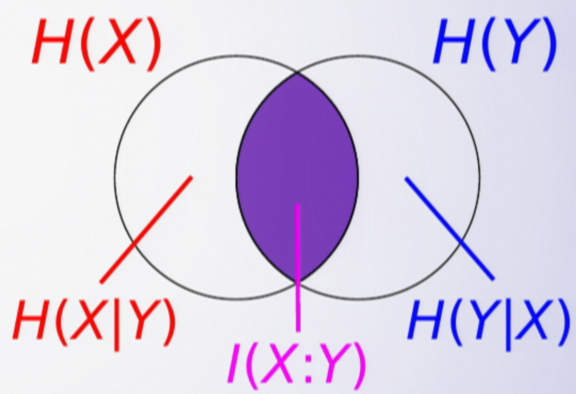

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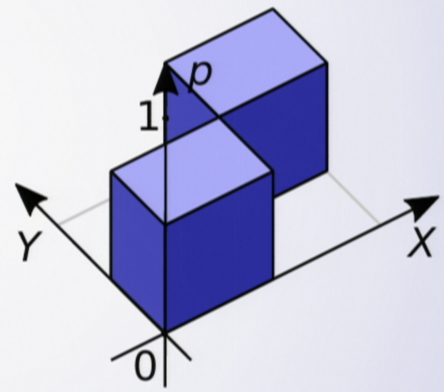
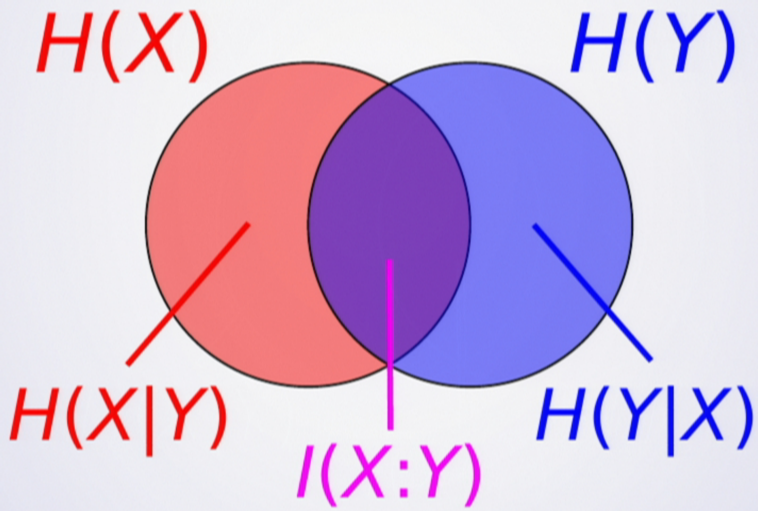


Information Decomposition

Set-theoretical picture:

"Statistical interactions are like intersections."



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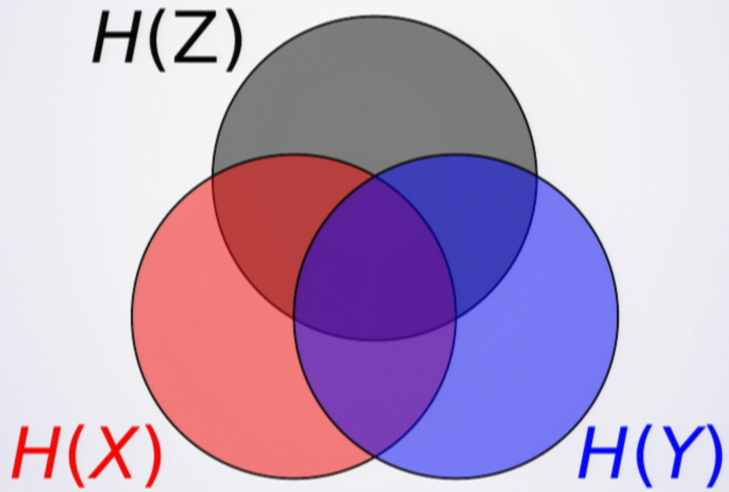

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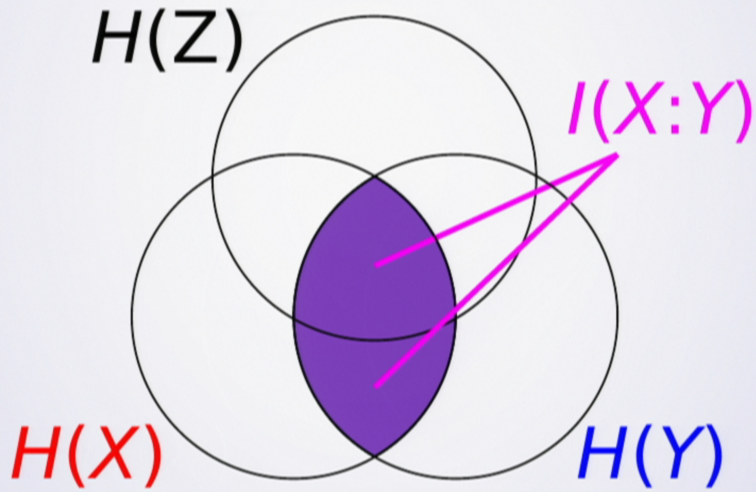

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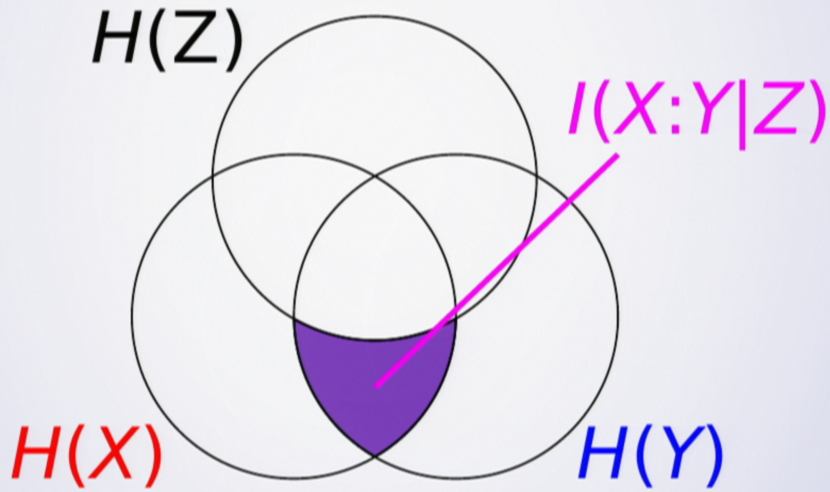

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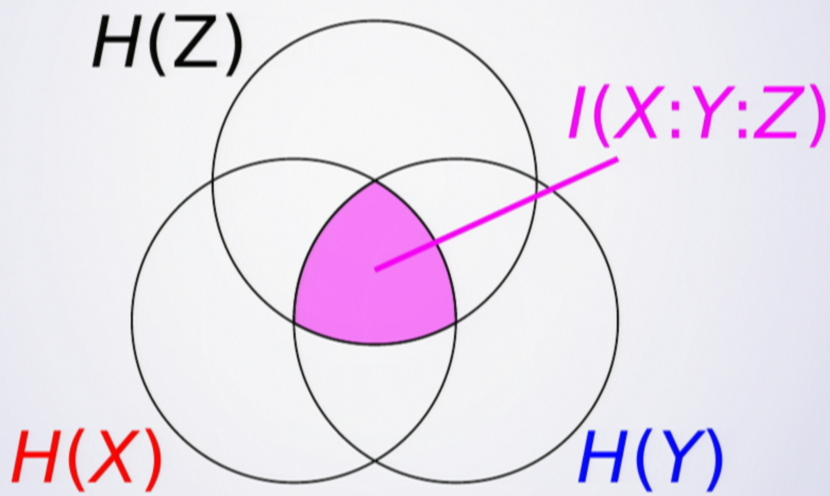

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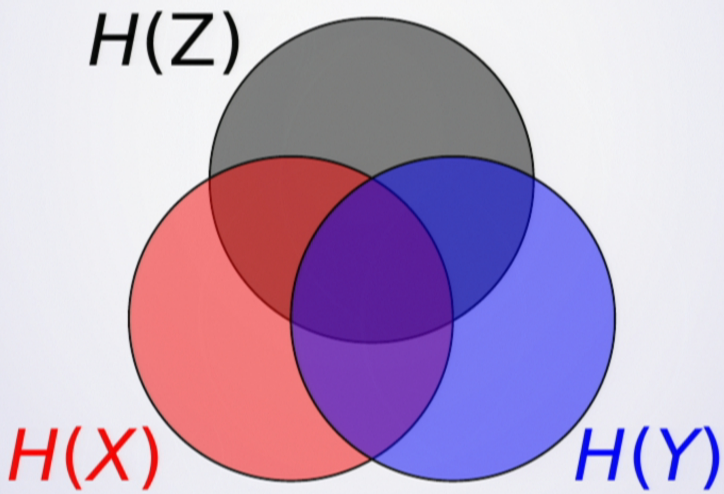

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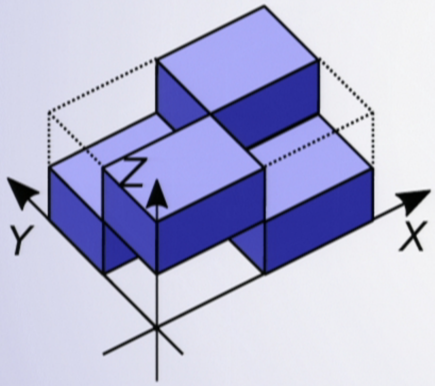

- For example, if $X = Y \oplus Z$:

$$I(X : Y) = 0$$

$$I(X : Y \mid Z) = 1$$

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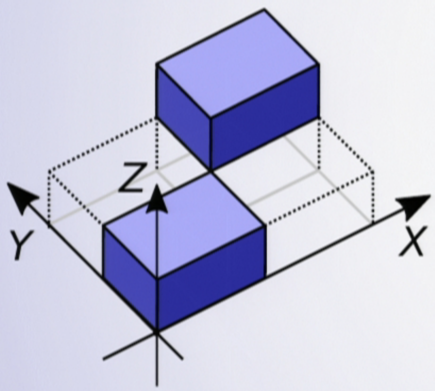

- For example, if $X = Y = Z$:

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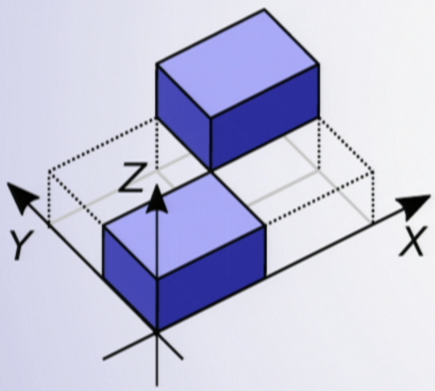

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Schneidmann, Bialek, Berry [1] and Williams, Beer [2].

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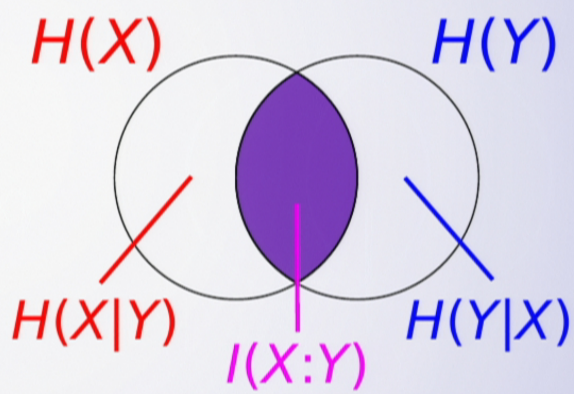

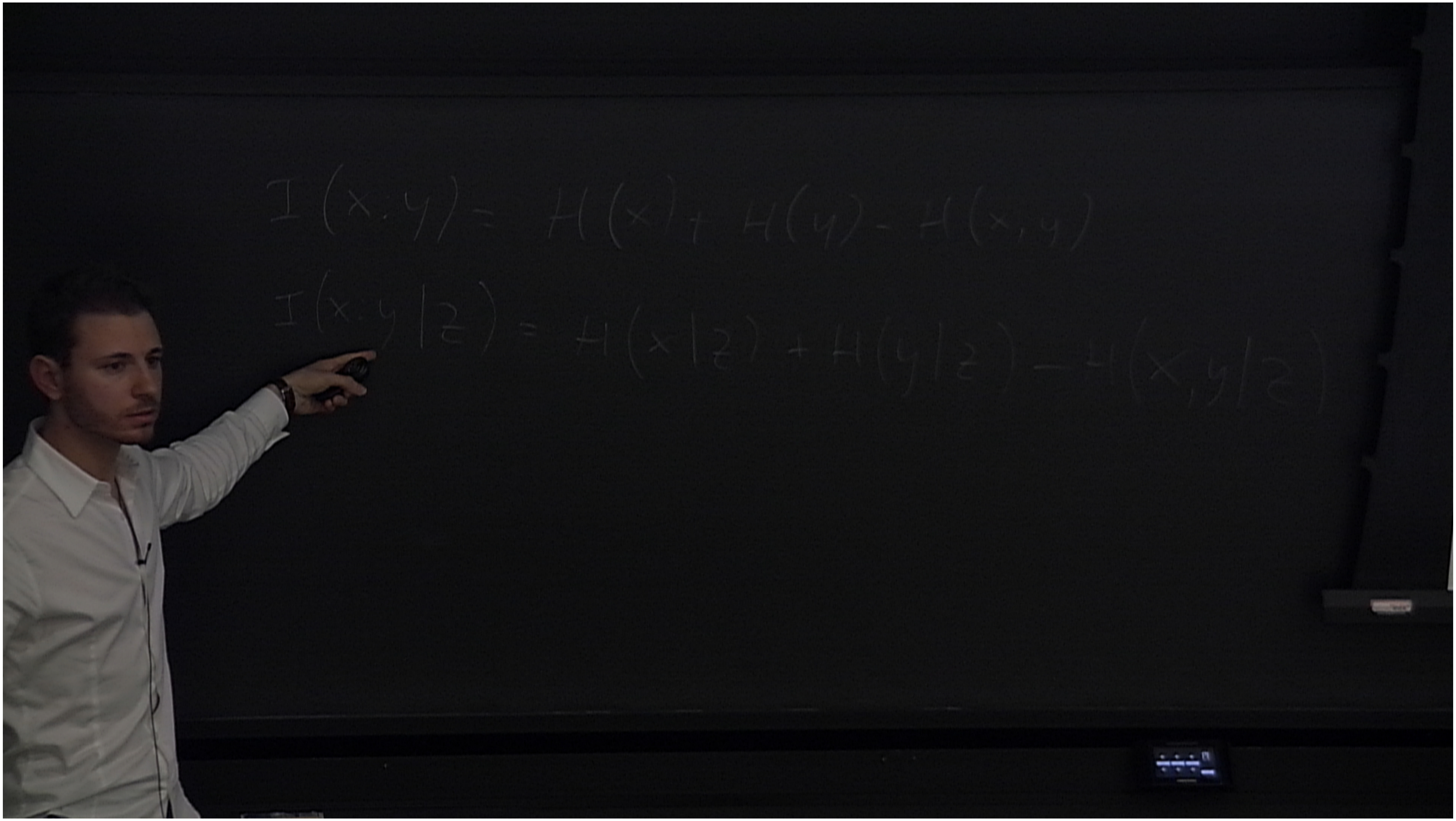
$I(x:y) = H(x) + H(y) - H(x,y)$
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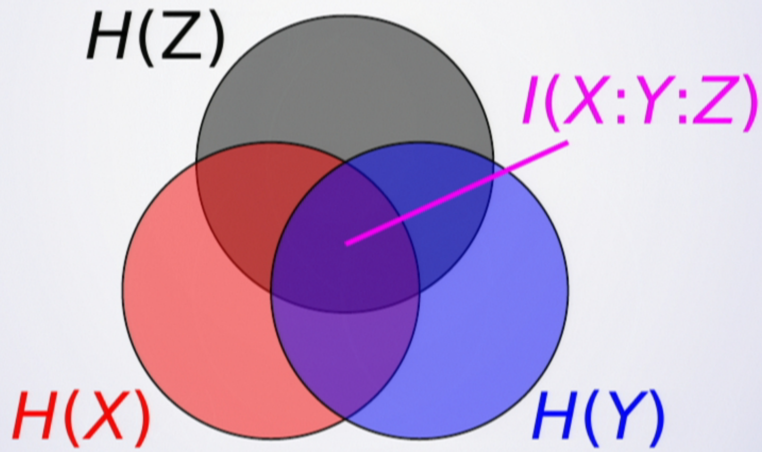

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Given a set U, we take a finite set of subsets $L \subseteq P(U)$ closed under union and intersection.

