Title: PSI 2016/2017 Quantum Field Theory II - Lecture 12

Date: Nov 22, 2016 09:00 AM

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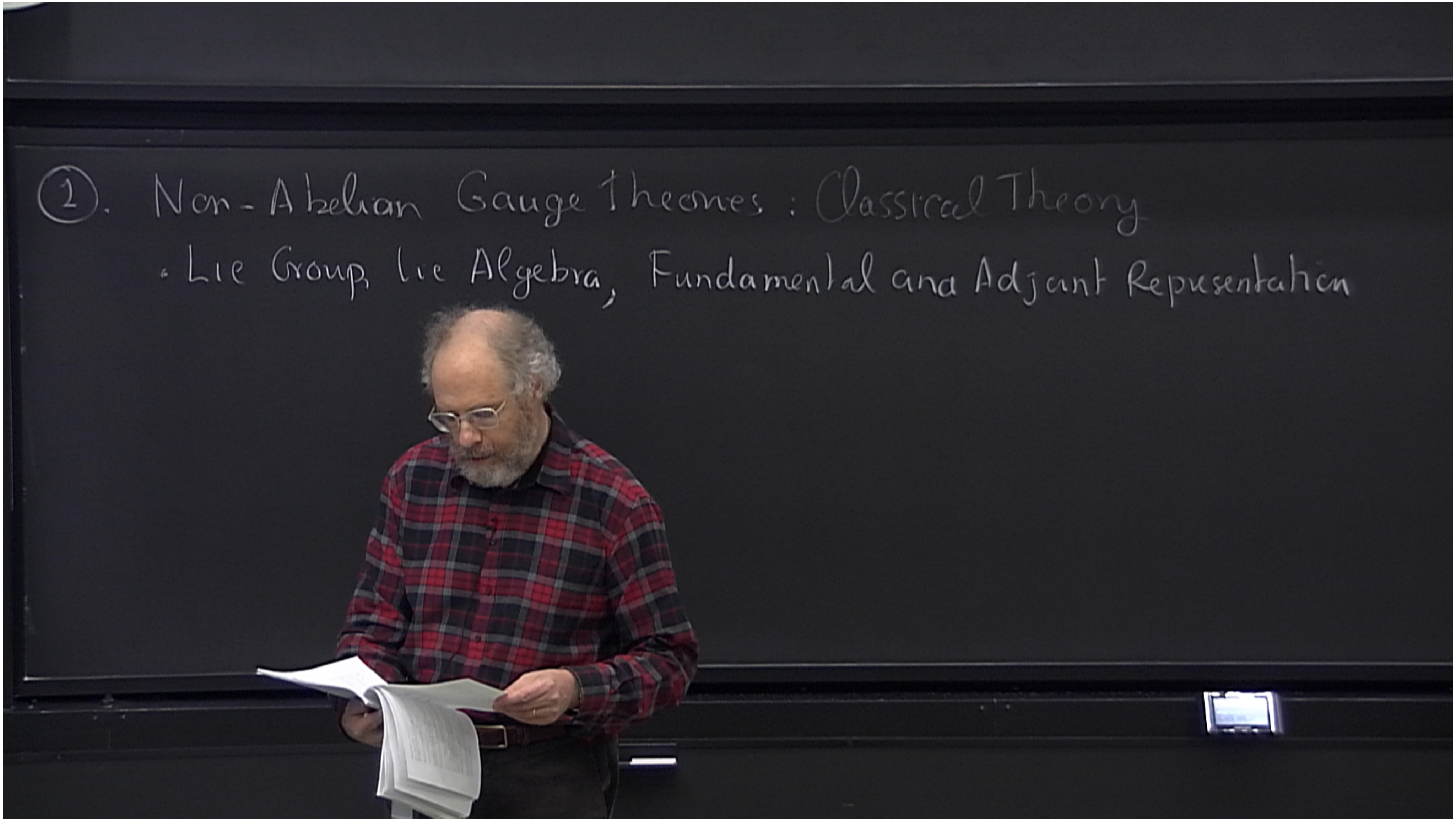
Abstract:

- Fermionic Path Integral
  apply to non-relativistic fermions
  Cond. Matter
- Leibnitz rule $\frac{\partial}{\partial\theta}(g_1 \cdot g_2) = \left(\frac{\partial}{\partial\theta} g_1\right) \cdot g_2 - g_1 \left(\frac{\partial}{\partial\theta} g_2\right)$

  $a \; \theta$


1. Non-Abelian Gauge theories : Classical Theory
· Lie Group, Lie Algebra, Fundamental and Adjoint Representation

① . Non-Abelian Gauge theories : Classical Theory

- Lie Group, Lie Algebra, Fundamental and Adjoint Representation
- Group $SU(2)$ 2×2 unitary matrices

dim 3 group 3 generators for its lie algebra $su(2)$

$t_a = \frac{1}{2}\sigma_a$ σ_a Pauli Matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Lie Bracket $[t_a, t_b] = i f_{ab}^{c} t_c$ $f_{ab}^{c} = \epsilon^{abc}$ antisym. Levi-Civita tensor

