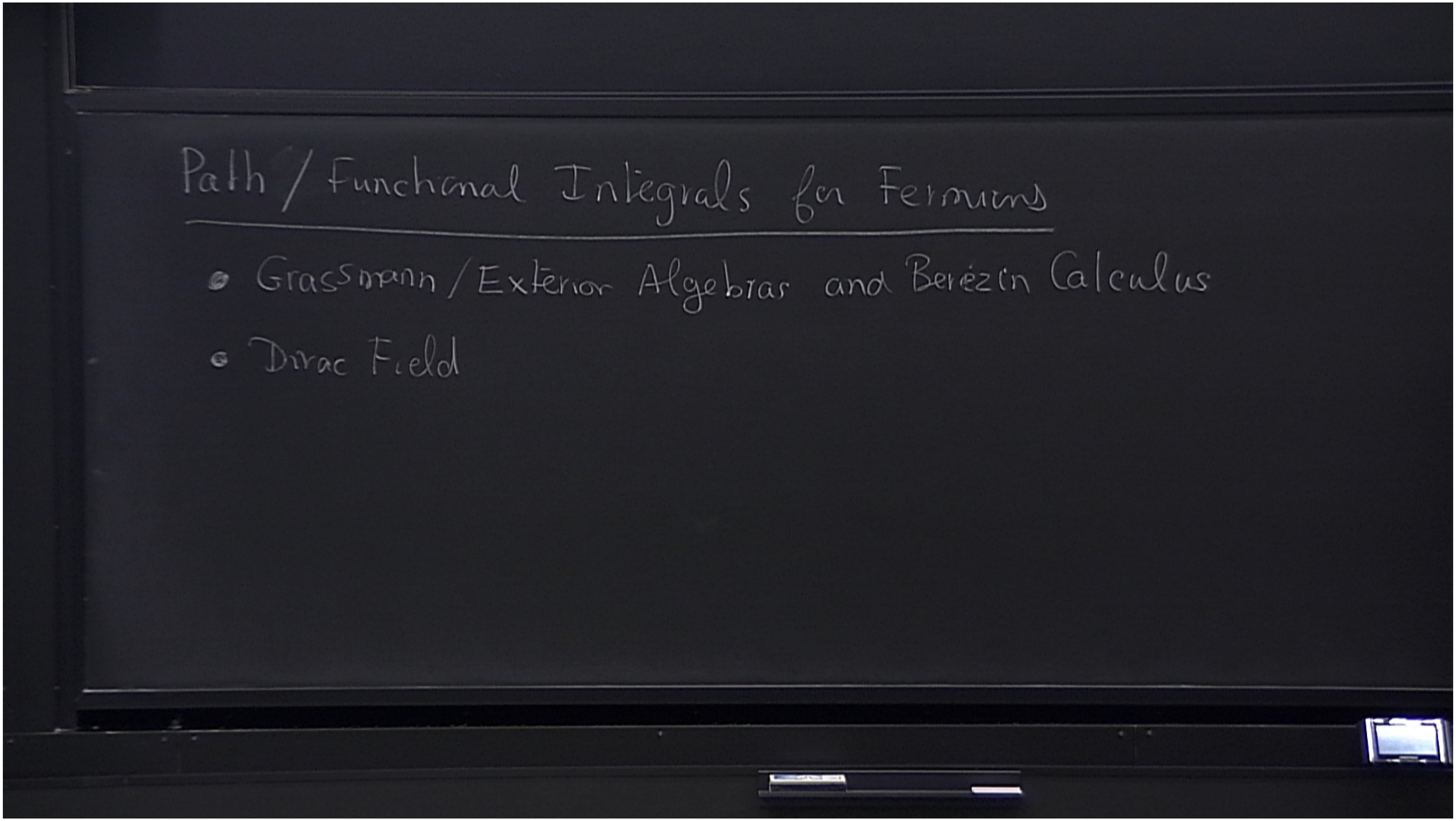
Title: PSI 2016/2017 Quantum Field Theory II - Lecture 11

