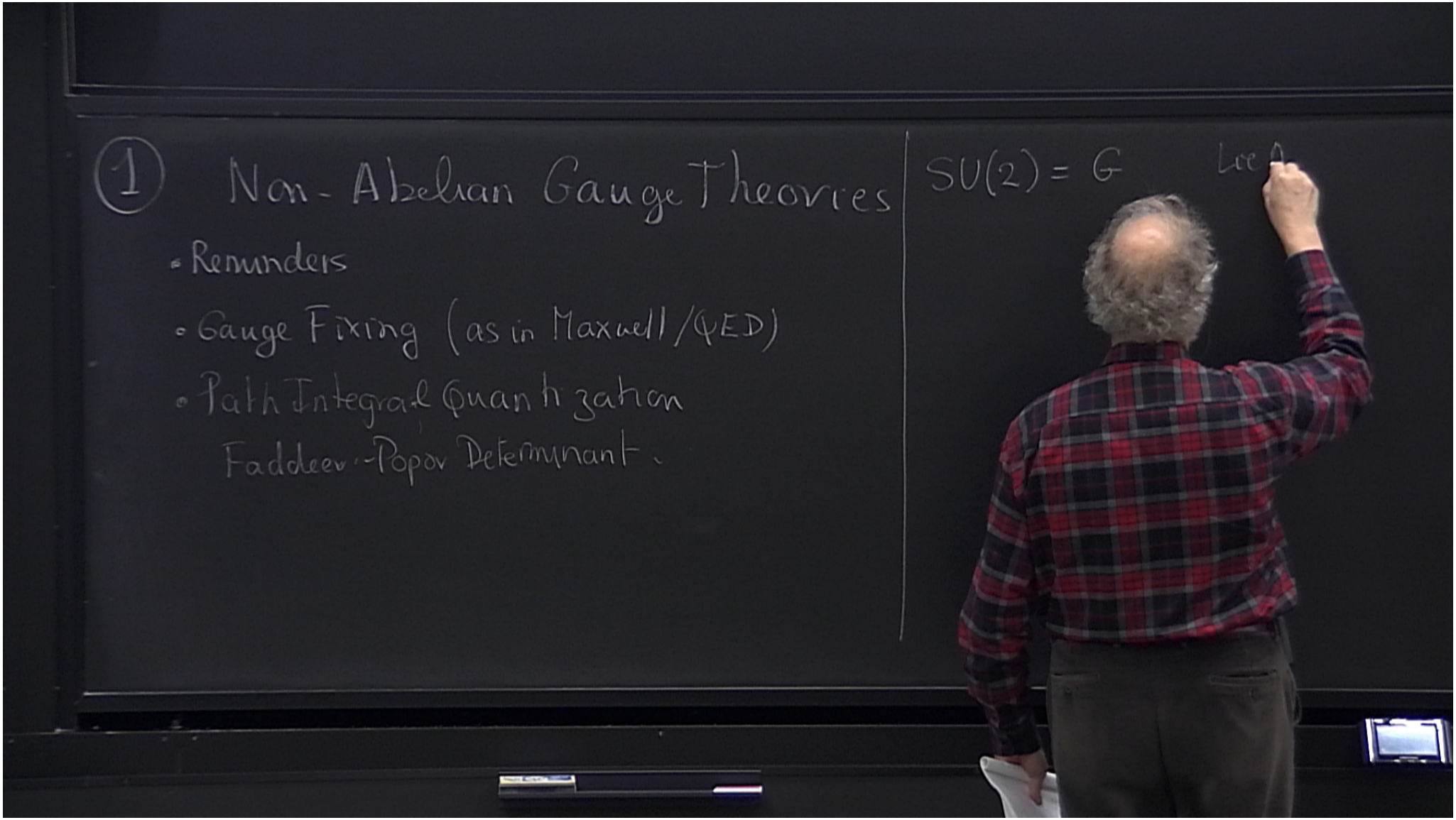
Title: PSI 2016/2017 Quantum Field Theory II - Lecture 13

Date: Nov 23, 2016 09:00 AM

URL: http://pirsa.org/16110013

Abstract:


1 Non-Abelian Gauge Theories
• Reminders
• Gauge Fixing (as in Maxwell/QED)
• Path Integral Quantization
Faddeev-Popov Determinant.
SU(2) = G
Lie A

$SU(2) = G$ Lie Algebra $t_a = \frac{1}{2}\sigma_a$ Pauli Matrices $a = 1, 2, 3$

Traceless Herm. Matrices 3 Gauge Fields since 3 charges

A_μ^a ← Lie Alg

↑ Lorentz

$A_\mu = A_\mu^a t_a$ 2×2 ... matrix $\in$ Adj Representation

Field Strength

$$F_{\mu\nu} = i[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

Gauge transformation $\alpha(x) \in$ Lie Algebra $= \alpha'^a(x) t_a$

$$A'_\mu(x) \to A_\mu(x) + D_\mu \alpha(x) \qquad D_\mu \alpha(x) = \partial_\mu \alpha(x) - i[A_\mu(x), \alpha(x)]$$

covariant derivative acting on $\alpha \in$ Adj Repr

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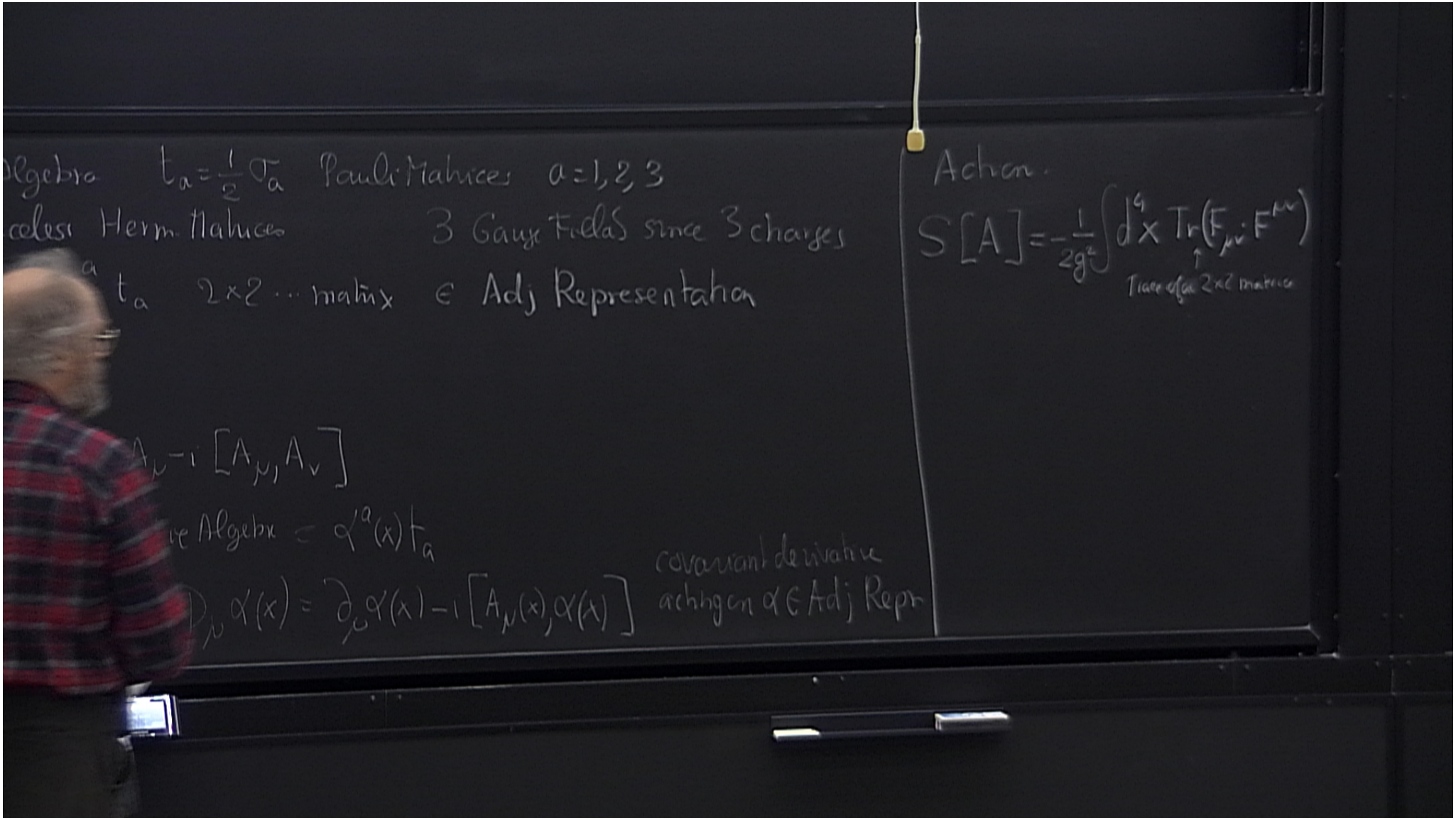
Field Strength

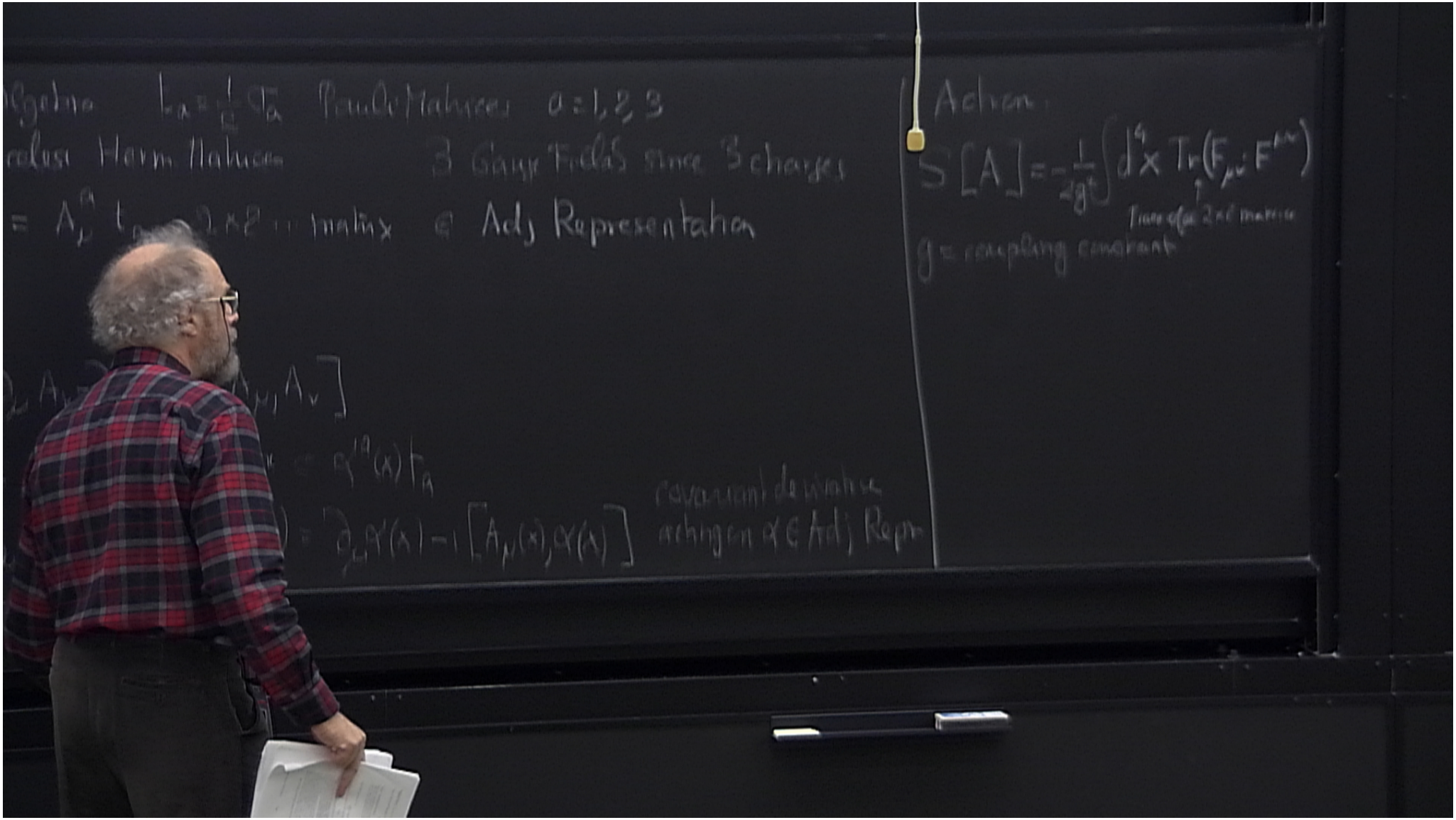
$$F_{\mu\nu} = i[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

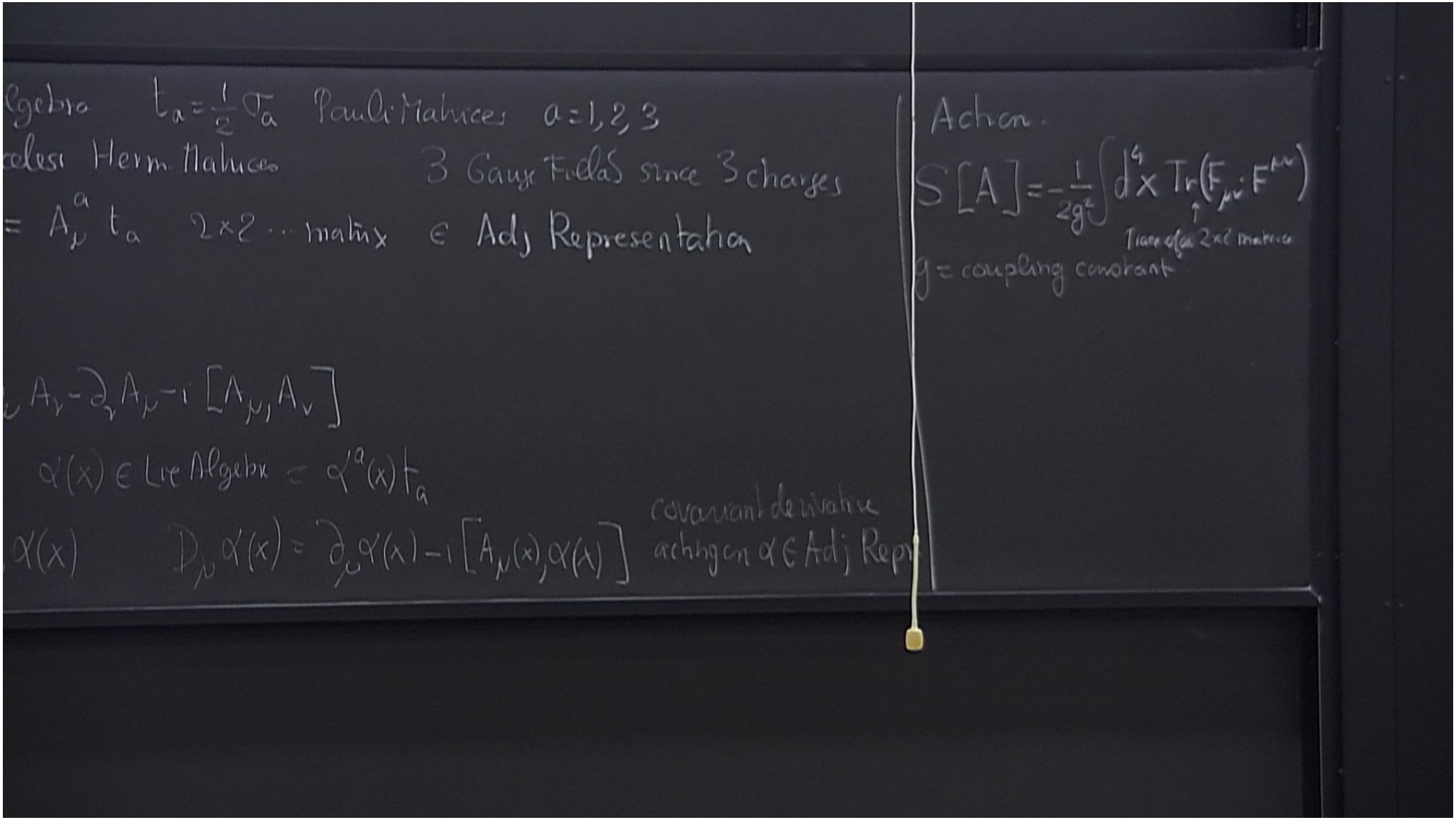
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lgebra $t_a = \frac{1}{2}\sigma_a$ Pauli Matrices $a = 1, 2, 3$
coles Herm. Matrices 3 Gauge Fields since 3 charges
t_a 2×2 ... matrix ∈ Adj Representation
$A_\mu - i[A_\mu, A_\nu]$
ve Algebra $= \alpha'^a(x) t_a$
$D_\mu \alpha(x) = \partial_\mu \alpha(x) - i[A_\mu(x), \alpha(x)]$
covariant derivative
acting on $\alpha \in$ Adj Repr
Action
$S[A] = -\frac{1}{2g^2}\int d^4x \, \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu})$
Trace of a 2×2 matrix

