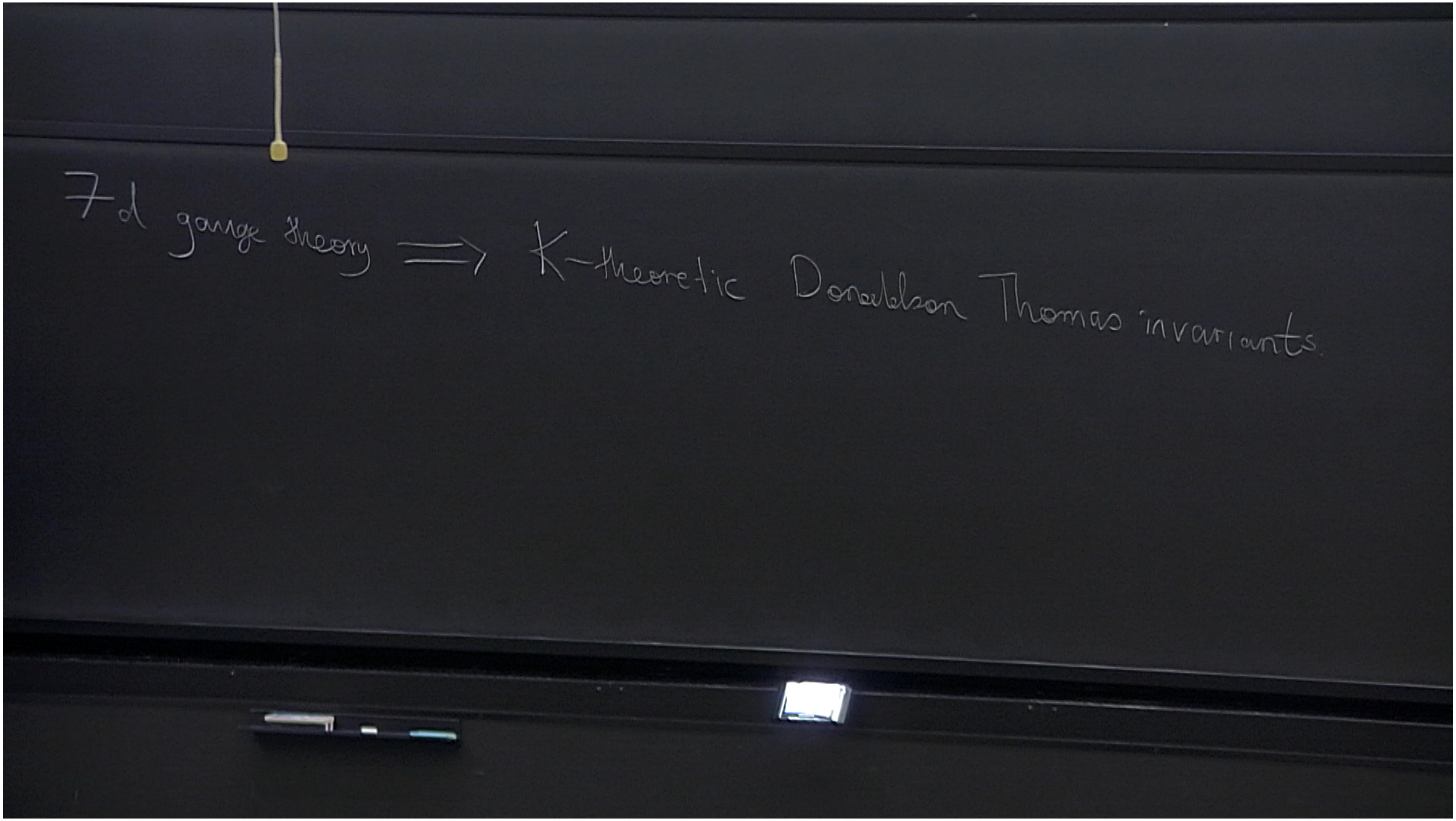
Title: Supersymmetric Field Theories for Mathematicians

Date: Nov 16, 2016 03:30 PM

URL: http://pirsa.org/16110023

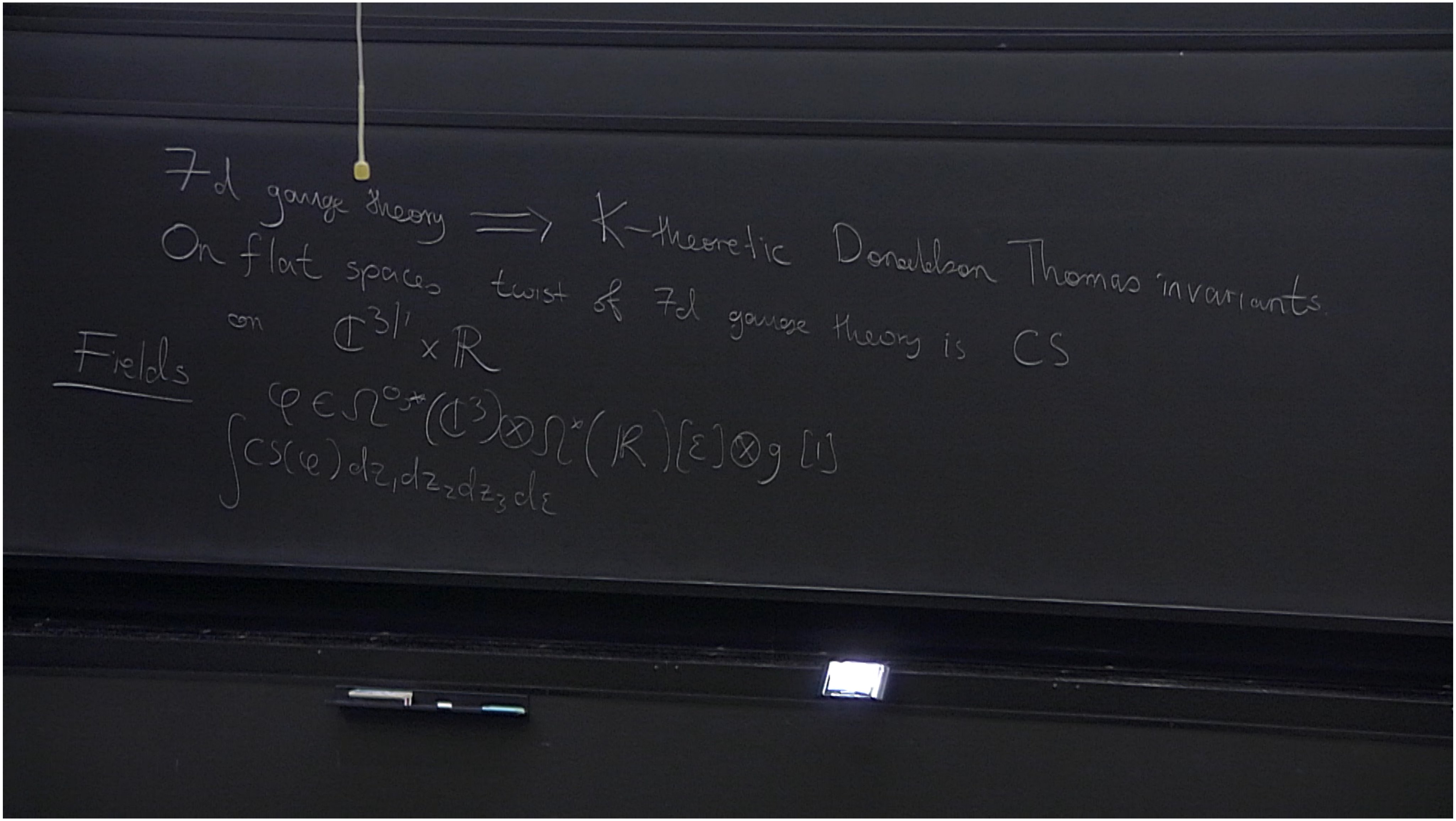
Abstract:


7d gauge theory ⟹ K-theoretic Donaldson Thomas invariants.