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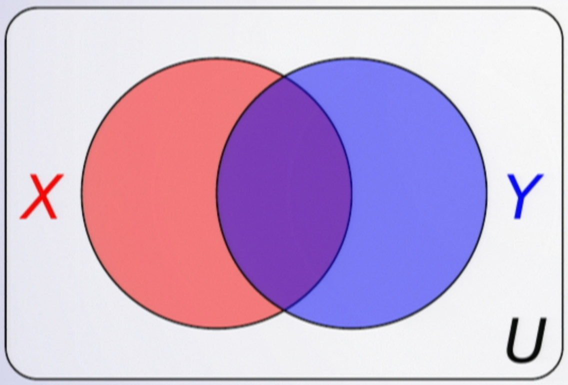

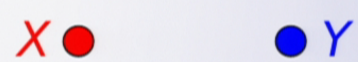

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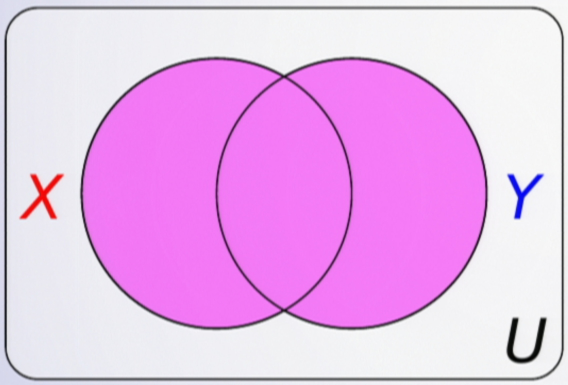

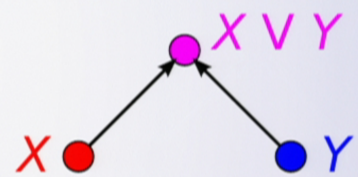

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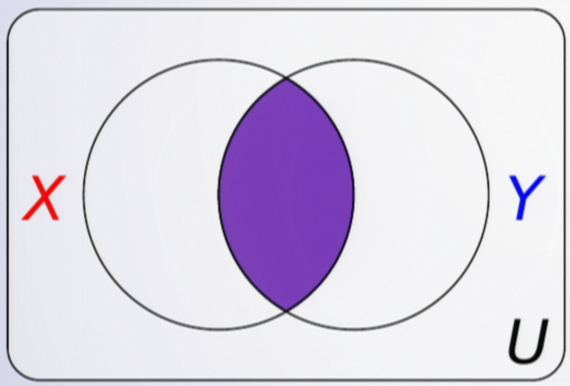

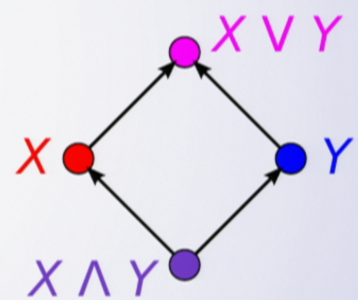

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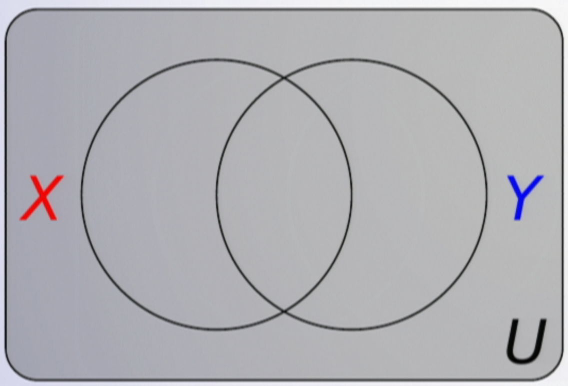

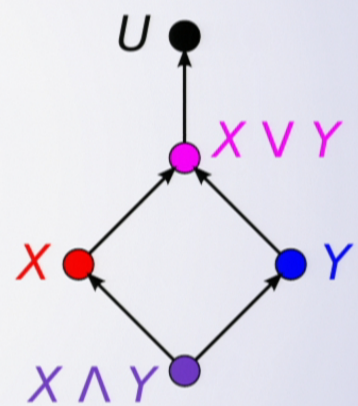

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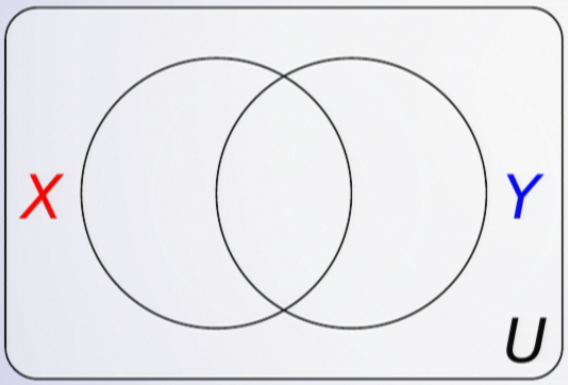

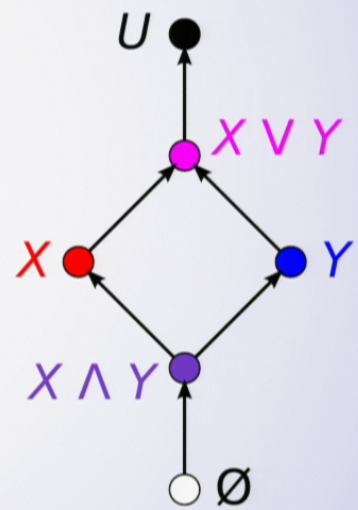

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Definition:
Let (U, Σ, μ) be a measure space. A *lattice of sets* in (U, Σ, μ) is a finite subset of Σ closed under union and intersection.

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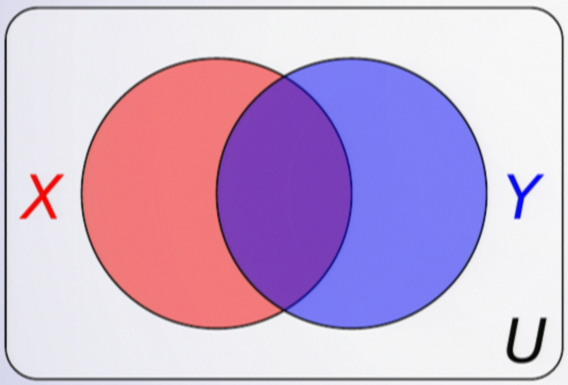

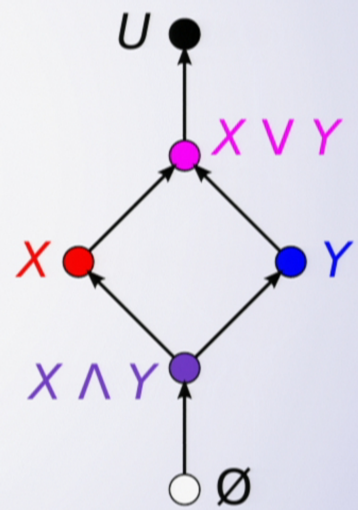

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This way, the measure μ restricted to L is a monotone function.

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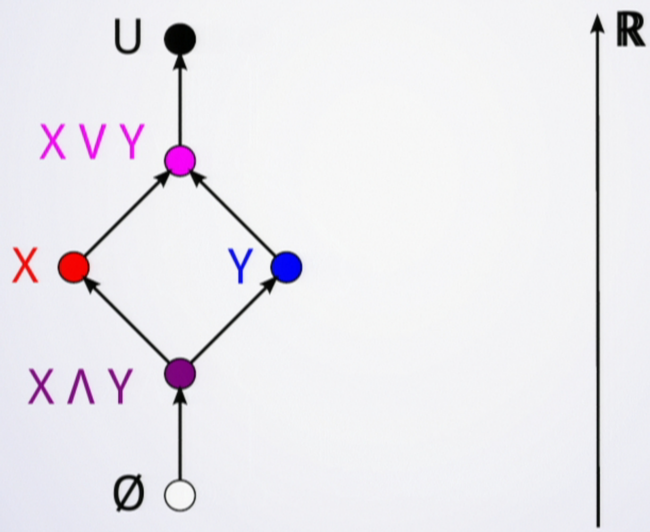

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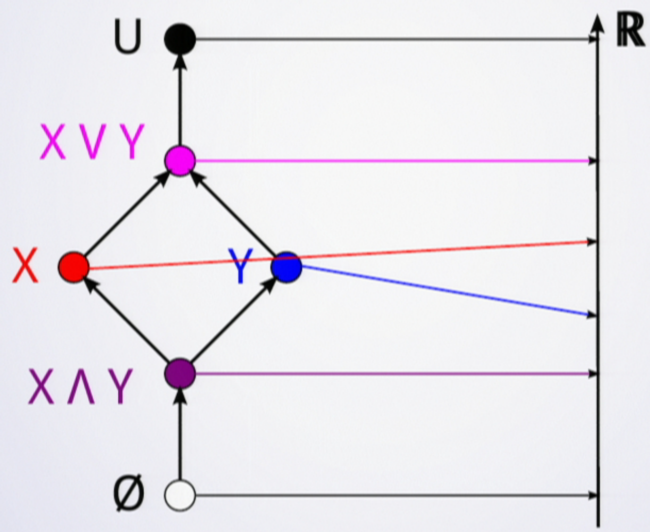

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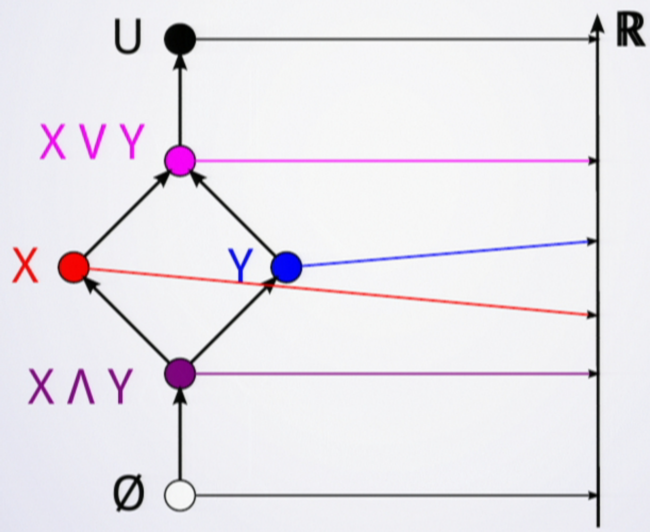

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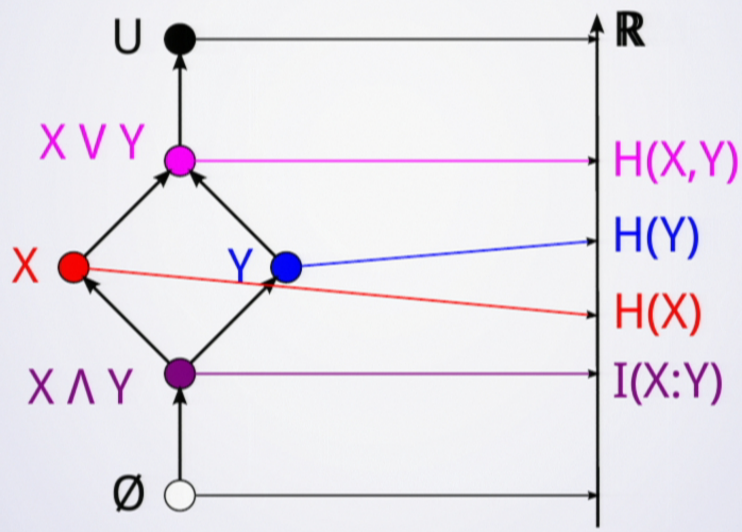

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Does information naturally form a lattice?

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Idea:

An observable Y is *more informative* than an observable X if it allows us to distinguish between more possible states.

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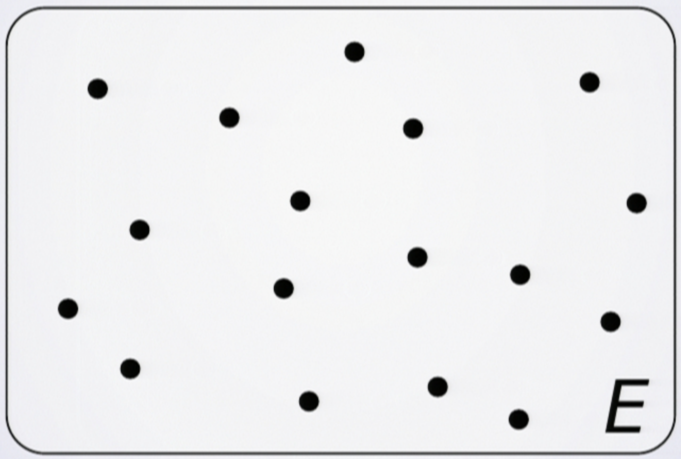

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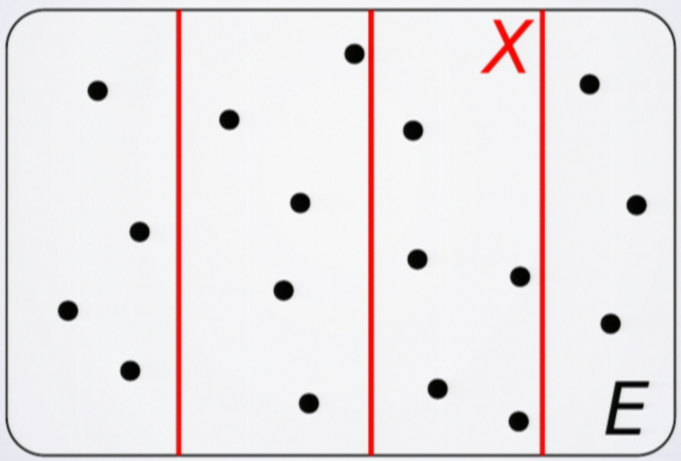

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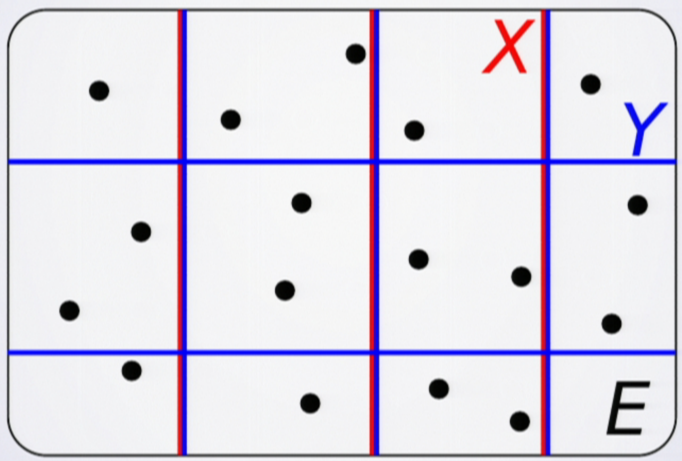

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Idea (refined):

An observable Y is *more informative* than an observable X if it induces a finer partition on the outcome space.