structure constant

Why: spin-1 vector fields carrying non-abelian charges

+ matter $\longleftarrow$ bigger sym. group

coupling involves the current $J_\mu(x)$ $A^\mu(x)$

$A^\mu \to A^\mu + \partial^\mu \alpha$ $\int J_\mu \partial^\mu \alpha$

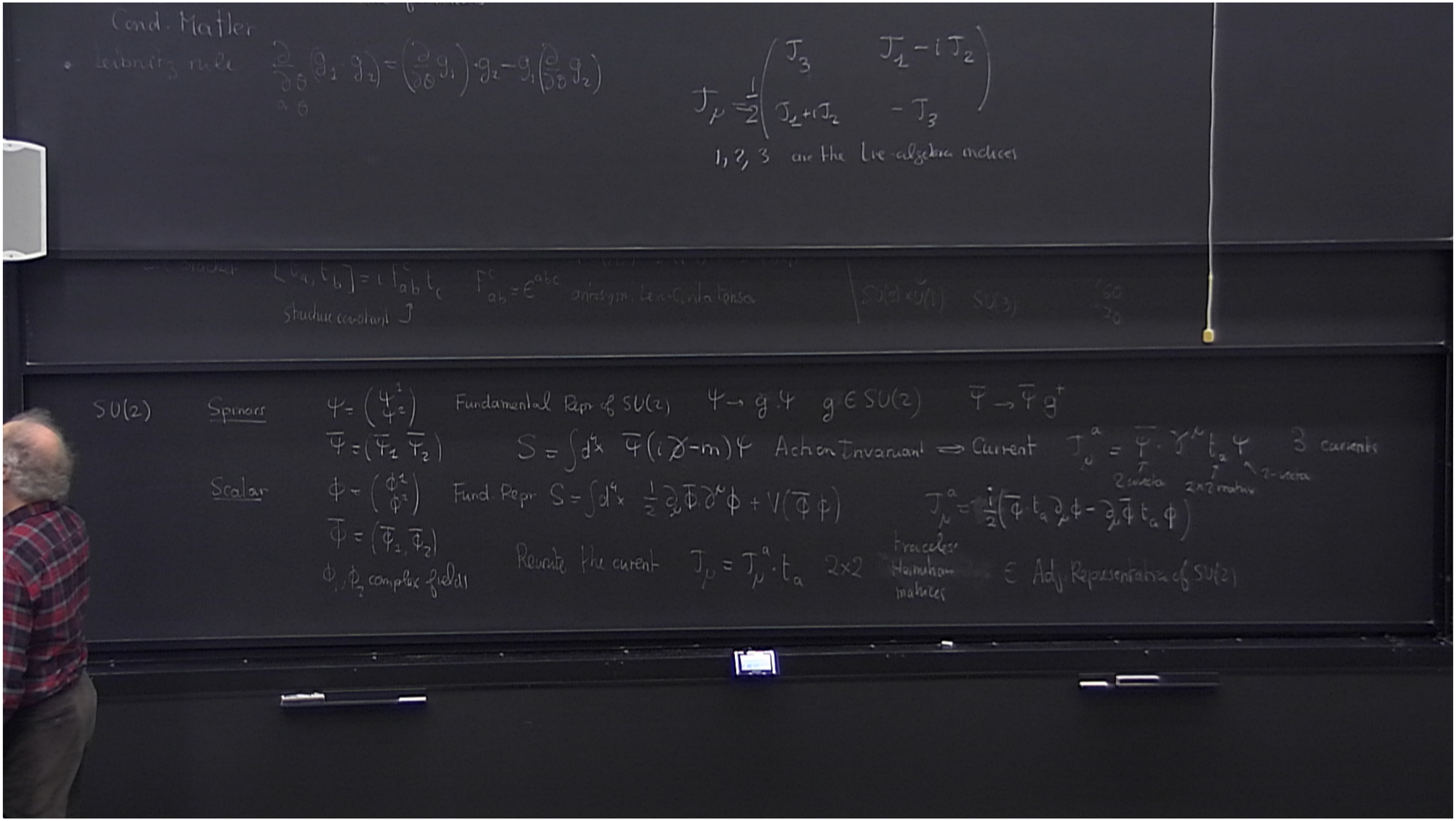
$= \alpha \partial^\mu J_\mu = 0$

Redundancy $\Rightarrow$ Gauge Invarianc

Yang Mills $SU(2)$ Isospin '50

$\downarrow$

$SU(2) \times U(1)$ $SU(3)$ '60 '70


Cond. Matter
Leibnitz rule $\frac{\partial}{\partial\theta}(g_1 \cdot g_2) = \left(\frac{\partial}{\partial\theta} g_1\right) \cdot g_2 - g_1 \left(\frac{\partial}{\partial\theta} g_2\right)$
$J_\mu = \frac{1}{2}\begin{pmatrix} J_3 & J_1 - iJ_2 \\ J_1 + iJ_2 & -J_3 \end{pmatrix}$
1, 2, 3 are the Lie-algebra indices
$[t_a, t_b] = i f_{ab}^{\ \ c} t_c$ $f_{ab}^{\ \ c} = \epsilon^{abc}$ antisym. Levi-Civita tensor
structure constant
SU(2)×U(1) SU(3)
SU(2) Spinors $\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$ Fundamental Repr of SU(2) $\psi \to g \cdot \psi$ $g \in SU(2)$ $\bar\psi \to \bar\psi g^\dagger$
$\bar\psi = (\bar\psi_1 \ \bar\psi_2)$ $S = \int d^4x \, \bar\psi (i\partial\!\!\!/ - m)\psi$ Action Invariant ⟹ Current $J_\mu^a = \bar\psi \cdot \gamma^\mu t_a \psi$ 3 currents
2 vector 2×2 matrix 2-vector
Scalar $\phi = \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix}$ Fund. Repr $S = \int d^4x \, \frac{1}{2} \partial_\mu \bar\Phi \partial^\mu \phi + V(\bar\Phi \phi)$ $J_\mu^a = \frac{i}{2}(\bar\Phi t_a \partial_\mu \phi - \partial_\mu \bar\Phi t_a \phi)$
$\bar\Phi = (\bar\Phi_1, \bar\Phi_2)$
ϕ_1, ϕ_2 complex fields
Rewrite the current $J_\mu = J_\mu^a \cdot t_a$ 2×2 traceless Hermitian matrices ∈ Adj. Representation of SU(2)