7d gauge theory $\Longrightarrow$ K-theoretic Donaldson Thomas invariants.

On flat spaces twist of 7d gauge theory is CS

on $\mathbb{C}^{3|1} \times \mathbb{R}$


7d gauge theory $\Longrightarrow$ K-theoretic Donaldson Thomas invariants
On flat spaces twist of 7d gauge theory is CS
on $\mathbb{C}^{3|1} \times \mathbb{R}$
Fields
$\varphi \in \Omega^{0,*}(\mathbb{C}^3) \otimes \Omega^*(\mathbb{R})[\varepsilon] \otimes \mathfrak{g}[1]$
$\int CS(\varphi)\, dz_1 dz_2 dz_3 d\varepsilon$

$\int \ldots dz_1 dz_2 dz_3 d\xi$

X a CY3

What's Hilbert space on $X \times \mathbb{R}$?

$\int \ldots(\varepsilon)\, dz_1 dz_2 dz_3 d\varepsilon$

X a CY3   What's Hilbert space on $X \times \mathbb{R}$?

$\underline{1^*}$ Understand phase space

Phase space $=$ sol$^{\text{n}}$s to EOM on a small interval $X \times (-\varepsilon, \varepsilon)$

Always symplectic.

$\int \ldots(\varepsilon)\, dz_1 dz_2 dz_3 d\varepsilon$

X a CY3 What is Hilbert space on $X \times \mathbb{R}$?

1st Understand phase space

Phase space = Sol'ns to EOM on a small interval $X \times (-\varepsilon, \varepsilon)$

Always symplectic.

Phase space = Sol'ns to EOM which are independent of t and dt

Phase space:

$\varphi \in \Omega^{0,*}(X)[\varepsilon]\otimes \mathfrak{g}[1]$

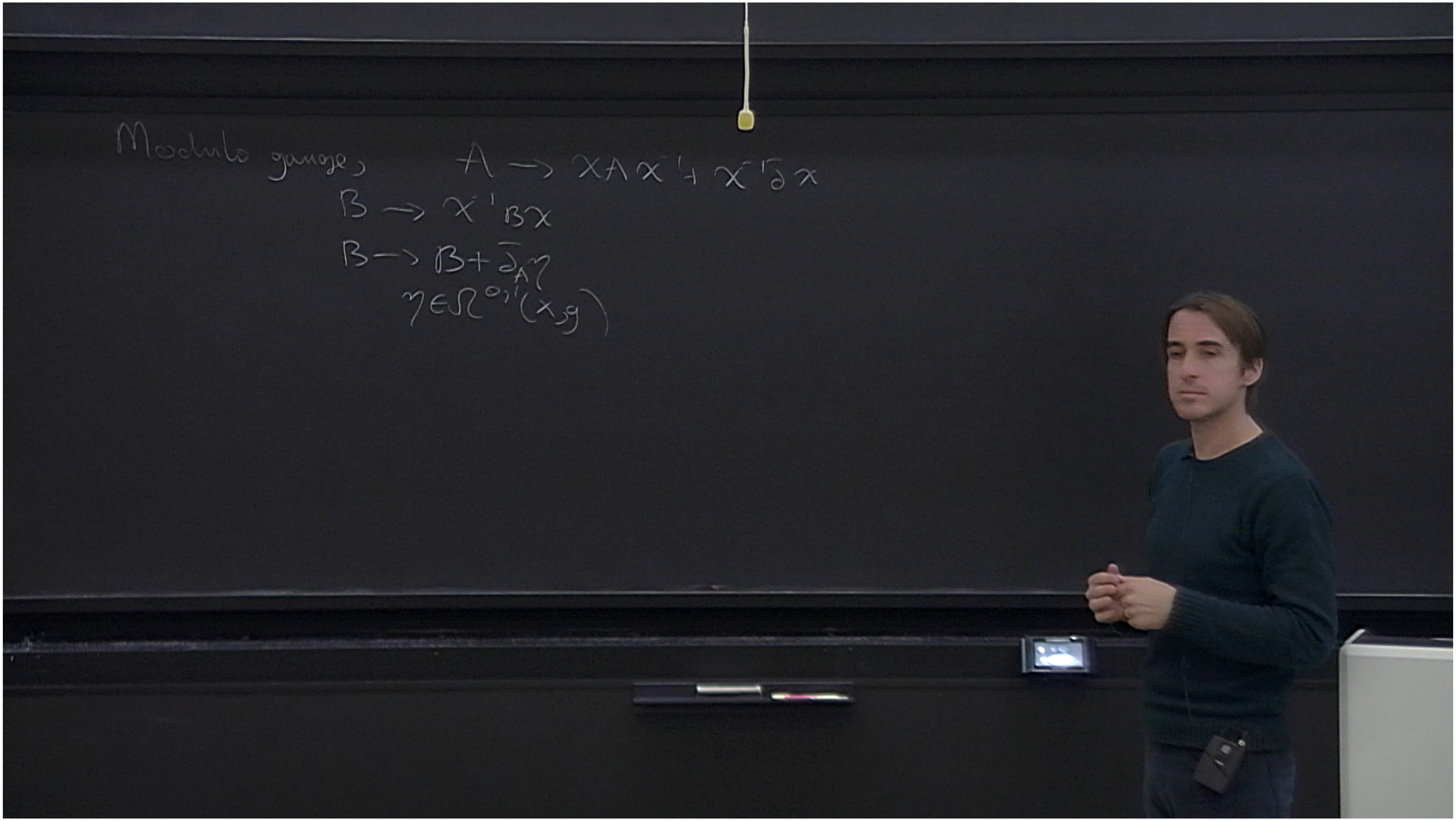
Satisfying an equation

ε is of degree -1, then degree 0 part is

$A \in \Omega^{0,1}(X)\otimes \mathfrak{g}$

$B \in \Omega^{0,2}(X)\otimes \mathfrak{g}$ (coeff. of ε)

Satisfying $F^{0,2}(A)=0$, $\bar{\partial}_A B = 0$


Modulo gauge, $A \to \chi A \chi^{-1} + \chi^{-1}\bar{\partial}\chi$
$B \to \chi^{-1} B \chi$
$B \to B + \bar{\partial}_A \eta$
$\eta \in \Omega^{0,1}(X, g)$

Modulo gauge, $A \to xAx^{-1} + x^{-1}\bar{\partial} x$

$B \to x^{-1}Bx$

$B \to B + \bar{\partial}_A \eta$

$\eta \in \Omega^{0,1}(X, \mathfrak{g})$

A: holomorphic bundle on X

B: element of $H^2_{\bar{\partial}}(X, \mathfrak{g}_{ad})$

↳ adjoint bundle

Phase space:

