Title: Quantum groups from character varieties

Date: Nov 10, 2016 04:00 PM

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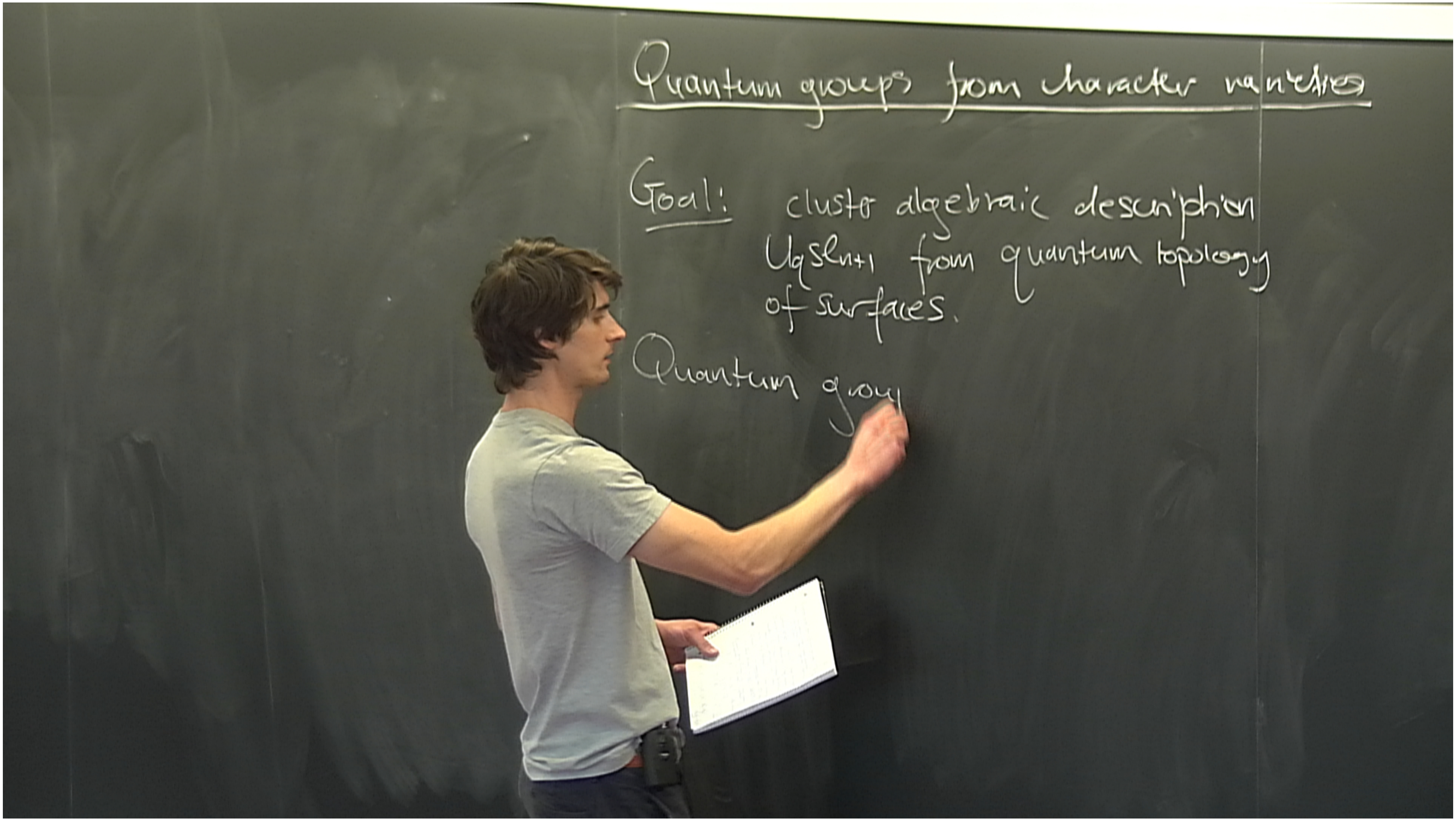
Abstract: <p>Quantum groups from character varieties
Abstract: The moduli spaces of local systems on marked surfaces enjoy many nice properties. In particular, it was shown by Fock and Goncharov that they form examples of cluster varieties, which means that they are Poisson varieties with a positive atlas of toric charts, and thus admit canonical quantizations. I will describe joint work with A. Shapiro in which we embed the quantized enveloping algebra U_q(sl_n) into the quantum character variety associated to a punctured disk with two marked points on its boundary. The construction is closely related to the (quantized) multiplicative Grothendieck-Springer resolution for SL_n. I will also explain how the R-matrix of U_q(sl_n) arises naturally in this topological setup as a (half) Dehn twist. Time permitting, I will describe some potential applications to the study of positive representations of the split real quantum group U_q(sl_n,R)</p>


Quantum groups from character varieties
Goal: cluster algebraic description
$U_q sl_{n+1}$ from quantum topology
of surfaces.
Quantum grou

$= \delta_{ij}$

Quantum groups from character varieties

Goal: cluster algebraic description $U_q\mathfrak{sl}_{n+1}$ from quantum topology of surfaces.

Quantum groups: $U_q\mathfrak{sl}_{n+1}$ is a Hopf algebra deformation of the enveloping algebra $U\mathfrak{sl}_{n+1}$.

Generators: $X_1^{\pm}, \dots, X_n^{\pm}, K_1, \dots, K_n$

...+1 is quasi-triangular

(q)

Relations: $[X_i^+, X_j^-] = \delta_{ij}(q-q^{-1})(K_i - K_i^{-1})$

$K_i X_j^{\pm} = q^{\pm a_{ij}} X_j^{\pm} K_i$,

$a_{ij} = \begin{cases} 2 & i=j \\ -1 & |i-j|=1 \end{cases}$

& cubic q-Serre relations.

Coproduct: $\Delta X_i^+ = X_i^+ \otimes 1 + K_i \otimes X_i^+$

$\Delta X_i^- = X_i^- \otimes K_i^{-1} + 1 \otimes X_i^-$

$\Delta K_i = K_i \otimes K_i$

↳ NOT co-commutative!

$\Delta \neq \Delta^{op} = \mathrm{flip} \circ \Delta$

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Goal: clust...
$U_q sl_n$...
of su...

Quantum g...

a Hopf algeb...
enveloping alge...

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$\mathbb{C}(q)$

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Quantum groups from character varieties

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Generators: $X_1^{\pm}, \dots, X_n^{\pm}, K_1, \dots, K_n$

But: $U_q sl_{n+1}$ is quasi-triangular:

there is $R \in U_q sl_{n+1}^{\otimes 2}$

such that $\Delta^{op} = Ad_R \circ \Delta$

& satisfies Yang-Baxter eq^n in $U_q^{\otimes 3}$

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

$\Longrightarrow$ Rep $U_q sl_{n+1}$ is braided tensor category:

$$V \otimes W \xrightarrow{\text{flip} \circ R} W \otimes V$$

Relations:

K_i

& cubic

Coproduct:

↳ NO

$\Delta \neq \Delta^{op} = \text{flip}$

Kirillov-Reshetikhin:

- fix a normal ordering on positive roots Δ_+.