- Fermionic Path Integral
  apply to non-relativistic fermions
  Cond. Matter
- Leibnitz rule $\frac{\partial}{\partial\theta}(g_1 \cdot g_2) = \left(\frac{\partial}{\partial\theta} g_1\right) \cdot g_2 - g_1 \left(\frac{\partial}{\partial\theta} g_2\right)$

  $a \quad \theta$

$$J_\mu = \frac{1}{2}\begin{pmatrix} J_3 & J_1 - i J_2 \\ J_1 + i J_2 & -J_3 \end{pmatrix}$$

1, 2, 3 are the Lie-algebra indices

Lie bracket $[t_a, t_b] = i f_{ab}^{c} t_c$ $\quad f_{ab}^{c} = \epsilon^{abc}$ antisym. Levi-Civita tensor

structure constant

$SU(2) \times U(1)$ $\quad SU(3)$

$SU(2)$ Spinors $\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$ Fundamental Repr of $SU(2)$ $\quad \psi \to g \cdot \psi$ $\quad g \in SU(2)$ $\quad \bar{\psi} \to$

$$\psi^i = (\psi^i_\alpha)$$

$$\psi = (\psi^i)_{i=1,2}$$

i index of Fund $SU(2)$

$$(t_{a,ij}) = t_a \quad a = 1,2,3$$

a index of Adj Repr

$\gamma^\mu_{\alpha\beta}$ Dirac Matrices

$\mu = 1, .. D$ Lorentz

$\alpha = 1, .. \, 2^{[D/2]}$ Dirac

$$J^a_\mu = \bar{\psi}^i_\alpha \gamma^\mu_{\alpha\beta} t_{a,ij} \psi^i_\beta$$

$$J_\mu = \frac{1}{2}\begin{pmatrix} J_3^\mu & J_1^\mu - i J_2^\mu \\ J_1^\mu + i J_2^\mu & -J_3^\mu \end{pmatrix}$$

1, 2, 3 are the Lie-algebra indices

Lie Bracket $[t_a, t_b] = i f^c_{ab} t_c$ $\quad f^c_{ab} = \epsilon^{abc}$ antisym, Levi-Civita tensor

structure constant

Pauli Matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} & \\ i & 0 \end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

SU(2)

② SU(2)

Spinors $\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$ Dirac Spinor (4 components in 4 dim)

Fundamental Repr of SU(2) $\psi \to g \cdot \psi$ $\quad g \in SU(2$

$\overline{\psi} = (\overline{\psi}_1 \ \overline{\psi}_2)$

$S = \int d^4x \ \overline{\psi}(i\not{\partial} - m)\psi$ Action Invaria

Scalar $\phi = \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix}$ Fund. Repr $S = \int d^4x \ \frac{1}{2} \partial_\mu \overline{\phi} \partial^\mu \phi + V(\overline{\phi}\phi)$

$\overline{\phi} = (\overline{\phi}_1, \overline{\phi}_2)$

ϕ_1, ϕ_2 complex fields

Rewrite the current $\boxed{J_\mu = J^a_\mu \cdot t_a \quad 2\times2}$

$\sigma_1 = \begin{pmatrix} & \\ 1 & 0 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} & \\ i & 0 \end{pmatrix}$ $\sigma_3 = \begin{pmatrix} & \\ 0 & -1 \end{pmatrix}$

ϵ^{abc} antisym. Levi-Civita tensor

SU(2) Isospin '50

$\downarrow$

$SU(2) \times U(1)$ SU(3) '60

'70

...nor (4 components in 4 dim)