$\varphi \in \Omega^{0,*}(X)[\varepsilon] \otimes \mathfrak{g}[1]$

satisfying an equation

ε is of degree

$B \in \Omega(X) \otimes \mathfrak{g}$

Satisfying $F^{0,2}(A) = 0$, $\bar{\partial}_A B = 0$

This is $T^* \mathrm{Bun}_G(X)$

At a bundle P,

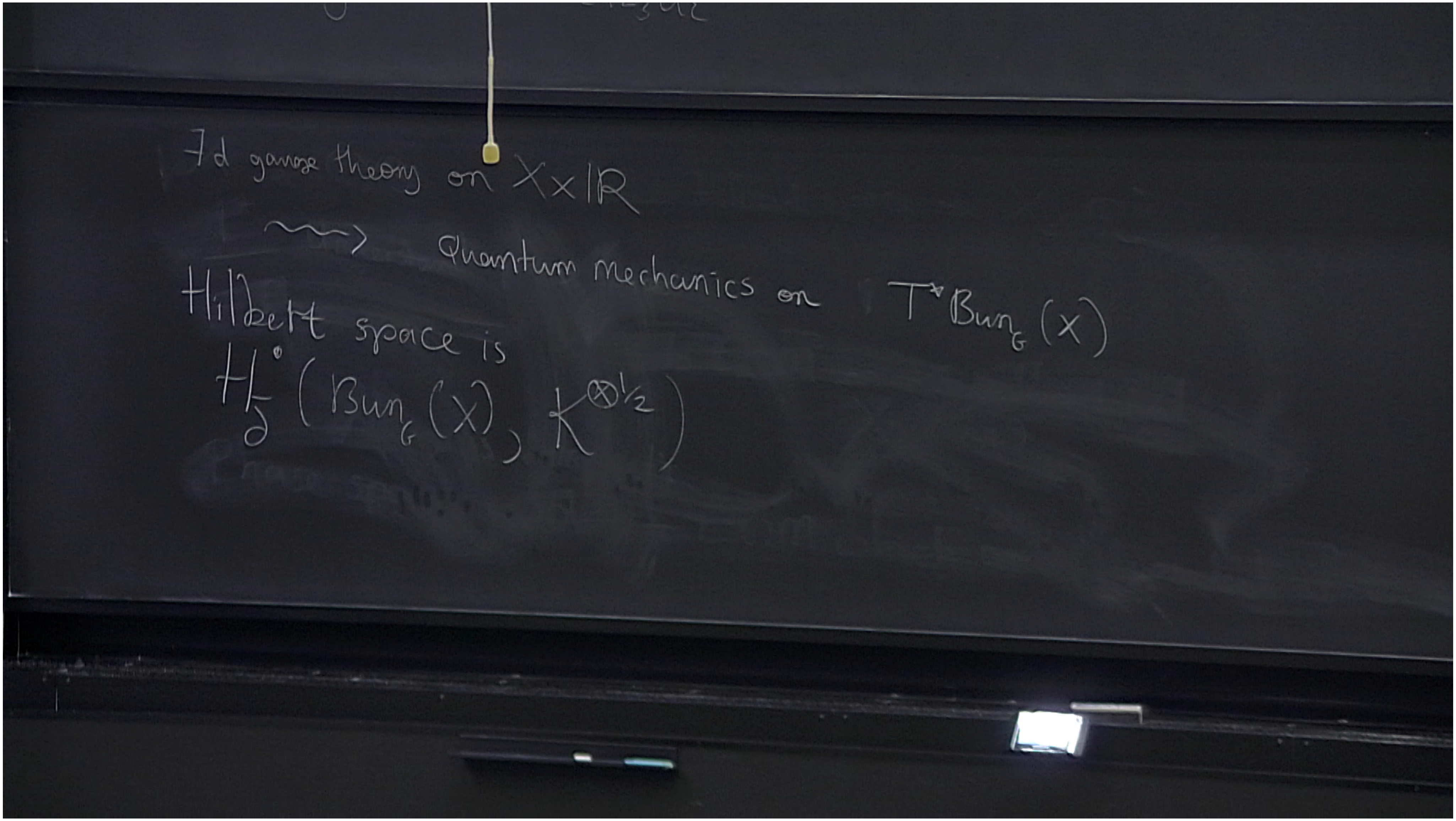
$$T_P \mathrm{Bun}_G(X) = H^1_{\bar{\partial}}(X, \mathfrak{g}_{ad})$$

dual by Serre duality to $H^2_{\bar{\partial}}(X, \mathfrak{g}_{ad})$

$$K_X = 0$$

7d gauge theory on $X \times \mathbb{R}$

$\rightsquigarrow$ Quantum mechanics on $T^*\mathrm{Bun}_G(X)$


7d gauge theory on $X \times \mathbb{R}$
$\rightsquigarrow$ Quantum mechanics on $T^* \mathrm{Bun}_G(X)$
Hilbert space is
$H_{\bar\partial}^{\bullet}(\mathrm{Bun}_G(X), K^{\otimes 1/2})$

7d gauge theory on $X \times \mathbb{R}$

$\rightsquigarrow$ Quantum mechanics on $T^* \mathrm{Bun}_G(X)$

Hilbert space is

$$H^{\bullet}_{\bar\partial}\left(\mathrm{Bun}_G(X), K^{\otimes 1/2}\right)$$

$Z(X \times S^1) =$ Euler char. of this

By RR, $Z(X \times S^1) = \int_{\mathrm{Bun}_G(X)} \mathrm{Td}(\mathrm{Bun}_G(X))\, e^{\frac{1}{2}c_1}$

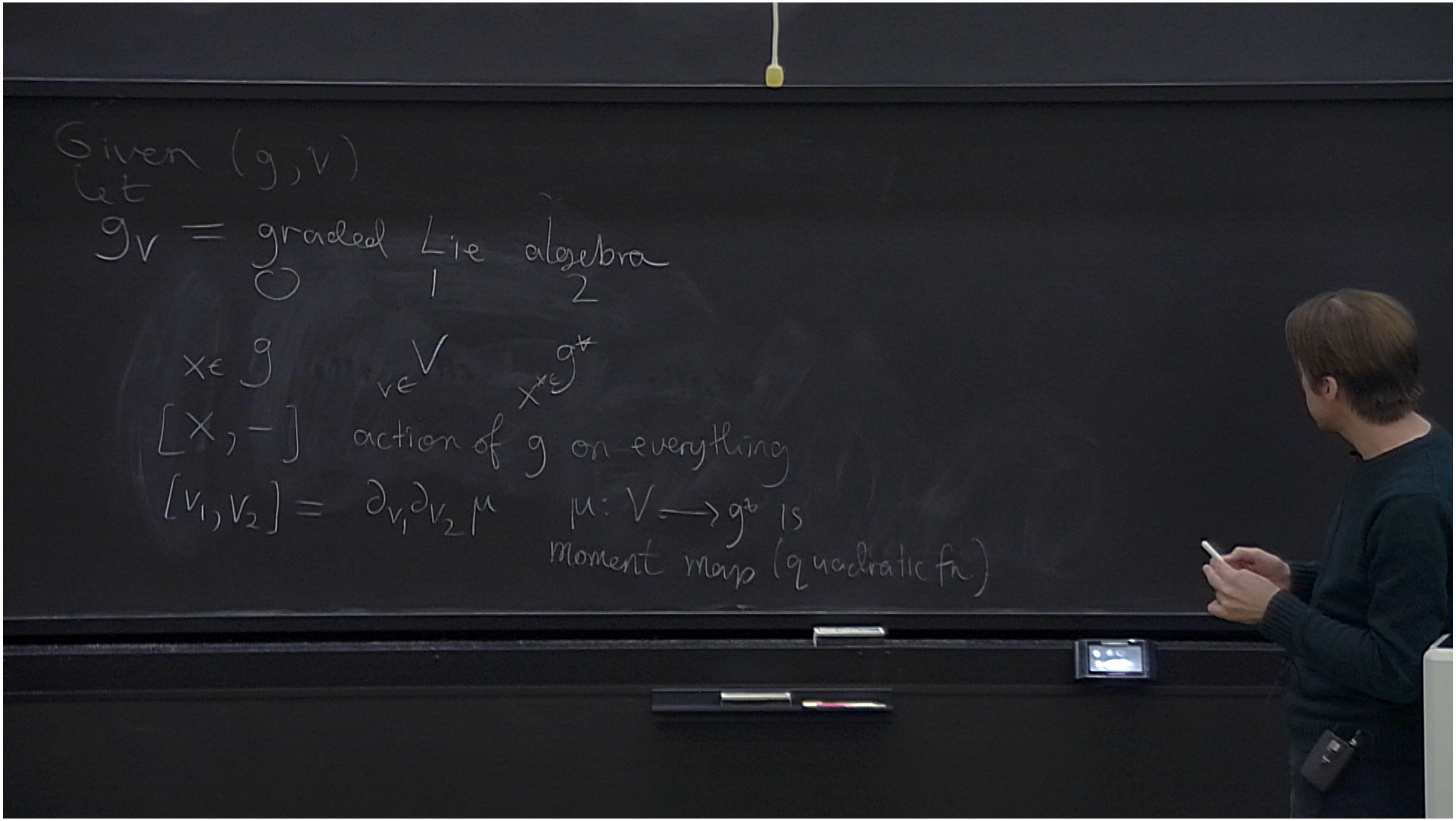
$=$ K-theoretic DT invariants

8 supercharges

To give a theory we need
a Lie alg. $\mathfrak{g}$ and a symplectic representation V.