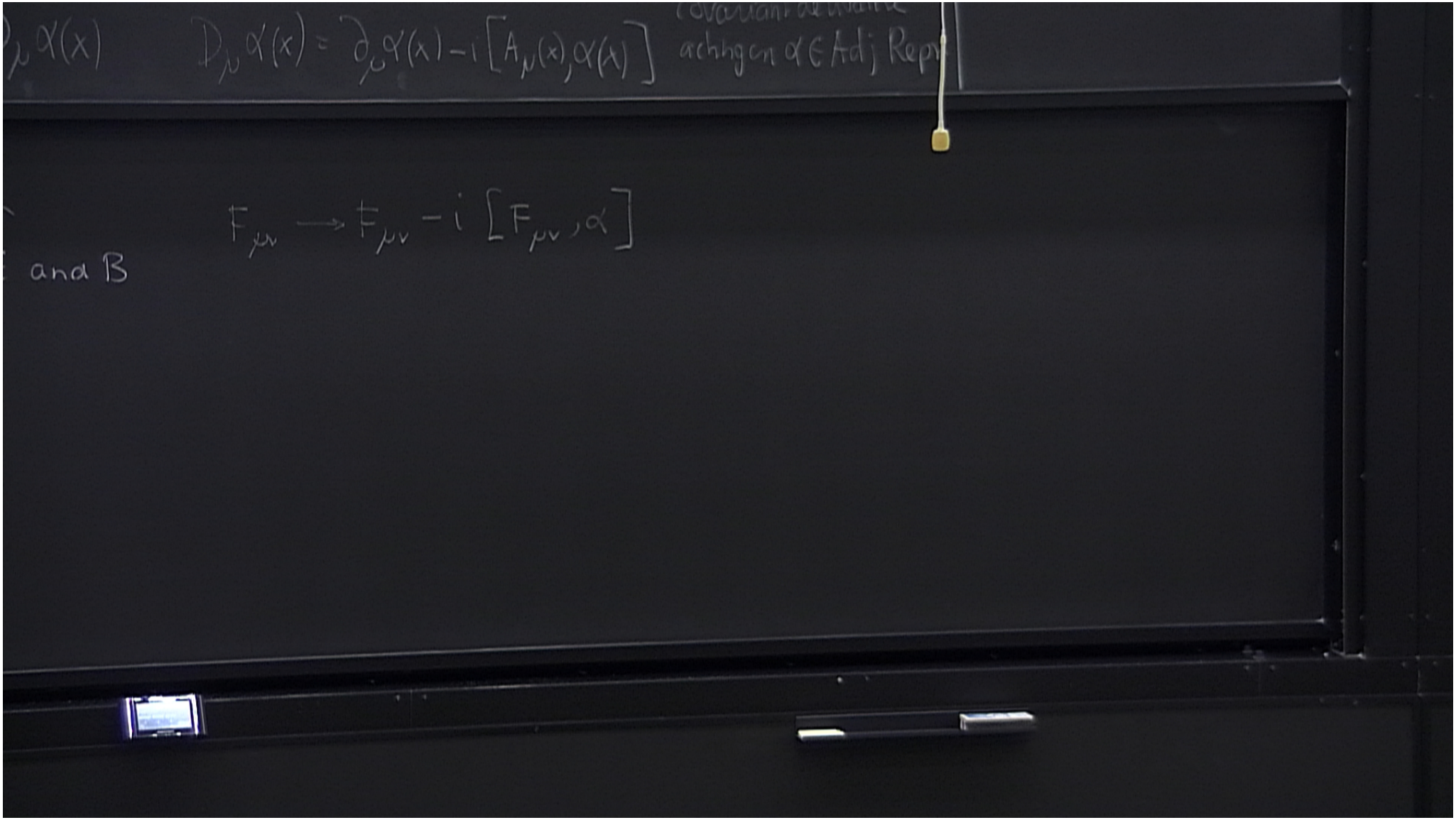

Pauli Matrices a = 1, 2, 3
3 Gauge Fields since 3 charges
Adj Representation
Action:
g = coupling constant

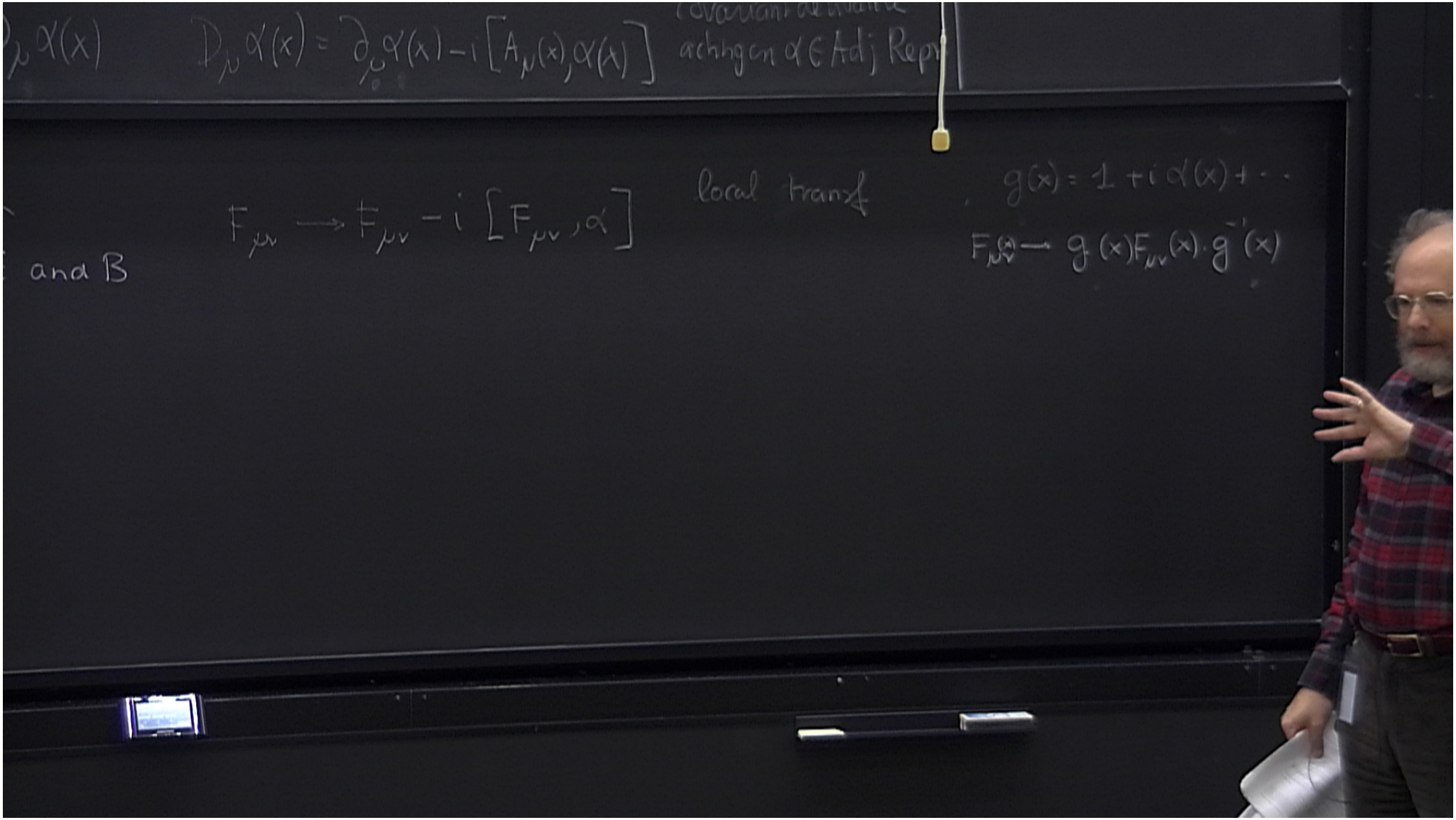

lgebra $t_a = \frac{1}{2}\sigma_a$ Pauli Matrices $a = 1, 2, 3$
oles: Herm. Matrices 3 Gauge Fields since 3 charges
$= A_\mu^a t_a$ 2×2 ... matrix $\in$ Adj Representation
$A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$
$\alpha(x) \in$ Lie Algebra $= \alpha^a(x) t_a$
$\alpha(x)$ $D_\mu \alpha(x) = \partial_\mu \alpha(x) - i[A_\mu(x), \alpha(x)]$ covariant derivative
acting on $\alpha \in$ Adj Repn
Action
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Trace of a 2×2 matrix
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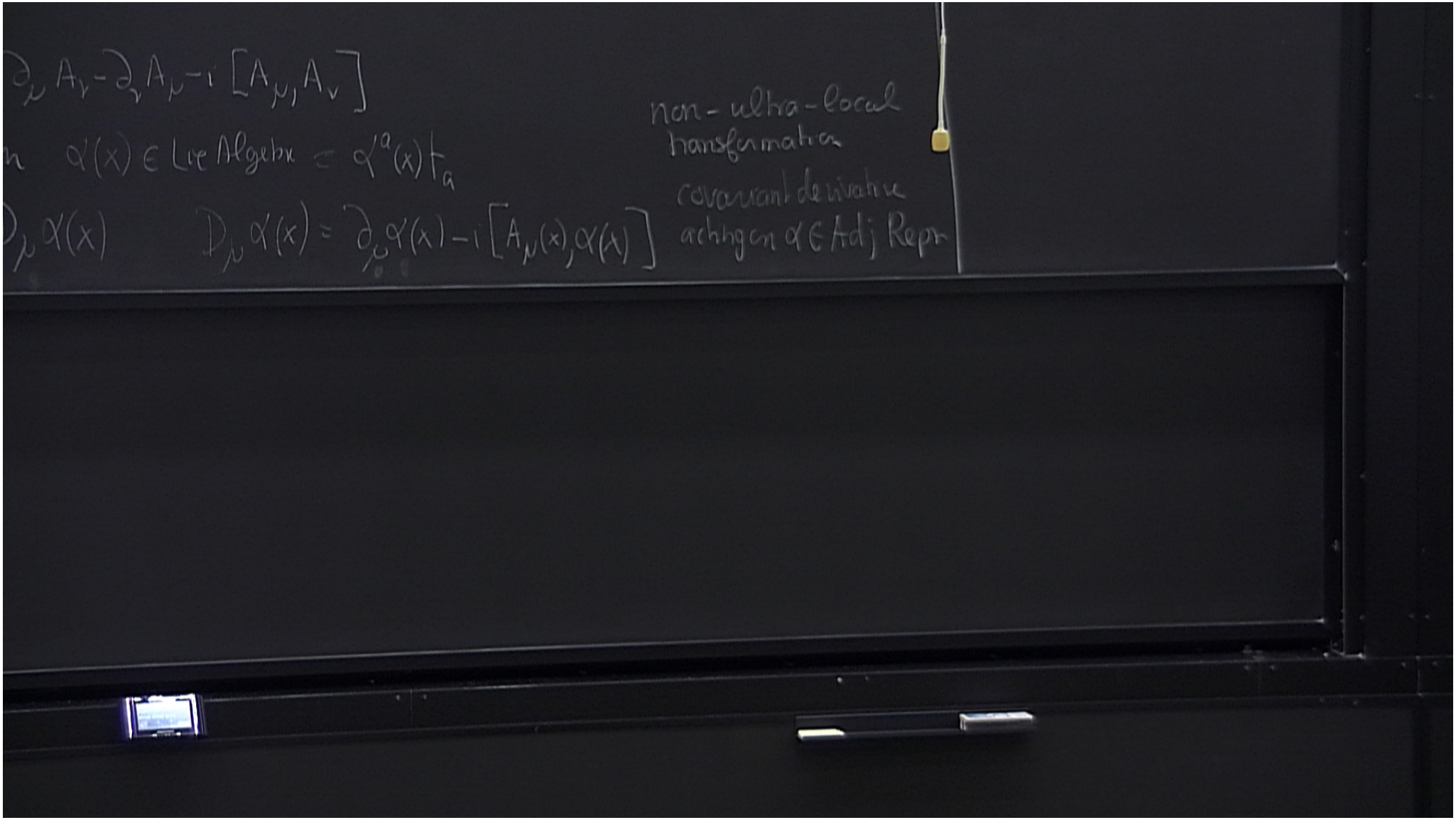
$A_\mu(x) \to A_\mu(x) + D_\mu q(x)$

Gauge Fixing. . Physics do not depend on Gauge Transformation

Big redundancy through using A_μ instead of E and B


$D_\mu \alpha(x)$
$D_\mu \alpha(x) = \partial_\mu \alpha(x) - i[A_\mu(x), \alpha(x)]$
acting on $\alpha \in$ Adj Rep
$F_{\mu\nu} \longrightarrow F_{\mu\nu} - i[F_{\mu\nu}, \alpha]$
and B


$D_\mu \alpha(x) = \partial_\mu \alpha(x) - i[A_\mu(x), \alpha(x)]$
covariant derivative
acting on $\alpha \in$ Adj Repr
local transf
$g(x) = 1 + i\alpha(x) + \cdots$
$F_{\mu\nu} \longrightarrow F_{\mu\nu} - i[F_{\mu\nu}, \alpha]$
$F_{\mu\nu} \longrightarrow g(x) F_{\mu\nu}(x) \cdot g^{-1}(x)$
and B


$\partial_\nu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$
$\alpha(x) \in$ Lie Algebra $= \alpha'^a(x) t_a$
$D_\mu \alpha(x)$
$D_\mu \alpha(x) = \partial_\mu \alpha(x) - i[A_\mu(x), \alpha(x)]$
non-ultra-local transformation
covariant derivative acting on $\alpha \in$ Adj Repr

$A_\mu(x) \to A_\mu(x) + D_\mu \alpha(x)$

Gauge Fixing. . Physics do not depend on Gauge Transformation

Big redundancy through using A_μ instead of E and B

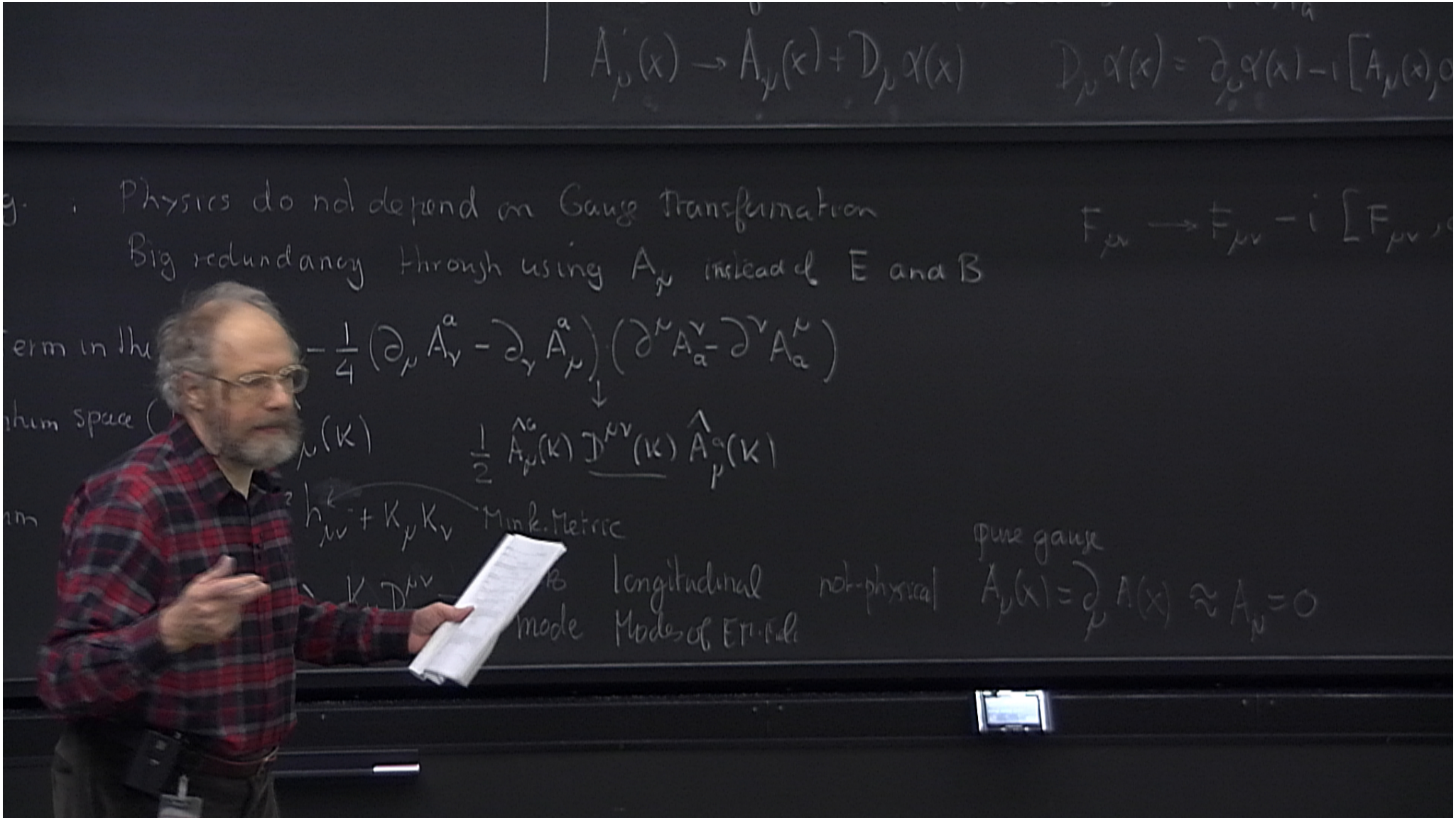
Quadratic Term in the Action $-\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a)$

momentum space (F.Tr) $\hat{A}_\mu(k)$ $\frac{1}{2}\hat{A}^a_\mu(k)\, D^{\mu\nu}(k)\, \hat{A}^a_\mu(k)$

quadratic form $D_{\mu\nu}(k) = -k^2 h_{\mu\nu} + k_\mu k_\nu$ Mink. Metric

4×4 matrix $\hat{A}_\mu(k) = k_\mu \cdot A(k)$ $k_\mu D^{\mu\nu} = 0$ Zero longitudinal mode Modes of EM.