Date: Nov 21, 2016 09:00 AM

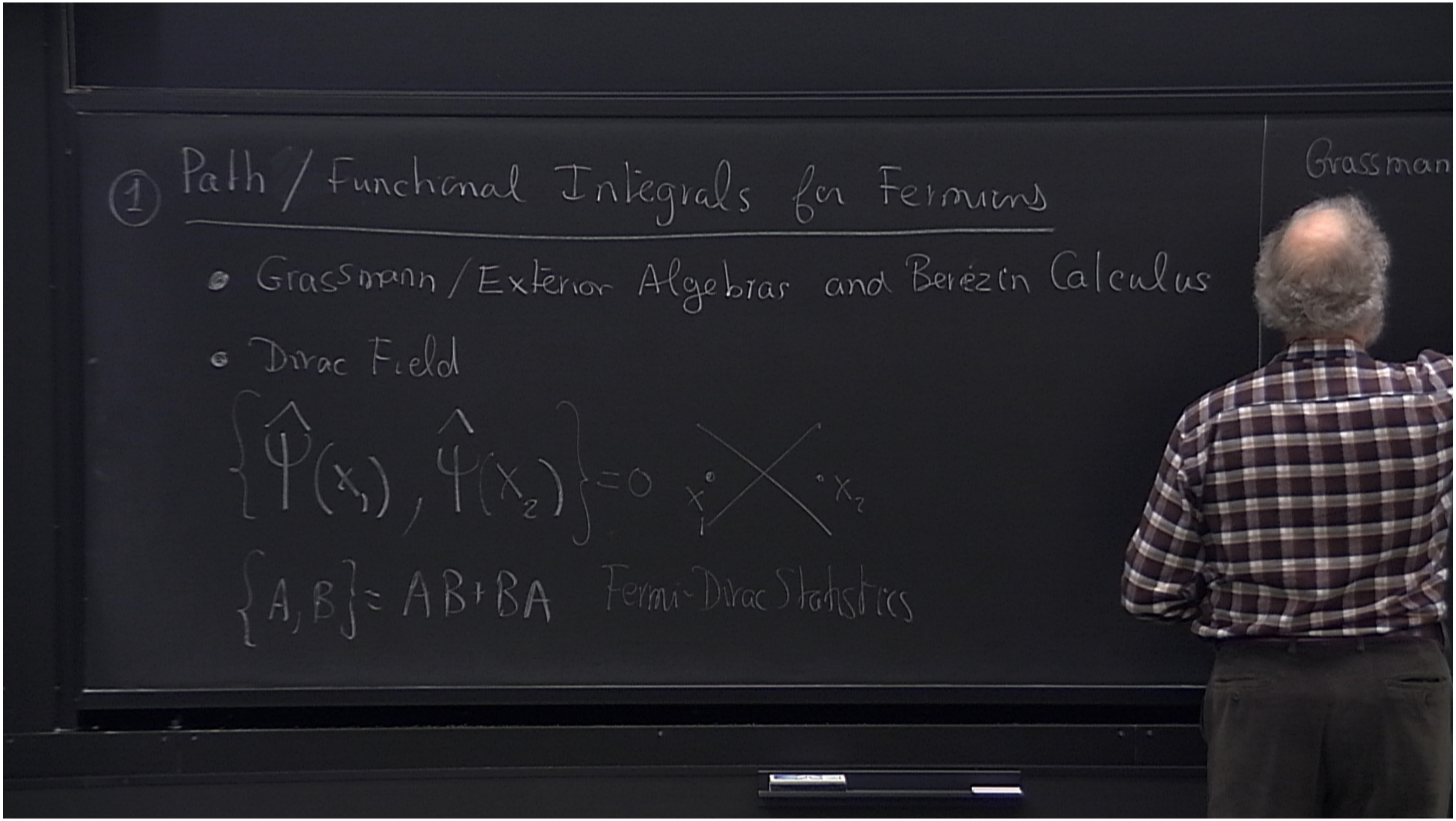
URL: http://pirsa.org/16110011

Abstract:



Path / Functional Integrals for Fermions

- Grassmann / Exterior Algebras and Berezin Calculus
- Dirac Field



① Path / Functional Integrals for Fermions

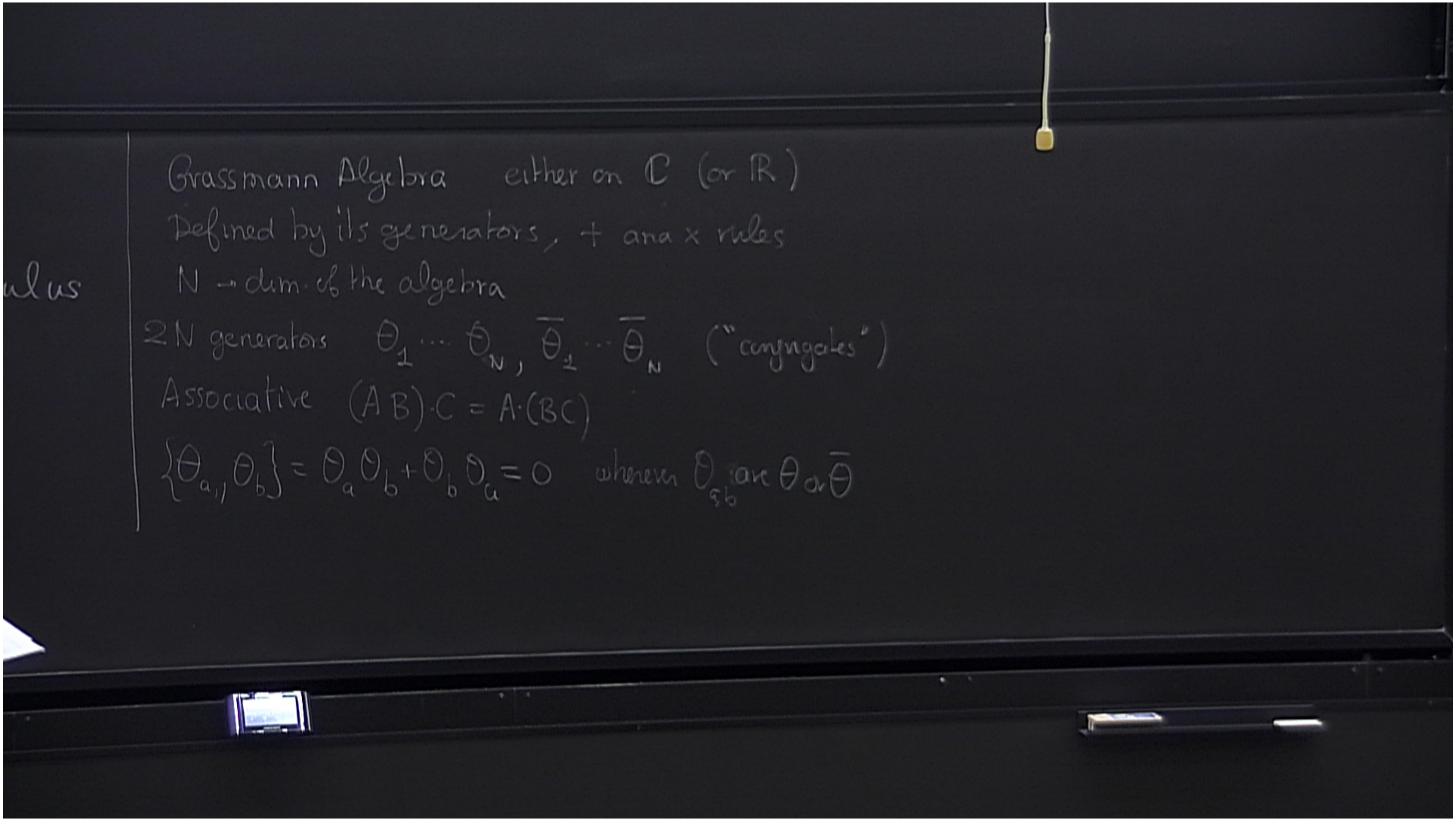
- Grassmann / Exterior Algebras and Berezin Calculus
- Dirac Field

$$\{\hat{\Psi}(x_1), \hat{\Psi}(x_2)\} = 0$$

$$\{A, B\} = AB + BA$$

Fermi-Dirac Statistics

Grassman



ulus

Grassmann Algebra either on $\mathbb{C}$ (or $\mathbb{R}$)