$\hookrightarrow$ $\exists$ quantum root vectors

$$X^{\pm}_{\alpha}\ , \ \alpha \in \Delta_+$$

and $\mathcal{R}$ = (prefactor in K_i's)

$$\cdot \prod_{\alpha \in \Delta_+} \Phi^q(X^+_\alpha \otimes X^-_\alpha)$$

where $\Phi^q(x) = \prod_{n\geq 1}^{\infty} \frac{1}{1+q^{2n+1}x}$ is q-dilogarithm.

But: $U_q sl_{n+1}$

there is $\mathcal{R}$

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& satisfies

$R_{12}R_{13}R$

$\Longrightarrow$ Rep $(U$

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Character varieties (following Fock-Goncharov)

A marked surface $\hat{S}$ is a compact oriented surface S, together with a finite set $\{x_1, \ldots, x_k\} \subset \partial S$ of boundary marked points.

$G =$

The punctured boundary of $\hat{S}$ is $\partial \hat{S} = \partial S \setminus \{x_1, \ldots, x_k\}$

(union of S^1 & intervals)

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The punctured boundary of $\hat{S}$ is $\partial \hat{S} = \partial S \setminus \{x_1, \dots x_k\}$ (union of S^1 & intervals)

Fix $G = PGL_{n+1}\mathbb{C}$.

$$\left\{ \begin{matrix} G\text{-local systems} \\ \text{on } S \end{matrix} \right\} \Big/ \text{iso} \longleftrightarrow \mathrm{Hom}(\pi_1(S) \to G) / \mathrm{Ad}\, G$$

Fix $B \subset G$ Borel subgroup, so G/B is the flag variety

$\mathcal{L}$ local system $\rightsquigarrow \mathcal{L} \times_G G/B$ associated flag bundle

A

$)/\mathrm{Ad}G$

ssociated
flag bundle

A framed G-local system on $\widehat{S}$ is:

1) a G-local system $\mathcal{L}$ on S

2) a flat section of $\left(\mathcal{L} \times_G G/B\right)\Big|_{\partial \widehat{S}}$

Concretely: framing data means:

- for each interval component of $\partial \widehat{S}$, a flag $\mathcal{F} \in G/B$.

- for each S^1 component γ, a flag $\mathcal{F}$ fixed by monodromy around γ.

Def: $\mathcal{X}_{G,\hat{S}}$

$\shortparallel$

moduli of framed G-local systems on $\hat{S}$.

Character var

A marked

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boundary w

$[X_i^+, X_j^-] = \delta_{ij}(q-q^{-1})(K_i - K_i^{-1})$

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q-Serre relations.

$\Delta X_i^+ = X_i^+ \otimes 1 + K_i \otimes X_i^+$

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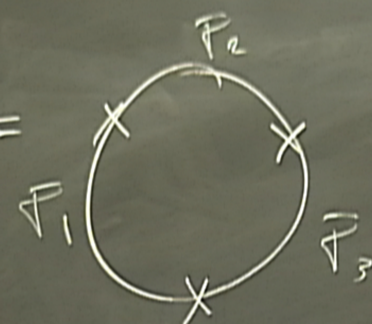
$\Delta K_i = K_i \otimes K_i$

co-commutative!

$p \circ \Delta$

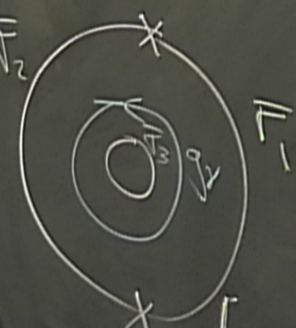
Examples: 1) $\hat{S} =$



$\chi_{G,\hat{S}} = \left(G/B\right)^{\times 3} / G$

$= \mathrm{Conf}_3(G/B).$

2) $\hat{S} = \bar{F}_2$



$\chi_{G,\hat{S}} = \{ g_\gamma, \bar{F}_1, \bar{F}_2, \bar{F}_3 \mid g_\gamma \cdot \bar{F}_3 = \bar{F}_2 \} / G$

$U_q^{\otimes 3}$

(q)

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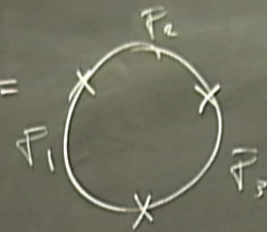

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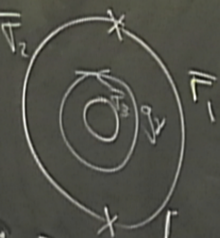

$\chi_{G,\hat S} = \{ g_r, \bar F_1, \bar F_2, \bar F_3 \mid g_r \cdot \bar F_3 = \bar F_2 \} / G$

Closely related to Grothendieck-Springer resolution $\tilde G = \{(g, \mathcal{F}) \mid g\mathcal{F} = \mathcal{F}\}$

$\downarrow$

G

Def: χ

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...es of $\mathcal{X}_{G,\hat{S}}$:

...s are rational varieties with ...isson bracket $\{\cdot,\cdot\}: \mathcal{O}_{\mathcal{X}} \otimes \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}$

...$_{G,\hat{S}}$ is a cluster Poisson variety.

This means: $\mathcal{X}_{G,\hat{S}}$ is covered up to codim 2 by an atlas of toric charts

$$T_\Sigma : (\mathbb{C}^\times)^d \longrightarrow \mathcal{X}_{G,\hat{S}}$$

labelled by quivers Q_Σ.

Let $\varepsilon_{jk} = (\# j \to k) - (\# k \to j)$.