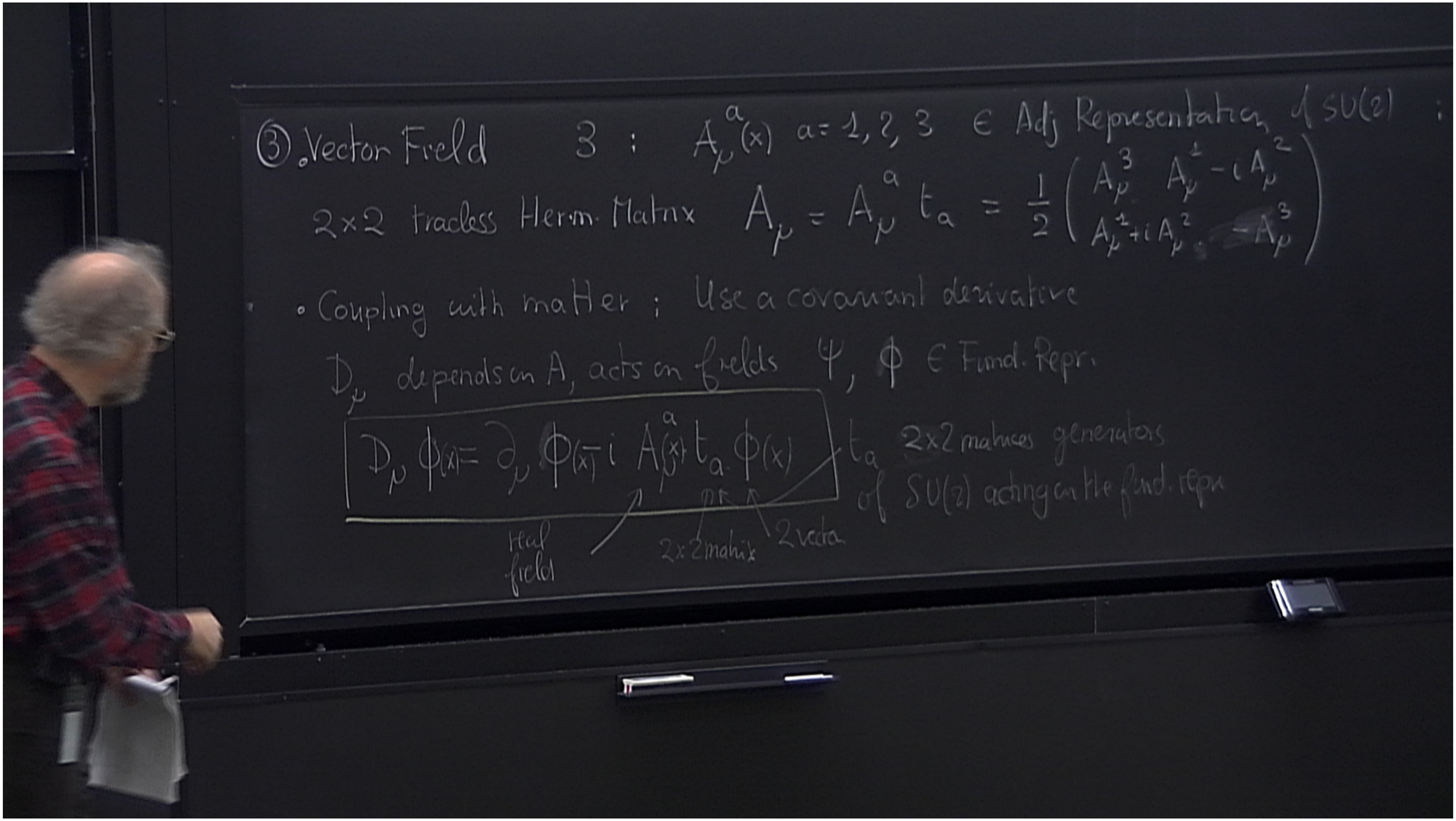
...dal Repr of SU(2) $\Psi \to g \cdot \Psi$ $g \in SU(2)$ $\bar{\Psi} \to \bar{\Psi} g^{+}$

$S = \int d^4x \, \bar{\Psi}(i\not{\partial} - m)\Psi$ Action Invariant $\Longrightarrow$ Current $\boxed{J_\mu^a = \bar{\Psi} \cdot \gamma^\mu t_a \Psi}$ 3 currents

2 covector 2×2 matrix 2-vector

...or $S = \int d^4x \, \frac{1}{2}\partial_\mu \bar{\Phi} \partial^\mu \phi + V(\bar{\Phi}\phi)$ $\boxed{J_\mu^a = \frac{i}{2}(\bar{\phi} \cdot t_a \partial_\mu \phi - \partial_\mu \bar{\phi} \, t_a \phi)}$

Rewrite the current $\boxed{J_\mu = J_\mu^a \cdot t_a \quad 2\times 2}$ traceless Hermitian matrices $\in$ Adj. Representation of SU(2)


③ Vector Field 3 : $A_\mu^a(x)$ a = 1, 2, 3 ∈ Adj Representation of SU(2) :
2×2 traceless Herm. Matrix $A_\mu = A_\mu^a t_a = \frac{1}{2}\begin{pmatrix} A_\mu^3 & A_\mu^1 - i A_\mu^2 \\ A_\mu^1 + i A_\mu^2 & -A_\mu^3 \end{pmatrix}$
• Coupling with matter ; Use a covariant derivative
D_μ depends on A, acts on fields ψ, ϕ ∈ Fund. Repr.
$\boxed{D_\mu \phi(x) = \partial_\mu \phi(x) - i A_\mu^a(x) t_a \phi(x)}$
t_a 2×2 matrices generators of SU(2) acting on the fund. repr
real field
2×2 matrix
2 vector

③ Vector Field 3 : $A_\mu^a(x)$ $a = 1, 2, 3$ $\in$ Adj Representation of $SU(2)$

2×2 traceless Herm. Matrix $A_\mu = A_\mu^a\, t_a = \frac{1}{2}\begin{pmatrix} A_\mu^3 & A_\mu^1 - i A_\mu^2 \\ A_\mu^1 + i A_\mu^2 & -A_\mu^3 \end{pmatrix}$

- Coupling with matter ; Use a covariant derivative

D_μ depends on A, acts on fields ψ, ϕ $\in$ Fund. Repr.