16 supercharges is special case $V = T^*\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}^*$

Given $(\mathfrak{g}, V)$


Given (g, V)
Let
g_V = graded Lie algebra
0 1 2
$x \in g$ $v \in V$ $x^* \in g^*$
$[X, -]$ action of g on everything
$[v_1, v_2] = \partial_{v_1}\partial_{v_2}\mu$ $\mu : V \longrightarrow g^*$ is
moment map (quadratic fn)

An element $\alpha \in \mathfrak{g}_V$ of degree 1, α satisfies MC equation if $[\alpha,\alpha]=0$

An element $\alpha \in \mathfrak{g}_V$ of degree 1, α satisfies MC equation if $[\alpha, \alpha] = 0$

In this case $\alpha \in V$ satisfies MC $\Longleftrightarrow$ $\alpha \in \mu^{-1}(0)$

An element $\alpha \in \mathfrak{g}_V$ of degree 1, α satisfies MC equation if $[\alpha,\alpha]=0$

In this case $\alpha \in V$ satisfies MC $\iff \alpha \in \mu^{-1}(0)$

$x \in \mathfrak{g}_V^0$ act as symmetries of set of MC sol'ns

$\alpha \to \alpha + \varepsilon[x,\alpha]$

An element $\alpha \in \mathfrak{g}_V$ of degree 1, α satisfies MC equation if $[\alpha,\alpha]=0$

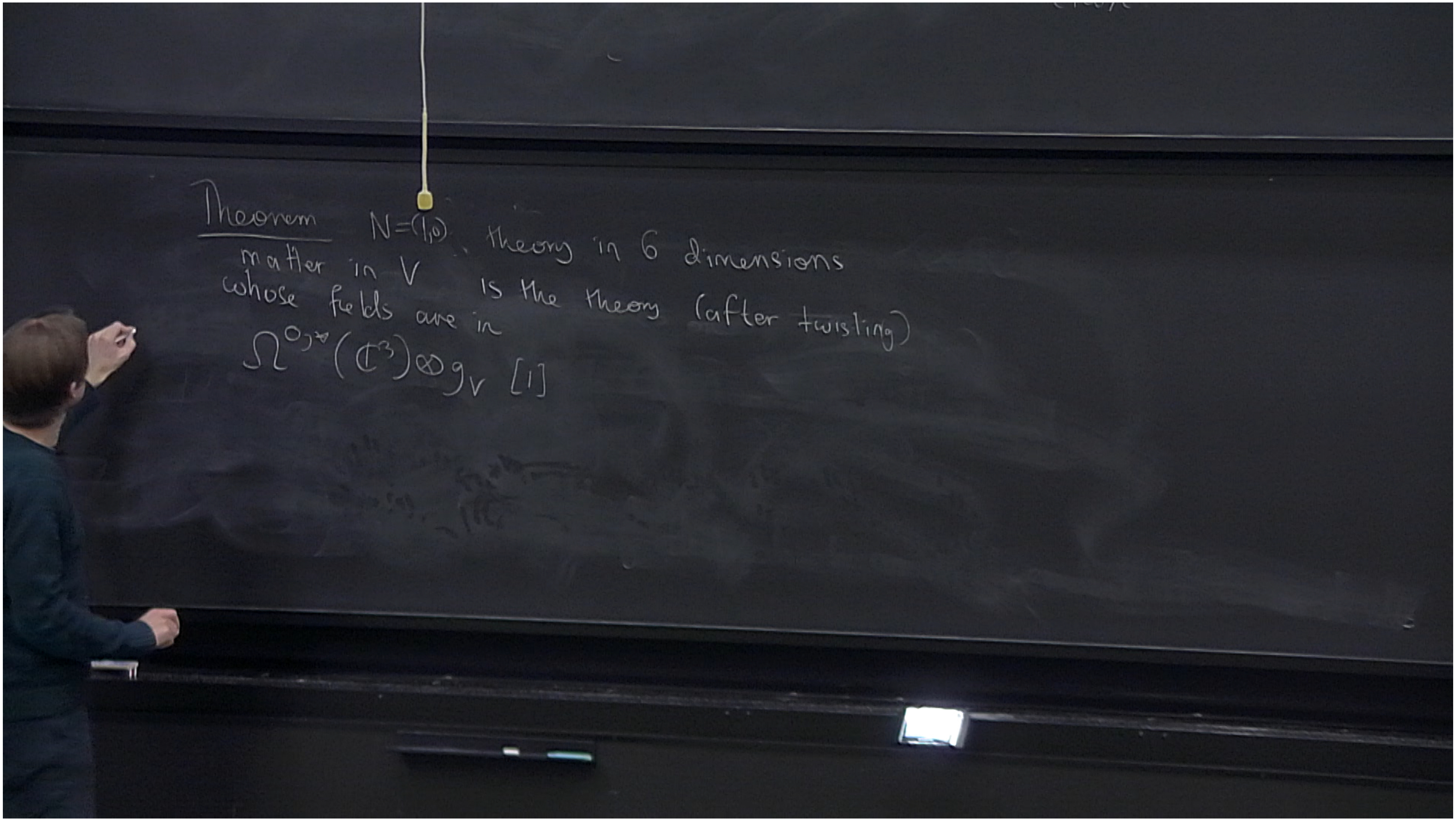
In this case $\alpha \in V$ satisfies MC $\iff \alpha \in \mu^{-1}(0)$

$x \in \mathfrak{g}_V^0$ act as symmetries of set of MC sol'ns

$$\alpha \to \alpha + \varepsilon[x,\alpha]$$

MC sol'ns/gauge $=$ $\mu^{-1}(0)/G$ $=$ symplectic reduction

7d gauge theory on $X \times \mathbb{R}$


Theorem N=(1,0) theory in 6 dimensions
matter in V is the theory (after twisting)
whose fields are in
$\Omega^{0,*}(\mathbb{C}^3)\otimes g_V [1]$