Then coord functions $x_1, \dots, x_d$ satisfy

$$\boxed{\{x_j, x_k\} = \varepsilon_{jk} x_j x_k}$$

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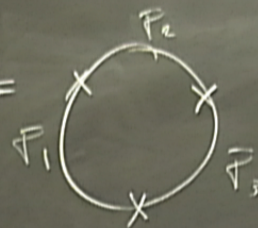
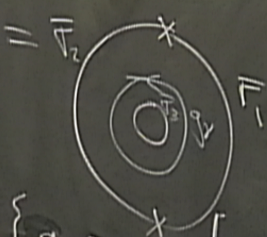


$$\mathcal{X}_{G,\hat{S}} = (G/B)^{\times 3} / G = \mathrm{Conf}_3(G/B)$$

2) $\hat{S} = $



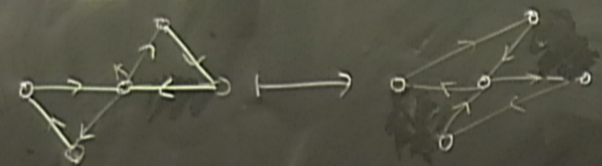
$$\mathcal{X}_{G,\hat{S}} = \{ g_\gamma, \bar{F}_1, \bar{F}_2, \bar{F}_3 \mid g_\gamma \cdot \bar{F}_3 = \bar{F}_2 \} / G$$

Closely related to Grothendieck-Springer resolution

$$\tilde{G} = \{(g, \bar{F}) \mid gF = F\} \longrightarrow G$$

Different quivers are related by mutations:

k vertex in $Q_\Sigma \xrightarrow{\mu_k} Q_{\mu_k(\Sigma)}$



Mutated coordinate functions:

$$x_j' = \begin{cases} x_k^{-1} & j=k \\ x_j\left(1+x_k^{\operatorname{sgn}(\varepsilon_{jk})}\right)^{\varepsilon_{jk}} & j\neq k \end{cases}$$

satisfy: $\{x_j', x_k'\} = \varepsilon_{jk}' x_j' x_k'$.

Fix $G = PGL_{n+1}\mathbb{C}$.

$$\text{iso}\Big\backslash\left\{\begin{matrix} G\text{-local systems} \\ \text{on } S \end{matrix}\right\} \longleftrightarrow \mathrm{Hom}(\pi_1(S) \longrightarrow G)\big/\mathrm{Ad}\,G$$

Fix $B \subset G$ Borel subgroup, so G/B is the flag variety

$\mathcal{L}$ local system $\rightsquigarrow \mathcal{L} \times_G G/B$ associated flag bundle

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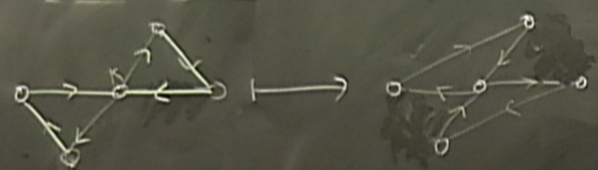

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Cluster combinatorics for

$\mathcal{X}_{G,\hat{S}}$ has a family of toric charts indexed by ideal triangulations of $\hat{S}$.

picture S^1 components of $\partial \hat{S}$ as "punctures" / "cusps"

ideal triangulation has vertices are at marked points or punctures

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Let S

Then coord

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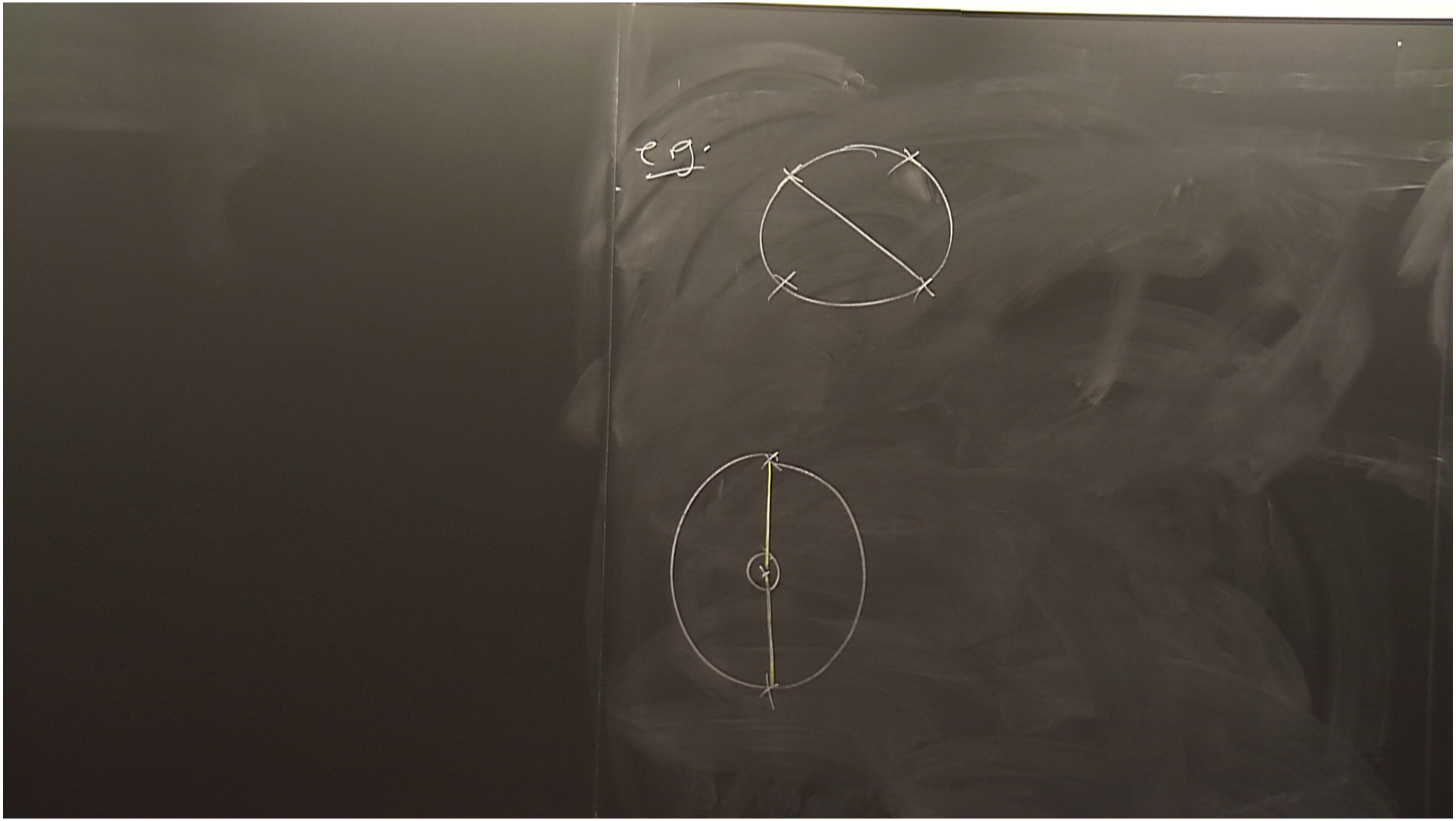
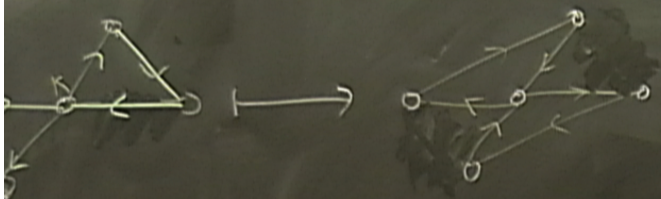

e.g.