Fund. Repr $\boxed{D_\mu \phi(x) = \partial_\mu \phi(x) - i A_\mu^a(x)\, t_a\, \phi(x)}$ t_a 2×2 matrices generators of $SU(2)$ acting on the fund. repr

Real field 2×2 matrix 2 vector

dj Representation of $SU(2)$; $SU(2)$ $D_\mu \phi^i = \partial_\mu \phi^i - i A_\mu^a t_{a,ij} \phi^j$

$$\frac{1}{2}\begin{pmatrix} A_\mu^3 & A_\mu^1 - i A_\mu^2 \\ A_\mu^1 + i A_\mu^2 & -A_\mu^3 \end{pmatrix}$$

rve

Repr.

generators

g on the fund. repr

$\phi \in$ Representation R ; Generator T_a^R of $SU(2)$ in R

dim d_R $\qquad$ $d_R \times d_R$ matrix

... SU(2) : SU(2) $D_\mu \phi^i = \partial_\mu \phi^i - i A_\mu^a \, t_{a,ij} \phi^j$

$\phi_R \in$ Representation R ; Generator T_a^R of SU(2) in R

dim d_R ; $d_R \times d_R$ matrix

$D_\mu \phi_R = \partial_\mu \phi_R - i A_\mu^a \, T_a^R \, \phi_R$; $a = 1, .. 3$ for SU(2)

real ; $d_R \times d_R$ matrix ; d_R vector

Fund Repr

$$\boxed{D_\mu \phi(x) = \partial_\mu \phi(x) - i A^a_\mu(x)\, t_a\, \phi(x)}$$

real field 2x2 matrix 2 vector

t_a 2x2 matrices generators of SU(2) acting on the fund. repr

④: For Field $\phi_A \in$ Adj representation

2x2 traceless Sym. matrix $\phi_A = \begin{bmatrix} \phi_3 & \phi_1 - i\phi_2 \\ \phi_1 + i\phi_2 & -\phi_3 \end{bmatrix}$

then $\boxed{D_\mu \phi_A = \partial_\mu \phi_A - i\,[A_\mu, \phi_A]}$ ← commutator

2x2 matrix 2x2 matrix

$SU(2)$: $SU(2)$ $D_\mu \phi^i = \partial_\mu \phi^i - i A_\mu^a t_{a,ij} \phi^j$

$\phi_R \in$ Representation R ; Generator T_a^R of $SU(2)$ in R

dim d_R

$d_R \times d_R$ matrix

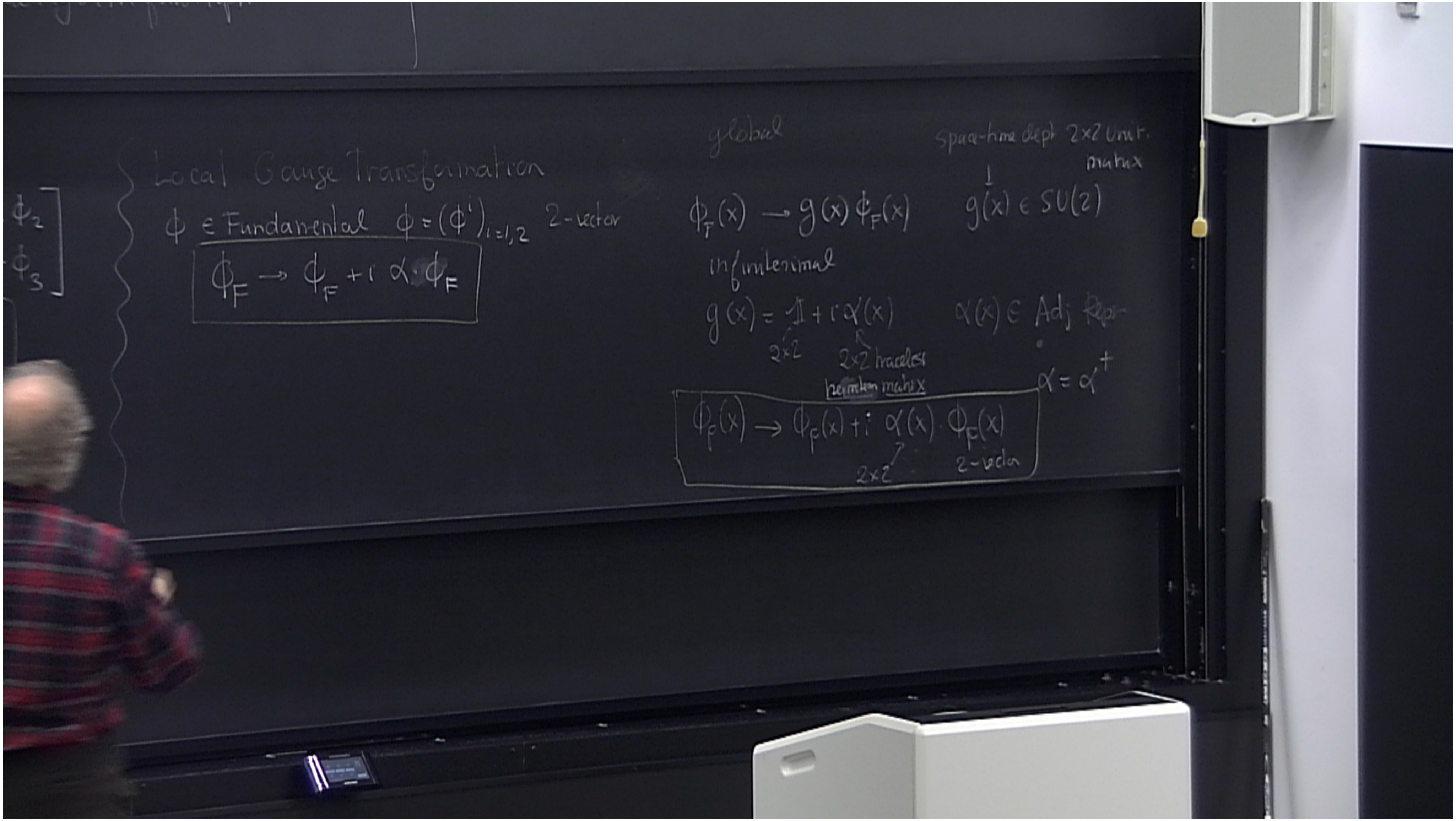
$$D_\mu \phi_R = \partial_\mu \phi_R - i A_\mu^a T_a^R \phi_R$$