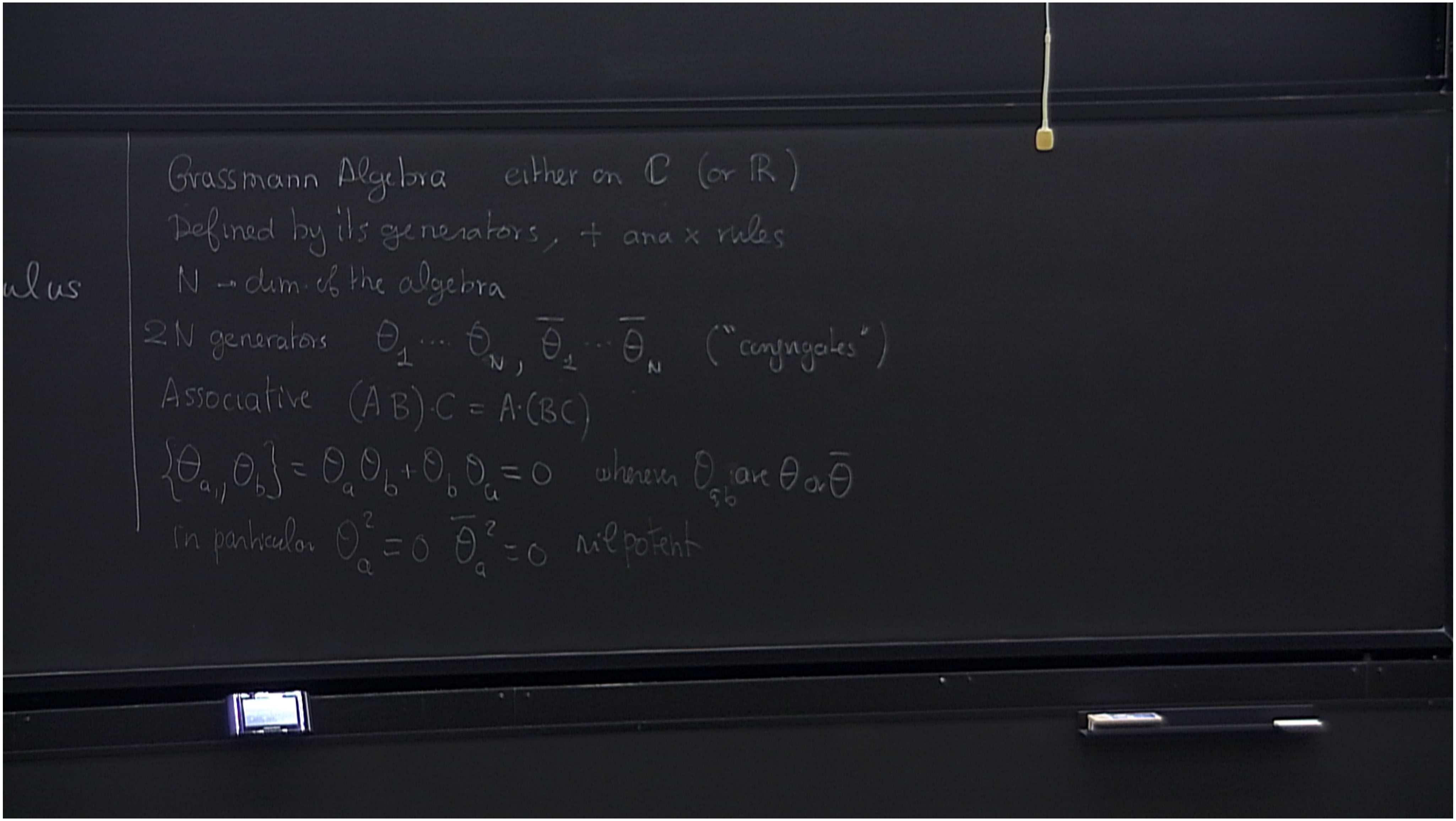
Defined by its generators, + and × rules

$N \rightarrow$ dim. of the algebra

$2N$ generators $\theta_1 \cdots \theta_N, \bar{\theta}_1 \cdots \bar{\theta}_N$ ("conjugates")

Associative $(AB)\cdot C = A\cdot(BC)$

$\{\theta_a, \theta_b\} = \theta_a\theta_b + \theta_b\theta_a = 0$ wherever $\theta_{a,b}$ are θ or $\bar{\theta}$