quivers are related by mutations:

vertex in $Q_\Delta \xrightarrow{\mu_k} Q_{\mu_k(\Delta)}$



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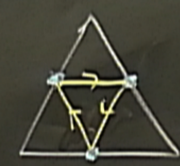
$$x'_j = \begin{cases} x_k^{-1} & j=k \\ x_j\left(1+x_k^{\mathrm{sgn}(\varepsilon_{kj})}\right)^{\varepsilon_{kj}} & j\neq k \end{cases}$$

$$\{x'_j, x'_k\} = \varepsilon'_{jk}\, x'_j x'_k .$$

Quiver from triangulation:

For PGL_2: • inside each triangle, draw "basic quiver"



• glue nodes of quivers by triangulation, cancel length 2 cycles

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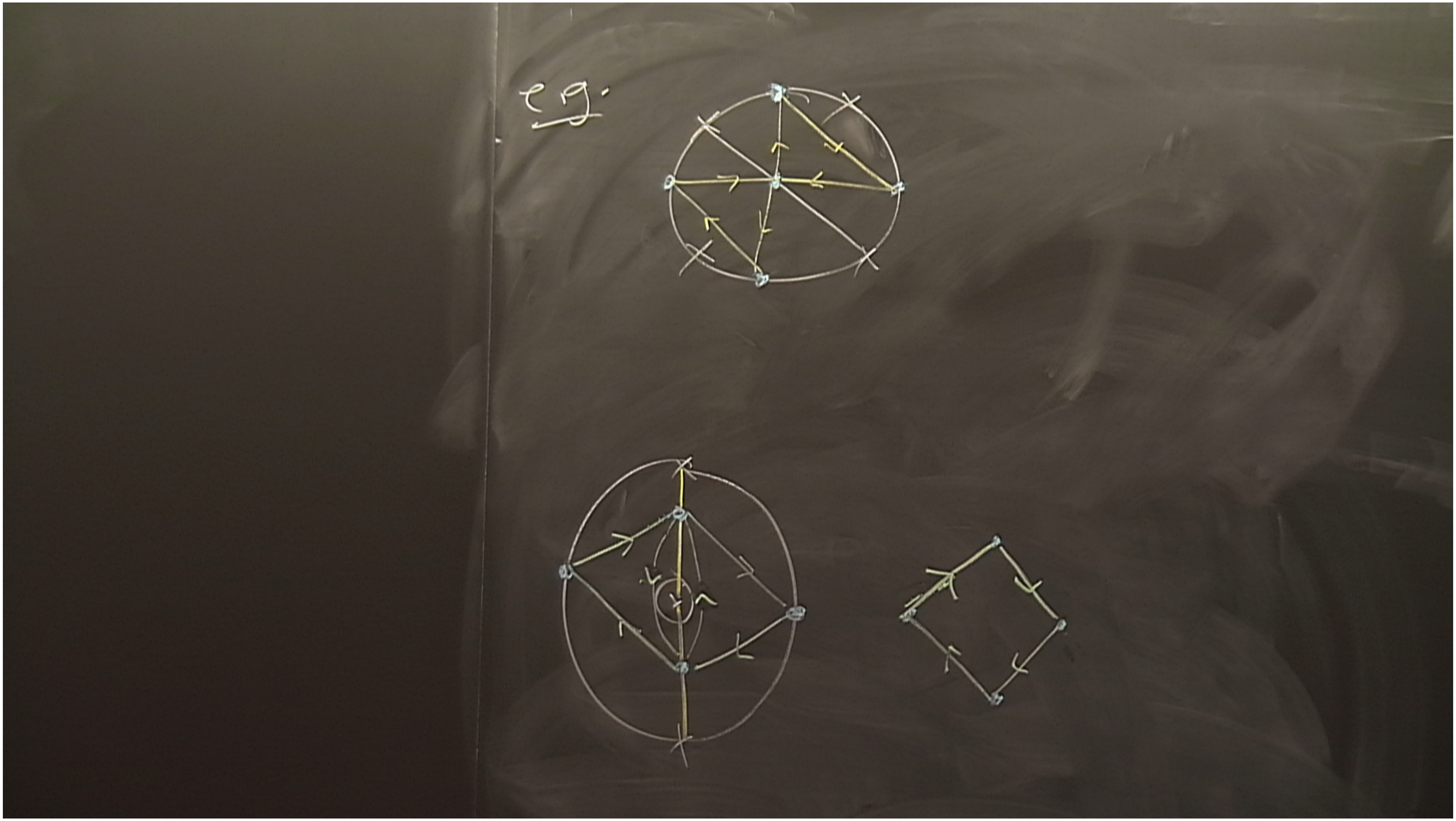
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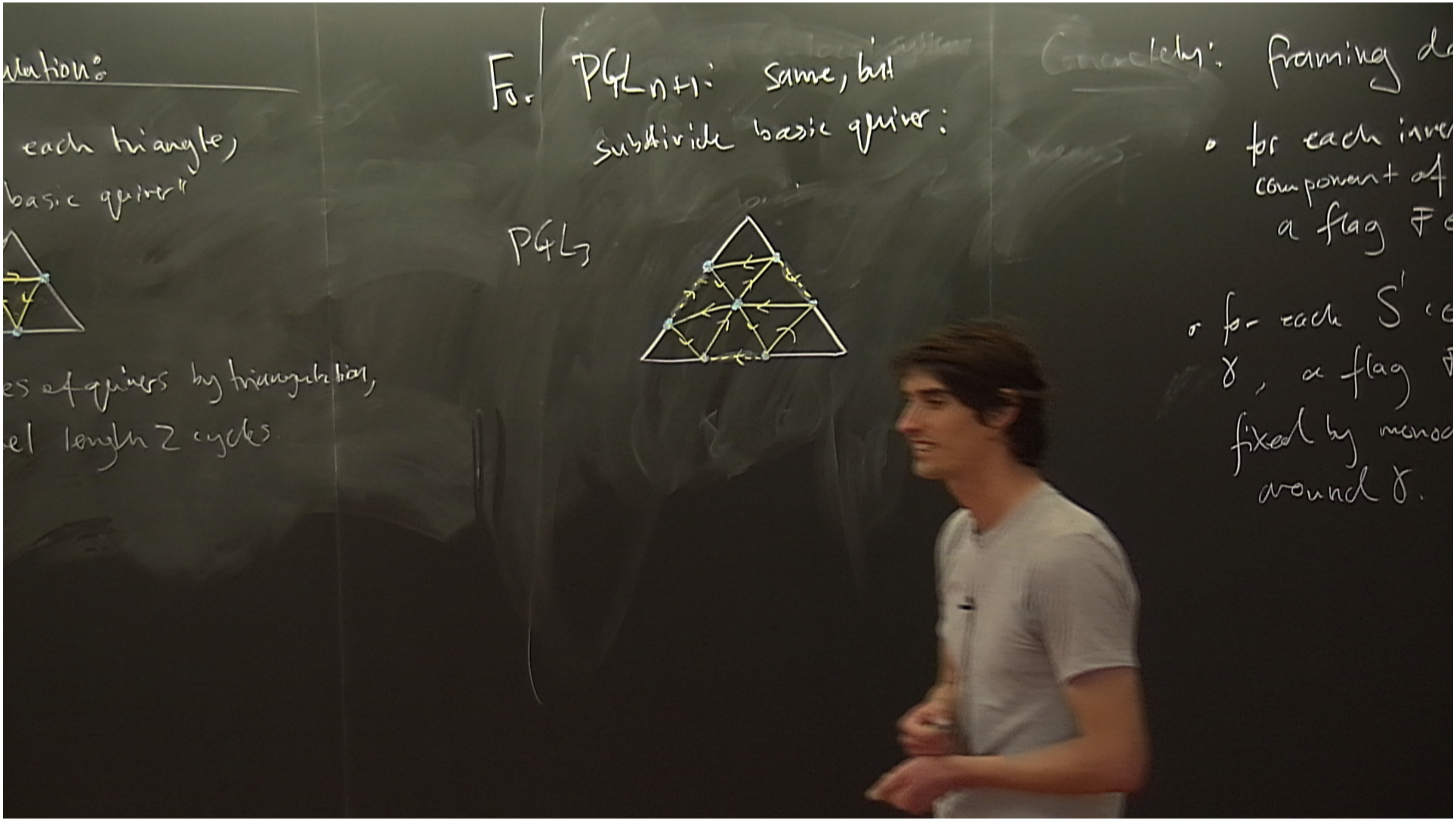