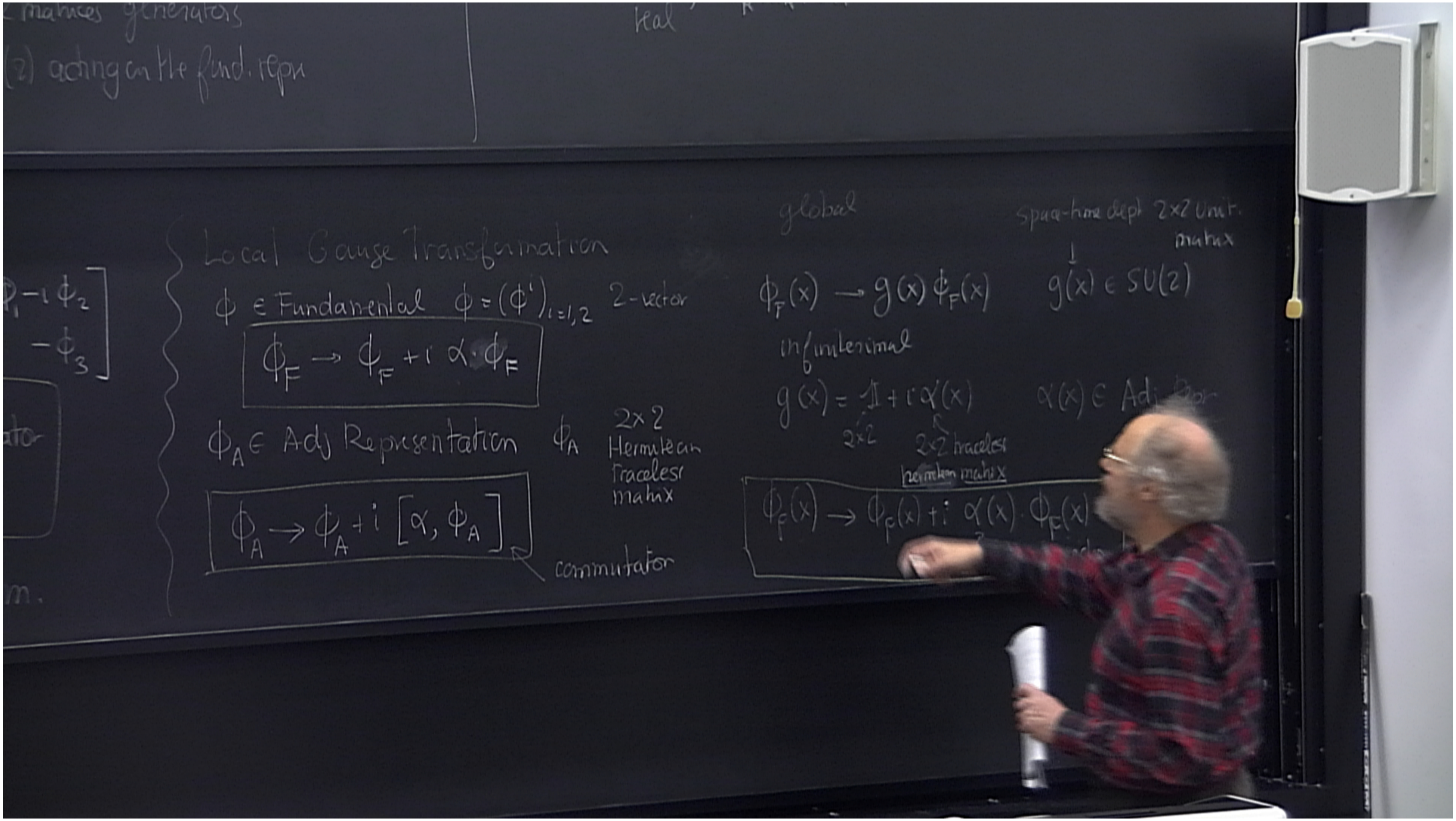
$a = 1, .. 3$ for $SU(2)$

real

$d_R \times d_R$ matrix

d_R vector


Local Gauge Transformation
φ ∈ Fundamental φ = (φ^i)_{i=1,2} 2-vector
φ_F → φ_F + i α·φ_F
global
space-time dept 2×2 Unit. matrix
φ_F(x) → g(x) φ_F(x)
g(x) ∈ SU(2)
infinitesimal
g(x) = 1 + i α(x)
2×2
2×2 traceless hermitean matrix
α(x) ∈ Adj Repr
α = α^†
φ_F(x) → φ_F(x) + i α(x) · φ_F(x)
2×2
2-vector