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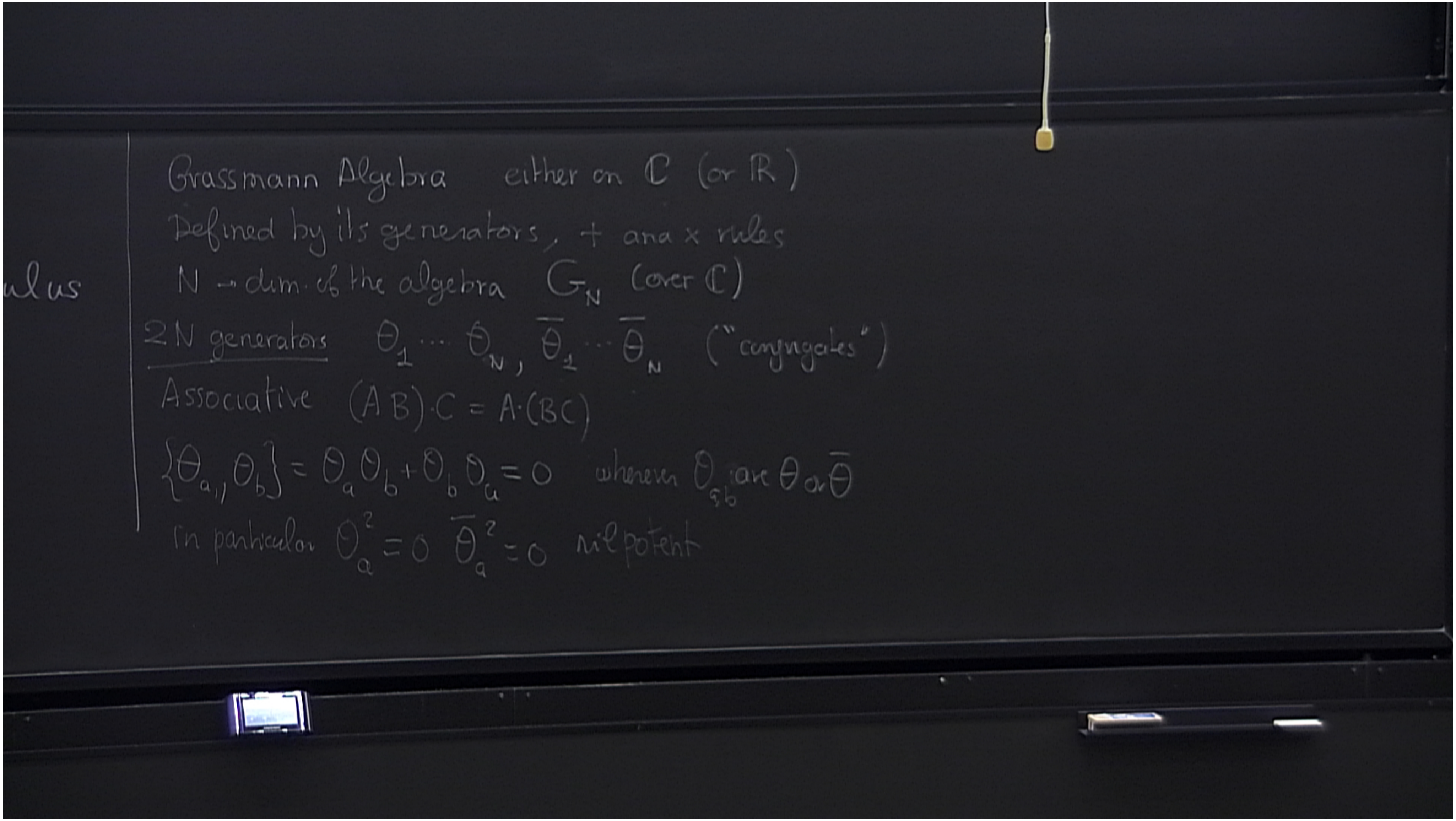
in particular $\theta_a^2 = 0$ $\bar{\theta}_a^2 = 0$ nilpotent



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Grassmann Algebra either on $\mathbb{C}$ (or $\mathbb{R}$)