each triangle,
basic quiver"
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For PGL_{n+1}: same, but
subdivide basic quiver:
PGL_3
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for each inter
component of
a flag
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fixed by monod
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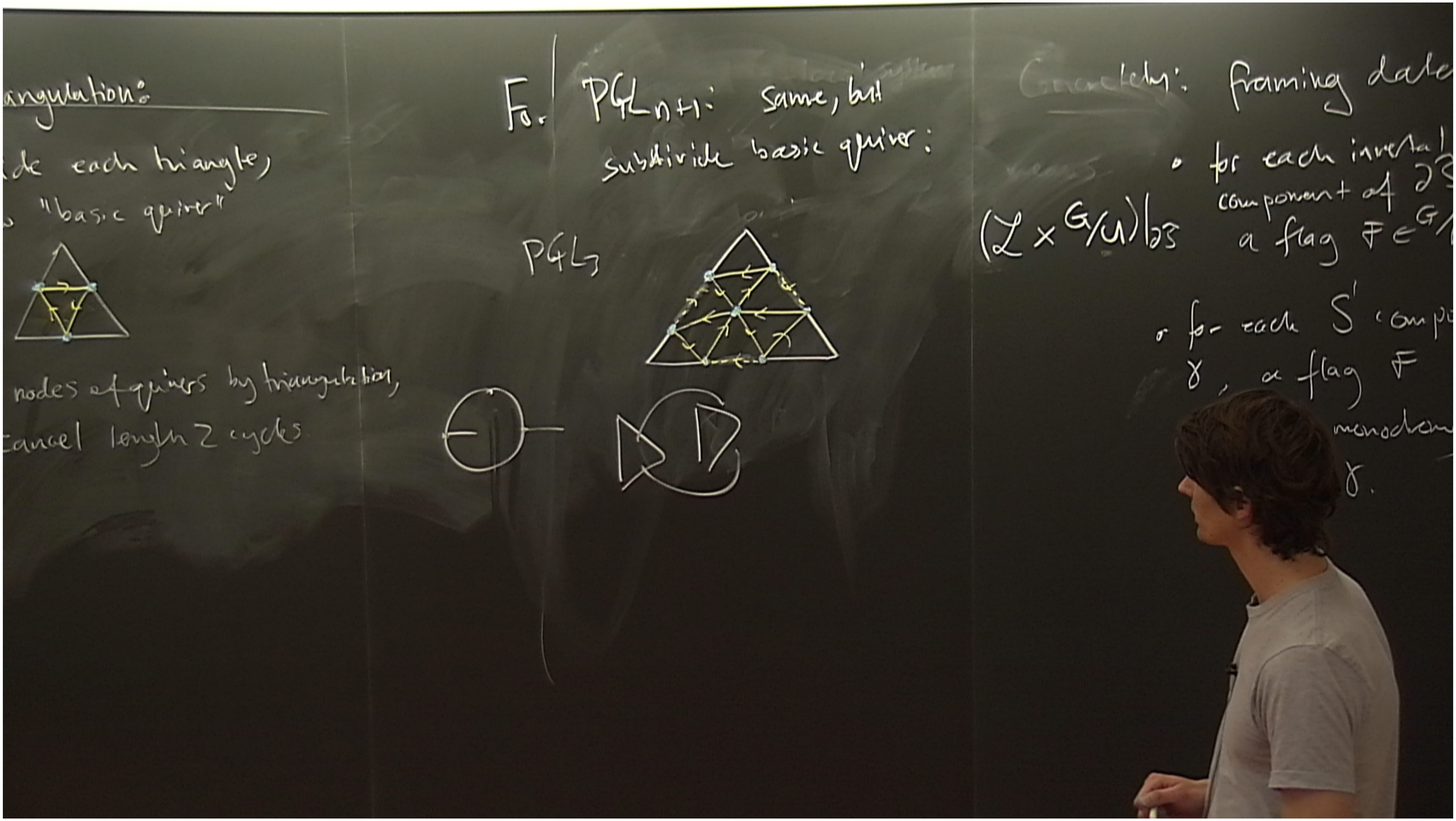
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for each inverted component of $\partial\Sigma$
a flag $F \in G/$
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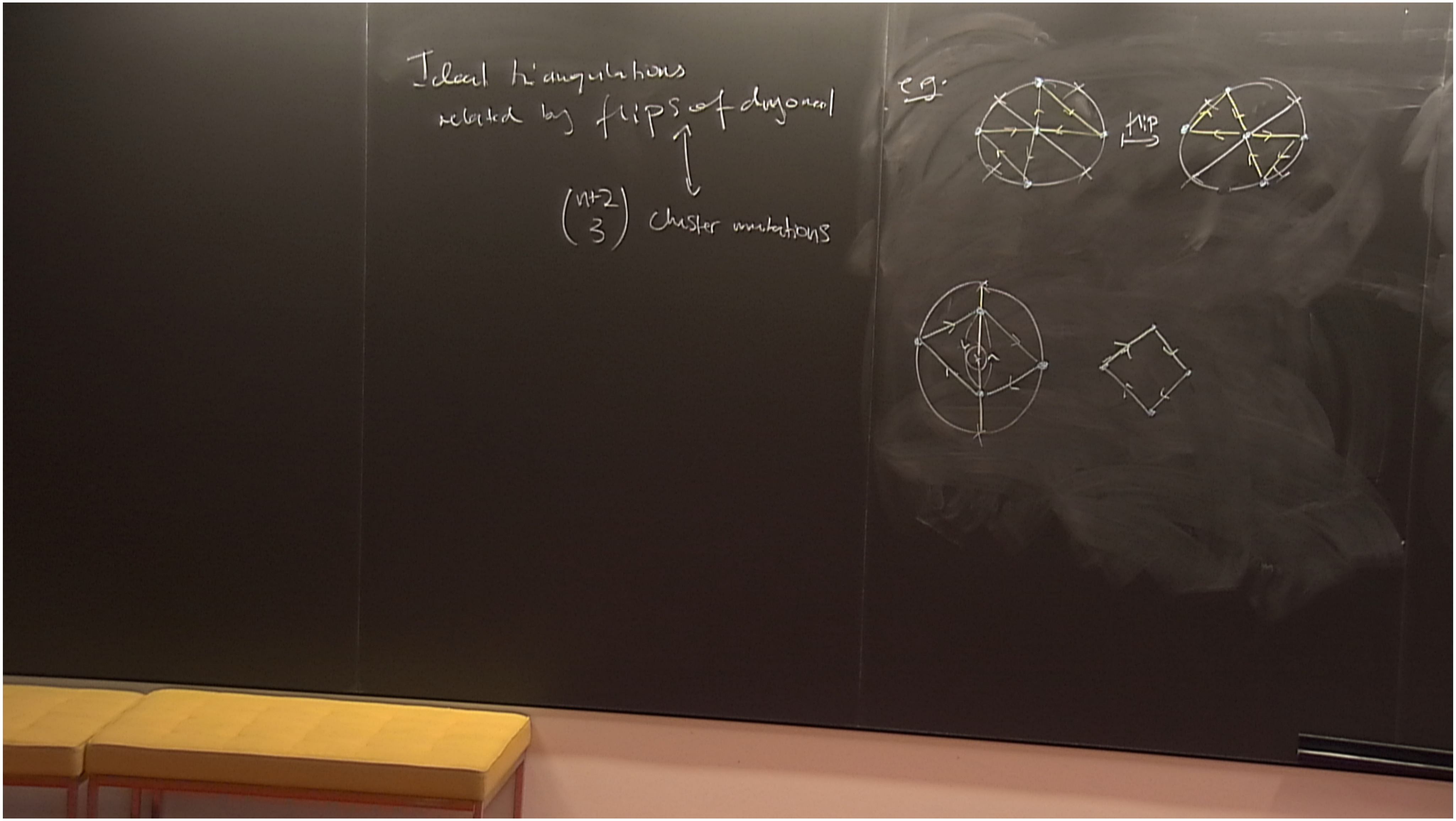
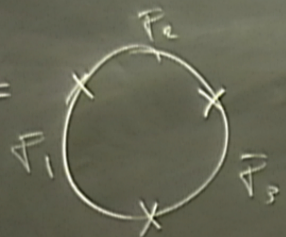
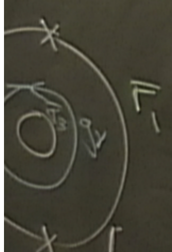