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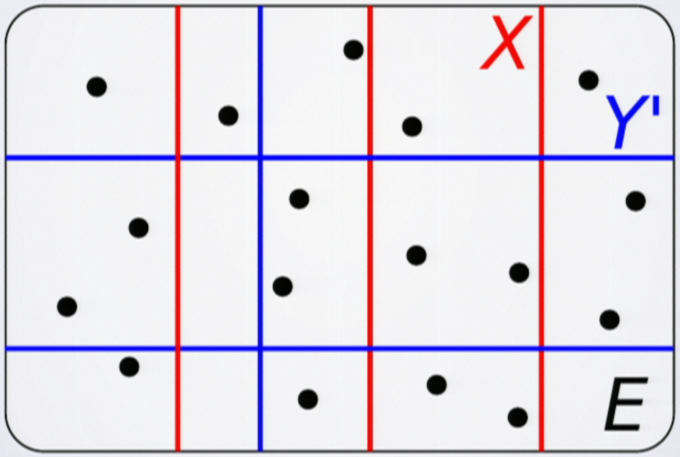

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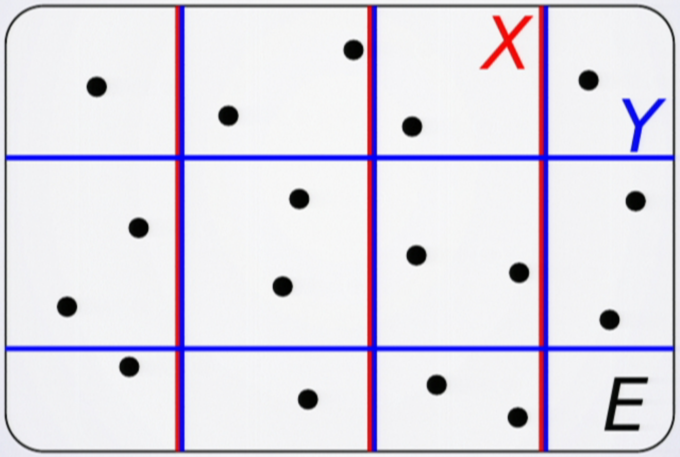

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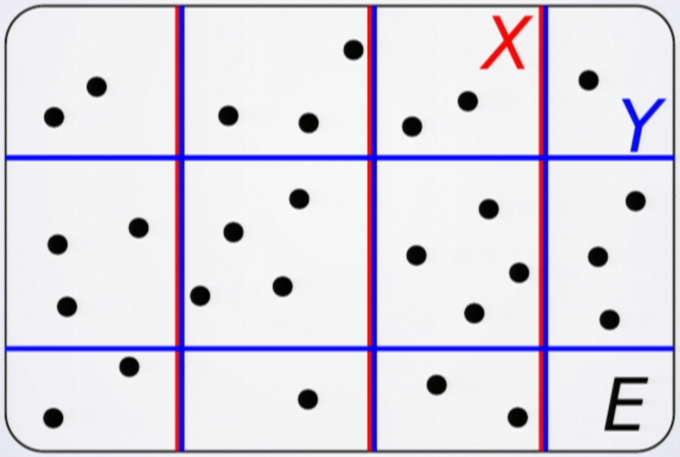

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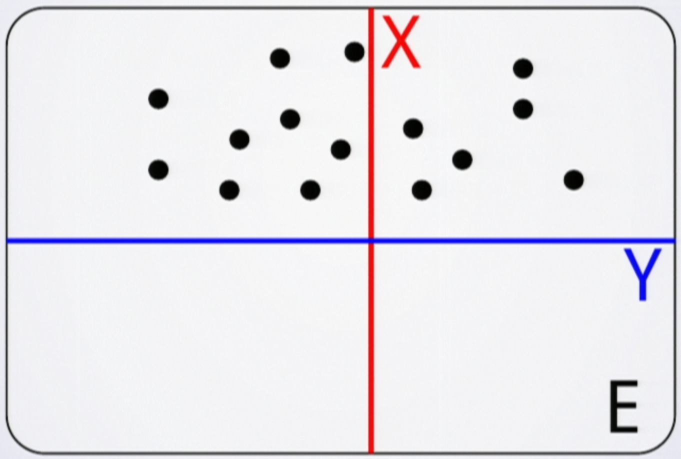

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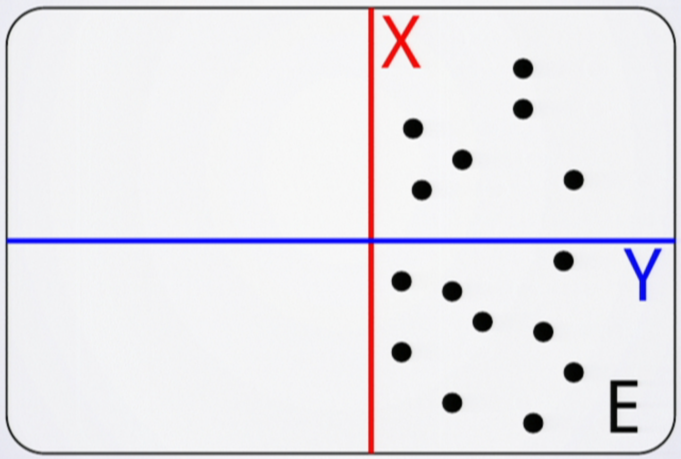

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- The relation "more informative" forms only a *partial* order!
- This order relation does not depend on the sample distribution.
- We will write in this case $Y \geq X$.

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The *information lattice* of a set of observables $X, Y, \ldots$ is the sublattice of the partition lattice of E generated by the partitions induced by $X, Y, \ldots$.

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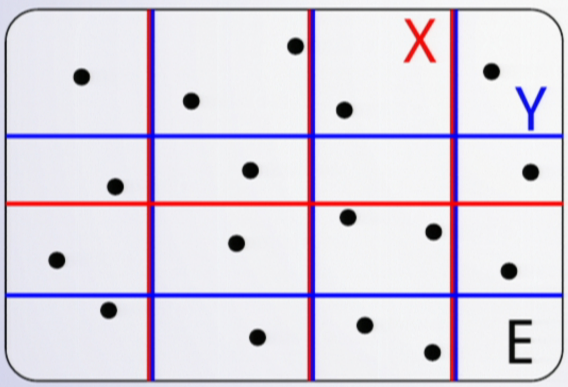

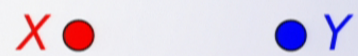

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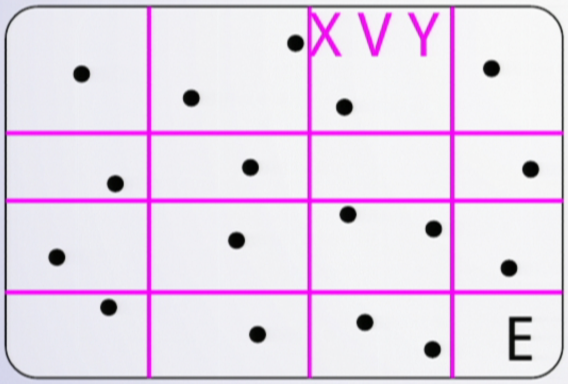

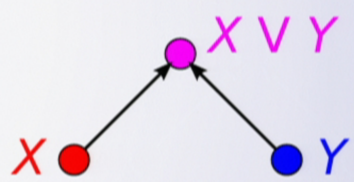

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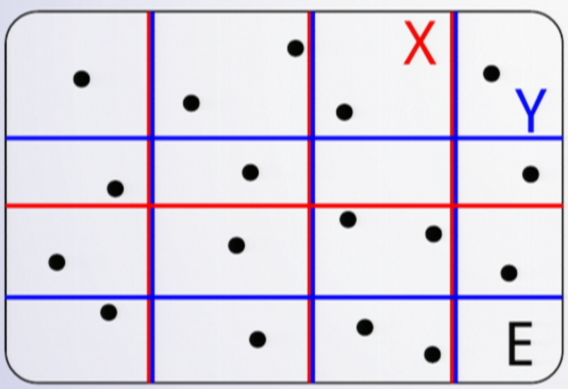

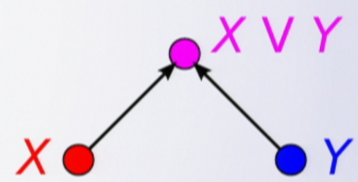

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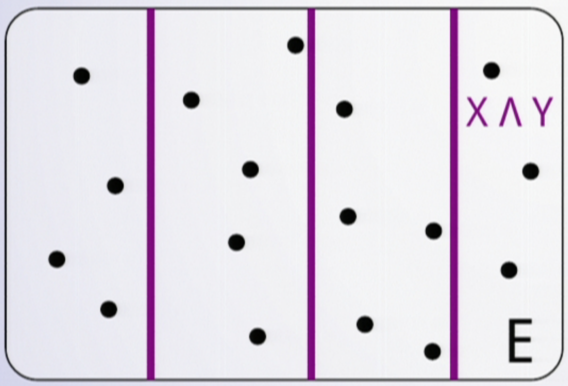

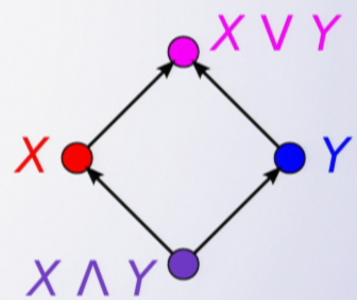

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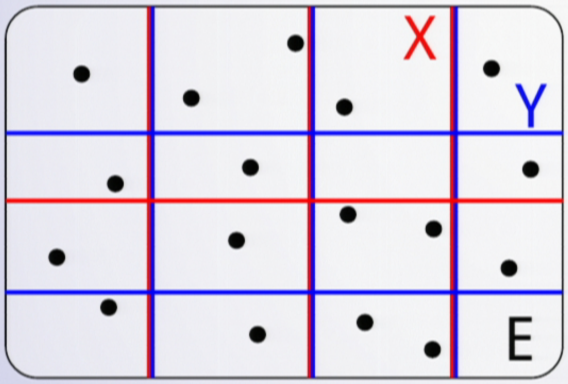

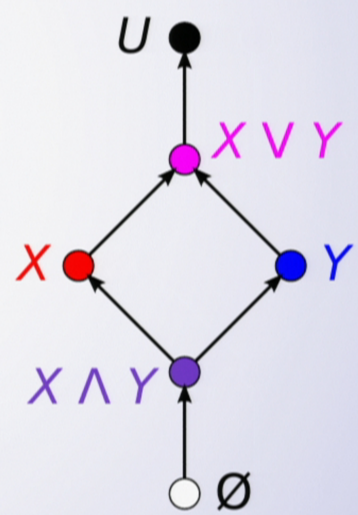

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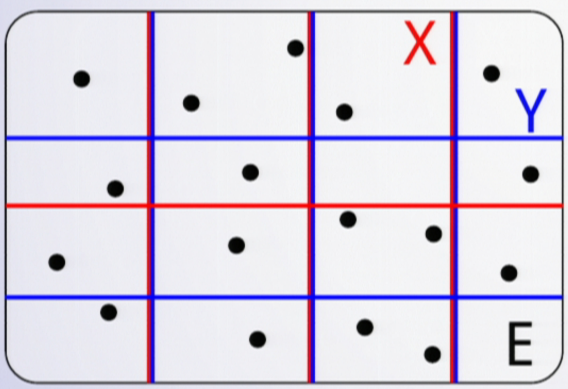

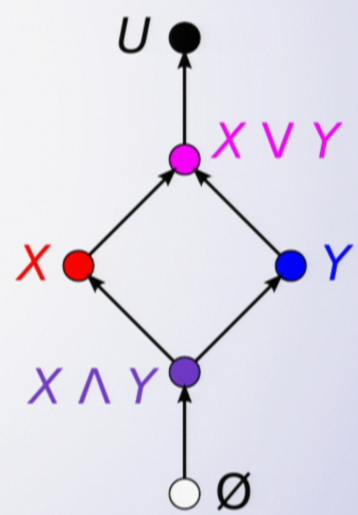

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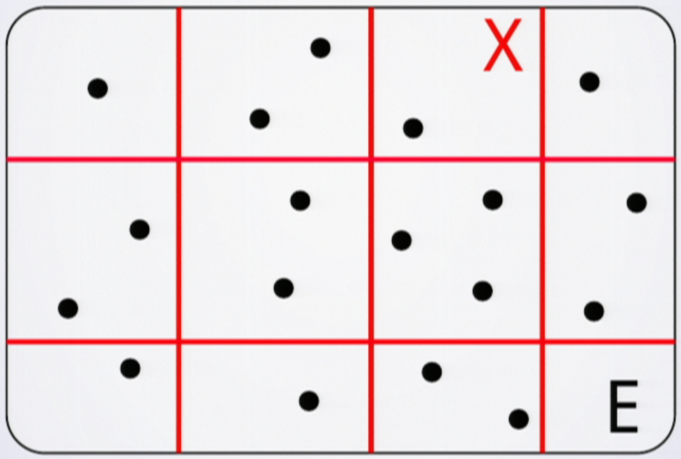

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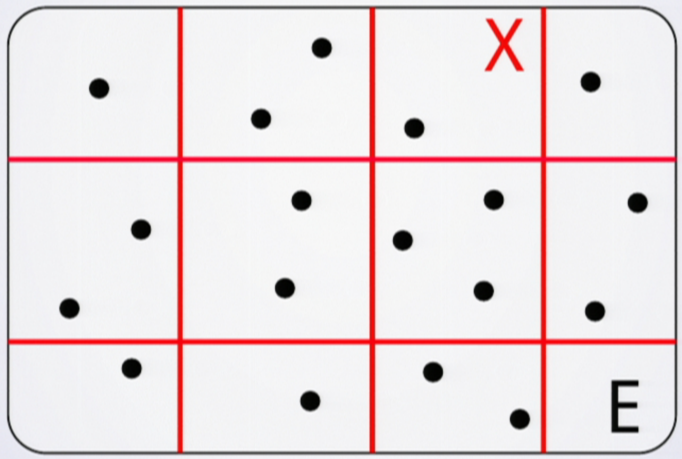

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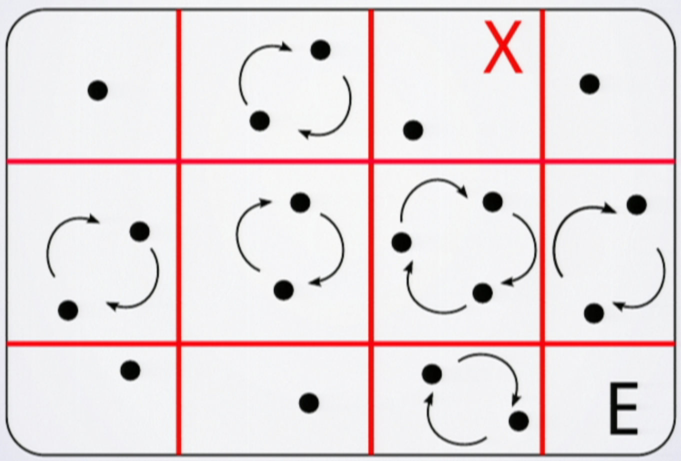

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The permutation group $\mathrm{Aut}(X)$ of a partition $p : E \to X$ is given by:

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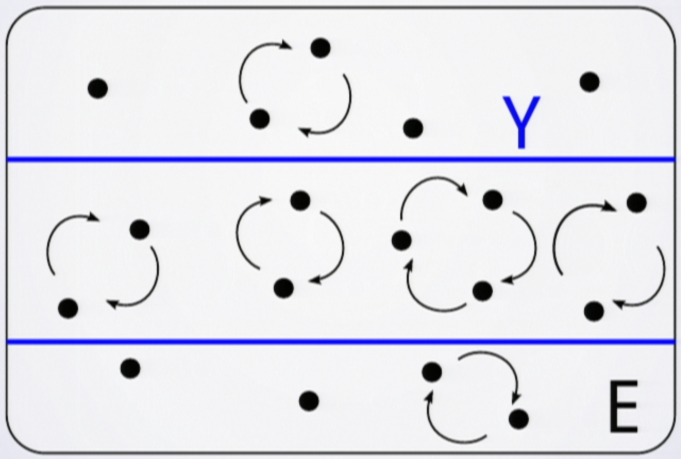

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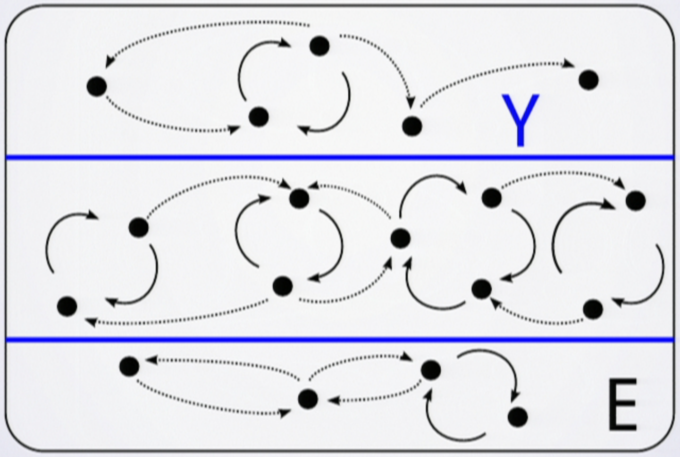

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If $X \geq Y$, $\mathrm{Aut}(X)$ is a subgroup of $\mathrm{Aut}(Y)$.