Dim = 4


$A_\mu(x) \to A_\mu(x) + D_\mu \alpha(x)$
$D_\mu \alpha(x) = \partial_\mu \alpha(x) - i[A_\mu(x), \alpha$
Physics do not depend on Gauge Transformation
Big redundancy through using A_μ instead of E and B
$F_{\mu\nu} \to F_{\mu\nu} - i[F_{\mu\nu}$
$-\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a)$
$\frac{1}{2} \hat{A}^a_\mu(k) D^{\mu\nu}(k) \hat{A}^a_\mu(k)$
$h^k_{\mu\nu} + K_\mu K_\nu$ Mink. Metric
longitudinal
mode
not-physical
Modes of EM field
pure gauge
$A_\mu(x) = \partial_\mu \Lambda(x) \approx A_\mu = 0$

$(x) + D_\mu \alpha(x)$ $\quad D_\mu \alpha(x) = \partial_\mu \alpha(x) - i\,[A_\mu(x), \alpha(x)]$ covariant derivative acting on $\alpha \in$ Adj Rep

nation

d of E and B

A_a^μ)

$F_{\mu\nu} \longrightarrow F_{\mu\nu} - i\,[F_{\mu\nu}, \alpha]$ local transf $\quad g(x) = 1 + i\,\alpha(x) + \dots$

$F_{\mu\nu} \to g(x) F_{\mu\nu}(x) \cdot g^{-1}(x)$

As for Maxwell

Propagator $\mu \sim\sim\sim \nu = G_{\mu\nu} = (\mathcal{D}^{-1})_{\mu\nu}$ not well defined

pure gauge

not physical $\quad A_\mu(x) = \partial_\mu A(x) \approx A_\mu = 0$

Full

$D_\mu \alpha(x) = \partial_\mu \alpha(x) - i\,[A_\mu(x), \alpha(x)]$ covariant derivative

acting on $\alpha \in$ Adj Rep

$F_{\mu\nu} \longrightarrow F_{\mu\nu} - i\,[F_{\mu\nu}, \alpha]$

local transf. $g(x) = 1 + i\,\alpha(x) + \cdots$

$F_{\mu\nu} \to g(x) F_{\mu\nu}(x)\, g^{-1}(x)$

As for Maxwell

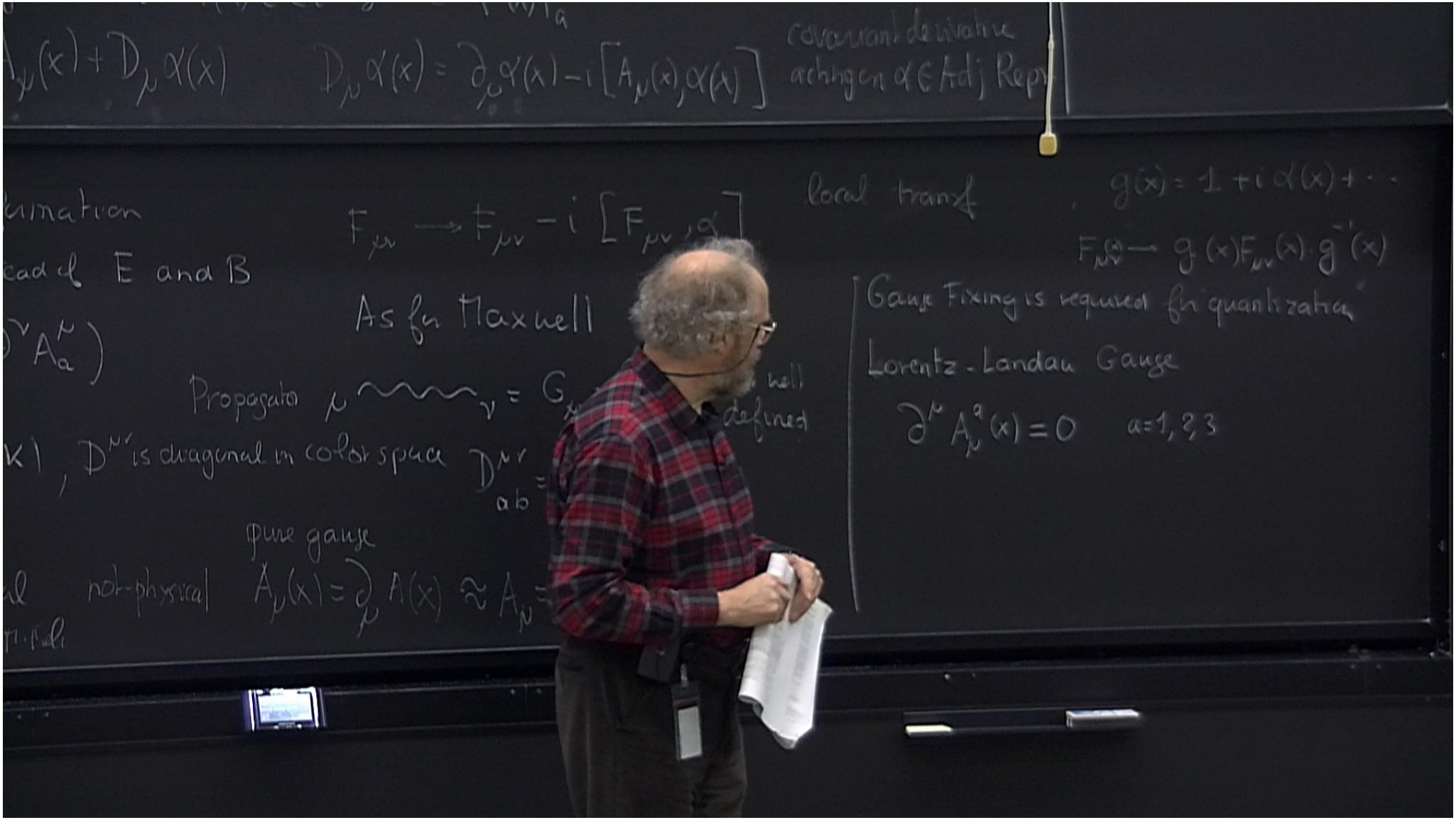
Gauge Fixing is required

Propagator $\mu \sim\sim\sim \nu = G_{\mu\nu} = (D^{-1})_{\mu\nu}$ not well defined

$D^{\mu\nu}$ is diagonal in color space $D^{\mu\nu}_{ab} = \delta_{ab} D^{\mu\nu}$

pure gauge

not-physical $A_\mu(x) = \partial_\mu \Lambda(x) \approx A_\mu = 0$


covariant derivative
local transf
Gauge Fixing is required for quantization
Lorentz - Landau Gauge
As for Maxwell
Propagator
pure gauge
not-physical

$D_\mu \alpha(x) = \partial_\mu \alpha(x) - i\,[A_\mu(x), \alpha(x)]$ covariant derivative

$\alpha \in$ Adj Rep

$F_{\mu\nu} \longrightarrow F_{\mu\nu} - i\,[F_{\mu\nu}, \alpha]$

As for Maxwell

local transf. $g(x) = 1 + i\,\alpha(x) + \dots$

$F_{\mu\nu} \to g(x) F_{\mu\nu}(x)\, g^{-1}(x)$

Propagator $\mu \sim\sim\sim \nu = G_{\mu\nu} = \left(D^{-1}\right)_{\mu\nu}$ not well defined

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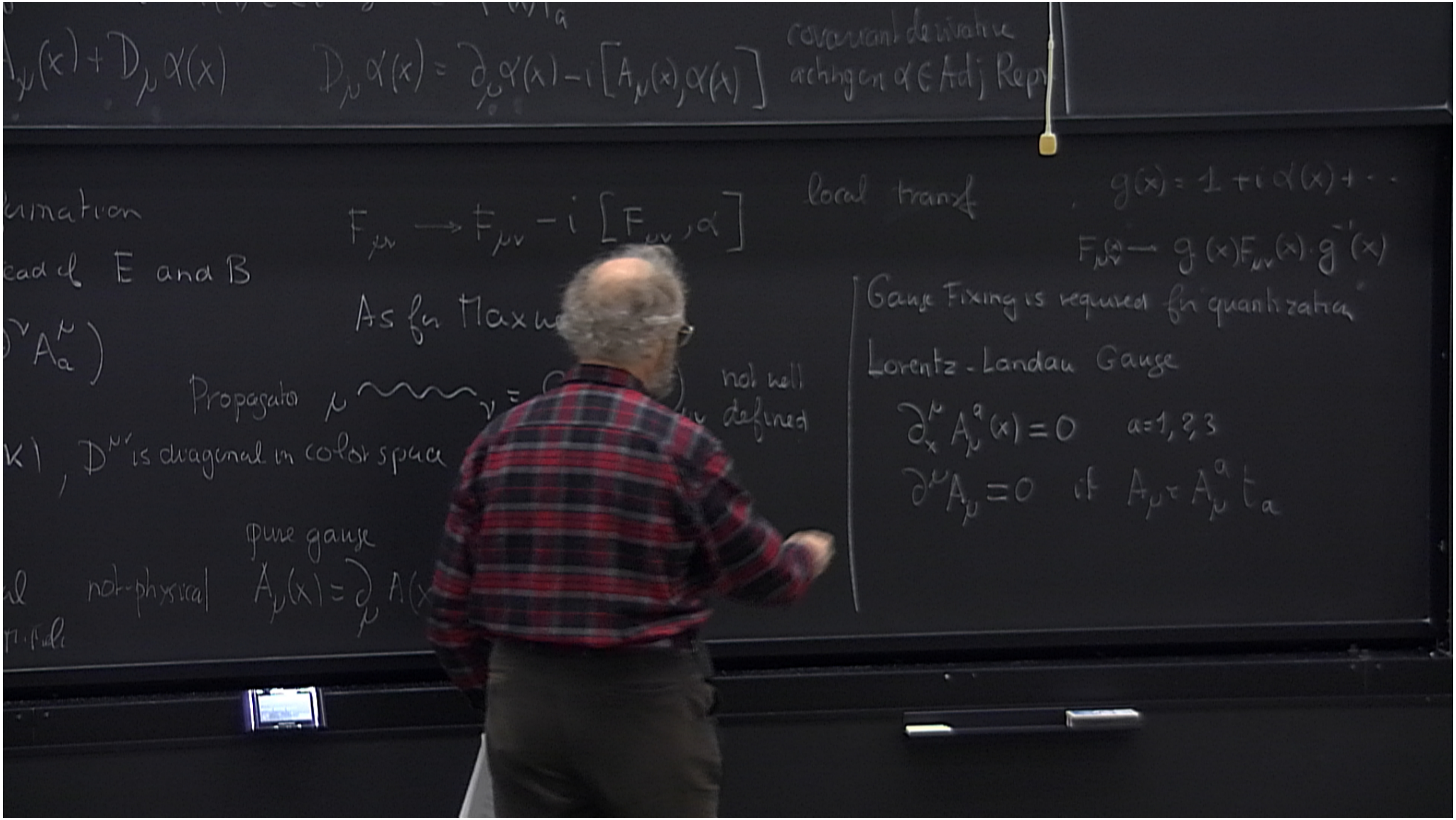
Lorentz - Landau Gauge

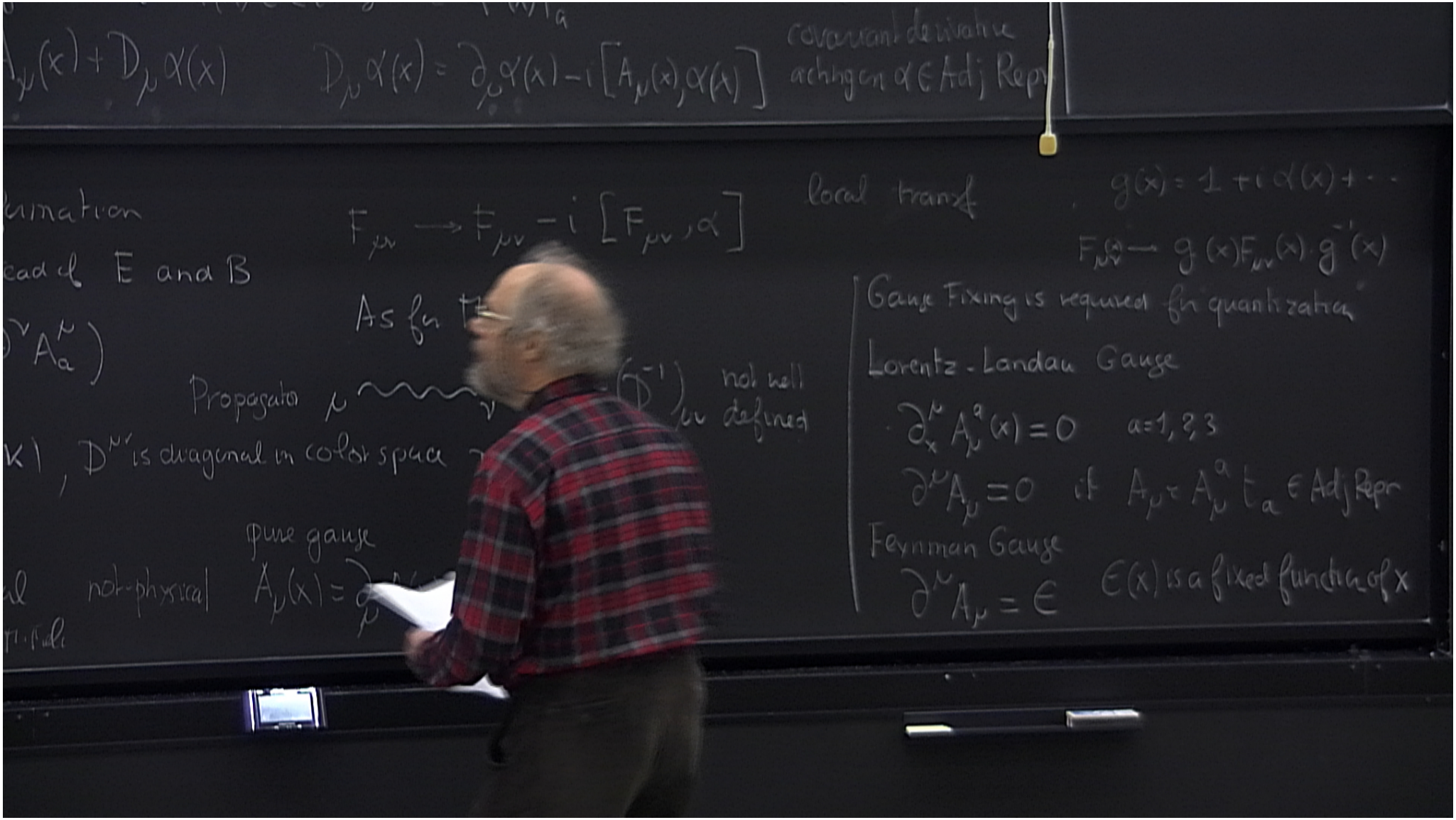
$\partial_x^\mu A_\mu^a(x) = 0 \quad a = 1, 2, 3$