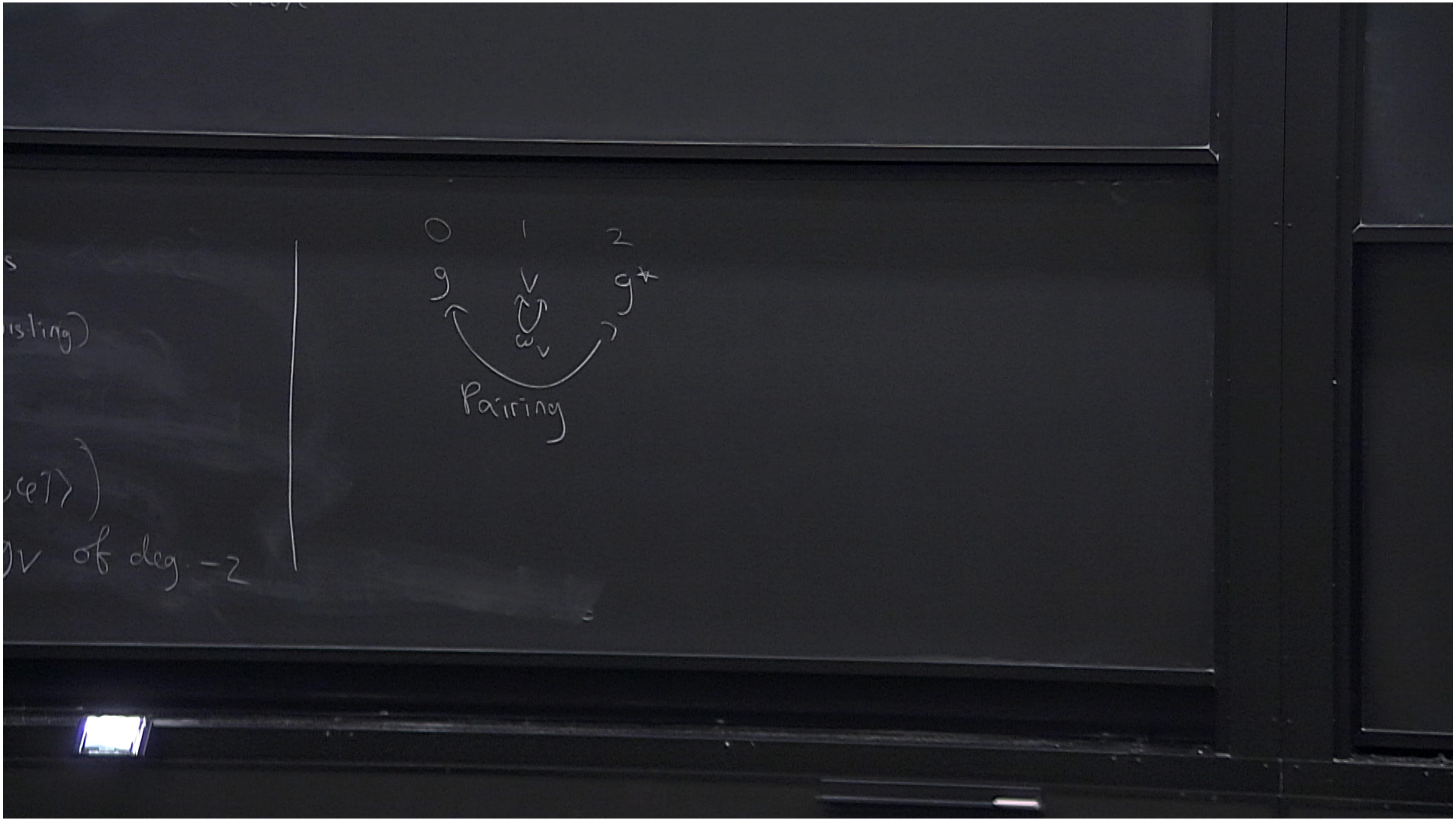
Theorem $N=(1,0)$ theory in 6 dimensions
matter in V is the theory (after twisting)
whose fields are in

$$\varphi \in \Omega^{0,*}(\mathbb{C}^3) \otimes \mathfrak{g}_V [1]$$

$$S(\varphi) = \int dz_1 dz_2 dz_3 \left(\frac{1}{2} \langle \varphi, \bar\partial \varphi \rangle + \frac{1}{6} \langle \varphi, [\varphi, \varphi] \rangle \right)$$

where $\langle \ \rangle$ is the invariant pairing on $\mathfrak{g}_V$ of deg -2

0	1
$\mathfrak{g}$	V


0 1 2
g V g*
ω_V
Pairing
isting)
V of deg. -2

<u>Theorem</u> $N=(1,0)$ theory in 6 dimensions
matter in V is the theory (after twisting)
whose fields are in

$$\varphi \in \Omega^{0,*}(\mathbb{C}^3) \otimes \mathfrak{g}_V [1]$$

$$S(\varphi) = \int dz_1 dz_2 dz_3 \left(\frac{1}{2} \langle \varphi, d\varphi \rangle + \frac{1}{6} \langle \varphi, [\varphi, \varphi] \rangle \right)$$

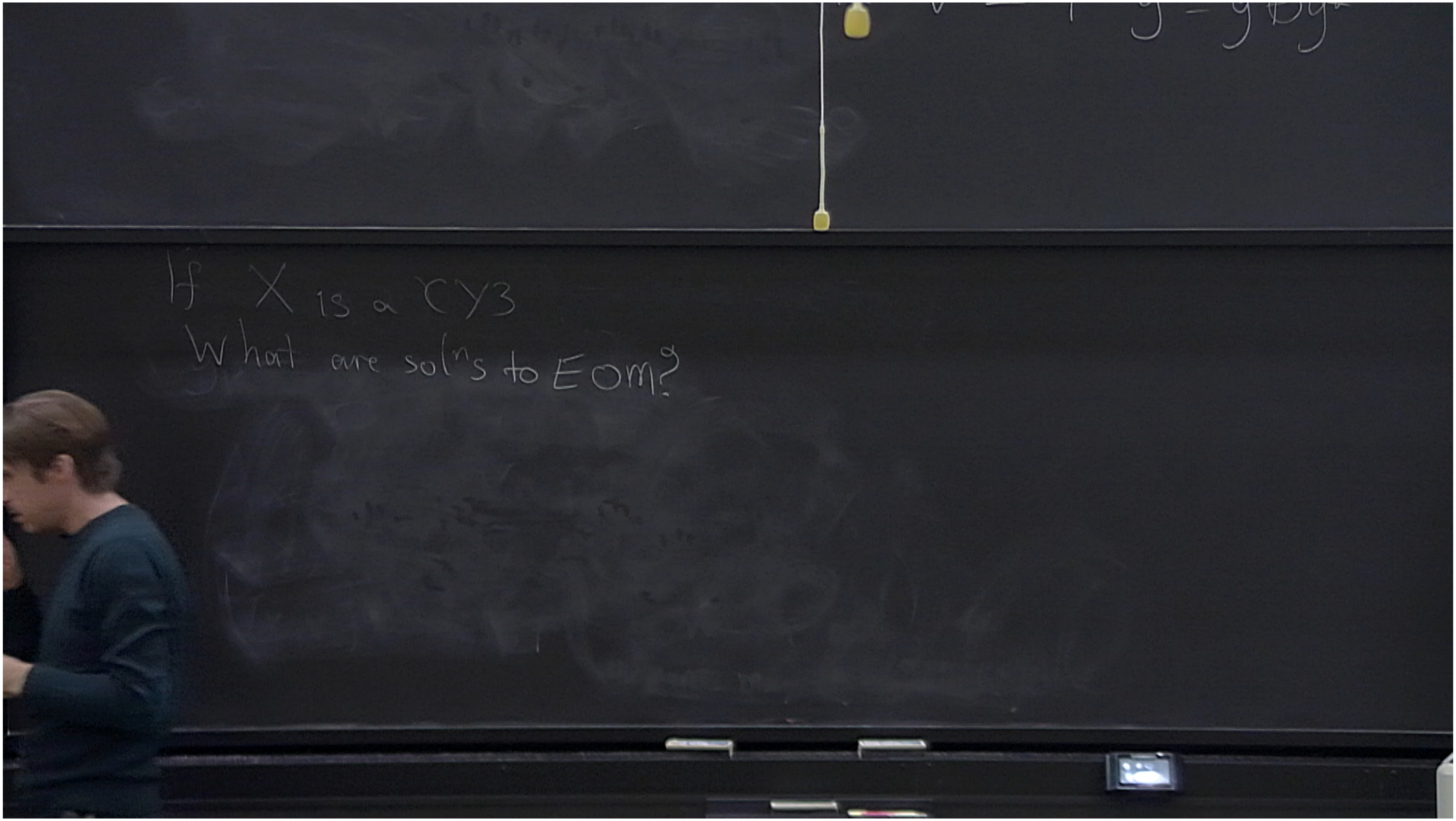
where $\langle \ \rangle$ is the invariant pairing on $\mathfrak{g}_V$ of deg -2

This is holomorphic Rozansky-Witten theory (gauged)

O

g

Pai


If X is a CY3
What are sol'ns to EOM?