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- If $X \geq Y$, $\mathrm{Aut}(X) \leq \mathrm{Aut}(Y)$.

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- We therefore take the lattice of *quotients*.

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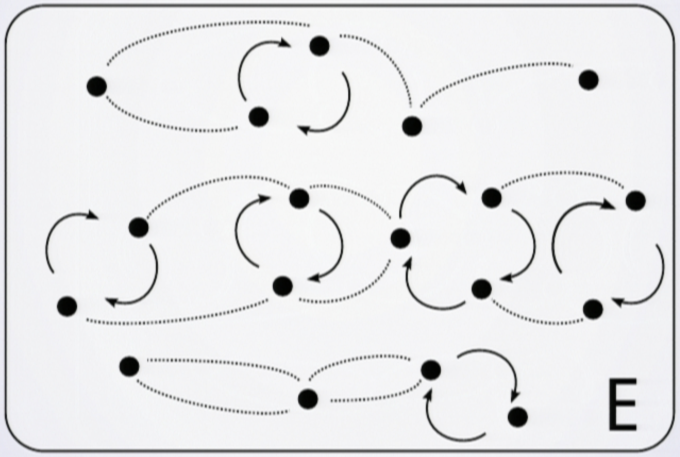

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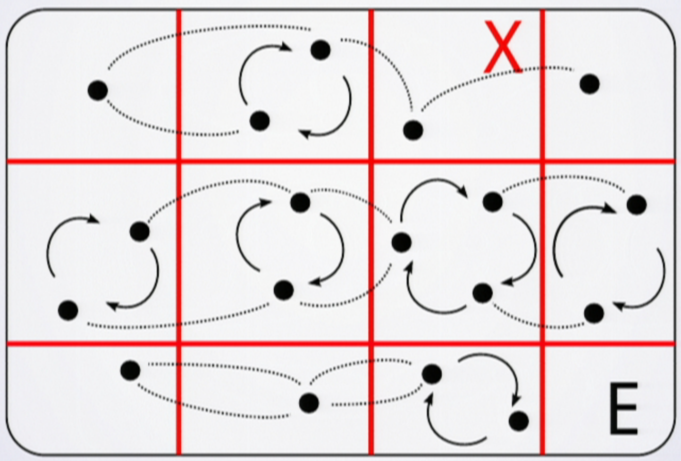

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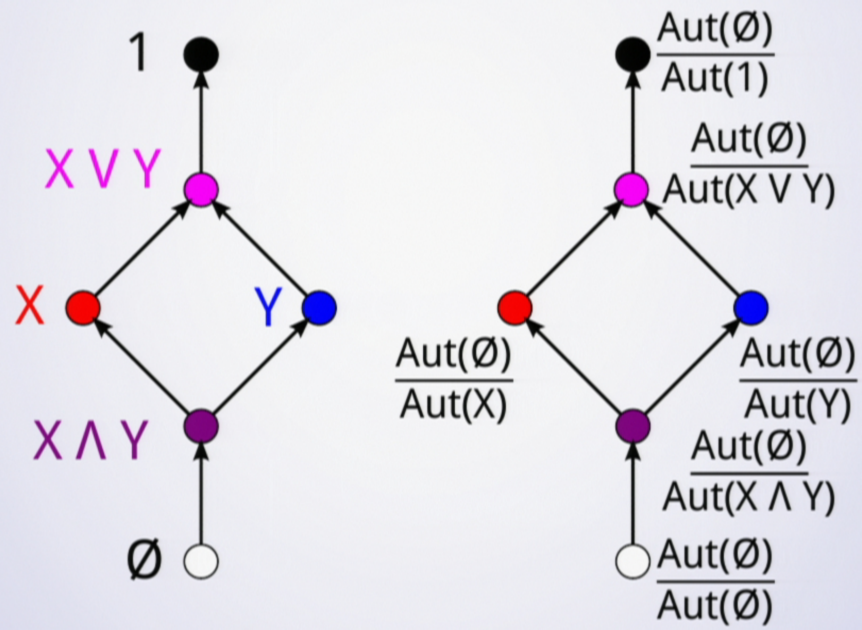

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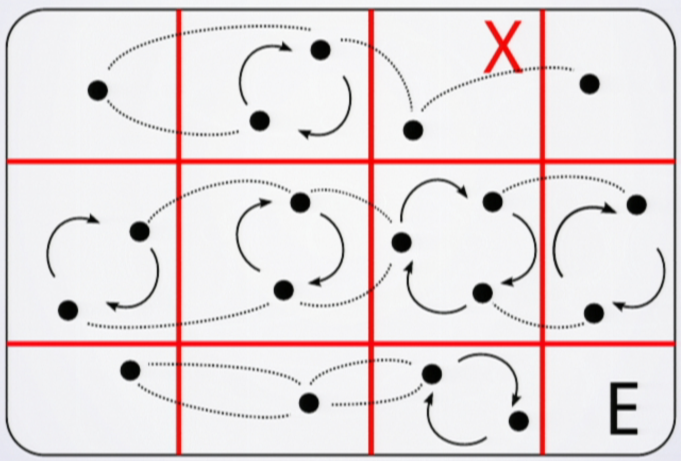

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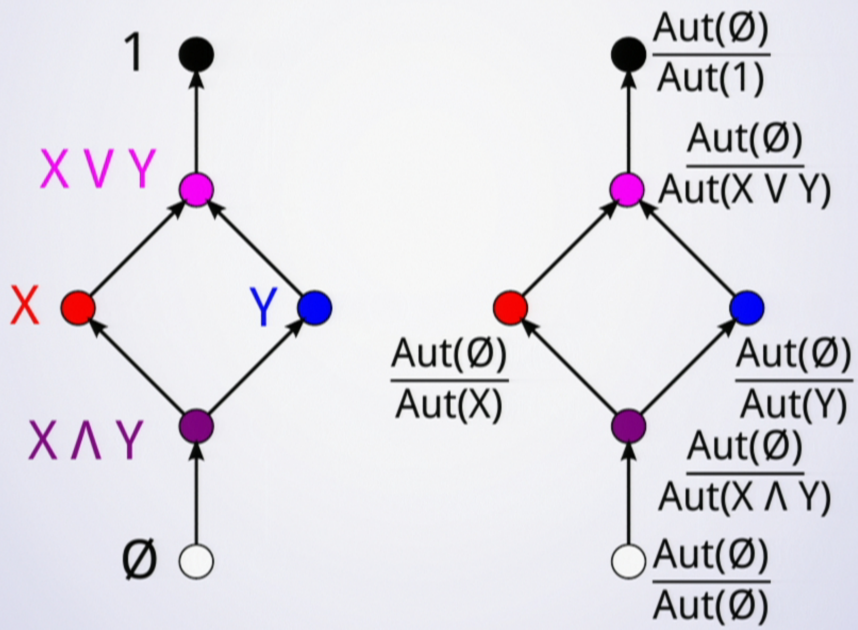

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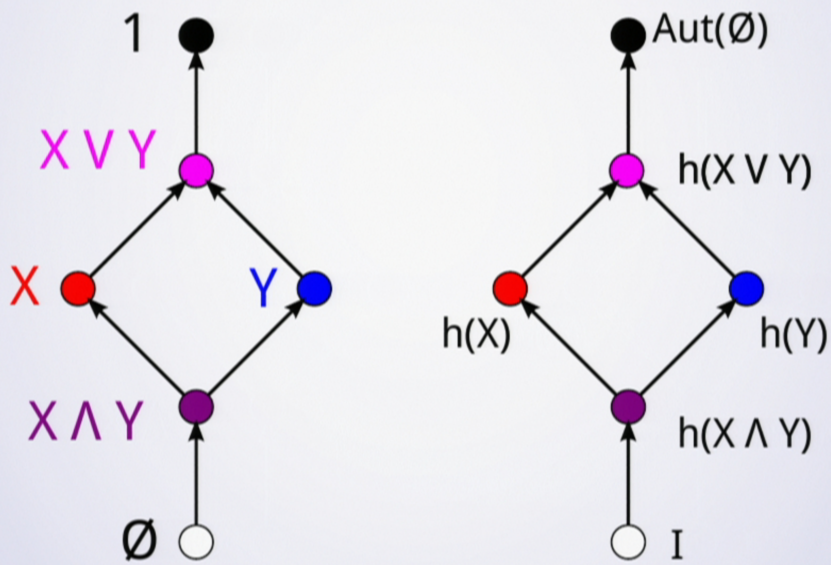

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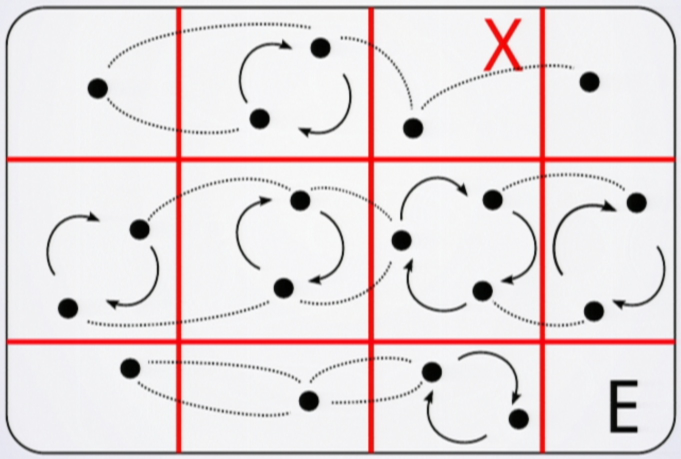