Ideal triangulations
related by flips of diagonal
$\binom{n+2}{3}$ cluster mutations
e.g.
flip

$\hat{S} =$ 

$/G$

(G/B)



$\{\ldots \mid g_\gamma \cdot F_3 = F_3\}/G$

Grothendieck-Springer

$\{(g, F) \mid gF = F\}$

Canonical quantization of cluster varieties:

promote tori T_Σ to quantum torus algebra T^q_Σ

$$x_j x_k = q^{2\varepsilon_{jk}} x_k x_j$$

Quantum mutation:

$$\mu_k = \mu_k' \circ \mathrm{Ad}\, \Phi^q(x_k)$$

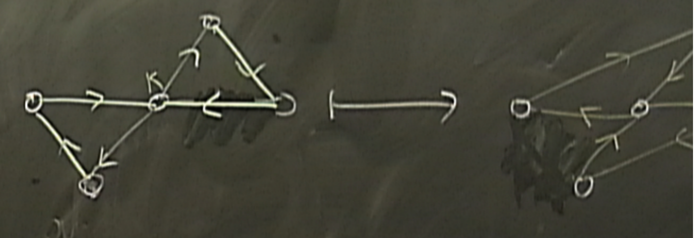
μ_k' = monomial hom

$T^q_{\mu_k(\Sigma)}$

Using: $\Phi^q(q^2 x) = (1+qx)\Phi^q(x)$

Different quivers are related by mu

k vertex in $Q_\Sigma \xrightarrow{\mu_k} Q_{\mu_k(\Sigma)}$



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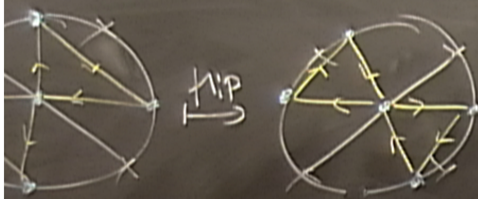
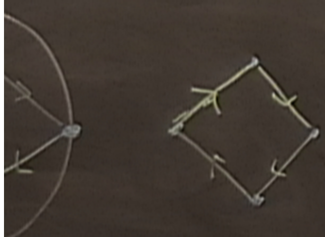
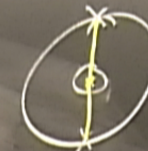
$$x_j' = \begin{cases} x_k^{-1} & j=k \\ x_j\left(1 + x_k^{\mathrm{sgn}(\varepsilon_{kj})}\right)^{\varepsilon_{kj}} & j \neq \end{cases}$$

Satisfying: $\{x_j', x_k'\} = \varepsilon_{jk}' x_j' x_k'$





joint w. A. Shapiro

Thm. (S.-Shapiro '16) $\hat{S} =$ 

Let T^q be the quantum torus corresponding to the triangulation of $\hat{S}$ above. There is algebra embedding $U_q sl_{n+1} \hookrightarrow T^q$ such that for each $X_i^{\pm}$, there is a cluster in which $X_i^{\pm}$ is cluster monomial.