If X is a CY3

What are sol^{n}s to EOM?

$$\Omega^{0,*}(X) \otimes \begin{pmatrix} \mathfrak{g} & V & \mathfrak{g}^* \\ -1 & 0 & 1 \end{pmatrix}$$

$\mathfrak{g}$: we will find a hol. G-bundle on X

V part: $\rightsquigarrow$ a section φ of assoc. G-bundle

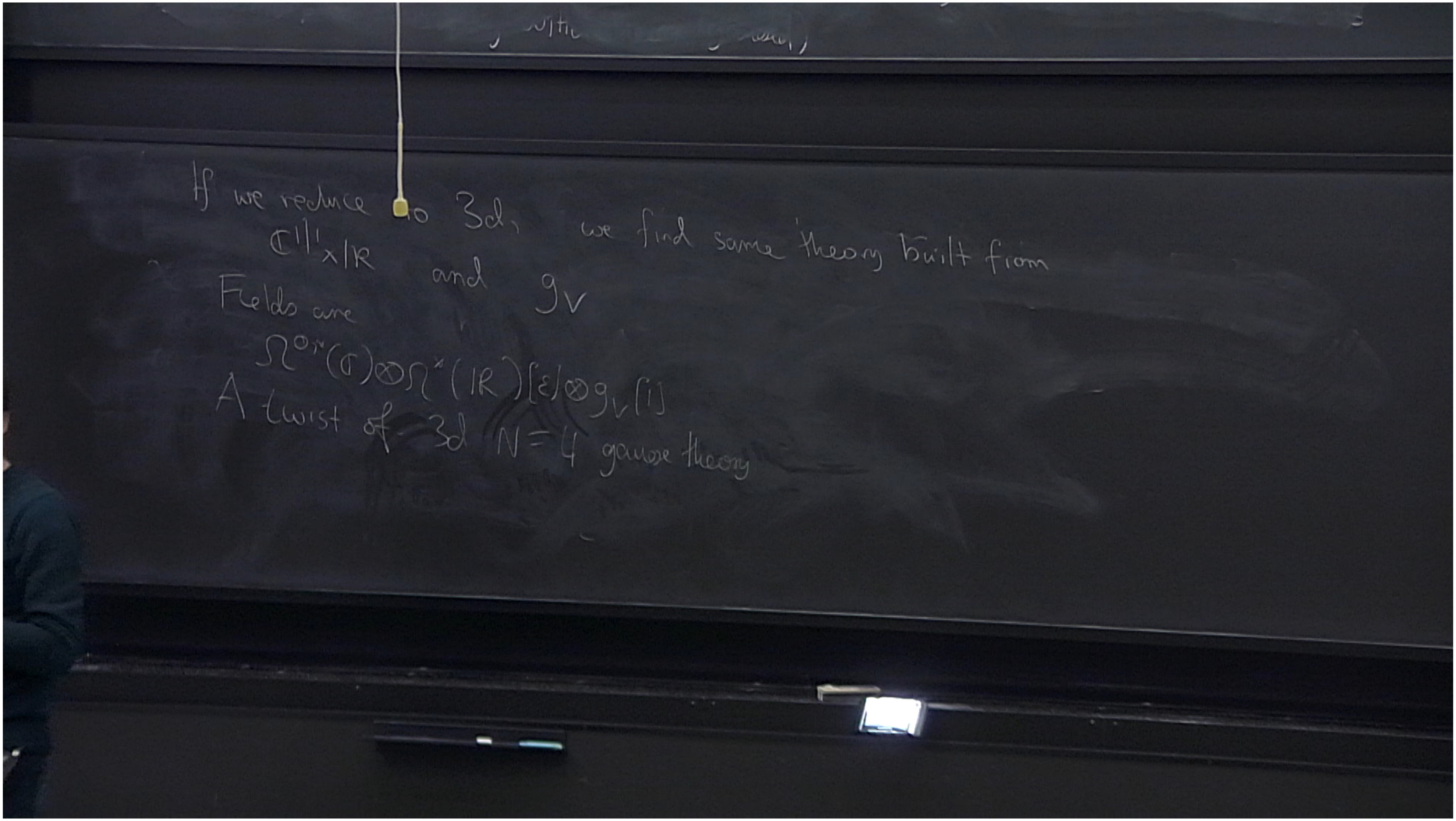
$\bar{\partial}_A \varphi = 0$

<u>Also</u> $\mu(\varphi) = 0 \in \Omega^{0,0}(X, \mathfrak{g}^*)$

If $U \subseteq V$ preserved by G, and G acts freely. If φ lands in U, then

Sol^n's to EOM $=$ hol. maps X to $U /\!/ G$.

"$V /\!/ G$ = Higgs branch"


If we reduce to 3d, we find same theory built from
$\mathbb{C}^{1|1} \times \mathbb{R}$ and $\mathfrak{g}_V$
Fields are
$\Omega^{0,*}(\mathbb{C}) \otimes \Omega^*(\mathbb{R})[\varepsilon] \otimes \mathfrak{g}_V[1]$
A twist of 3d $N=4$ gauge theory

me theory built from

Remaining SUSY
is given by the odd v. fields
$\varepsilon \partial_z$ and ∂_ε on $\mathbb{C}^{1|1}$

...me theory built from

Remaining SUSY
is given by the odd v. fields
$\varepsilon \partial_z$ and ∂_ε on $\mathbb{C}^{1|1}$