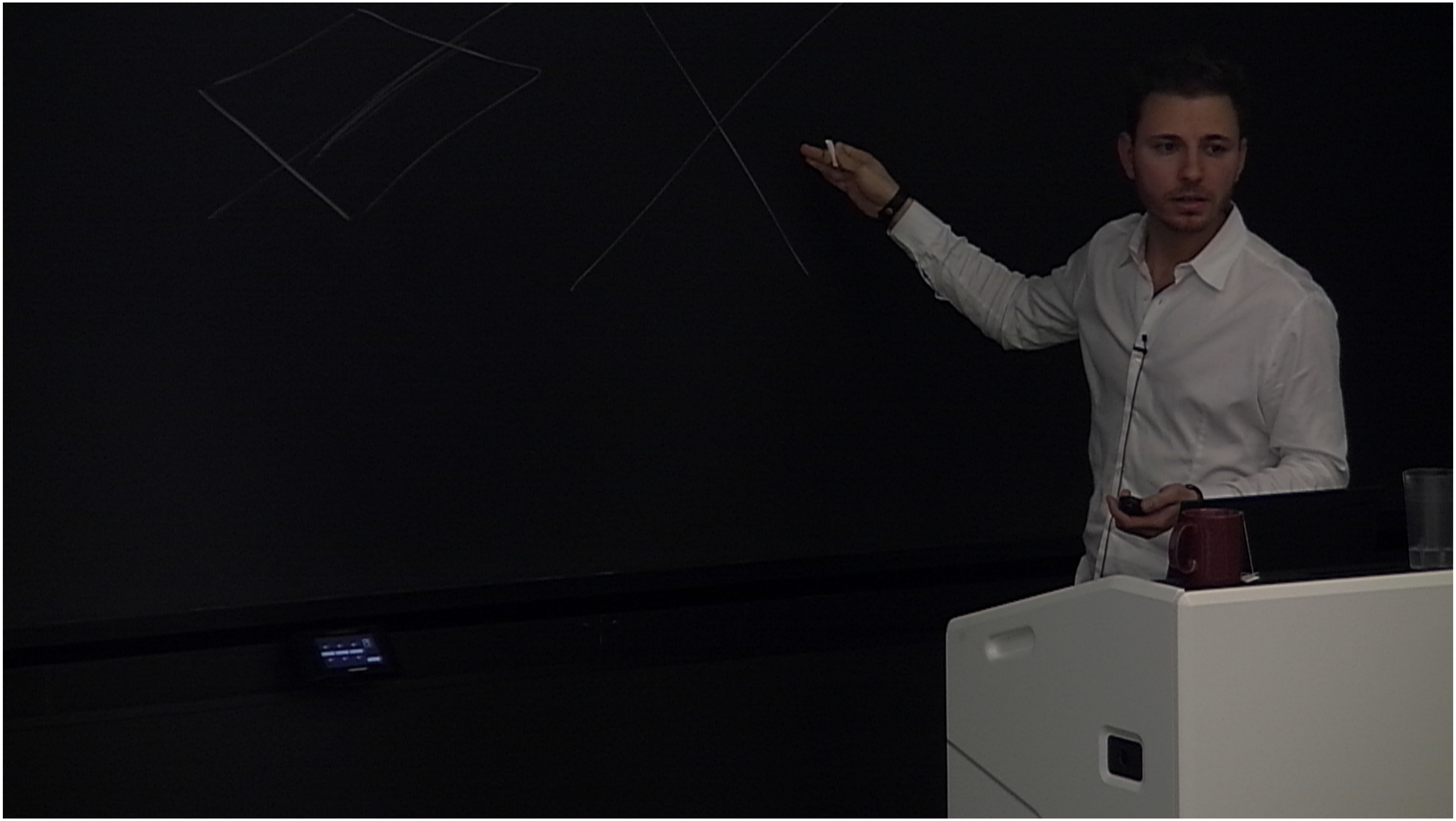

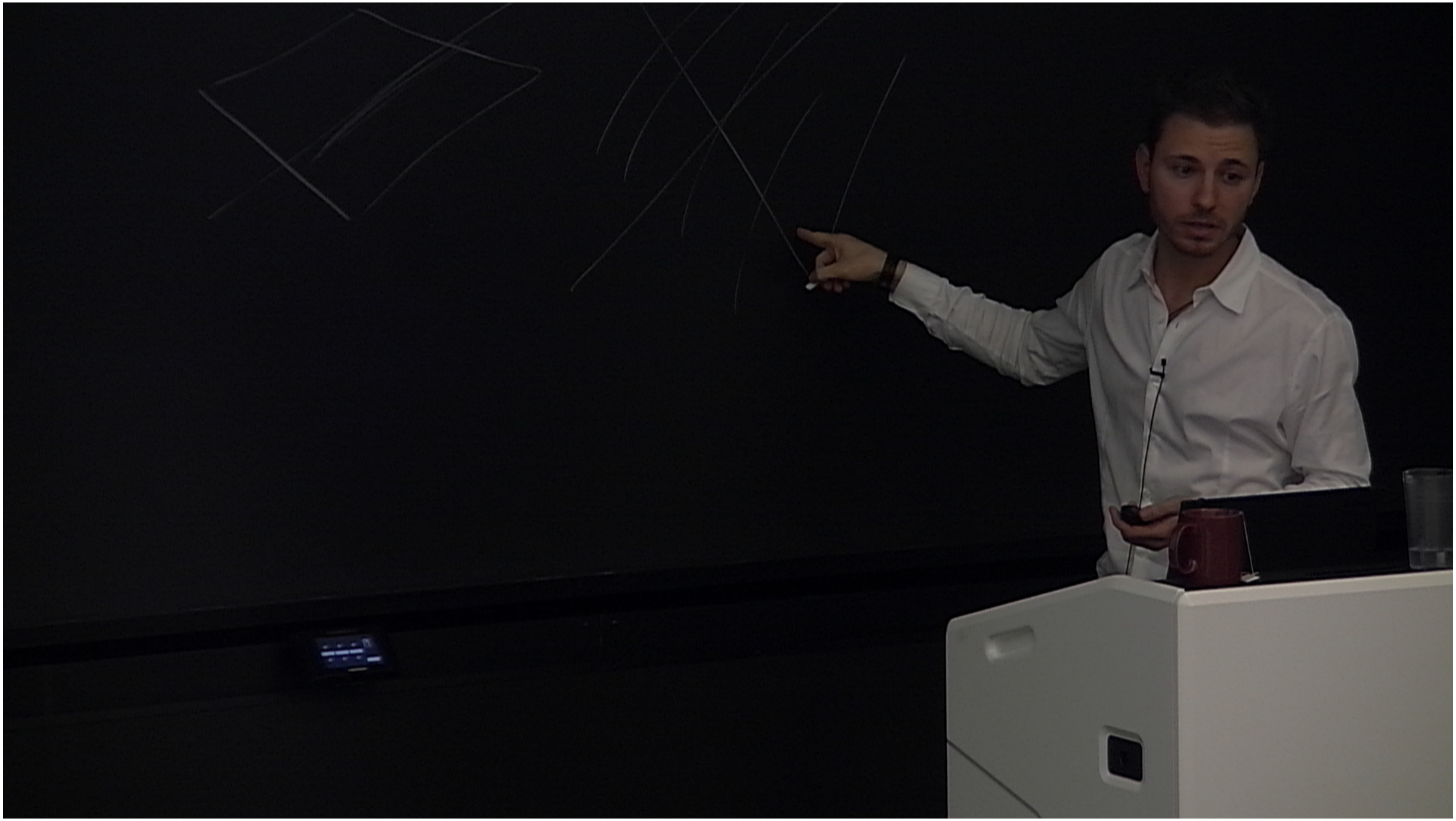

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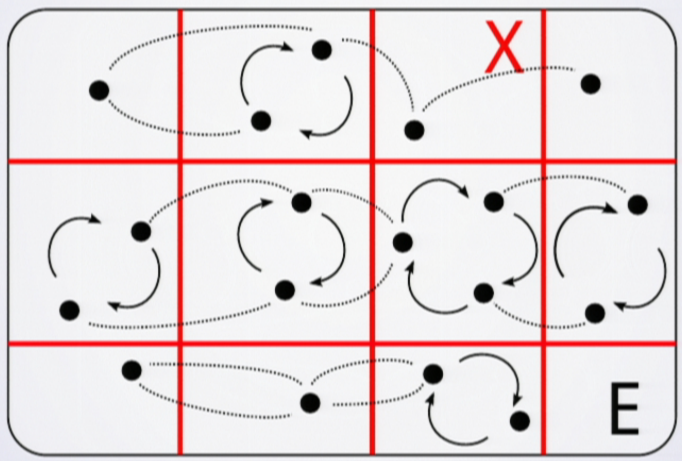

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(Even if the elements of the lattice are sets, the lattice of quotients is not a lattice of sets as defined before, the join is not the union.)

Definition:

The *coarse entropy* of X is the log-cardinality of the quotient $h(X)$:

$$H_c(X) := \log |h(X)| = \log \frac{|E|!}{\prod_{x \in X} |p^{-1}(x)|!}$$

This is a natural monotone function on the information lattice.

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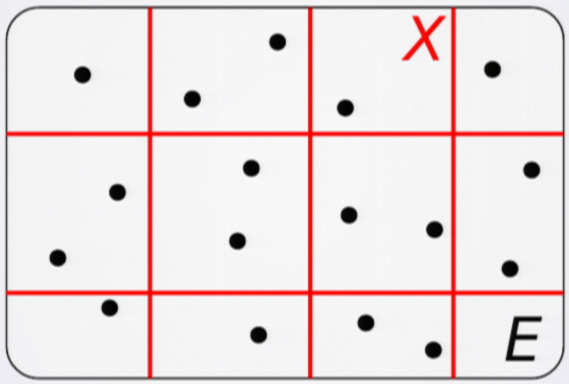

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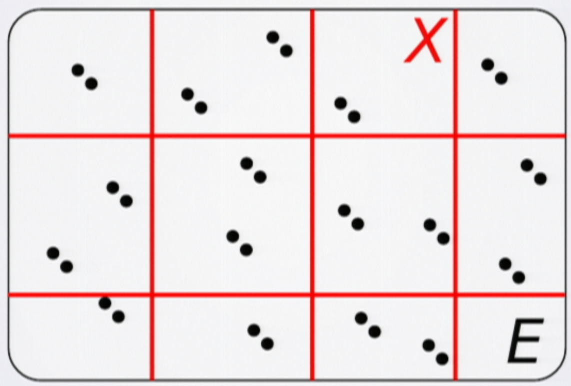

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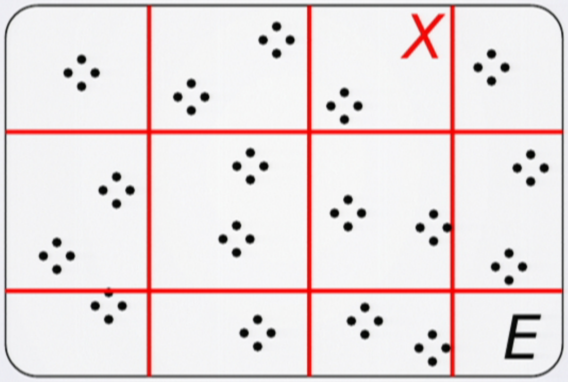

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$$\lim_{n\to\infty} \frac{H_c(X)}{|E|} = H(X)$$

Up to the constant, the limit rate is the Shannon entropy of X.

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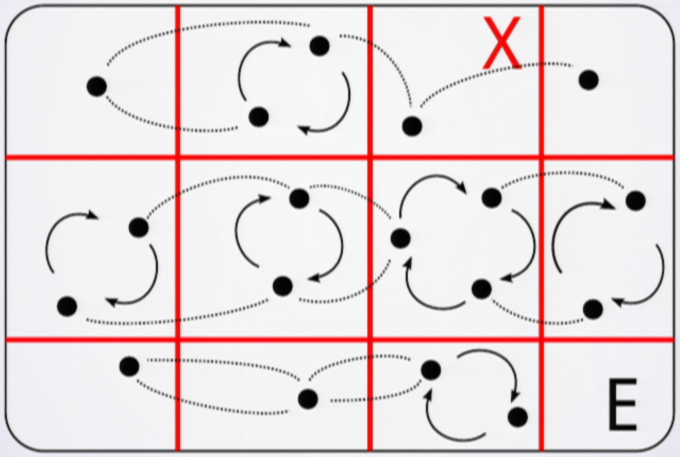

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Shannon-type inequalities do not depend on the distribution.
(Rigorous mathematical treatment in Chan [4] and Yeung [5].)

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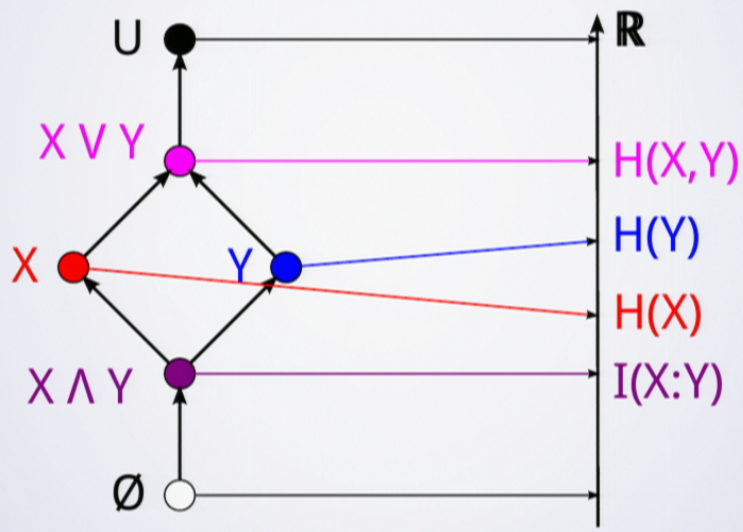

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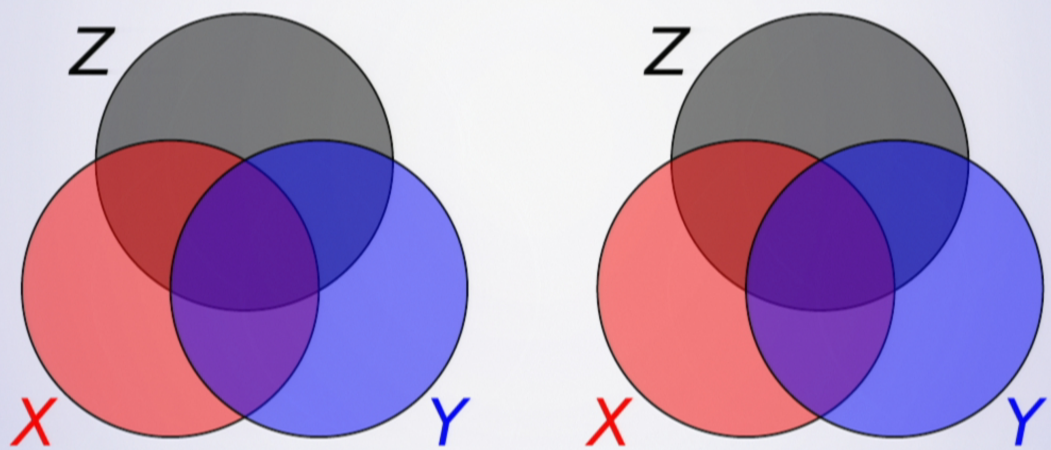

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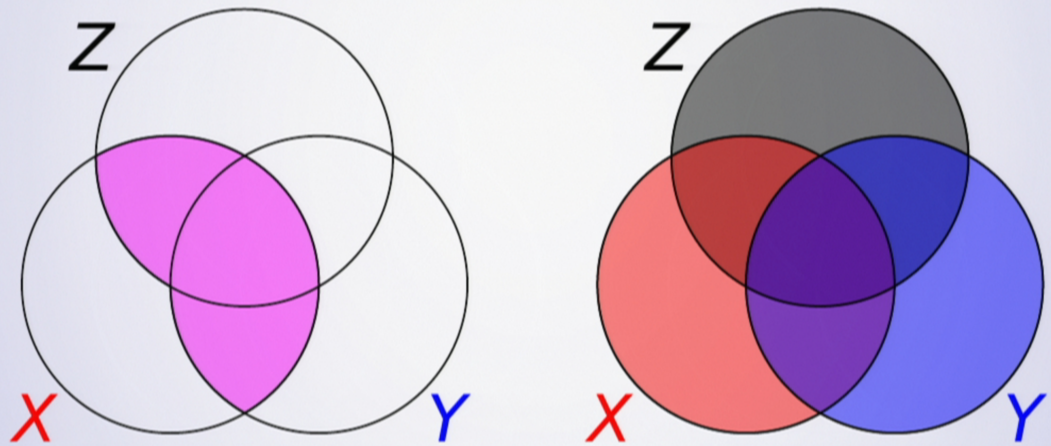

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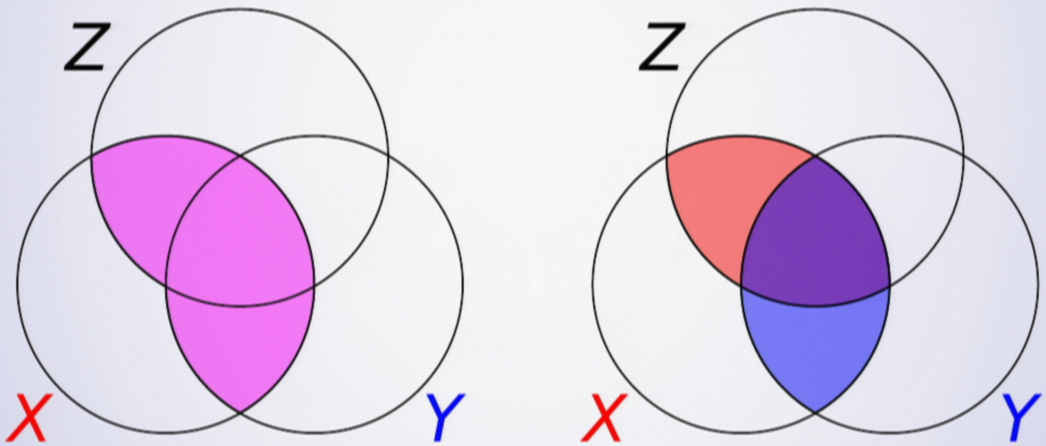

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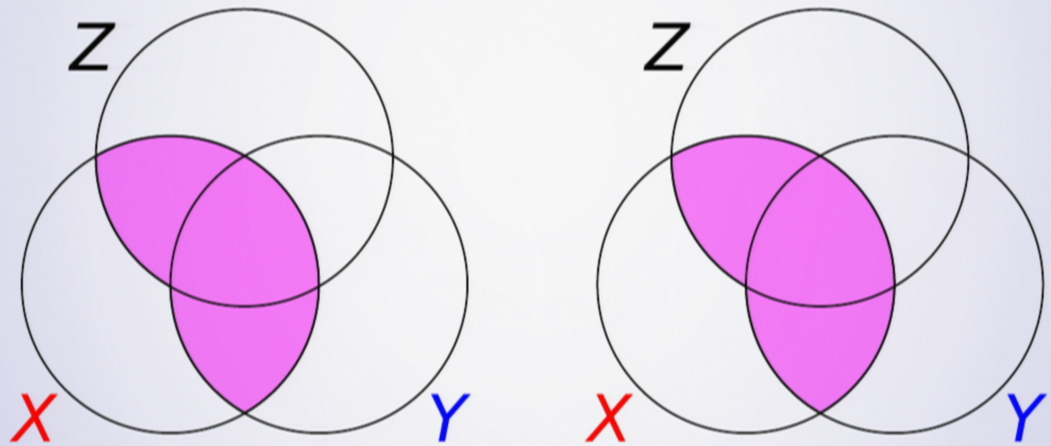

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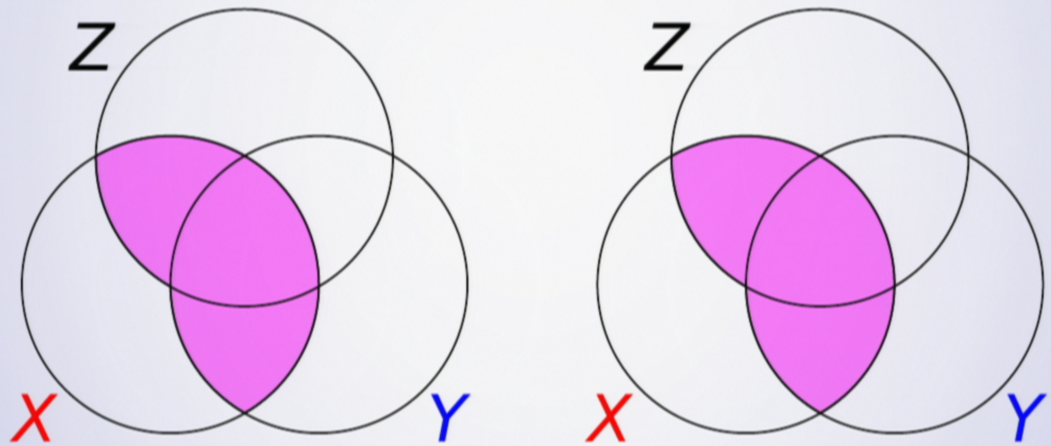

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Theorem (Birkhoff):

A finite lattice is isomorphic to a lattice of sets if and only if it is distributive. (Sometimes called "representable lattice".)

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Every finite lattice can be embedded in a finite partition lattice.

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Since not every finite lattice is distributive, *some* sublattice of a partition lattice will not be representable.