matrices generators
(2) acting on the fund. repn
real
Local Gauge Transformation
ϕ ∈ Fundamental ϕ = (ϕ^i)_{i=1,2} 2-vector
ϕ_F → ϕ_F + i α · ϕ_F
ϕ_A ∈ Adj Representation ϕ_A 2×2 Hermitean Traceless matrix
ϕ_A → ϕ_A + i [α, ϕ_A]
commutator
global
ϕ_F(x) → g(x) ϕ_F(x)
space-time dept 2×2 Unit. matrix
g(x) ∈ SU(2)
infinitesimal
g(x) = 1 + i α(x)
2×2
2×2 traceless hermitean matrix
α(x) ∈ Adj
ϕ_F(x) → ϕ_F(x) + i α(x) · ϕ_F(x)

Maxwell $G = U(1)$

new for non-abelian sym.

$\boxed{\phi_A \rightarrow \phi_A + i[\alpha, \phi_A]}$

$U(1) \quad g(x) = e^{i\alpha(x)}$

Gauge Transformations for vector-field

global $\quad A_\mu(x) \longrightarrow g(x)\, A_\mu(x)\, g^\dagger(x) + i\, g(x)\left(\partial_\mu g^\dagger(x)\right)$

infinitesimal $\quad \boxed{A_\mu \longrightarrow A_\mu + D_\mu \alpha} \quad$ with $\boxed{D_\mu \alpha = \partial_\mu \alpha - i\,[A_\mu, \alpha]}$

covariant derivative

A_μ and $\alpha \in$ Adj Repr

2x2 matrix 2x2 matrix

Maxwell $G = U(1)$ new for non-abelian sym.

$\phi_A \to \phi_A + i\,[\alpha, \phi_A$

Field Stre

Gauge Transformations for vector-field

global $A_\mu(x) \longrightarrow g(x)\, A_\mu(x)\, g^\dagger(x) + i\, g(x) \left(\partial_\mu g^\dagger(x) \right)$

infinitesimal $\boxed{A_\mu \longrightarrow A_\mu + D_\mu \alpha}$ with $\boxed{D_\mu \alpha = \partial_\mu \alpha - i\,[A_\mu, \alpha]}$

covariant derivative

A_μ and $\alpha \in$ Adj Repr

Interlude $A_\mu dx^\mu$ 1-form in the Adj Repr

Connection $\longrightarrow$ curvature

Field Strength $F_{\mu\nu} \longrightarrow E$ and B fields

$$\boxed{F_{\mu\nu} = i\left[D_\mu, D_\nu\right] = \partial_\mu A_\nu - \partial_\nu A_\mu - i\left[A_\mu, A_\nu\right]}$$

Curvature of a connection defined by A_μ

2 Form

$F = F_{\mu\nu}\, dx^\mu \wedge dx^\nu$

$[A_\mu, \alpha]$

$F_{\mu\nu} \in$ Adj Repr

$2\times 2 \cdots$ matrix

For each $\mu, \nu \in 1, \cdot\cdot 4$

antisymmetric tensor w.r. Lorentz

$F_{\mu\nu} = -F_{\nu\mu}$

$F_{\mu\nu} = F^a_{\mu\nu}\, t_a \quad a = 1,2,3$; 3 E^a and B^a Fields $\leftarrow$ each charge