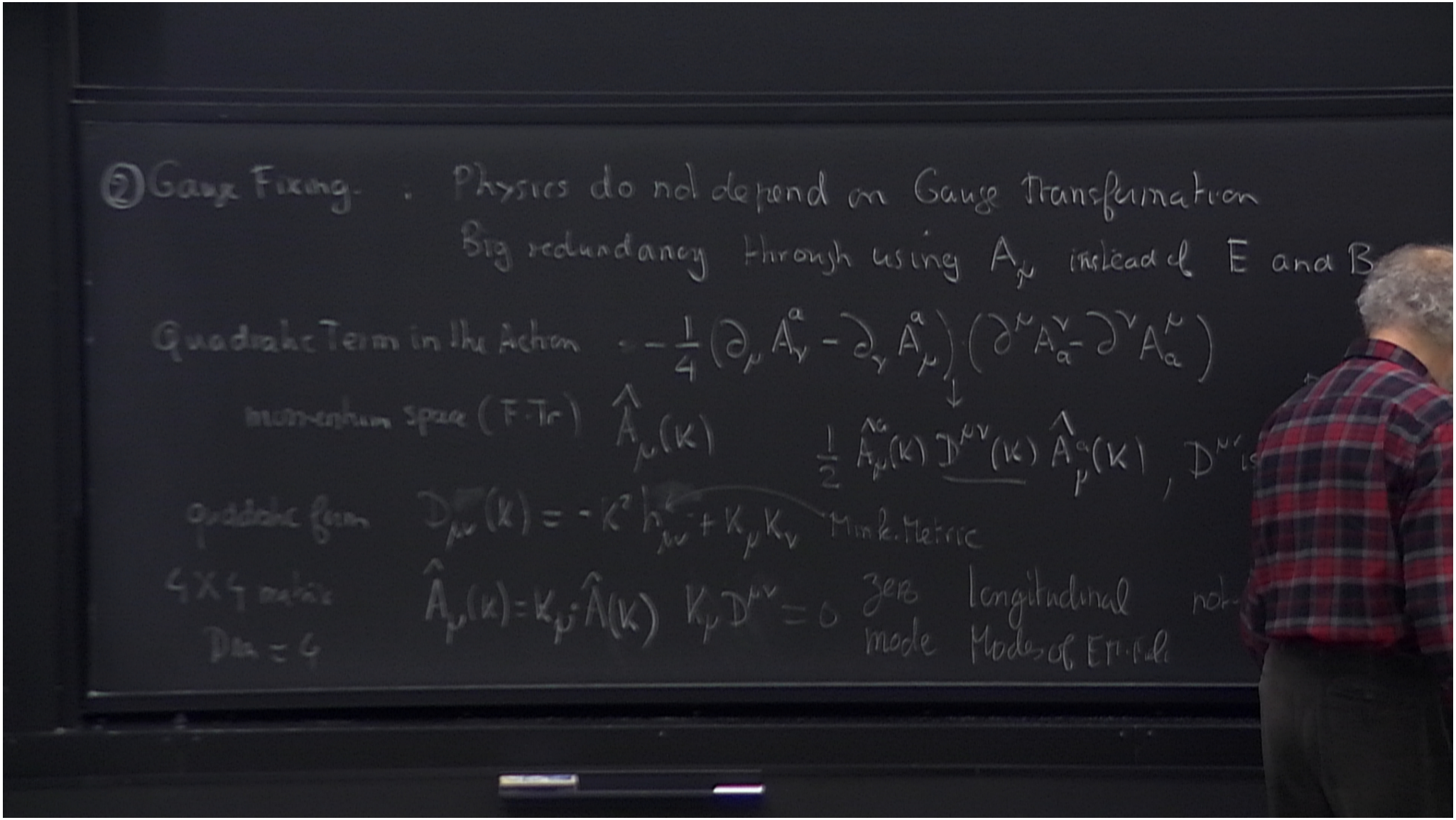
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pure gauge

not-physical $A_\mu(x) = \partial_\mu A(x) \approx A_\mu = 0$


covariant derivative
local transf
Gauge Fixing is required for quantization
Lorentz - Landau Gauge
Propagator
pure gauge
not physical


$D_\nu \alpha(x) = \partial_\nu \alpha(x) - i[A_\nu(x), \alpha(x)]$
covariant derivative
$F_{\mu\nu} \to F_{\mu\nu} - i[F_{\mu\nu}, \alpha]$
local transf
$g(x) = 1 + i\alpha(x) + \cdots$
$F_{\mu\nu} \to g(x) F_{\mu\nu}(x) g^{-1}(x)$
Gauge Fixing is required for quantization
Lorentz - Landau Gauge
$\partial_x^\mu A_\mu^a(x) = 0 \quad a = 1, 2, 3$
$\partial^\mu A_\mu = 0$ if $A_\mu = A_\mu^a t_a \in$ Adj Repr
Feynman Gauge
$\partial^\mu A_\mu = \epsilon$ $\epsilon(x)$ is a fixed function of x
Propagator
not well defined
pure gauge
not-physical


(2) Gauge Fixing. Physics do not depend on Gauge Transformation
Big redundancy through using A_μ instead of E and B
Quadratic Term in the Action $-\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a)$
momentum space (F.Tr) $\hat{A}_\mu(k)$
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Quadratic form $D_{\mu\nu}(k) = -k^2 h_{\mu\nu} + k_\mu k_\nu$ Mink. Metric
4X4 matrix
Det = 0
$\hat{A}_\mu(k) = k_\mu \hat{\Lambda}(k)$ $k_\mu D^{\mu\nu} = 0$ zero mode
longitudinal
Modes of EM. Fiel

4×4 matrix
Dim = 4

$\hat{A}_\mu(k) = k_\mu \cdot \hat{A}(k)$ $k_\mu D^{\mu\nu} = 0$ Zero mode Longitudinal Modes of EM. Fields non-physical $A_\mu(x) = \partial$

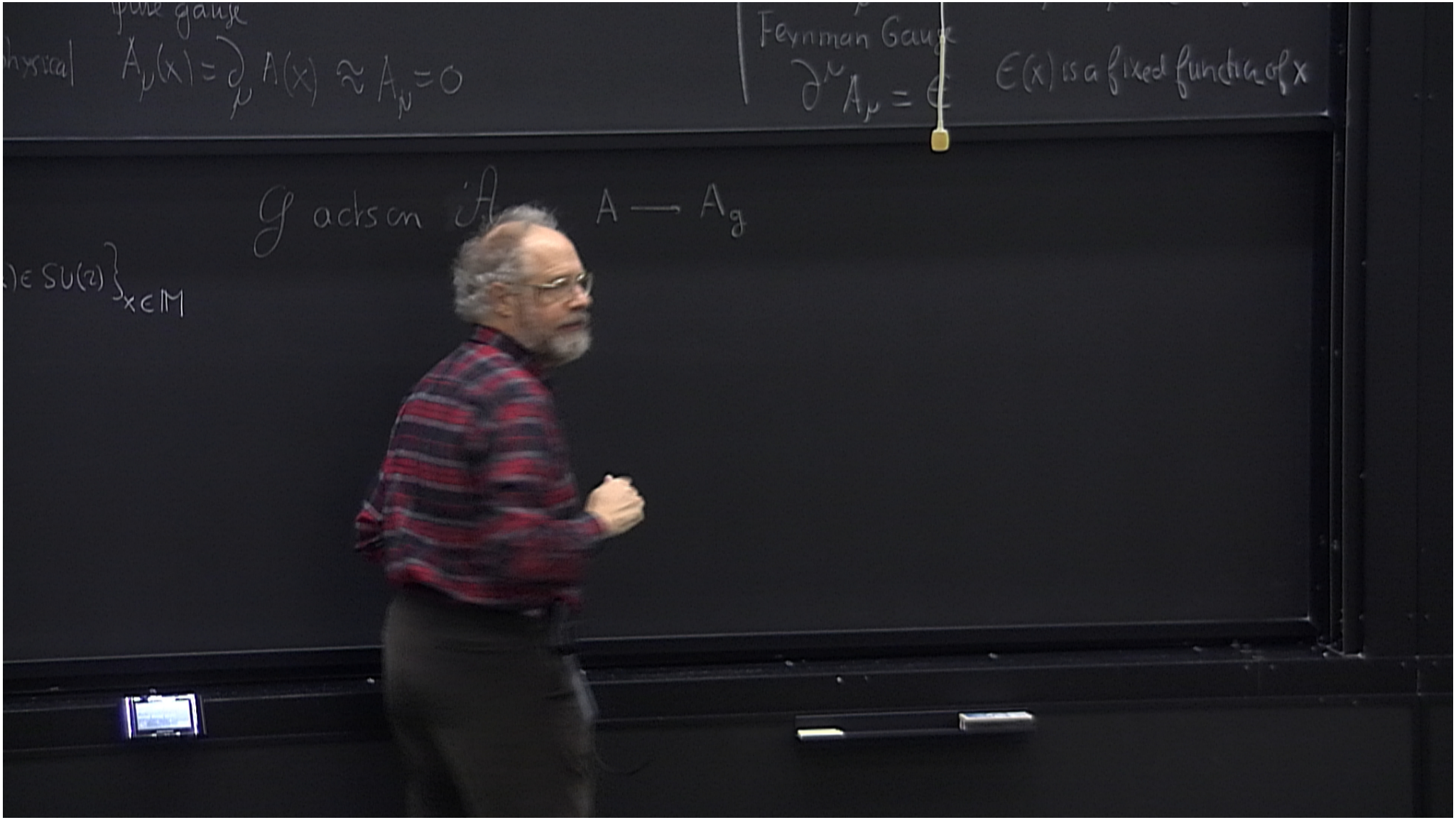
③ Geometrical Picture of Gauge Fixing

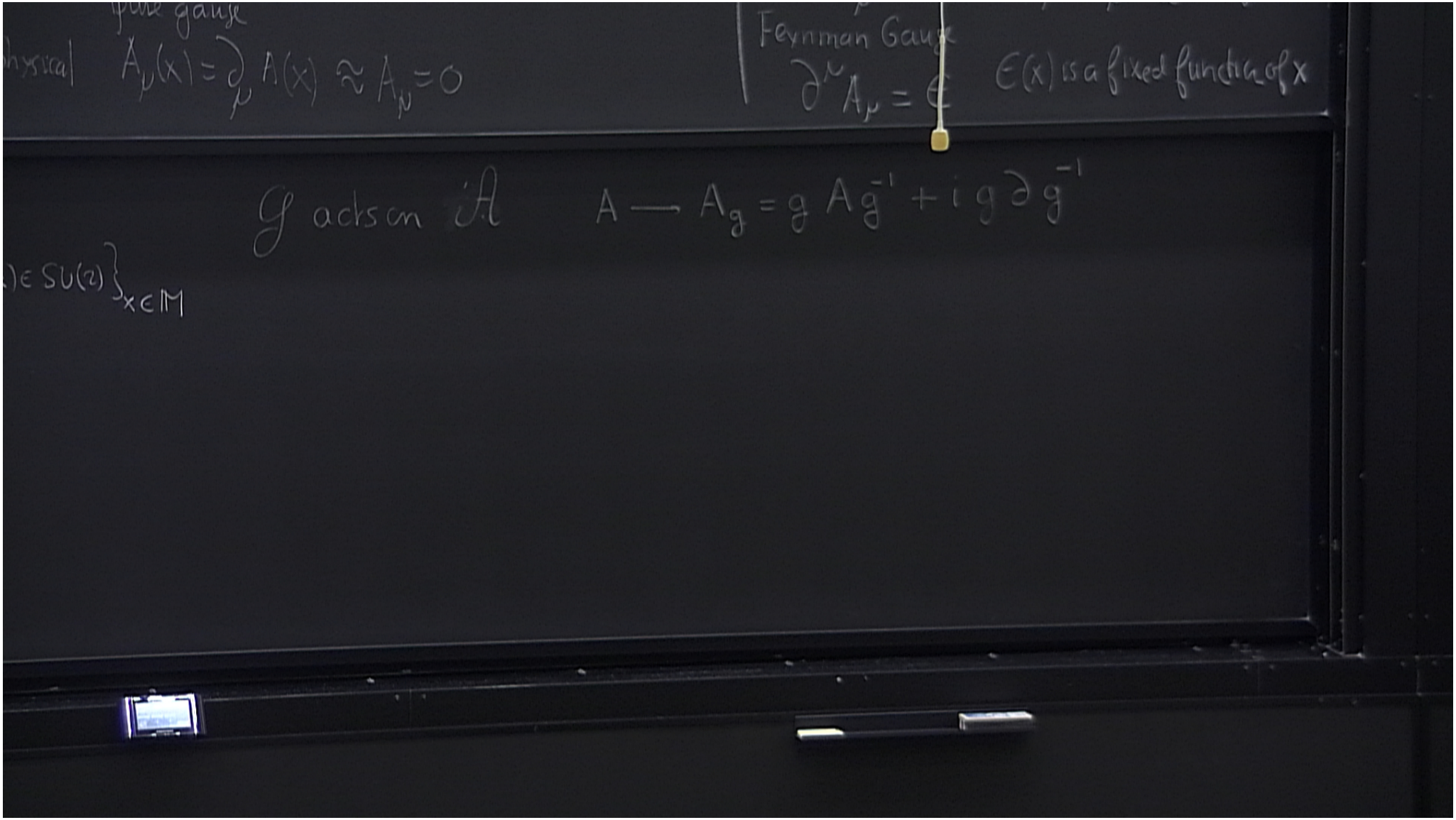


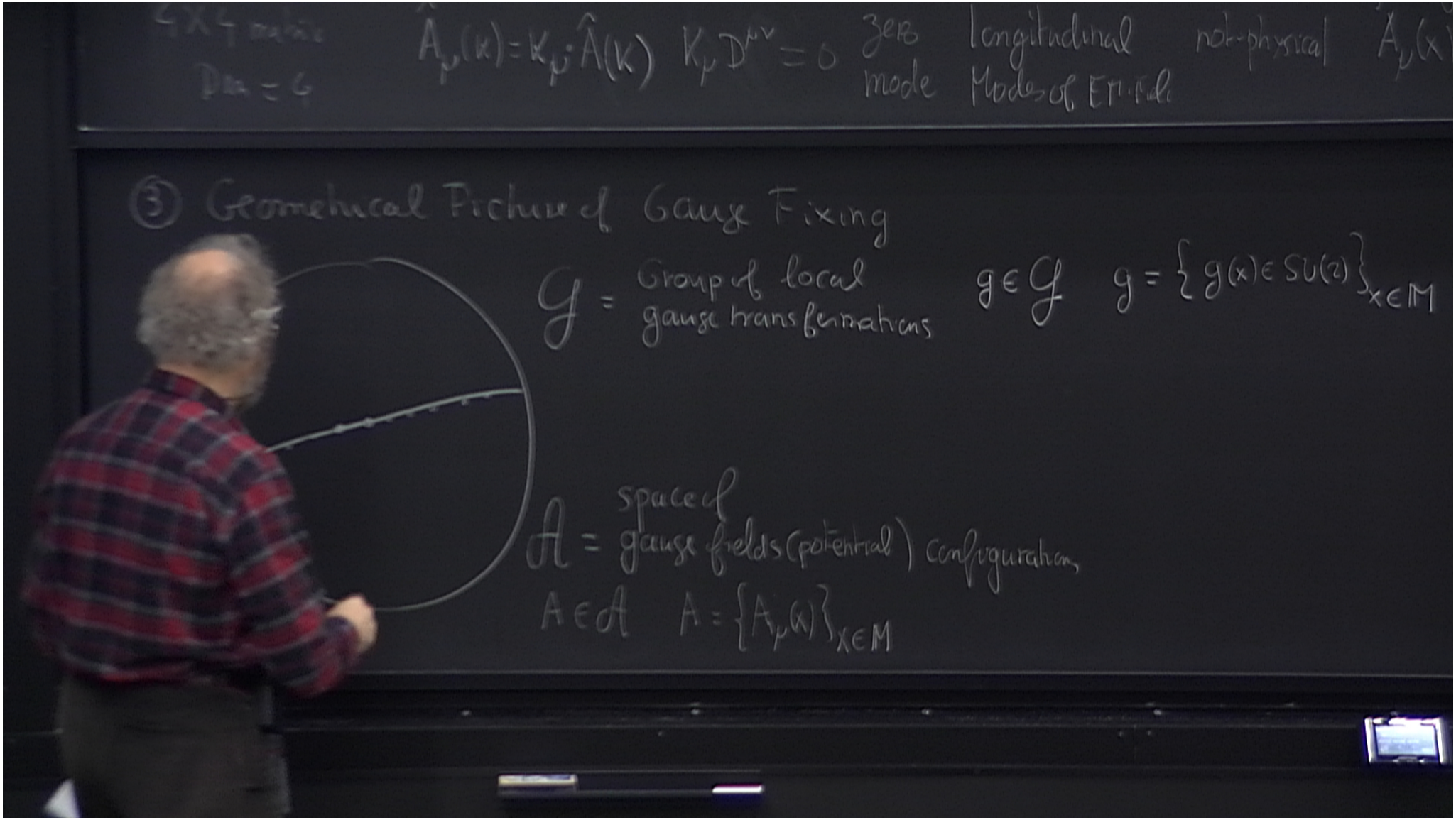
$\mathcal{G}$ = Group of local gauge transformations $g \in \mathcal{G}$ $g = \{g(x) \in SU(2)\}_{x \in M}$

$\mathcal{A}$ = space of gauge fields (potential) configurations

$A \in \mathcal{A}$ $A = \{A_\mu(x)\}_{x \in M}$


Feynman Gauge
$\epsilon(x)$ is a fixed function of x
$\mathcal{G}$ acts on
$A \longrightarrow A_g$
$\in SU(2)\}_{x \in M}$