$\varepsilon \partial_z \rightsquigarrow$ Rozansky-Witten twist

$\partial_\varepsilon \rightsquigarrow$ Twisted RW twist

...me theory built from

Remaining SUSY

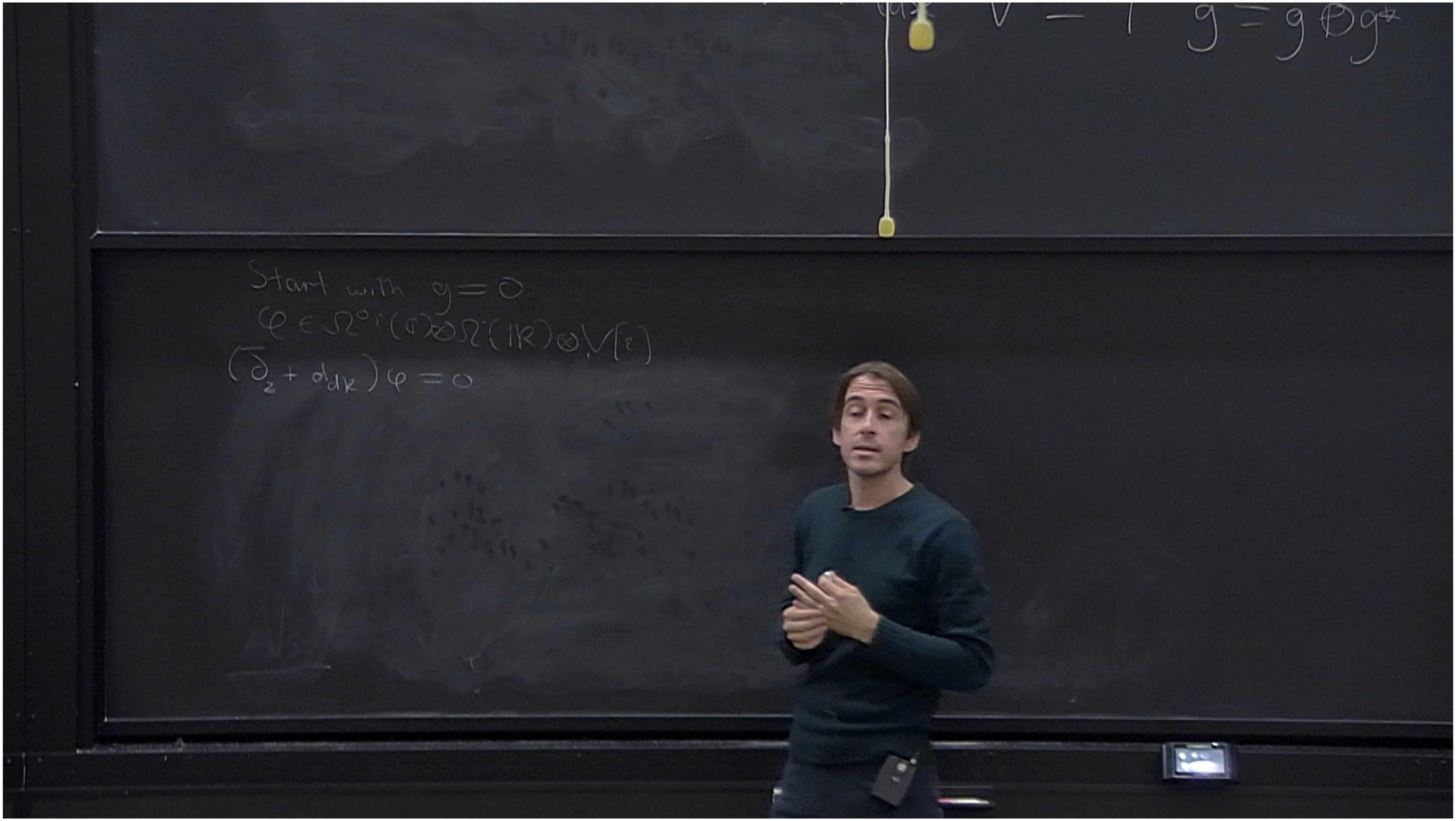
is given by the odd v. fields

$\varepsilon \partial_z$ and ∂_ε on $\mathbb{C}^{1|1}$

$\varepsilon \partial_z \rightsquigarrow$ Rozansky-Witten twist (B)

$\partial_\varepsilon \rightsquigarrow$ Twisted RW twist (A)

3d mirror symmetry


Start with g = 0
φ ∈ Ω^{0,•}(ℂ)⊗Ω^•(ℝ)⊗V(ε)
(∂̄_z + d_dR) φ = 0

Start with $y = 0$.

$\varphi \in \Omega^{0,*}(\mathbb{C}) \otimes \Omega^{*}(\mathbb{R}) \otimes V[\varepsilon]$

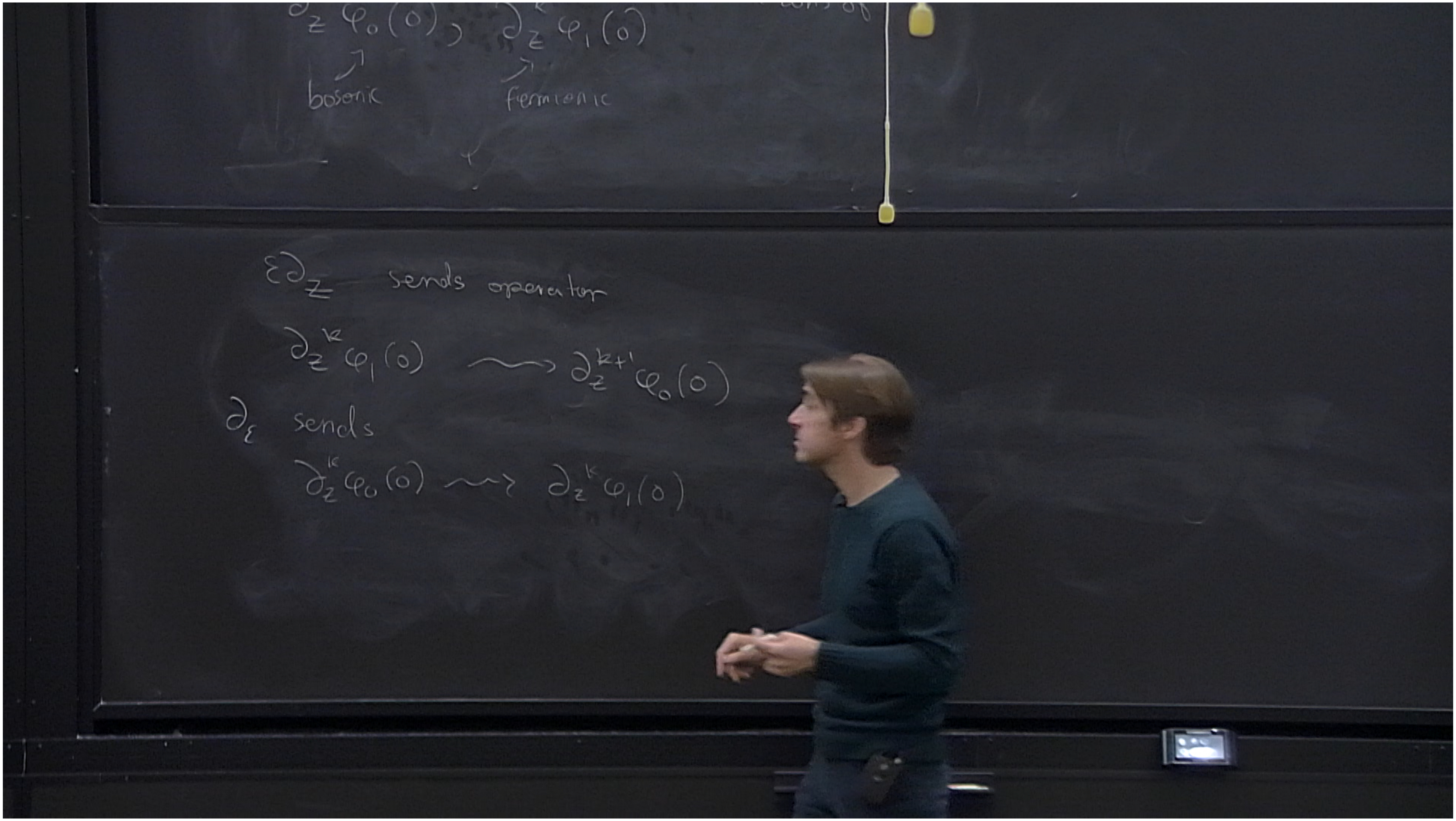
$(\bar{\partial}_z + d_{dR})\varphi = 0$