This means: $\mathcal{X}_G$ covered up to c

by an atlas of

$T_\Sigma : (\mathbb{C}^\times)^d \hookrightarrow$

labelled by quivers

Let $\varepsilon_{jk} = (\# j \to k$

Then coord functions $x_1, \dots$

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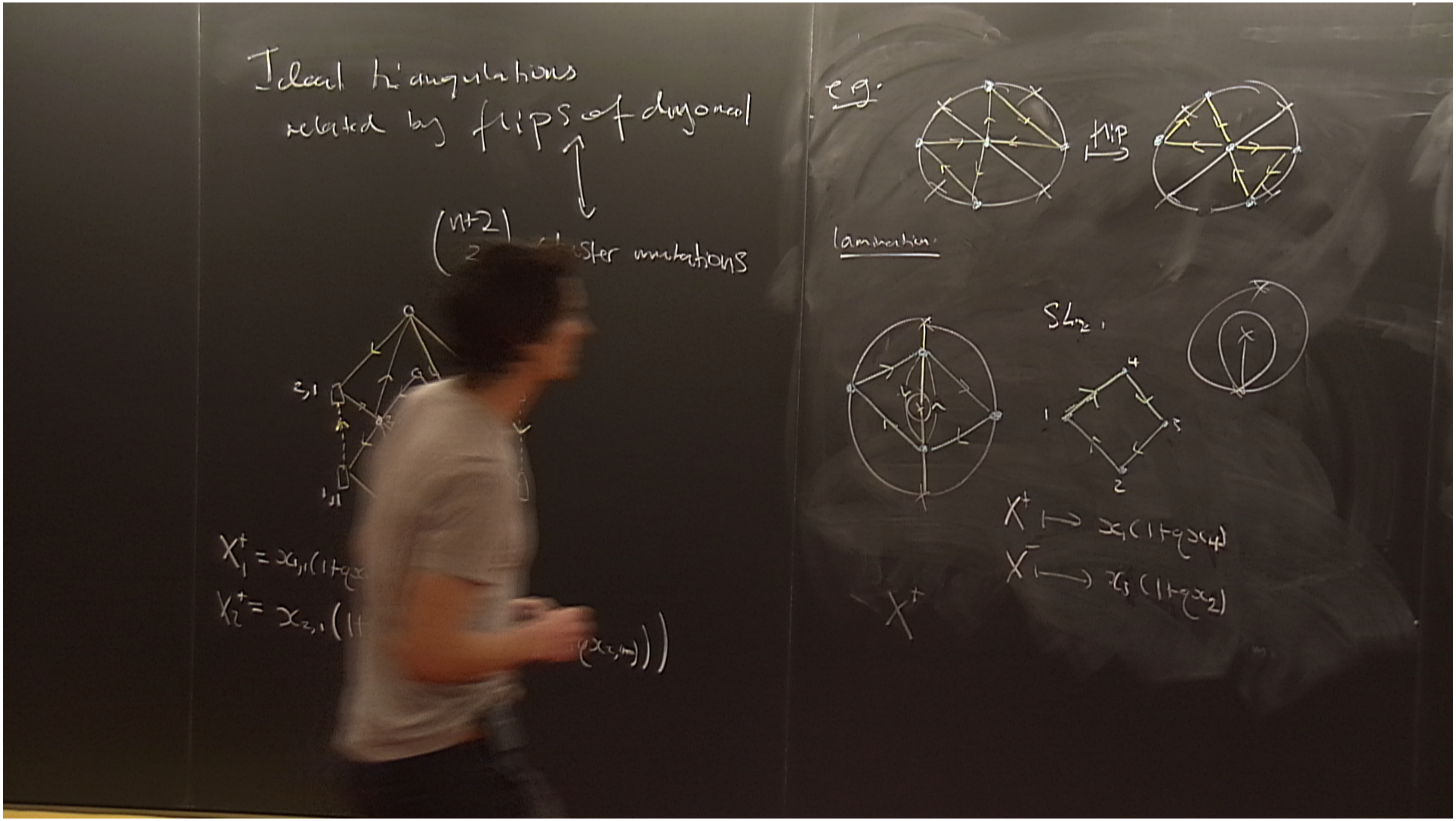
Ideal triangulations
related by flips of diagonal
cluster mutations
e.g.
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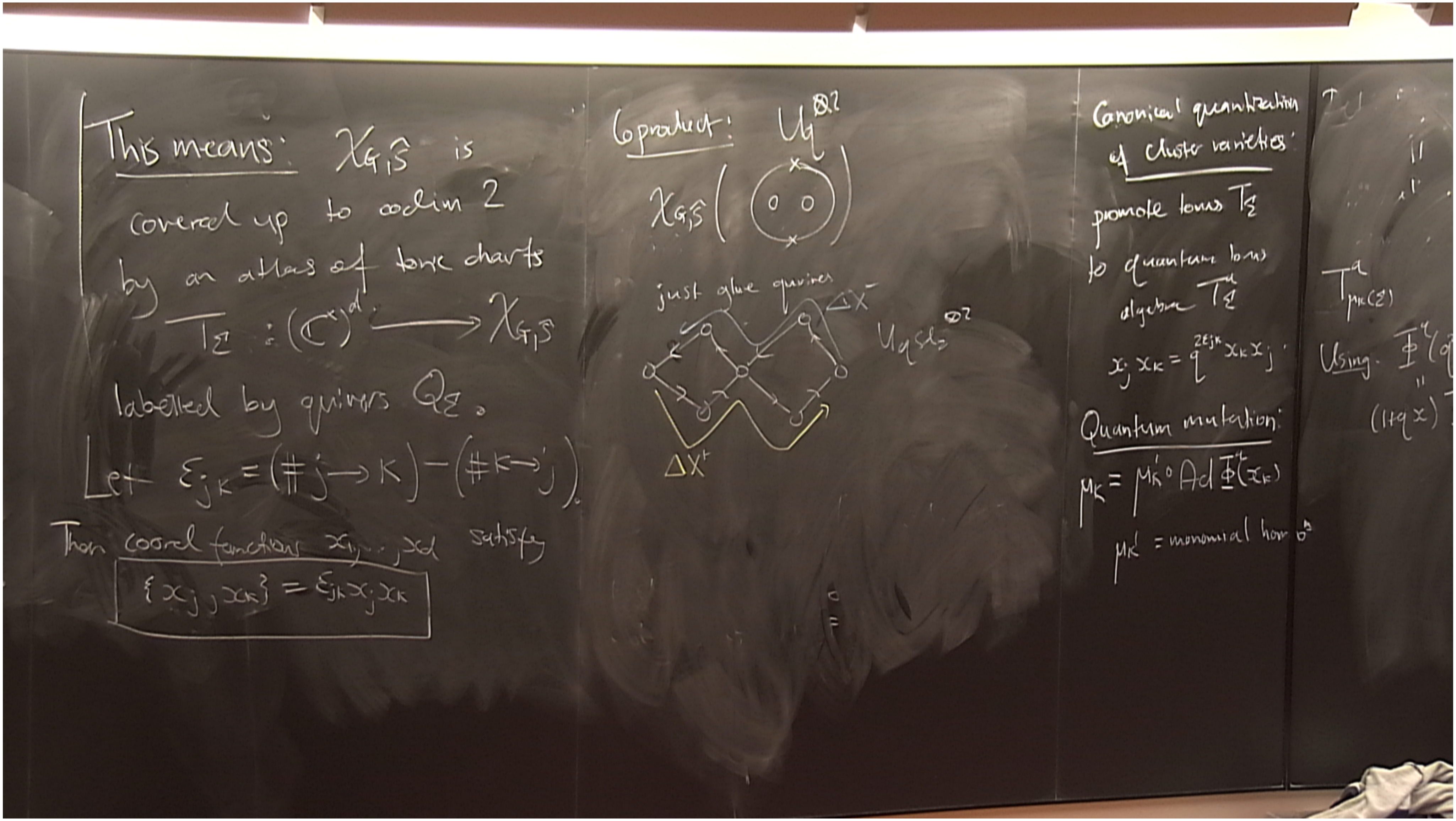
lamination
SL_2
$X^+ \longmapsto x_1(1+qx_4)$
$X^- \longmapsto x_3(1+qx_2)$
X^+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Coproduct: $U_q^{\otimes 2}$
$\mathcal{X}_{G,\hat{S}}(\;)$
just glue quivers
ΔX^-
ΔX^+
$U_q sl_2^{\otimes 2}$
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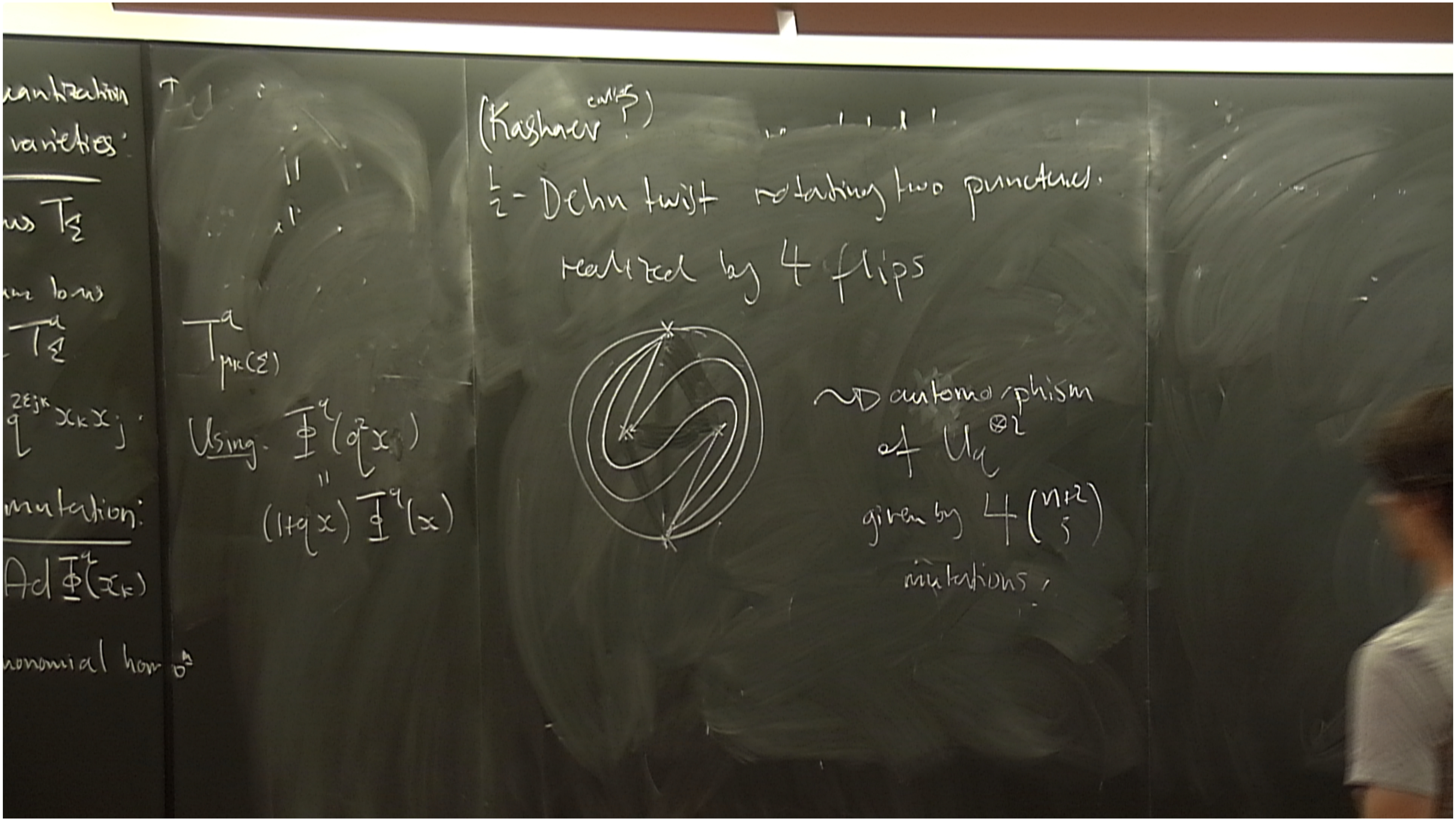