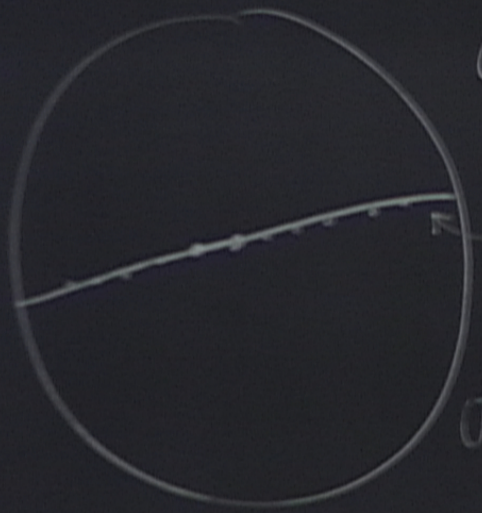

gauge
$A_\mu(x) = \partial_\mu \Lambda(x) \approx A_\mu = 0$
Feynman Gauge
$\partial^\mu A_\mu = \epsilon$
$\epsilon(x)$ is a fixed function of x
$\mathcal{G}$ acts on $\mathcal{A}$ $\quad A \longrightarrow A_g = g A g^{-1} + i g \partial g^{-1}$
$\in SU(2)\}_{x \in M}$


$\hat{A}_\mu(k) = k_\mu \hat{A}(k)$ $k_\mu D^{\mu\nu} = 0$ zero mode longitudinal Modes of Eq. of Mo non-physical
③ Geometrical Picture of Gauge Fixing
$\mathcal{G}$ = Group of local gauge transformations
$g \in \mathcal{G}$ $g = \{g(x) \in SU(2)\}_{x \in M}$
$\mathcal{A}$ = space of gauge fields (potential) configurations
$A \in \mathcal{A}$ $A = \{A_\mu(x)\}_{x \in M}$

$\hat{A}_\mu(k) = k_\mu \hat{A}(k)$ $k_\mu D^{\mu\nu} = 0$ zero mode longitudinal Modes of Eff. Pot. non-physical $A_\mu(x)$

③ Geometrical Picture of Gauge Fixing

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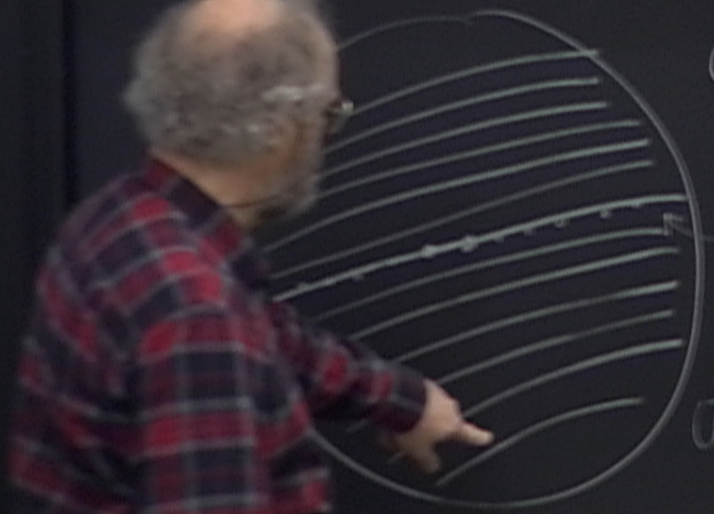


$\mathcal{A}$ = space of gauge fields (potential) configurations

$A \in \mathcal{A}$ $A = \{ A_\mu(x) \}_{x \in M}$

$A_\mu(k) = k_\mu \hat{A}(k)$ $k_\mu D^{\mu\nu} = 0$ zero mode longitudinal Modes of EM field non-physical $A_\mu(x)$

③ Geometrical Picture of Gauge Fixing



$\mathcal{G}$ = Group of local gauge transformations $g \in \mathcal{G}$ $g = \{ g(x) \in SU(2) \}_{x \in M}$

orbit of $\mathcal{G}$

$\mathcal{A}$ = space of gauge fields (potential) configurations

$A \in \mathcal{A}$ $A = \{ A_\mu(x) \}_{x \in M}$

$\mu\nu(k) = -k^2 h_{\mu\nu} + k_\mu k_\nu$ Mink. Metric

4×4 matrix

$Dm = 4$

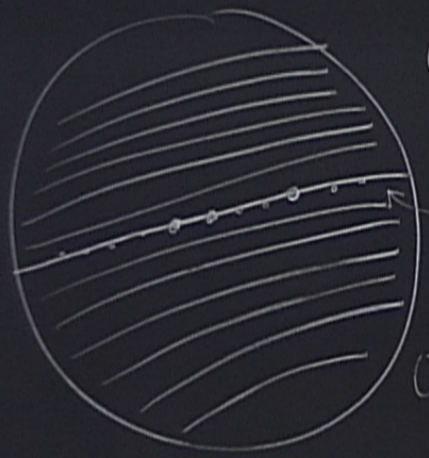
$\hat{A}_\mu(k) = k_\mu \cdot \hat{\Lambda}(k)$ $k_\mu D^{\mu\nu} = 0$ ZERO mode longitudinal Modes of EM. Fields non-physical

pure gauge

$A_\mu(x) = \partial_\mu \Lambda(x) \approx A_\mu = 0$

③ Geometrical Picture of Gauge Fixing

$\mathcal{G}$ acts on



$\mathcal{G}$ = Group of local gauge transformations $g \in \mathcal{G}$ $g = \{g(x) \in SU(2)\}_{x \in M}$

orbit of $\mathcal{G}$ A and A′ ∈ same orbit if there is a $g \in \mathcal{G}$ such that $A' = A_g$

$\mathcal{A}$ = space of gauge fields (potential) configurations

$A \in \mathcal{A}$ $A = \{A_\mu(x)\}_{x \in M}$

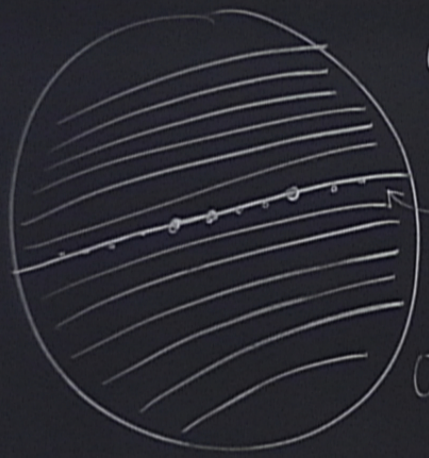
4×4 matrix

$D_{mn} = G$

$\hat{A}_\mu(k) = k_\mu \hat{A}(k)$ $k_\mu D^{\mu\nu} = 0$ zero mode longitudinal Modes of E.M. field not-physical

pure gauge $A_\mu(x) = \partial_\mu A(x) \approx A_\mu = 0$

③ Geometrical Picture of Gauge Fixing



$\mathcal{G}$ = Group of local gauge transformations $g \in \mathcal{G}$ $g = \{g(x) \in SU(2)\}_{x \in M}$

$\mathcal{G}$ acts on

orbit of $\mathcal{G}$

A and A′ ∈ same orbit if there is a $g \in \mathcal{G}$ such that $A' = A_g$

space of all A's equivalent by gauge transformations

$\mathcal{A}$ = space of gauge fields (potential) configurations

$A \in \mathcal{A}$ $A = \{A_\mu(x)\}_{x \in M}$


Space X, group G acts on X
G ↪ X

4×4 matrix

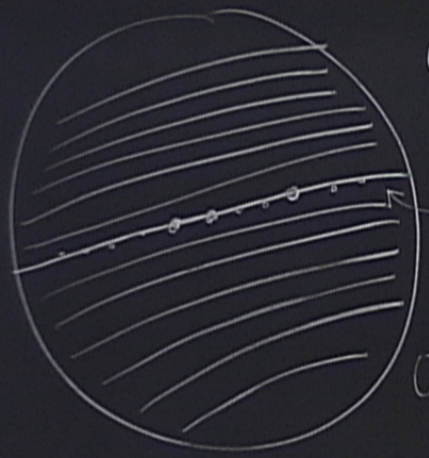
$D_{mn} = 4$

$\hat{A}_\mu(k) = k_\mu \hat{\Lambda}(k)$ $\quad k_\mu D^{\mu\nu} = 0$ zero mode longitudinal Modes of EM. field not-physical

pure gauge $\quad A_\mu(x) = \partial_\mu \Lambda(x) \approx A_\mu = 0$

③ Geometrical Picture of Gauge Fixing

$\mathcal{G}$ acts on



$\mathcal{G}$ = Group of local gauge transformations $\quad g \in \mathcal{G} \quad g = \{ g(x) \in SU(2) \}_{x \in M}$

↑ gauge group

orbit of $\mathcal{G}$ — A and A′ ∈ same orbit if there is a $g \in \mathcal{G}$ such that $A' = A_g$

space of all A's equivalent by gauge transformations

$\mathcal{A}$ = space of gauge fields (potential) configurations

$A \in \mathcal{A} \quad A = \{ A_\mu(x) \}_{x \in M}$

4×4 matrix
$D_{\mu\nu} = G$

$\hat{A}_\mu(k) = k_\mu \hat{\Lambda}(k)$ $k_\mu D^{\mu\nu} = 0$ zero mode longitudinal Modes of EM. Fields not-physical

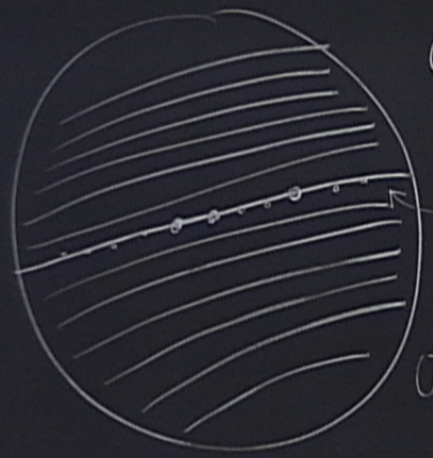
pure gauge
$A_\mu(x) = \partial_\mu \Lambda(x) \approx A_\mu = 0$

③ Geometrical Picture of Gauge Fixing

"smooth functions"

$\mathcal{G}$ = Group of local gauge transformations $g \in \mathcal{G}$ $g = \{ g(x) \in SU(2) \}_{x \in M}$ ← gauge group

$\mathcal{G}$ acts on



orbit of $\mathcal{G}$

A and A′ ∈ same orbit if there is a $g \in \mathcal{G}$ such that $A' = A_g$

space of all A's equivalent by gauge transformation

$\mathcal{A}$ = space of gauge fields (potential) configurations

$A \in \mathcal{A}$ $A = \{A_\mu(x)\}_{x \in M}$

4×4 matrix

$D_{mn} = 4$

$\hat{A}_\mu(k) = k_\mu \hat{\Lambda}(k)$ $\quad k_\mu D^{\mu\nu} = 0$ zero mode longitudinal Modes of E.M. field not-physical

Mink. Metric

pure gauge $\quad A_\mu(x) = \partial_\mu \Lambda(x) \approx A_\mu = 0$

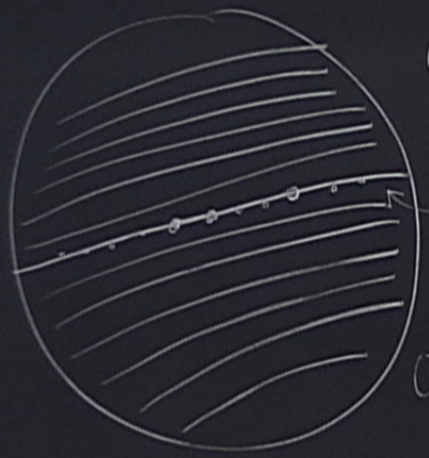
③ Geometrical Picture of Gauge Fixing

$\mathcal{G}$ = Group of local gauge transformations $\quad g \in \mathcal{G} \quad g = \{ g(x) \in SU(2) \}_{x \in M}$

"smooth functions"

gauge group

$\mathcal{G}$ acts on



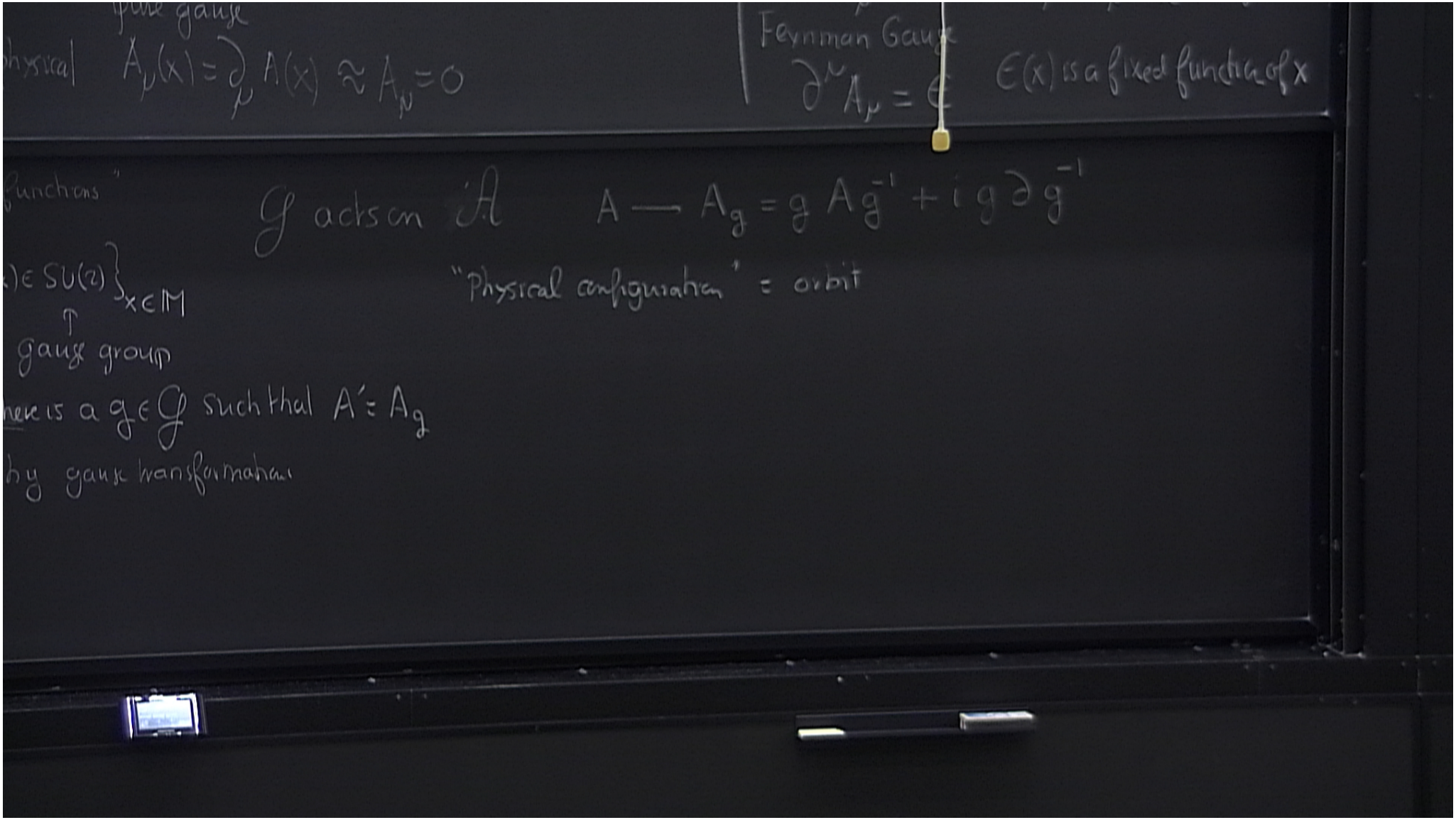
orbit of $\mathcal{G}$

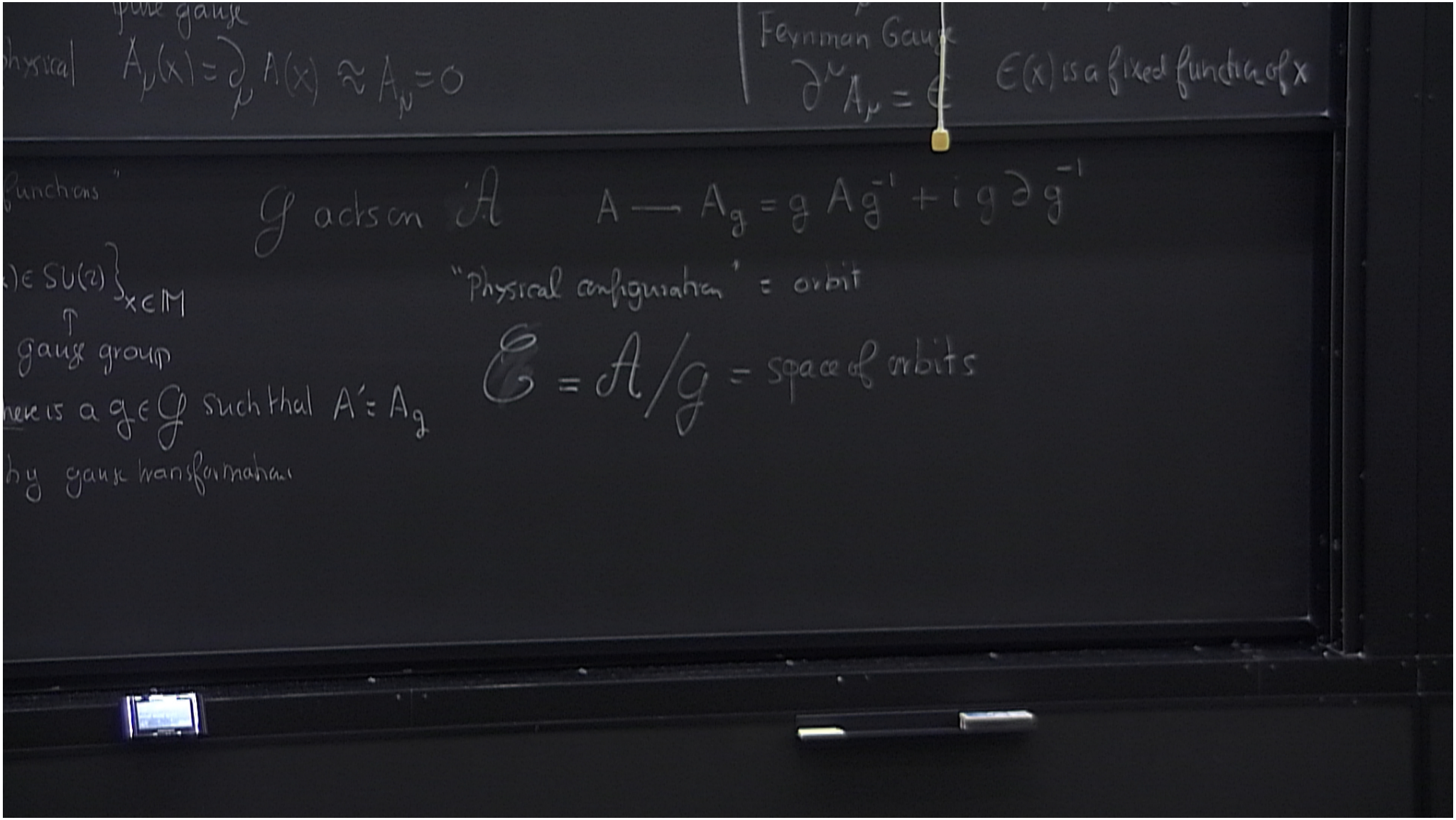
A and A' ∈ same orbit if there is a $g \in \mathcal{G}$ such that $A' = A_g$