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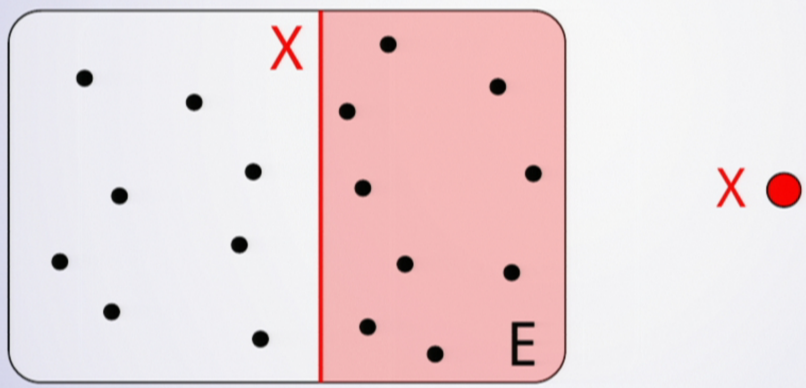

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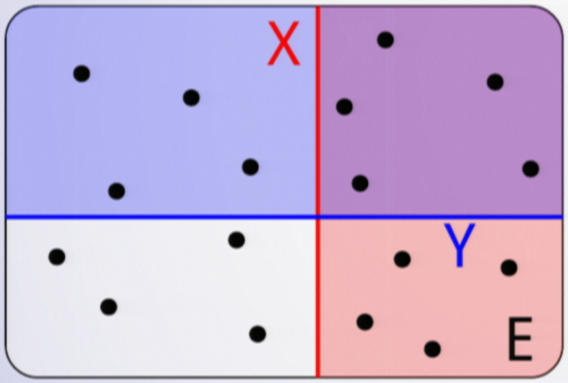

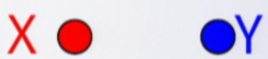

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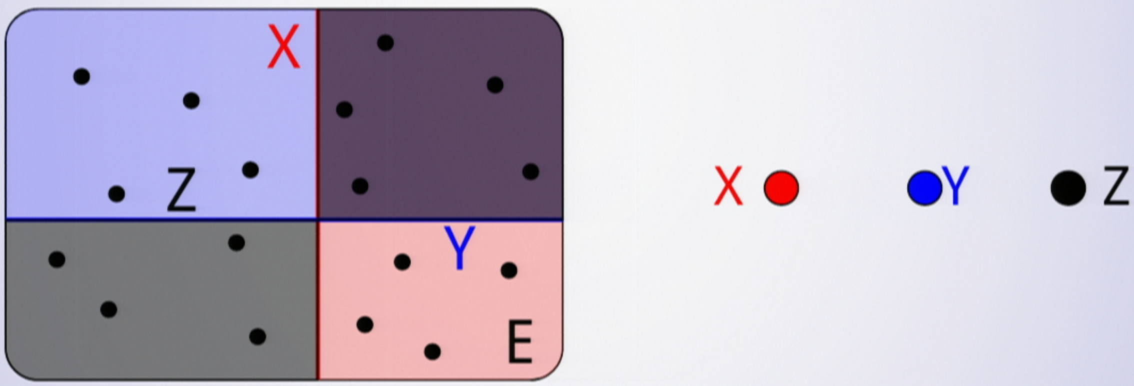

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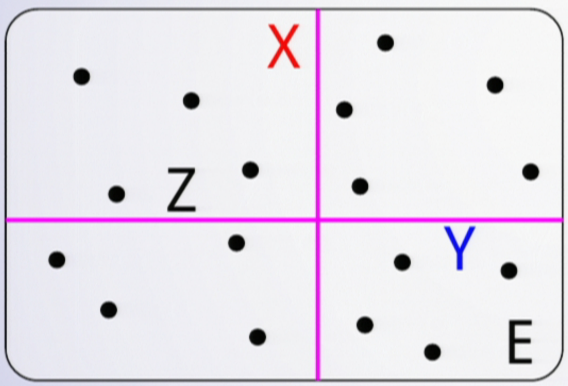

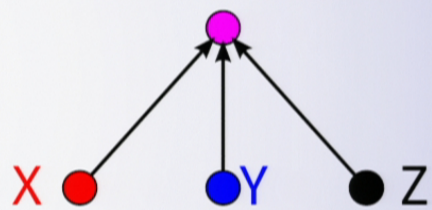

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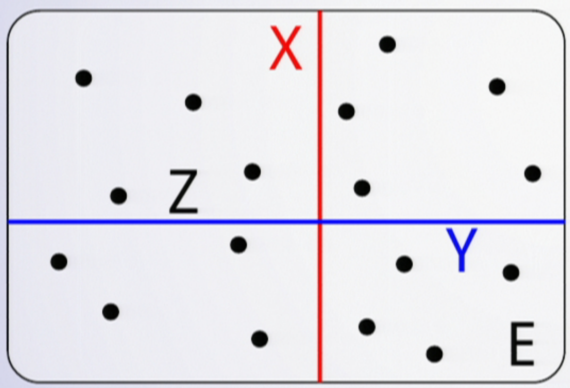

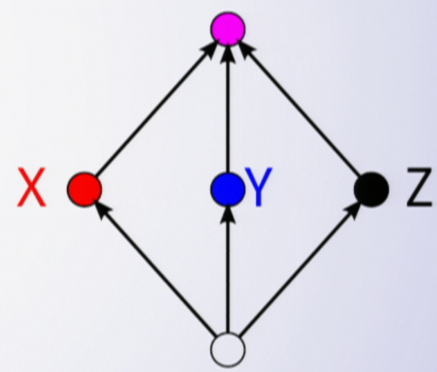

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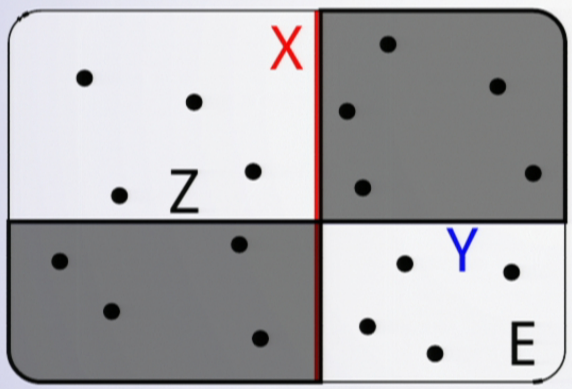

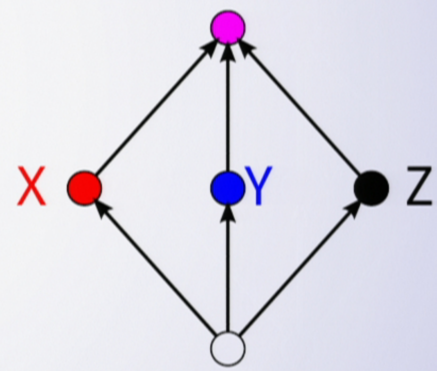

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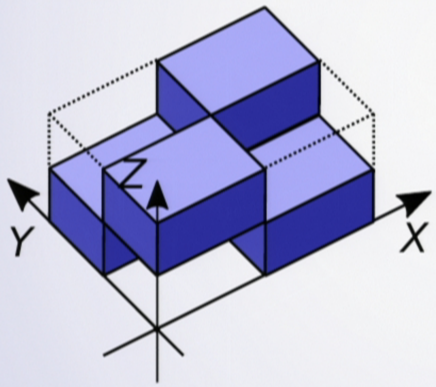

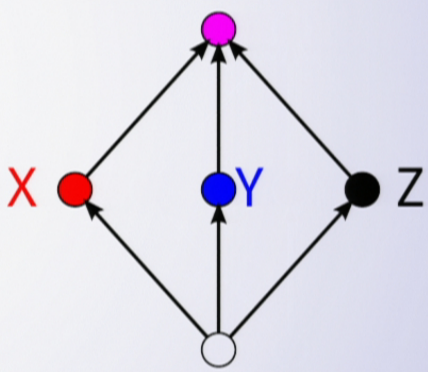

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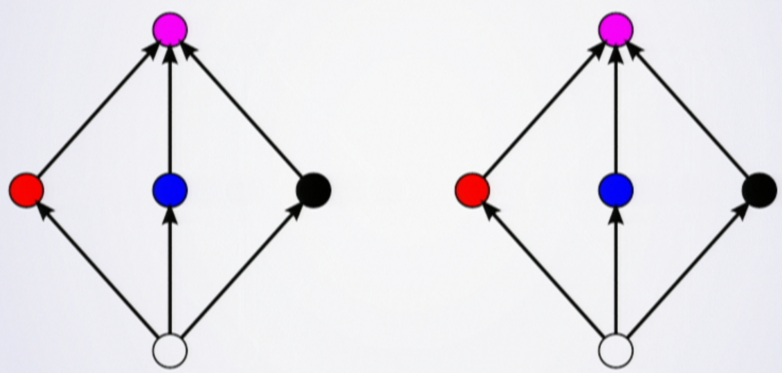

$X \wedge (Y \vee Z)$ $(X \wedge Y) \vee (X \wedge Z)$

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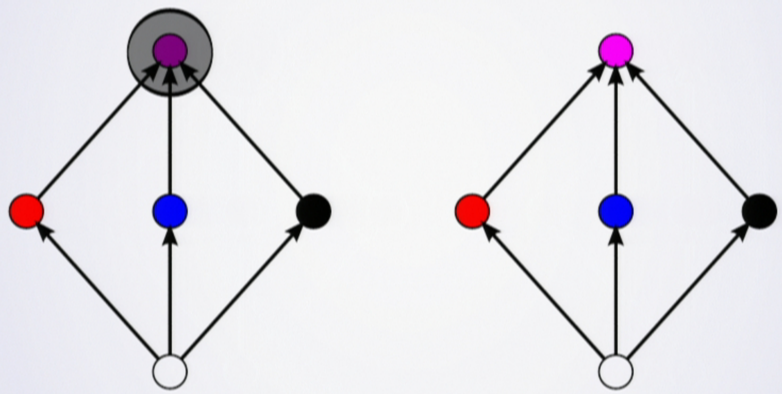

$X \wedge U$ $(X \wedge Y) \vee (X \wedge Z)$

Representability

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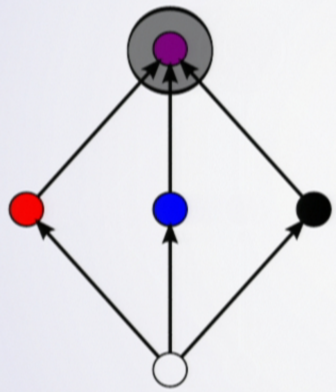
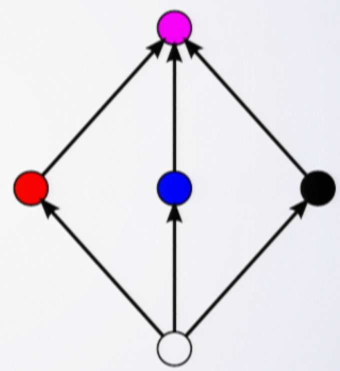


X ∧ U



(X ∧ Y) ∨ (X ∧ Z)

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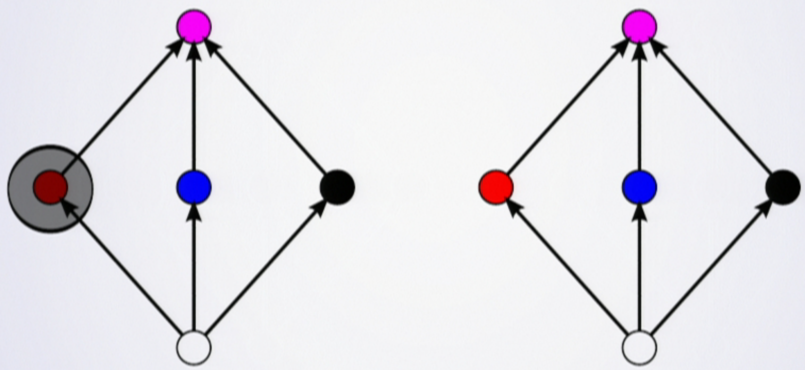

X $\qquad$ $(X \wedge Y) \vee (X \wedge Z)$

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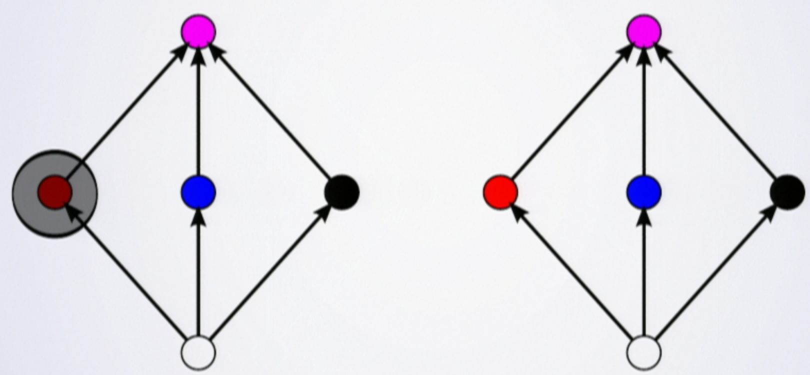

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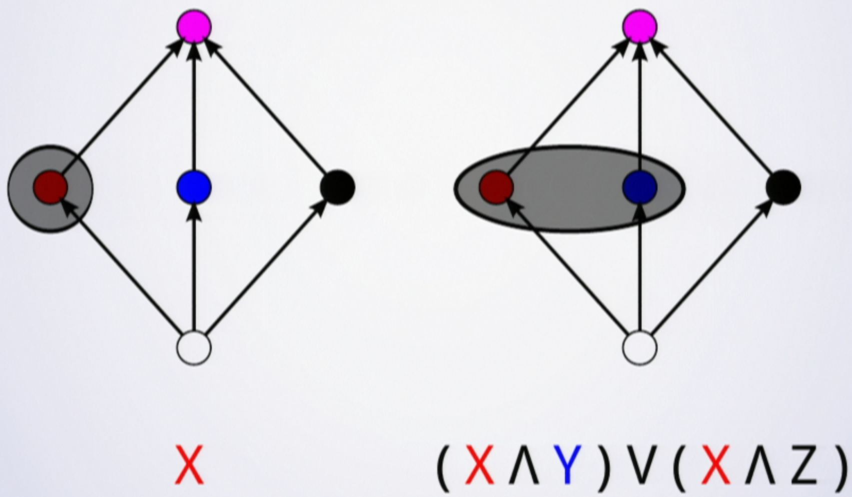

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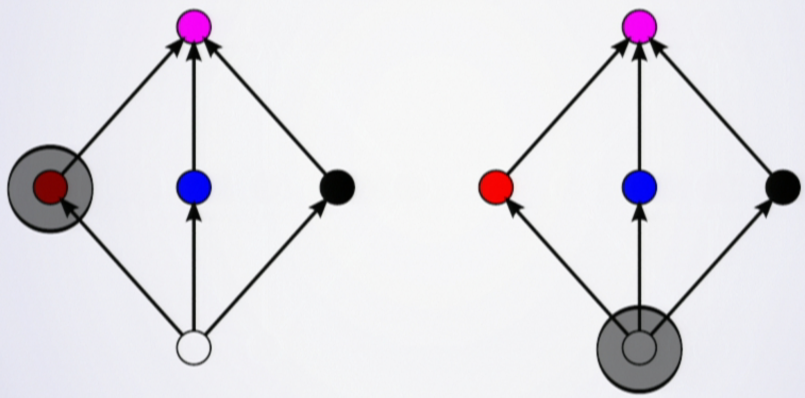

X $\quad\quad$ Ø $\vee$ (X $\wedge$ Z)

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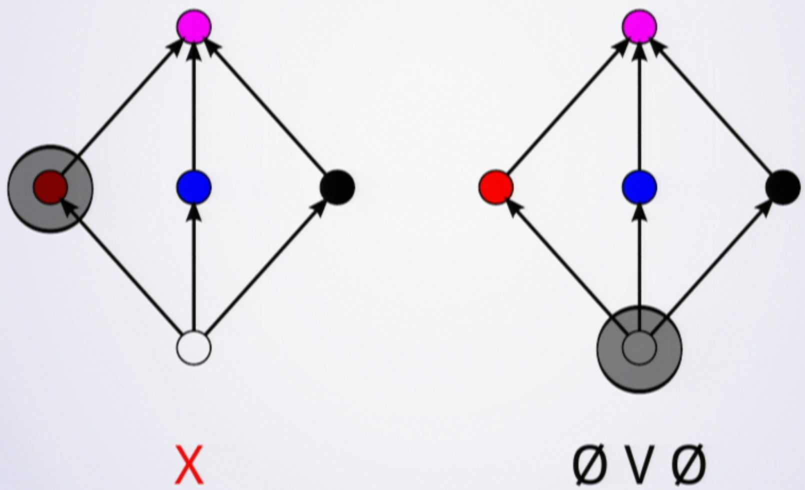

Representability

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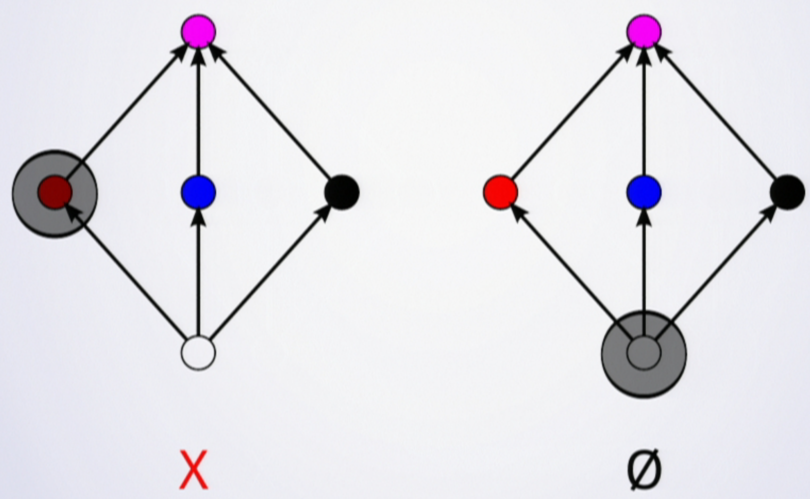

Conclusion

- The information of N observables forms naturally a lattice, as a sublattice of partitions of the outcome space.
- Entropy is a monotone function on the information lattice, obtained from permutations in the large numbers limit.
- The information lattice is not distributive for $N > 2$.
- Therefore, Shannon-type inequalities can be represented faithfully with Venn diagrams only for $N \leq 2$ random variables.