varieties:
mutation:
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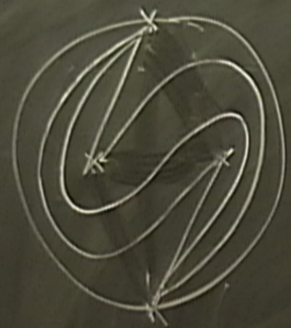
Using $\Phi^q(q^2x)$
$=$
$(1+qx)\,\Phi^q(x)$
(Kashaev ?)
$\frac{1}{2}$- Dehn twist rotating two punctures.
realized by 4 flips
$\rightsquigarrow$ automorphism of $U_q^{\otimes 2}$
given by $4\binom{n+2}{3}$ mutations.

(Kashaev ?)

$\frac{1}{2}$ - Dehn twist rotating two punctures.

realized by 4 flips



$\rightsquigarrow$ automorphism of $U_q^{\otimes 2}$ given by $4\binom{n+2}{3}$ mutations.

$)$

$^q(x)$

<u>Thm</u> $\frac{1}{2}$ Dehn twist automorphism coincides Ad_R.

$\hookrightarrow$ <u>Cor:</u> $\mathcal{R}$ factors into $4\binom{n+2}{3}$ q-dilog of cluster monomials.

SL_2: $\Phi^q(X^+ \otimes X^-)$

$\rightsquigarrow$

$\prod_{i=1}^{4} \Phi^q(x_i)$