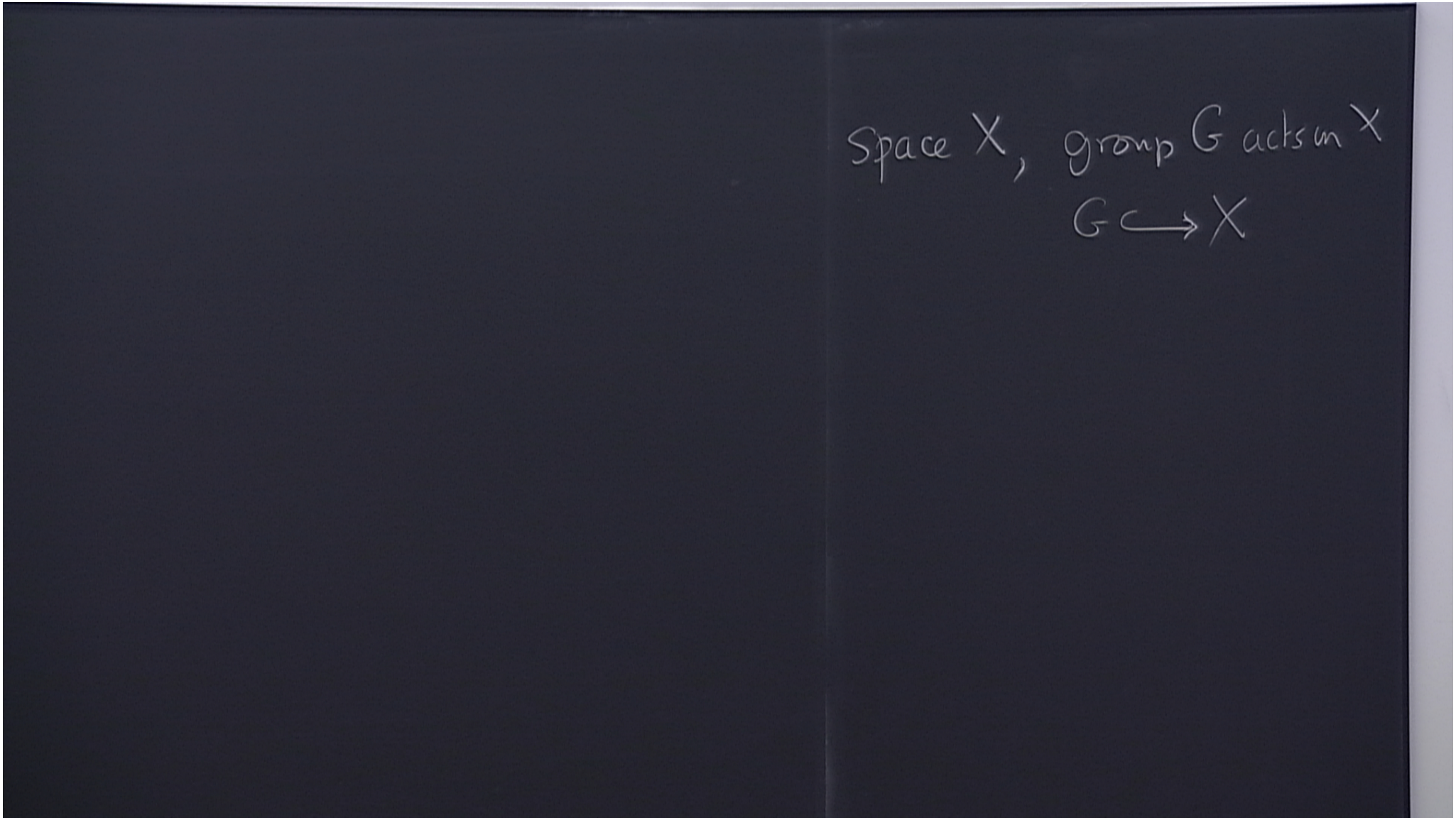
space of all A's equivalent by gauge transformations

$\mathcal{A}$ = space of gauge fields (potential) configurations

$A \in \mathcal{A} \quad A = \{ A_\mu(x) \}_{x \in M}$


gauge
physical $A_\mu(x) = \partial_\mu A(x) \approx A_\mu = 0$
Feynman Gauge
$\partial^\mu A_\mu = \epsilon$
$\epsilon(x)$ is a fixed function of x
$\mathcal{G}$ acts on $\mathcal{A}$ $A \longrightarrow A_g = g A g^{-1} + i g \partial g^{-1}$
"Physical configuration" = orbit
$\in SU(2)\}_{x \in M}$
gauge group
there is a $g \in \mathcal{G}$ such that $A' = A_g$
by gauge transformation


gauge
physical $A_\mu(x) = \partial_\mu \Lambda(x) \approx A_\mu = 0$
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gauge group
"Physical configuration" = orbit
there is a $g \in \mathcal{G}$ such that $A' = A_g$
$\mathcal{C} = \mathcal{A}/\mathcal{G}$ = space of orbits
by gauge transformation


Space X, group G acts on X
$G \hookrightarrow X$

pure gauge

physical $A_\mu(x) = \partial_\mu \Lambda(x) \approx A_\mu = 0$

Feynman Gauge

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functions"

$\mathcal{G}$ acts on $\mathcal{A}$ $\quad A \longrightarrow A_g = g A g^{-1} + i g \partial g^{-1}$

$\{ \in SU(2) \}_{x \in M}$

↑

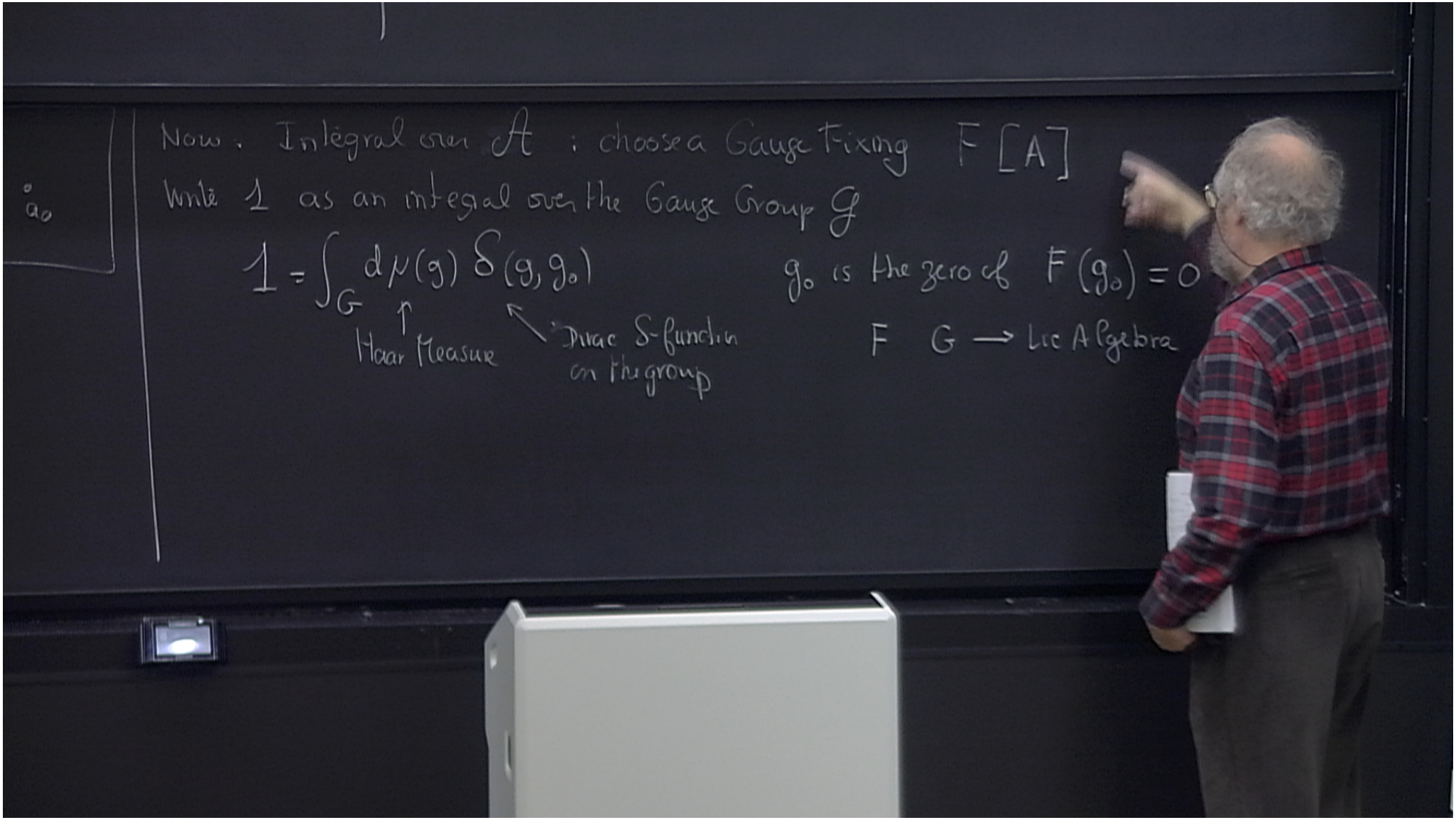
gauge group

"Physical configuration" = orbit

there is a $g \in \mathcal{G}$ such that $A' = A_g$

by gauge transformation

$\mathcal{C} = \mathcal{A}/\mathcal{G}$ = space of orbits = space of physically inequivalent configurations


Now. Integral over $\mathcal{A}$: choose a Gauge Fixing $F[A]$
Write 1 as an integral over the Gauge Group $\mathcal{G}$
$1 = \int_G d\mu(g)\, \delta(g, g_0)$
g_0 is the zero of $F(g_0) = 0$
Haar Measure
Dirac δ-function on the group
F G → Lie Algebra

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$F : G \longrightarrow$ Lie Algebra $\mathcal{G}$

$F(g) = F^a(g) t_a \quad a = 1, \dots \dim G = 3$

...bian Matrix of F

$(1 + i\alpha)$

...e Algebra $= \alpha^a t_a$

$$\left.\frac{\partial F^a(g(1+i\alpha))}{\partial \alpha^b}\right|_{\alpha=0} = \partial_b F^a(g) = F'(g)$$

matrix element of a $n \times m$ matrix $\quad n = \dim G = 3$

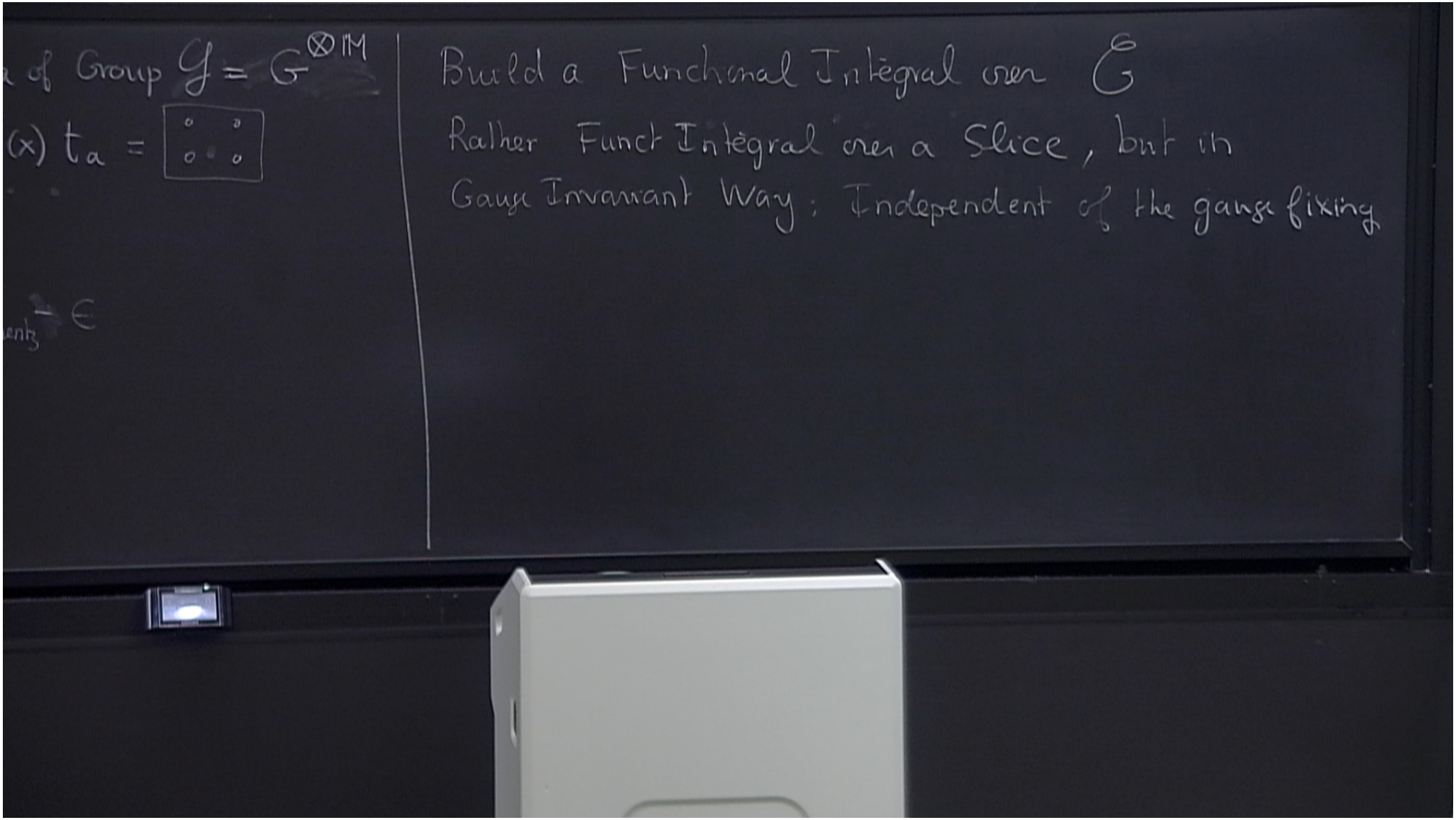
④ Gauge Fixing = choose a Function $F[A] \longrightarrow$ Lie Algebra of Group $\mathcal{G} = G^{\otimes \mathbb{M}}$