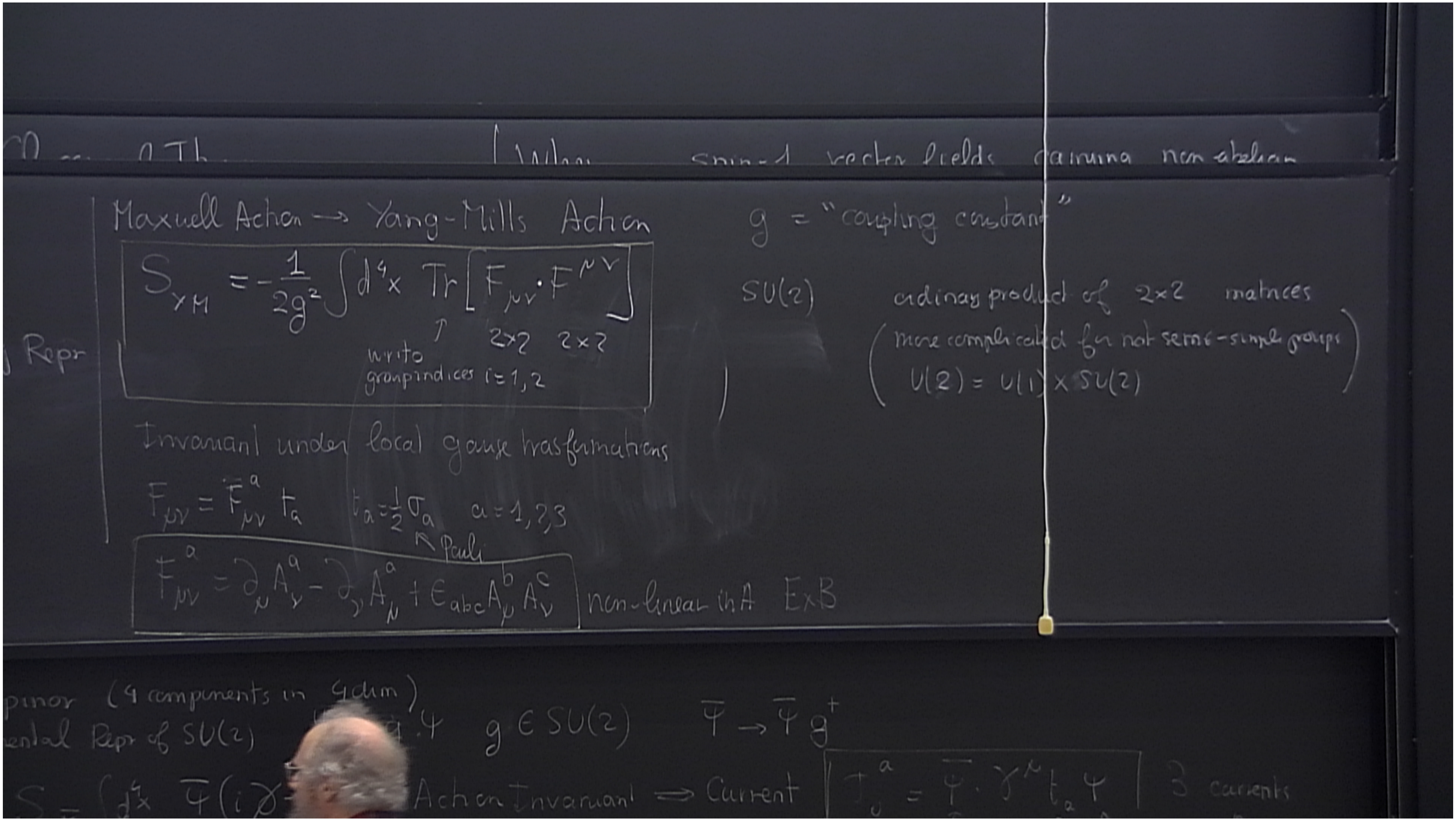
Gauge Transformation on $F_{\mu\nu}$

Infinitesimal $\quad A_\mu \to A_\mu + D_\mu \alpha$

$$F_{\mu\nu} \longrightarrow F_{\mu\nu} - i\left[F_{\mu\nu}, \alpha\right] \qquad F \in \text{Adj Repr}$$

$2\times2 \qquad 2\times2 \qquad 2\times2 \quad 2\times2$

Transforms covariantly but <u>locally</u>


Maxwell Action → Yang-Mills Action
$S_{YM} = -\frac{1}{2g^2}\int d^4x \, \mathrm{Tr}\left[F_{\mu\nu}\cdot F^{\mu\nu}\right]$
2×2 2×2
write group indices i = 1, 2
g = "coupling constant"
SU(2) ordinary product of 2×2 matrices
(more complicated for not semi-simple groups U(2) = U(1) × SU(2))
Invariant under local gauge trasformations
$F_{\mu\nu} = F^a_{\mu\nu} t_a$ $t_a = \frac{1}{2}\sigma_a$ a = 1, 2, 3
Pauli
$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon_{abc} A^b_\mu A^c_\nu$ non-linear in A E×B
Repr
pinor (4 components in 4dim)
ental Repr of SU(2)
g ∈ SU(2) $\bar\Psi \to \bar\Psi g^+$
Action Invariant ⟹ Current $J^a_\mu = \bar\Psi \cdot \gamma^\mu t_a \Psi$ 3 currents

$$A^a_\mu = g\,\tilde{A}^a_\mu$$ rescaling : so that

$$-\frac{1}{4}\left(\partial_\mu \tilde{A}^a_\nu - \partial_\nu \tilde{A}^a_\mu\right)\left(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a\right)$$ ← Maxwell Term

$$-g\,\frac{1}{2}\,\epsilon_{abc}\left(\partial_\mu \tilde{A}^a_\nu - \partial_\nu A^a_\mu\right)\tilde{A}^\mu_b\,\tilde{A}^\nu_c$$ ← non-linear terms

$$-g^2\,\frac{1}{4}\left(\tilde{A}^a_\mu\,\tilde{A}^b_\nu\,\tilde{A}^\mu_a\,\tilde{A}^\nu_b - A^a_\mu A^b_\nu A^\mu_b A^\nu_a\right)$$

$F_{\mu\nu} = F_{\mu\nu}^a t_a \quad t_a = \frac{1}{2}\sigma_a \quad a = 1,2,3$

Pauli

$$\boxed{F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon_{abc} A_\mu^b A_\nu^c}$$ non-linear in A E×B

le Term

linear

ns

Classical Action ⟶ YM equation of motion are non linear in A or E & B

The vector fields (photons) interact, not a free theory:

because they carry charges

Coupling with matter. As Usual

replace $\partial_\mu \to D_\mu$ covariant derivatives

in the action for Matter Fields ⟹ Gauge invariance

$\phi, \psi \in$ Fund. Repr.

$D_\mu \in$ Adj Repr

Good Transf Properties