$\varphi = \varphi_0 + \varepsilon \varphi_1$

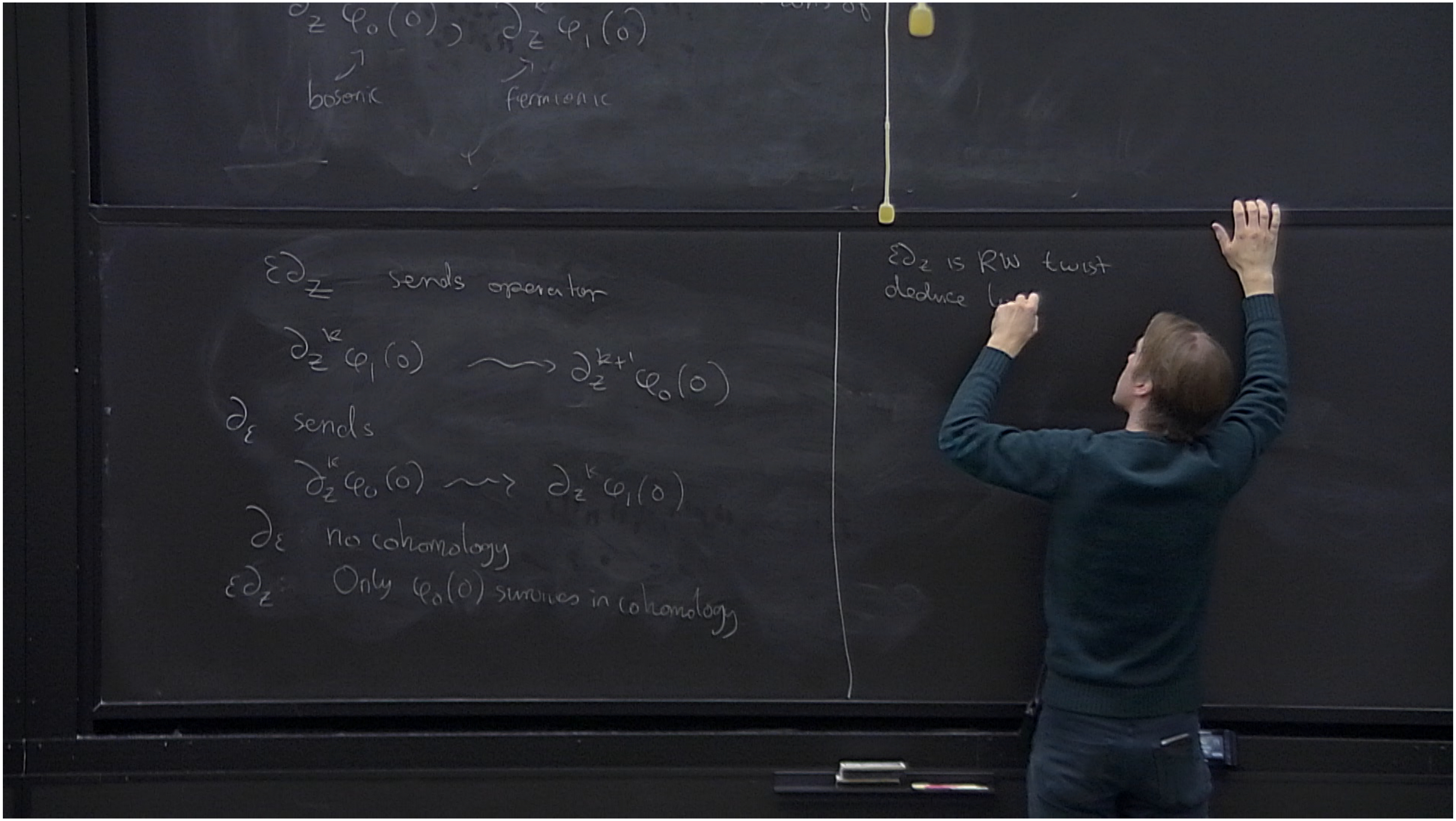
Operators we can build are functions of

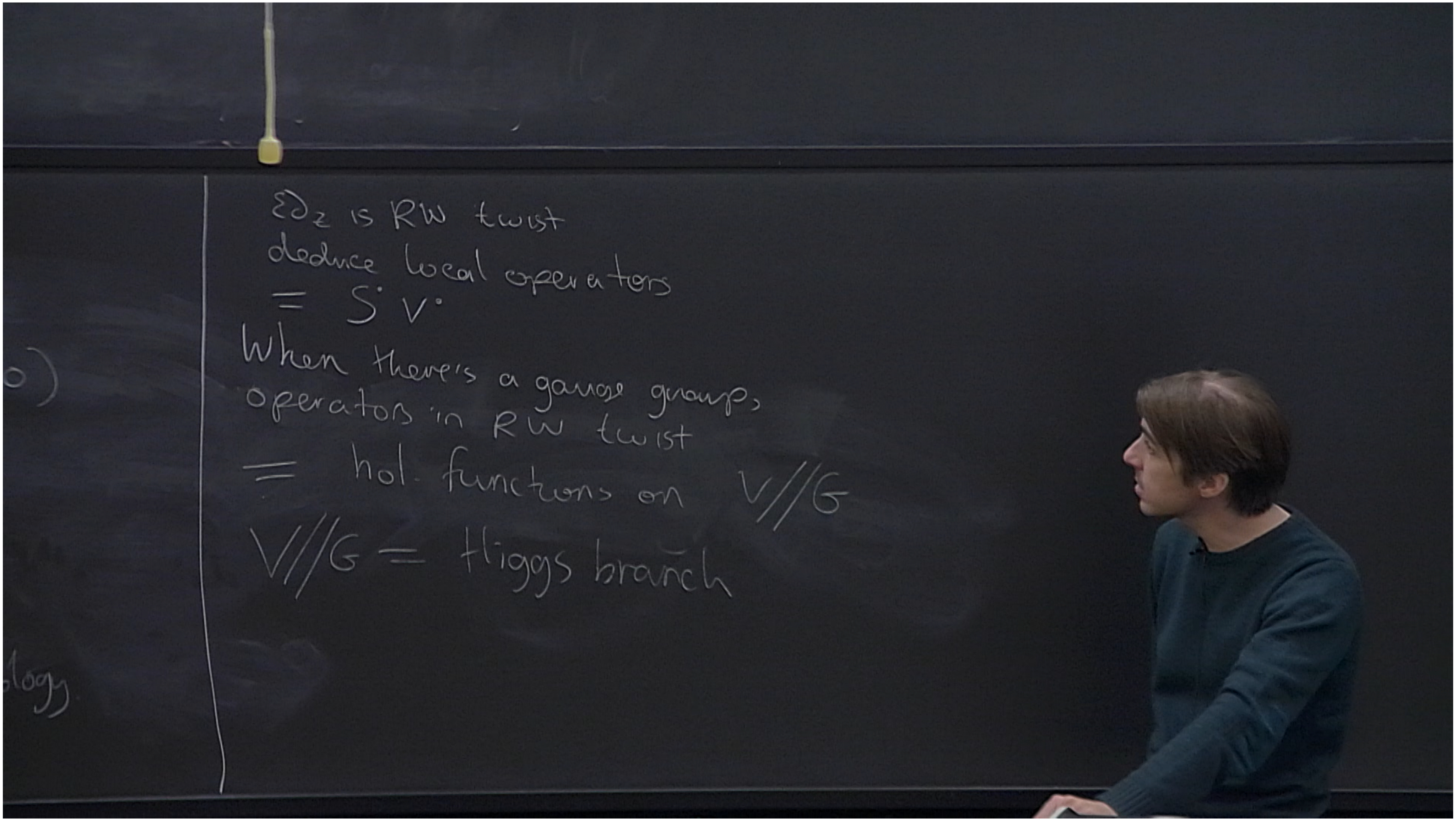
$\partial_z^k \varphi_0(0), \partial_z^k \varphi_1(0)$

bosenic fermionic



bosonic
fermionic
$\varepsilon\partial_z$ sends operator
$\partial_z^k \varphi_1(0) \rightsquigarrow \partial_z^{k+1} \varphi_0(0)$
∂_ε sends
$\partial_z^k \varphi_0(0) \rightsquigarrow \partial_z^k \varphi_1(0)$


$\partial_z \varphi_0(0)$, $\partial_z^k \varphi_1(0)$
bosonic
fermionic
$\varepsilon\partial_z$ sends operator
$\partial_z^k \varphi_1(0) \rightsquigarrow \partial_z^{k+1} \varphi_0(0)$
∂_ε sends
$\partial_z^k \varphi_0(0) \rightsquigarrow \partial_z^k \varphi_1(0)$
∂_ε no cohomology
$\varepsilon\partial_z$: Only $\varphi_0(0)$ survives in cohomology
$\varepsilon\partial_z$ is RW twist
deduce


ε∂z is RW twist
deduce local operators
= S· V·
When there's a gauge group,
operators in RW twist
= hol. functions on V//G
V//G = Higgs branch