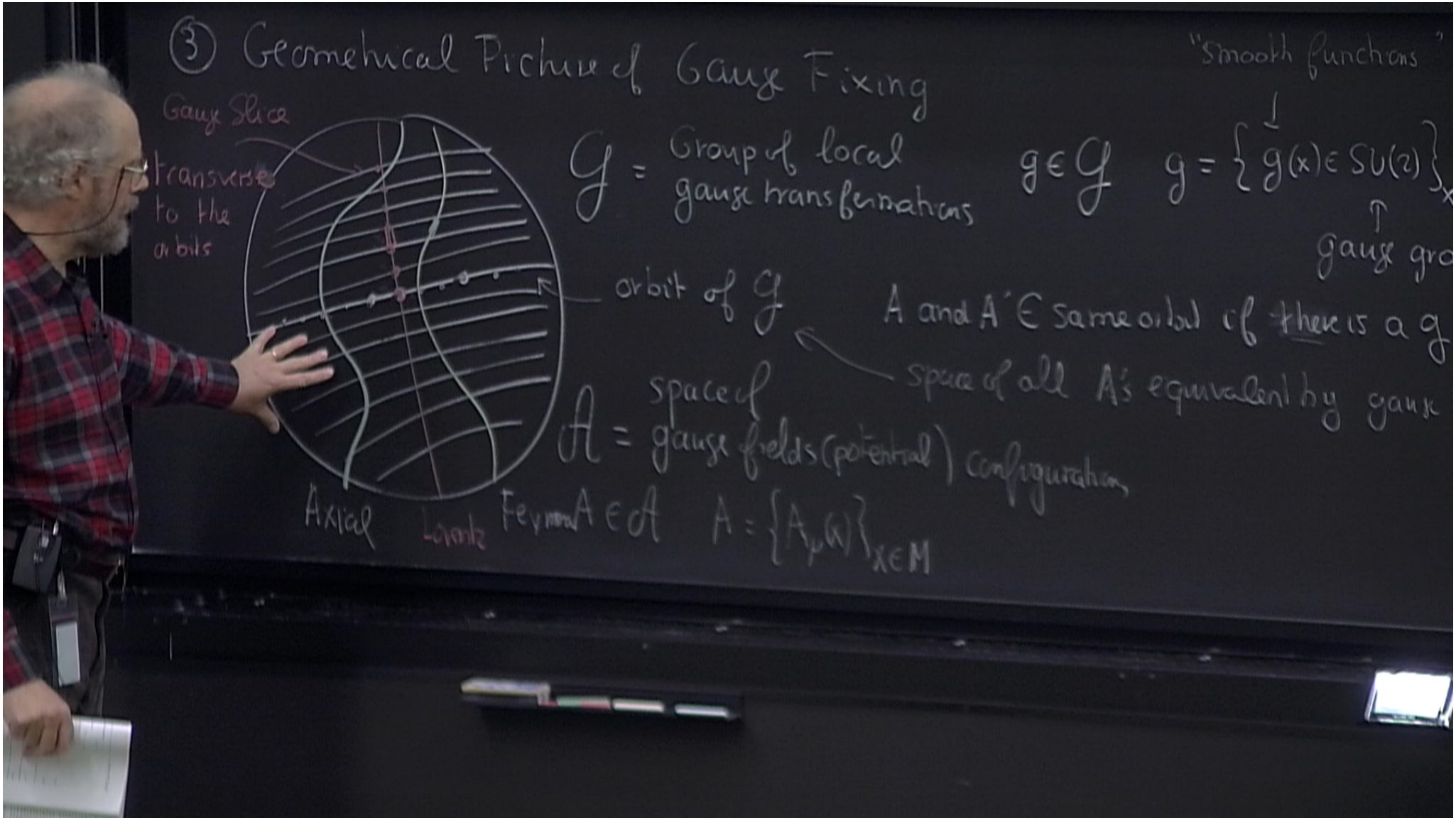
Example ; Lorentz Gauge $F_{\text{Lorentz}}[A](x) = \partial^\nu A_\nu(x) = \partial^\nu A^a_\mu(x)\, t_a =$ [ ]

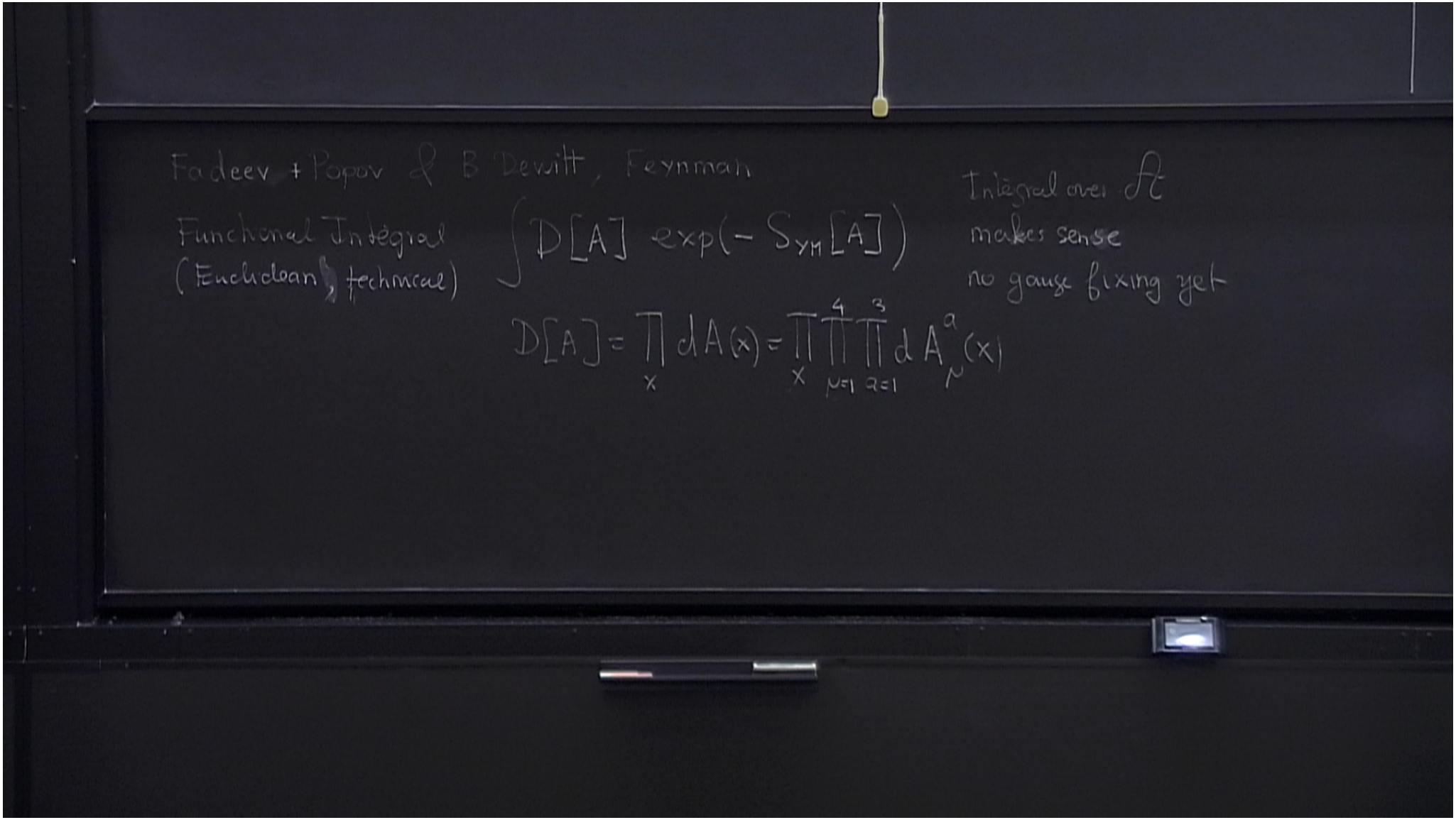
In general $F[A] = f(A, \partial_\nu A, \partial_\mu \partial_\nu A \dots)$

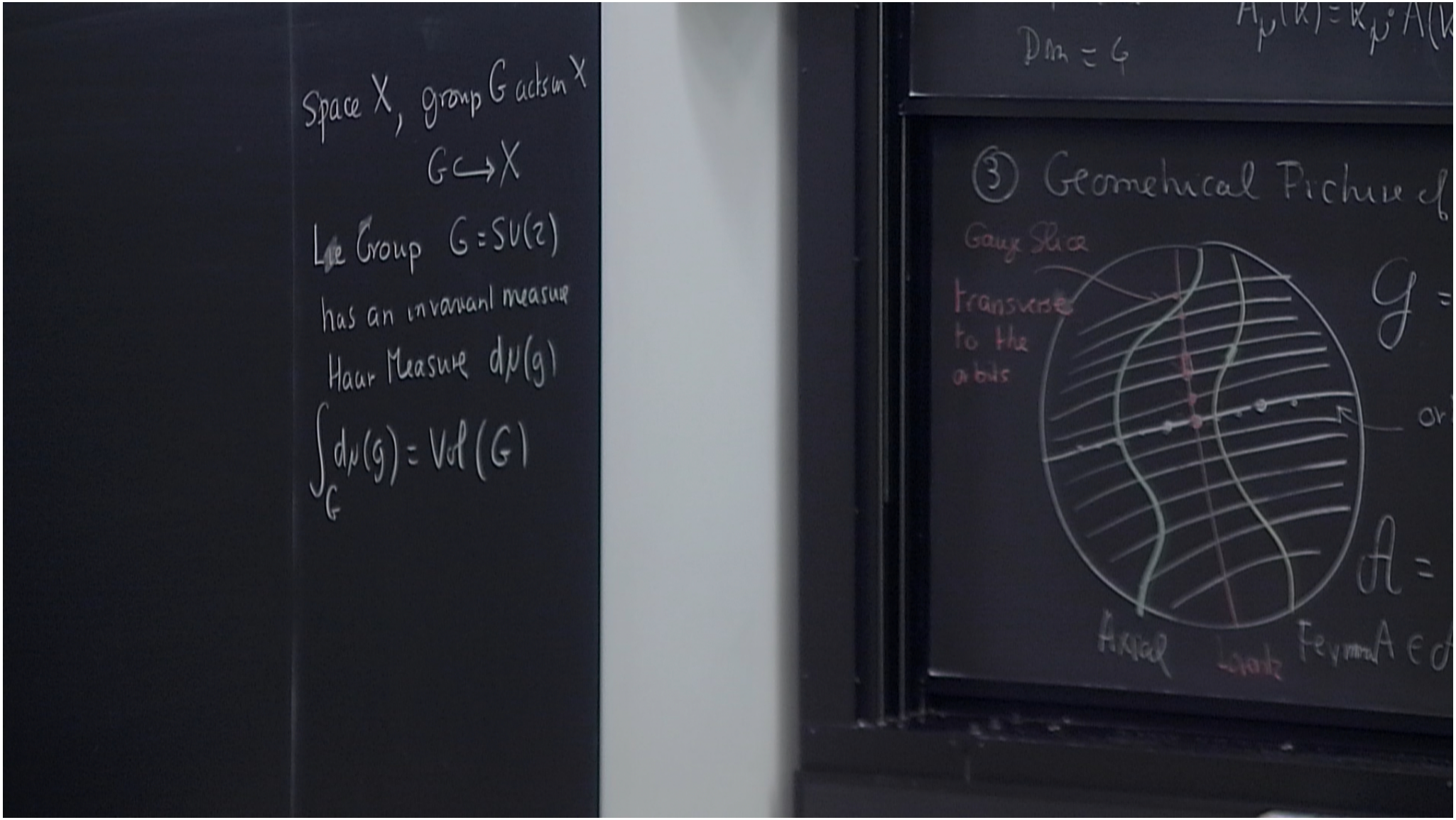
Feynman Gauge $F_{\text{Feynman}}[A](x) = \left(\partial^\nu A_\nu - \epsilon\right)(x) = F_{\text{Lorentz}} - \epsilon$

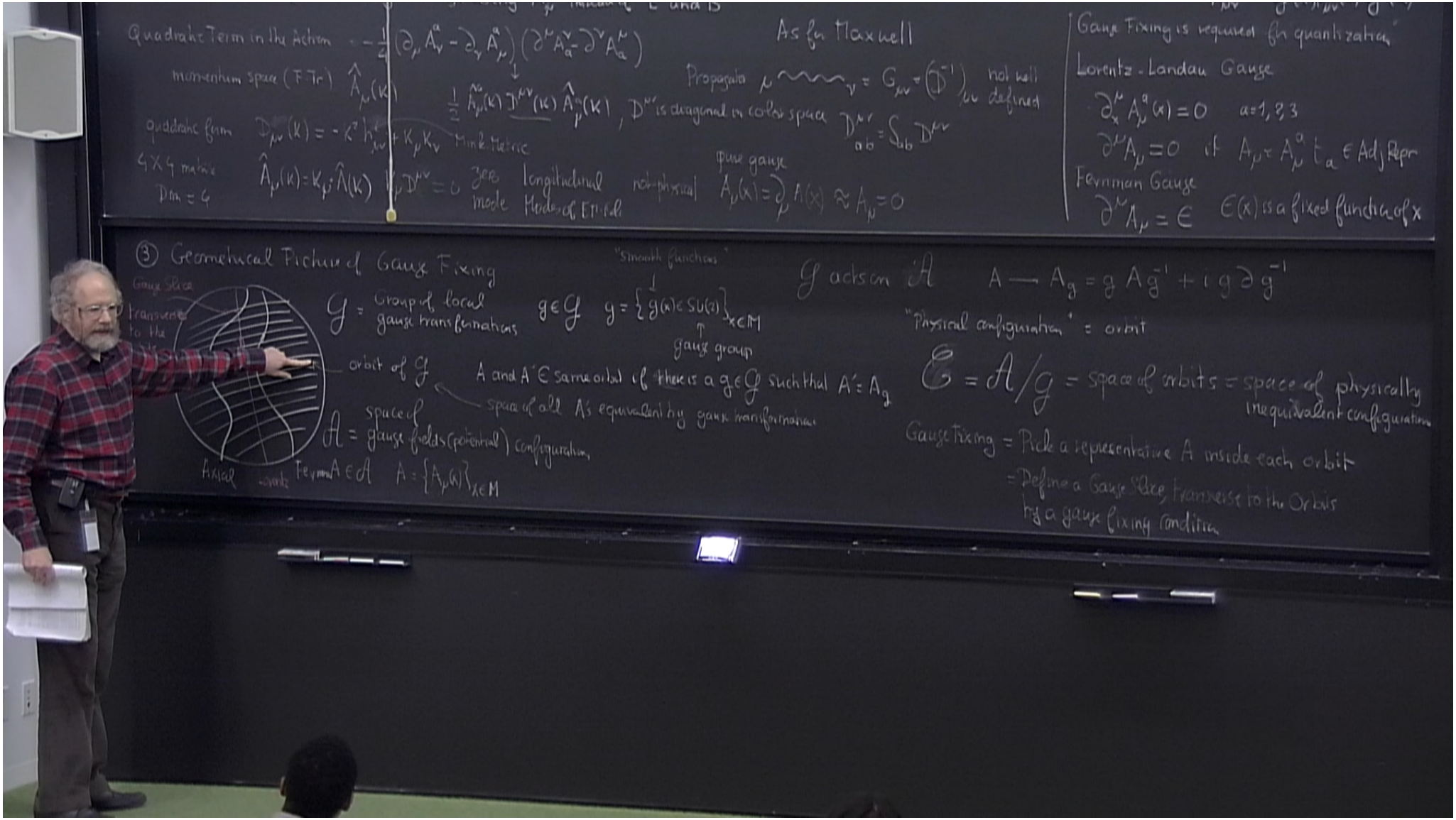
Different Gauge fixing $\Longleftrightarrow$ Different $F[A] \Longleftrightarrow$ Different slices

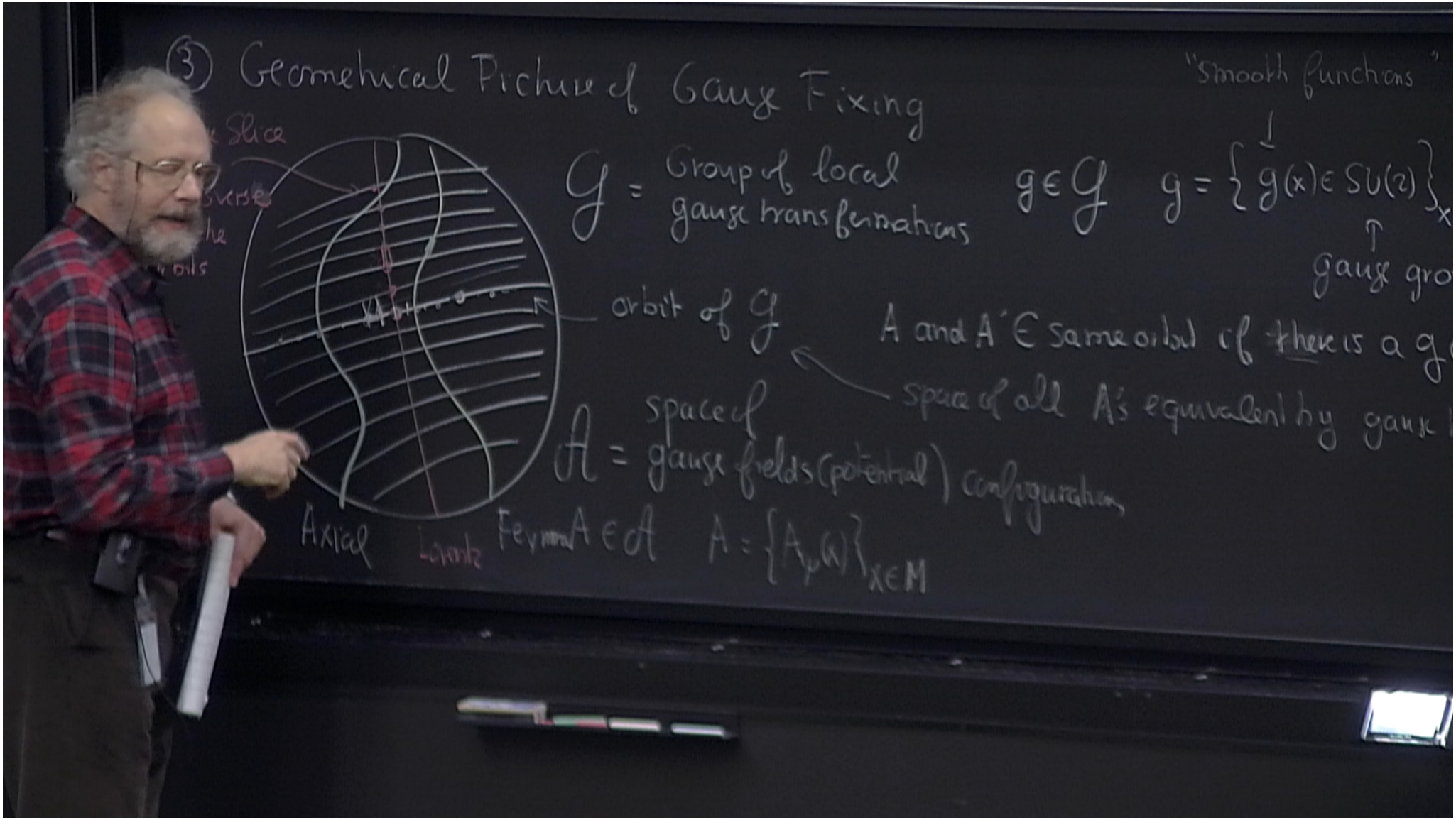

of Group $\mathcal{G} = G^{\otimes IM}$
(x) $t_a =$
Build a Functional Integral over $\mathcal{G}$
Rather Funct Integral over a Slice, but in
Gauge Invariant Way; Independent of the gauge fixing