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Inf. Lattice

Shannon Ineqs   Venn Diagrams

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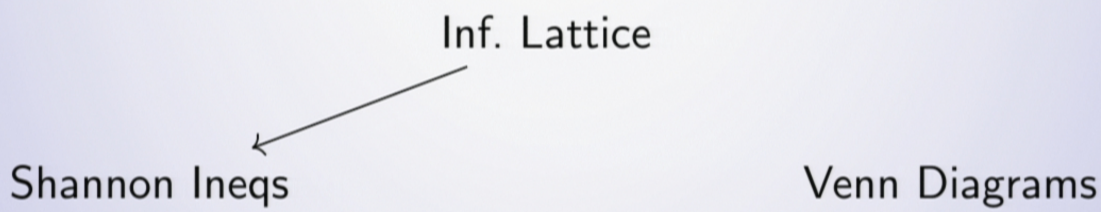

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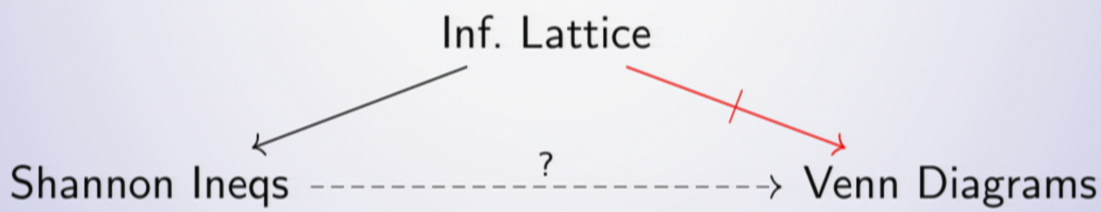
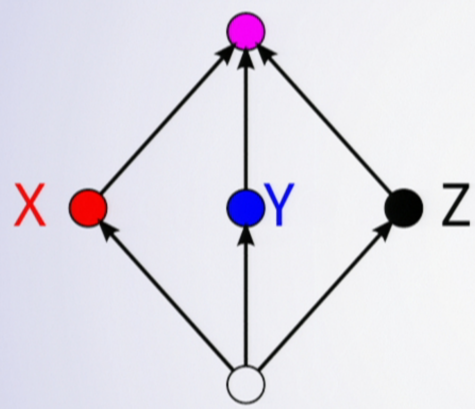
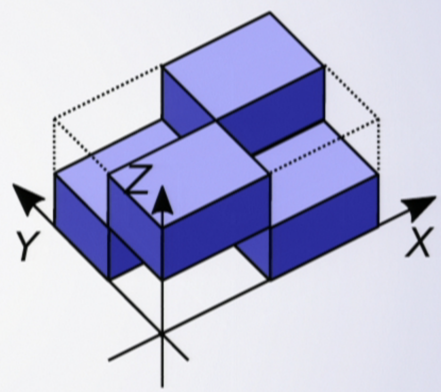


Further Questions

- Synergy and M_3: do they imply one another?





- More in general, is synergy linked to non-distributivity?

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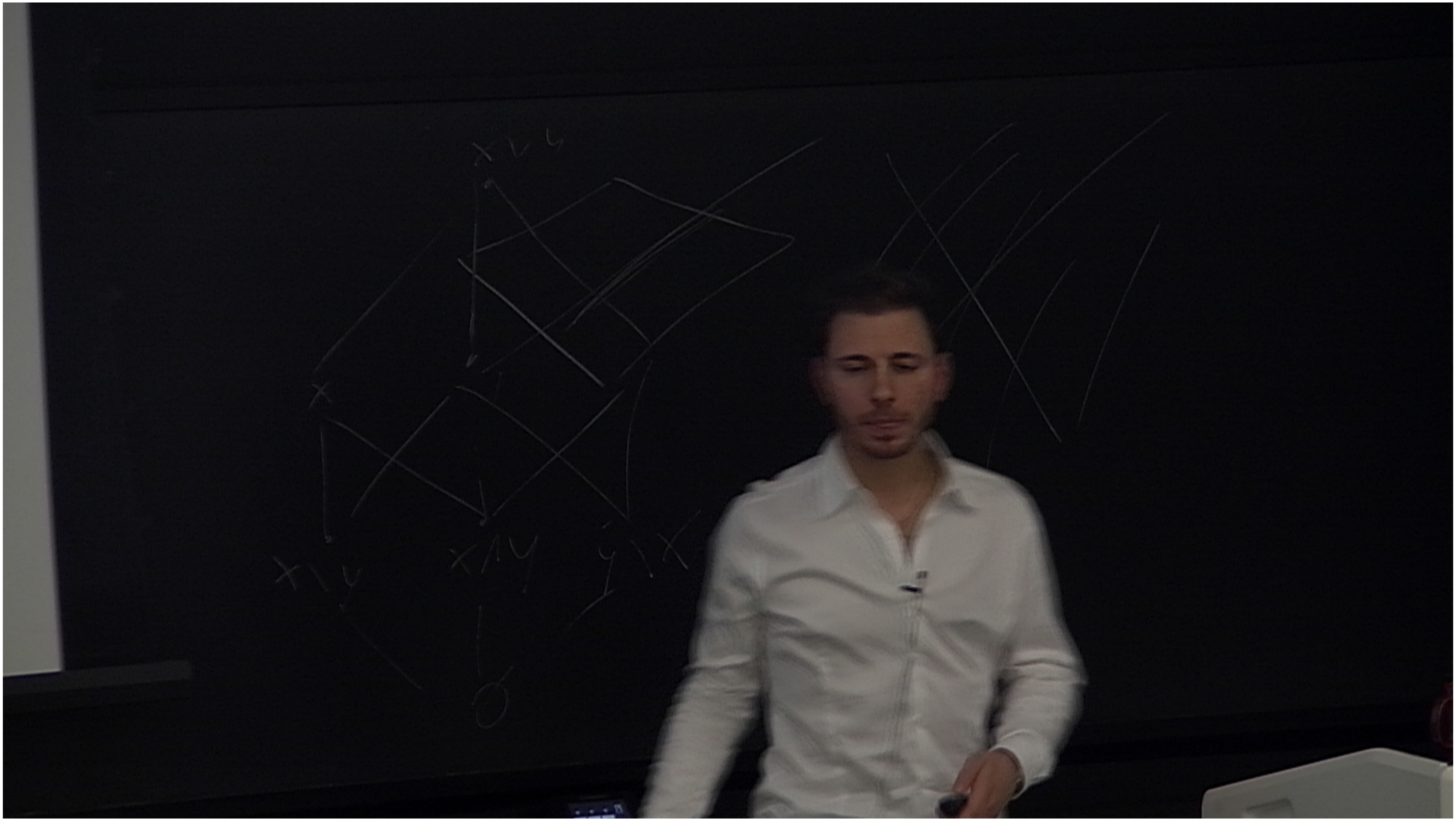
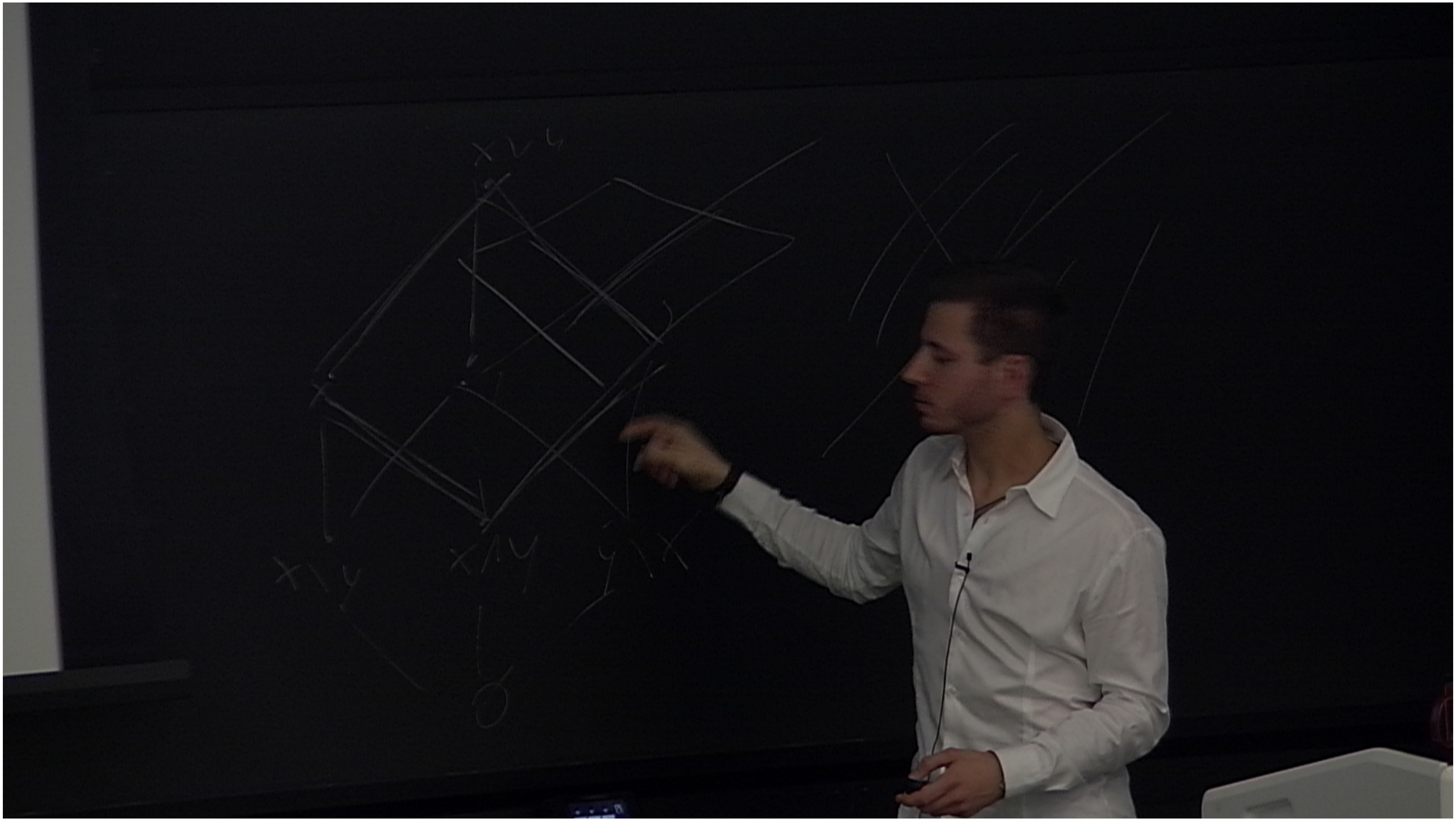
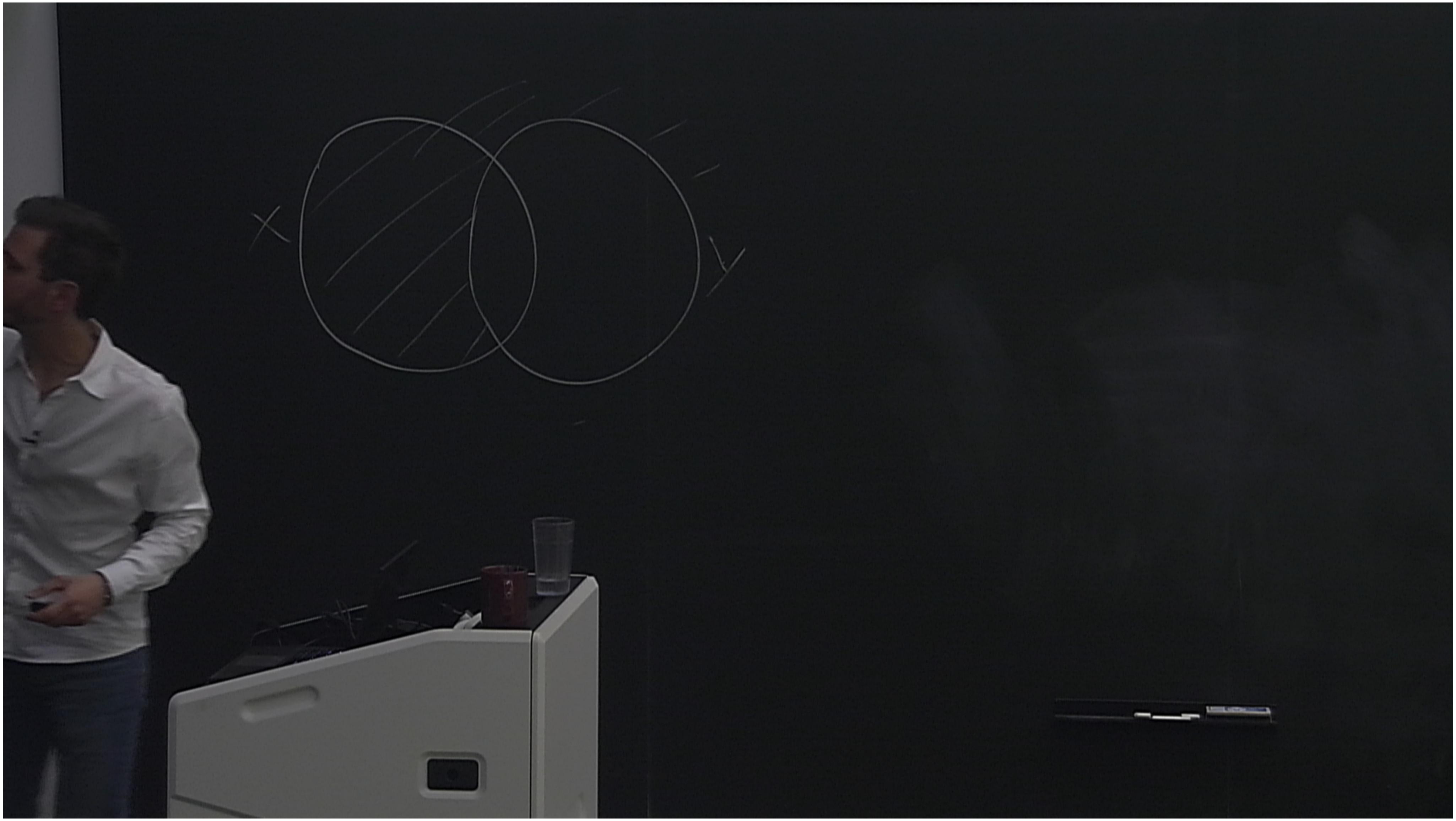
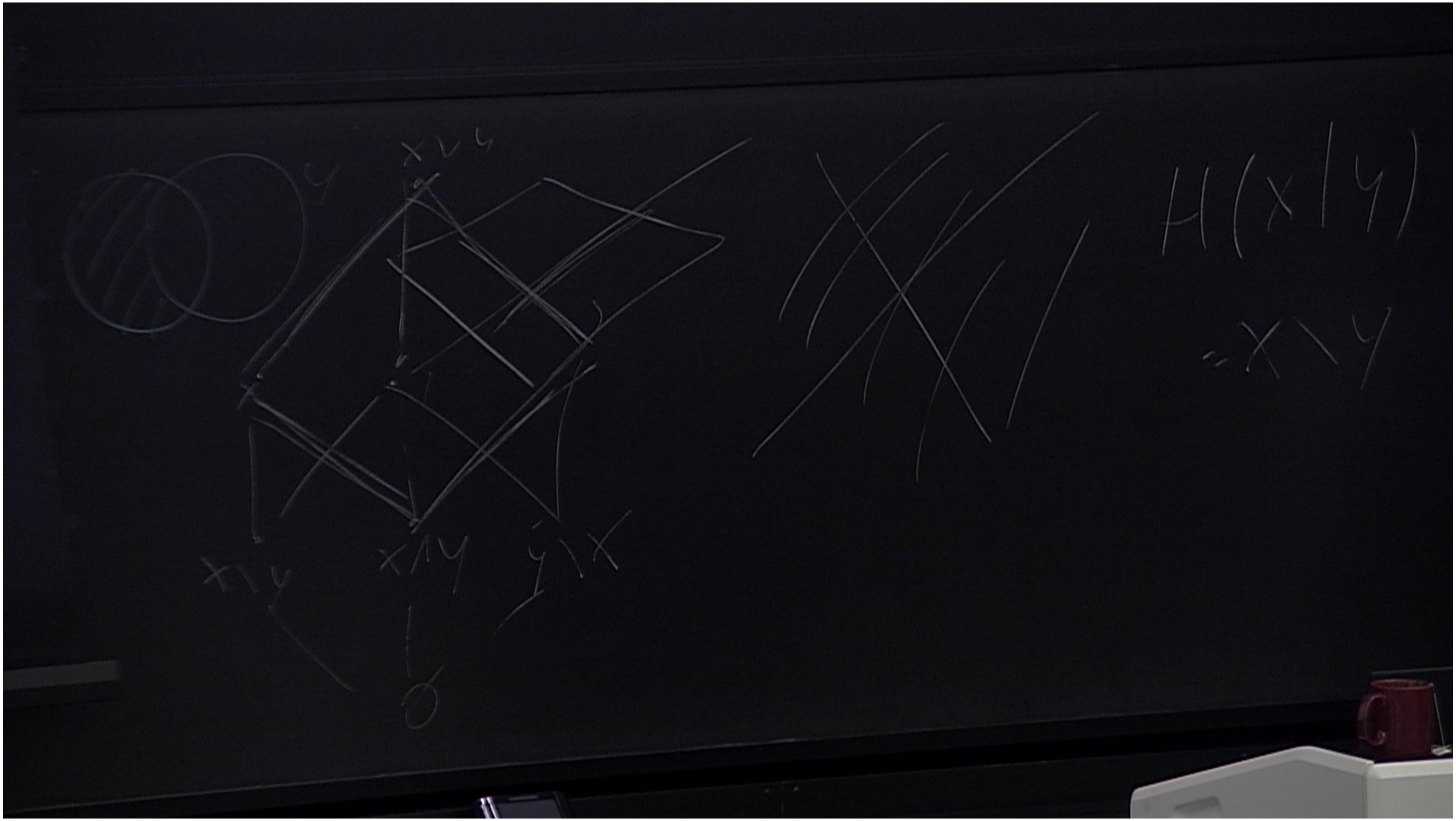
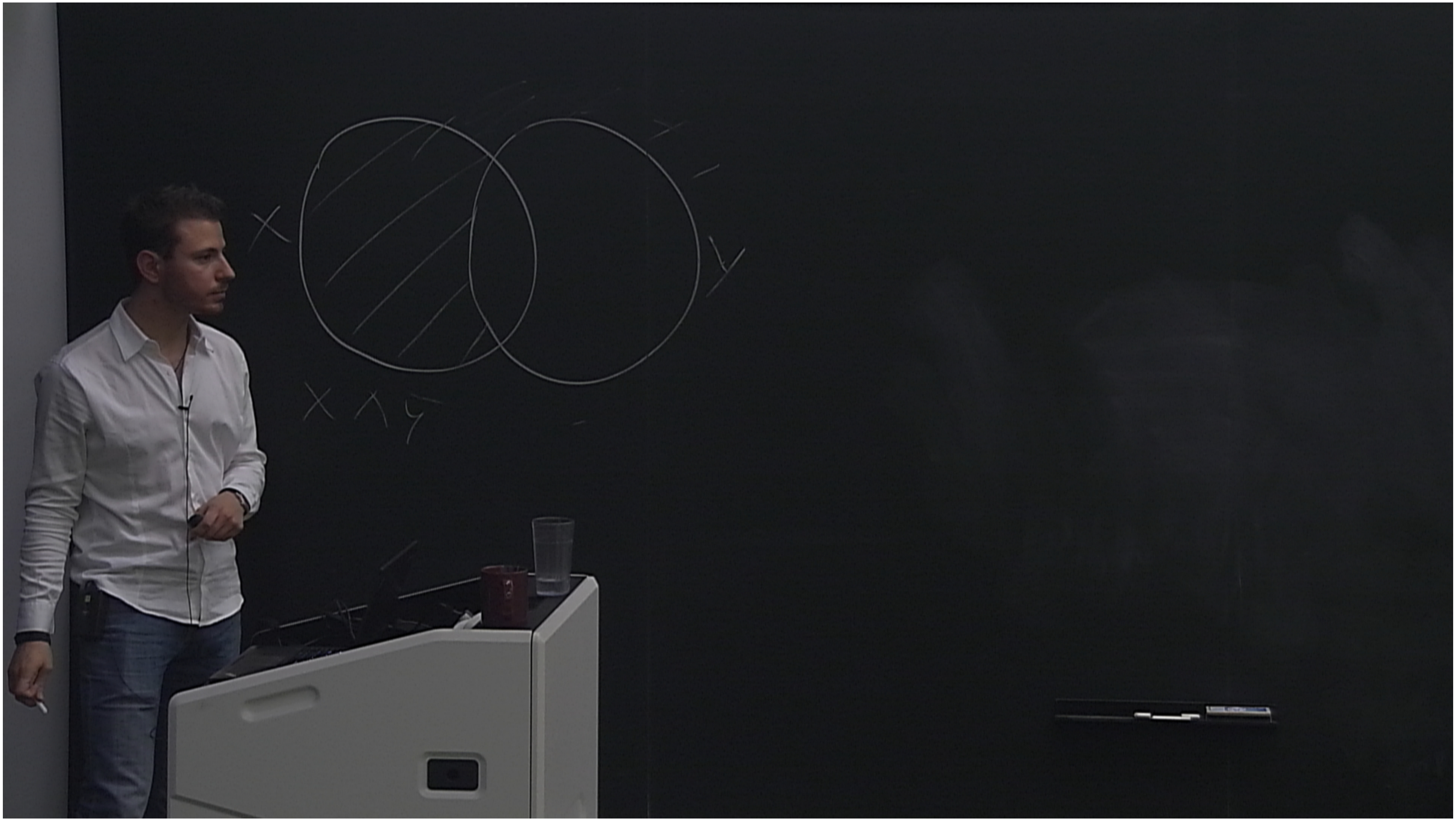
x∨y
x
x\y
x∧y
y\x
0