③ Geometrical Picture of Gauge Fixing
"Smooth functions"
Gauge Slice
transverse to the orbits
$\mathcal{G}$ = Group of local gauge transformations
$g \in \mathcal{G}$ $g = \{ g(x) \in SU(2) \}_x$
gauge gro
orbit of $\mathcal{G}$
A and A' ∈ same orbit if there is a g
space of all A's equivalent by gauge
$\mathcal{A}$ = space of gauge fields (potential) configurations
Axial
Lorentz
Feynman
$A \in \mathcal{A}$ $A = \{ A_\mu(x) \}_{x \in M}$


Fadeev + Popov & B Dewitt, Feynman
Functional Integral
(Euclidean, technical)
$\int D[A]\, \exp(-S_{YM}[A])$
Integral over $\mathcal{A}$
makes sense
no gauge fixing yet
$D[A] = \prod_x dA(x) = \prod_x \prod_{\mu=1}^{4} \prod_{a=1}^{3} d A^a_\mu(x)$


Space X, group G acts on X
G ↪ X
Lie Group G = SU(2)
has an invariant measure
Haar Measure dμ(g)
∫_G dμ(g) = Vol(G)
Dm = 4
③ Geometrical Picture
Gauge Slice
transverse to the orbits
Axial
Feynman


Quadratic Term in the Action
momentum space (F.Tr.)
quadratic form
4 × 4 matrix
Dim = 4
As for Maxwell
Propagator
not well defined
Minkowski Metric
pure gauge
Gauge Fixing is required for quantization
Lorentz - Landau Gauge
Feynman Gauge
③ Geometrical Picture of Gauge Fixing
Gauge Slice
transverse to the orbits
Group of local gauge transformations
"Smooth functions"
gauge group
orbit of G
space of all A's equivalent by gauge transformations
space of gauge fields (potential) configurations
Axial
Feynman
G acts on A
"Physical configuration" = orbit
= space of orbits = space of physically inequivalent configurations
Gauge Fixing = Pick a representative A inside each orbit
= Define a Gauge Slice transverse to the Orbits by a gauge fixing condition


(3) Geometrical Picture of Gauge Fixing
"smooth functions"
Slice
$\mathcal{G}$ = Group of local gauge transformations
$g \in \mathcal{G}$ $g = \{ g(x) \in SU(2) \}_x$
gauge gro
orbit of $\mathcal{G}$
A and A' ∈ same orbit if there is a g
space of all A's equivalent by gauge
$\mathcal{A}$ = space of gauge fields (potential) configurations
Axial
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Feynman
$A \in \mathcal{A}$ $A = \{ A_\mu(x) \}_{x \in M}$

Fadeev + Popov & B Dewitt, Feynman

Functional Integral (Euclidean, technical)

$$Z = \int_{\mathcal{A}} \mathcal{D}[A] \exp(-S_{YM}[A])$$

Integral over $\mathcal{A}$ makes sense

no gauge fixing yet

$$\mathcal{D}[A] = \mathcal{D}[A_g] \qquad g \in \mathcal{G}$$

$$\mathcal{D}[A] = \prod_x dA(x) = \prod_x \prod_{\mu=1}^{4} \prod_{a=1}^{3} dA^a_\mu(x)$$

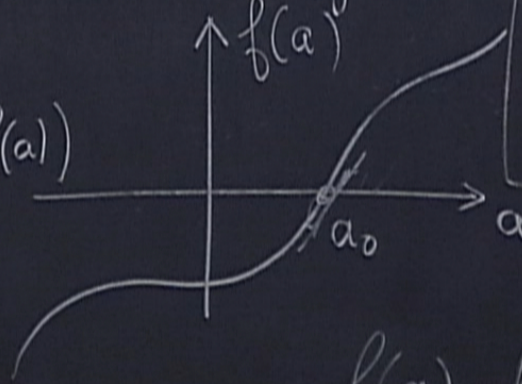
measure is invariant under gauge transformation

Can we rewrite $Z = \int_{\text{Slice}} \cdots$

Trick 1 How to write 1 as an integral

$$1 = \int_{-\infty}^{+\infty} da\ \delta(a - a_0) = \int_{-\infty}^{+\infty} da\ |f'(a)|\,\delta(f(a))$$

$$1 = \int da\ |f'(a)|\,\delta(f(a))$$



$f(a) = 0$ has only 1 solution

$f(a_0) = 0$

$f(a) = f'(a_0)(a - a_0)$

$\delta(x \cdot c) = \frac{1}{c}\delta(x)$

$\int_{\text{Slice}}$

$\vec{a} = (a_1 \dots a_N)$ $\quad \vec{f}(\vec{a}) = (f_1 \dots f_N)$ such that $\vec{f}(\vec{a}) = \vec{0}$ has only 1 solution $\vec{a}_0$



$$1 = \int_{\mathbb{R}^N} d^N\vec{a}\ \delta(\vec{a} - \vec{a}_0) = \int d^N\vec{a}\ \left|\frac{\partial f_i}{\partial a_j}\right| \delta(\vec{f}(\vec{a}))$$

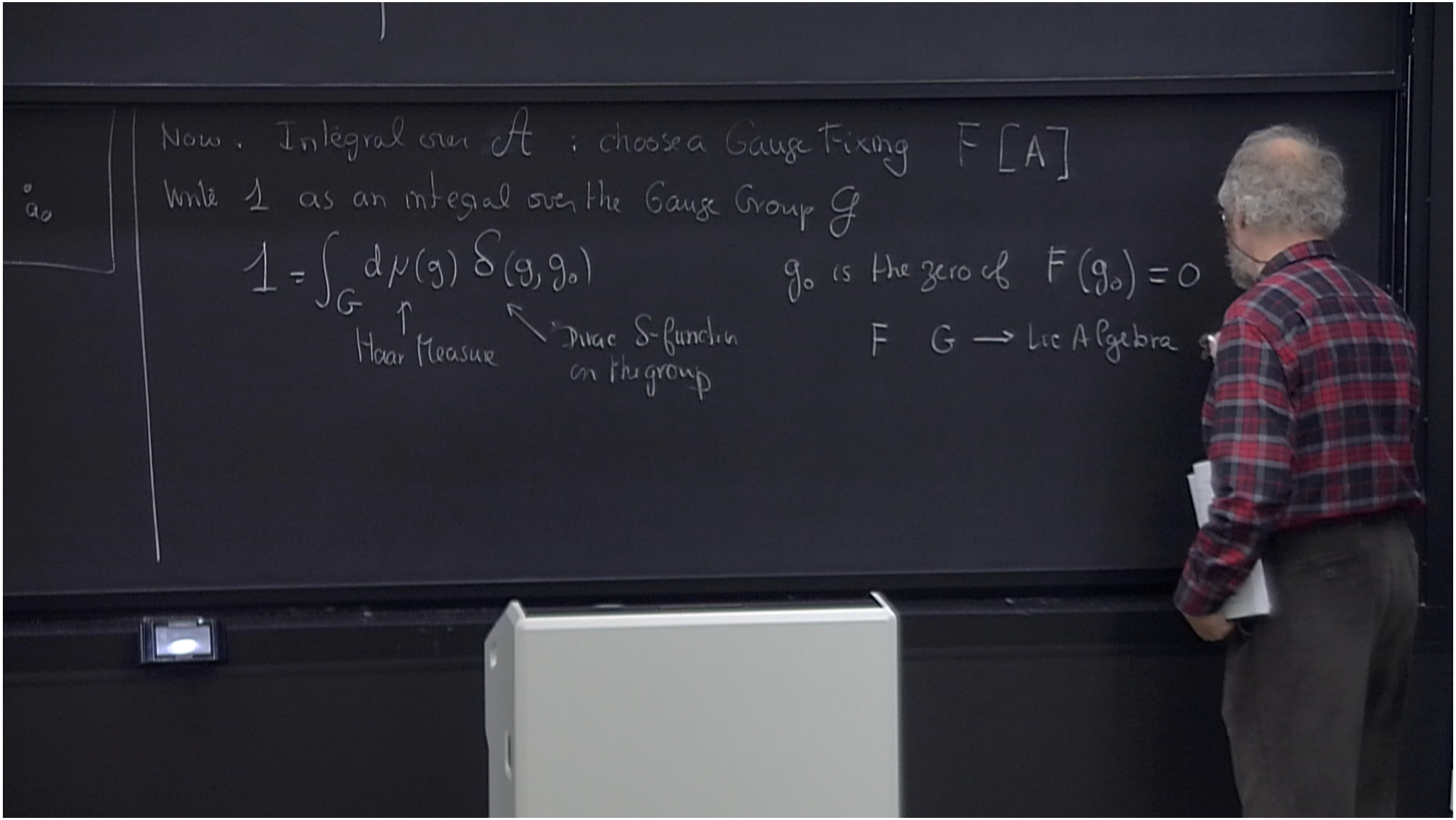
det of the Jacobian

$$\prod_{i=1}^{N} \delta(f_i(a_1 \dots a_N))$$

$$\left|\det\left(F'(a)\right)\right| = \left|\det\left(\frac{\partial f_i}{\partial a_j}(\vec{a})\right)\right|$$

a vector

F vector


Now. Integral over $\mathcal{A}$: choose a Gauge Fixing $F[A]$
Write 1 as an integral over the Gauge Group $\mathcal{G}$
$1 = \int_G d\mu(g)\, \delta(g, g_0)$
Haar Measure
Dirac δ-function on the group
g_0 is the zero of $F(g_0) = 0$
F G → Lie Algebra

Now. Integral over A : choose a Gauge Fixing $F[A]$

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$F : G \longrightarrow$ Lie Algebra $\mathfrak{g}$

$F(g) = F^a(g) t_a \quad a = 1, \dots \dim G = 3$

Jacobian Matrix of F

$g \to g(1 + i\alpha)$

④ Gauge Fixing = choose a Function $F[A] \longrightarrow$ Lie Algebra of Group $\mathcal{G}$

Example : Lorentz Gauge $F_{\text{Lorentz}}[A](x) = \partial^\mu A_\mu(x) = \partial^\mu A^a(x)\, t_a = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

In general $F[A] = f(A, \partial_\mu A, \partial_\mu \partial_\nu A \dots)$

Feynman Gauge $F_{\text{Feynman}}[A](x) = \left(\partial^\mu A_\mu - \epsilon\right)(x) = F_{\text{Lorentz}} = \epsilon$

Different Gauge fixing $\Longleftrightarrow$ Different $F[A] \Longleftrightarrow$ Different slices

Fadeev + Popov & B Dewitt, Feynman

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matrix element of a $n \times m$ matrix $n = \dim G = 3$

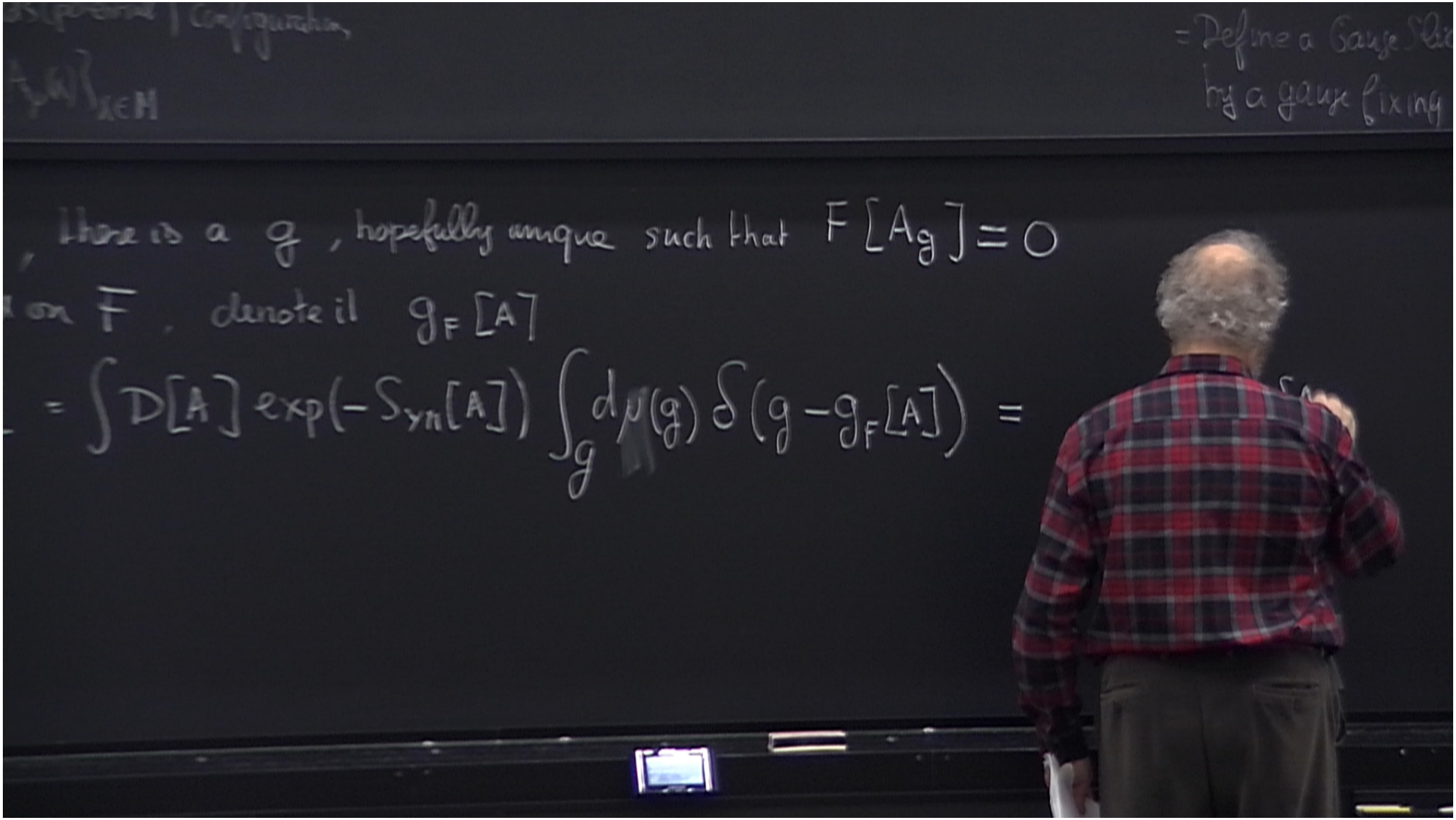
$= F'(g)$

$\mathcal{A}$ = gauge fields (potential) configurations

Feynman $A \in \mathcal{A}$ $A = \{A_\mu(x)\}_{x \in M}$

(7) For each $A \in \mathcal{A}$ in Z, there is a g, hopefully unique such th

thus g depends on A and on F. denote it $g_F[A]$


= Define a Gauge
by a gauge fixing
there is a g, hopefully unique such that F[A_g] = 0
on F, denote il g_F[A]
= ∫ D[A] exp(−S_ym[A]) ∫_g dμ(g) δ(g − g_F[A]) =

= Define a Gauge Slice, transverse to the Orbits by a gauge fixing condition

such that $F[A_g] = 0$

Gauge Invariance

Gauge Fixing

$$\int d\mu(g)\,\delta(g - g_{F[A]}) = \underbrace{\int_{\mathcal{G}} d\mu(g)}_{\text{Volume of gauge group } \mathcal{G}} \cdot \int \mathcal{D}[A] \exp(-S_{YM}) \left|\det(J)\right| \delta[F(A)]$$

↑ some jacobian