X
Y

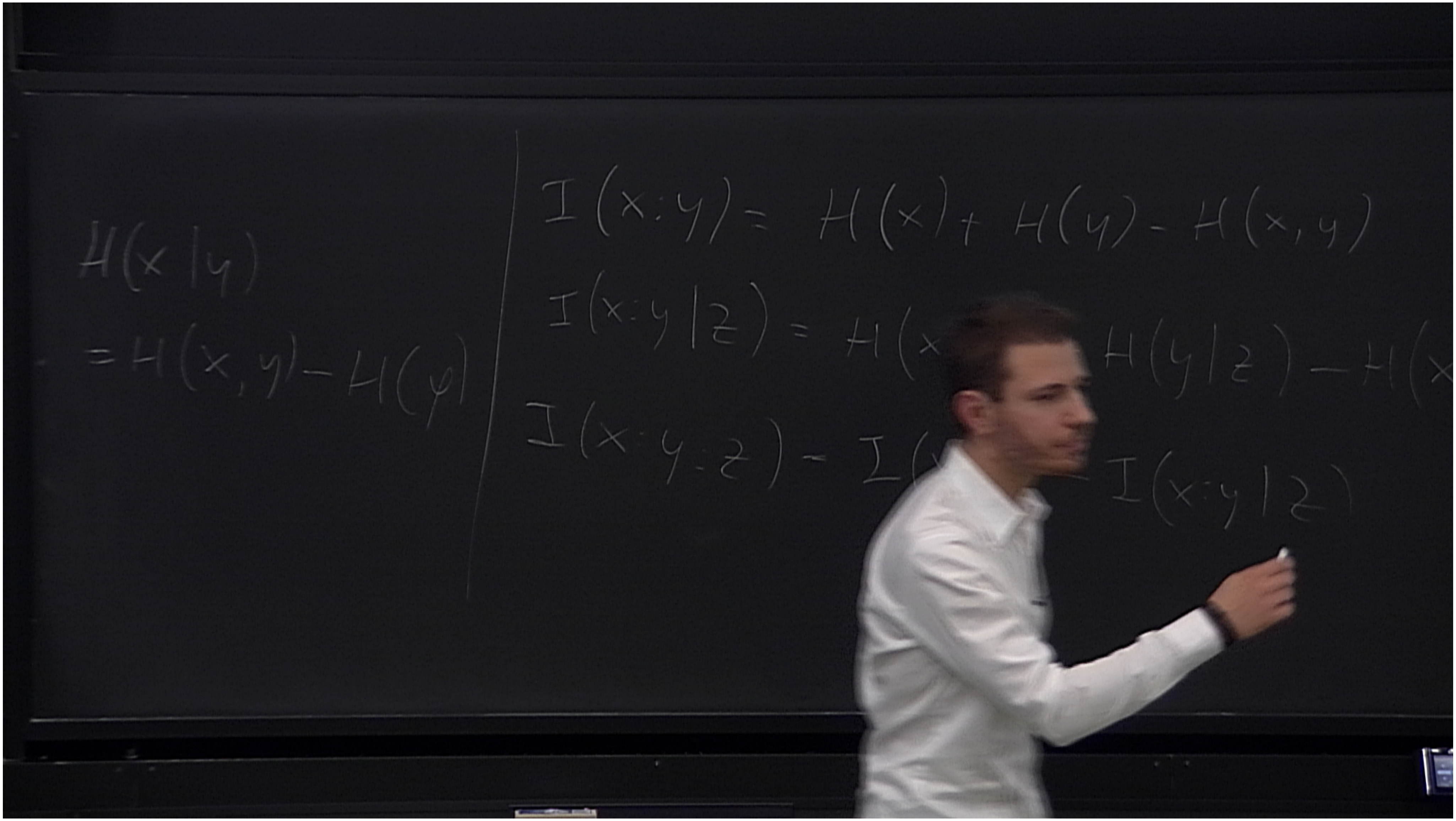

x∨y
y
x∧y
H(x|y)
= x \ y


X
Y
X ∧ Y

$$I(x:y) = H(x) + H(y) - H(x,y)$$

$$I(x:y|z) = H(x|z) + H(y|z) - H(x,y|z)$$

$$I(x:y:z) = I(x:y) - I(x:y|z)$$


H(x|y)
=H(x,y)-H(y)
I(x:y) = H(x) + H(y) - H(x,y)
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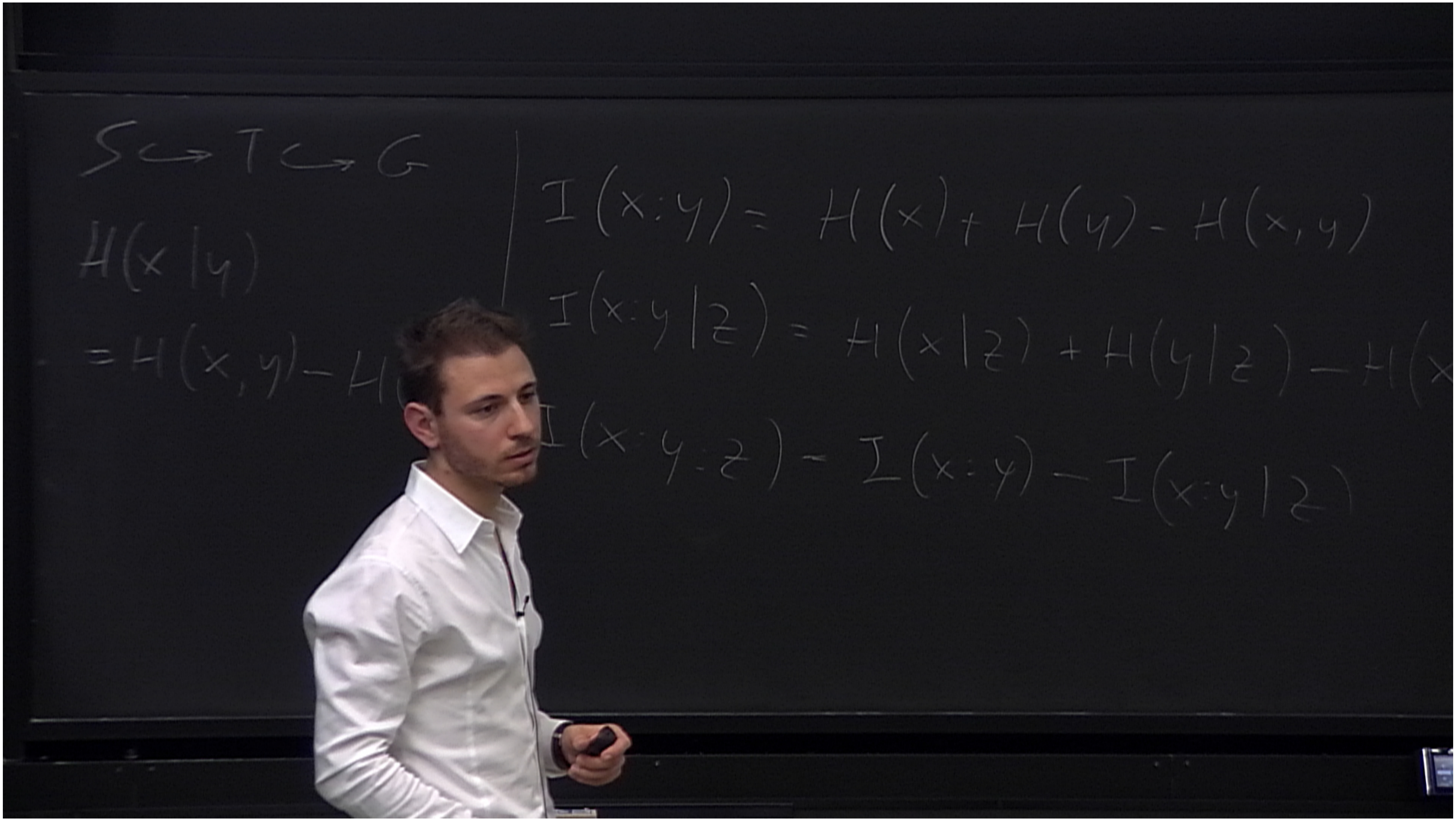
S ↪ T ↪ G
H(x|y)
= H(x,y) − H(
I(x:y) = H(x) + H(y) − H(x,y)
I(x:y|z) = H(x|z) + H(y|z) − H(x
I(x:y:z) = I(x:y) − I(x:y|z)