Defined by its generators, + ana × rules

N → dim. of the algebra G_N (over $\mathbb{C}$)

<u>2N generators</u> $\theta_1 \cdots \theta_N, \bar{\theta}_1 \cdots \bar{\theta}_N$ ("conjugates")

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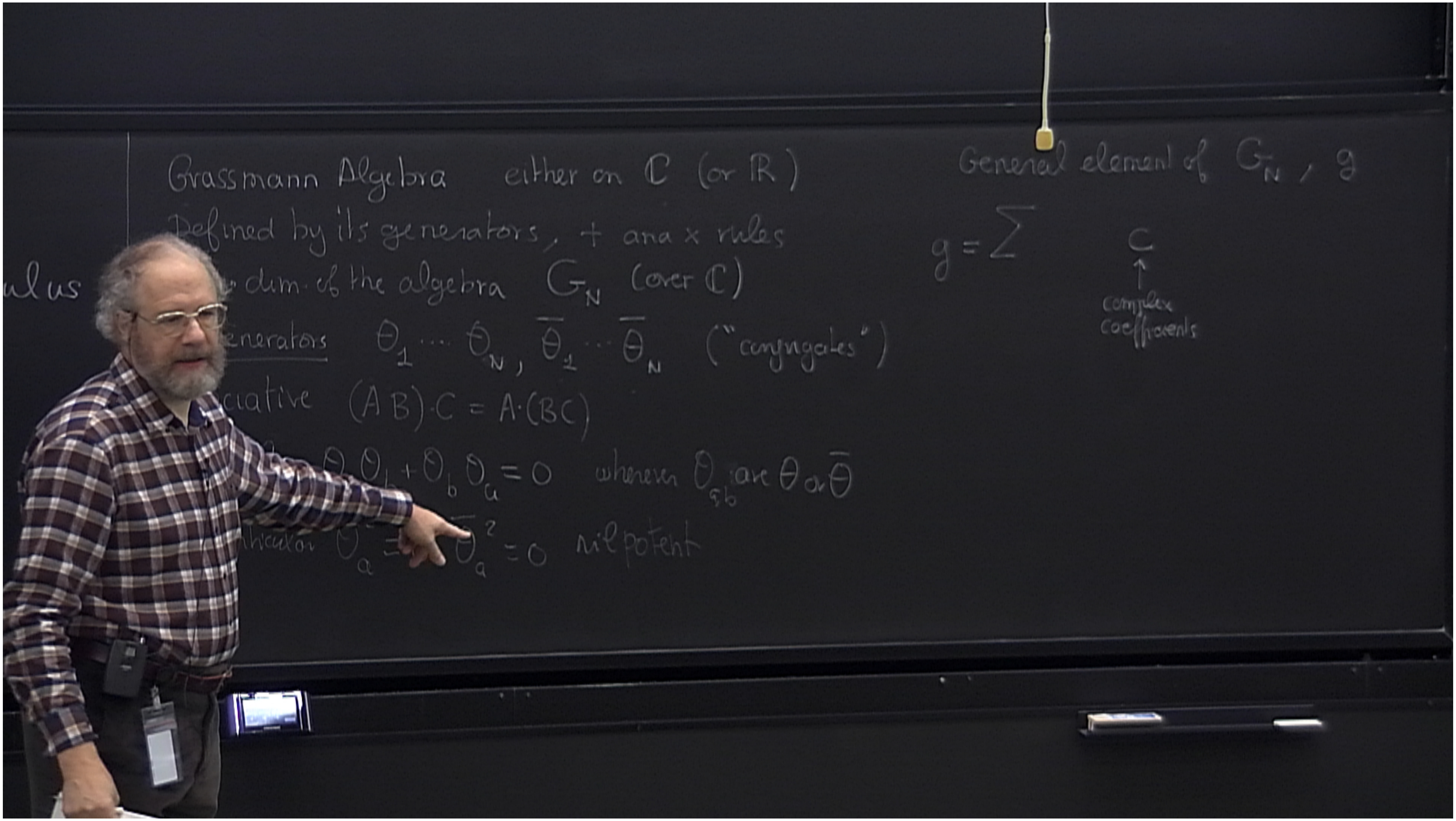


ulus

Grassmann Algebra either on ℂ (or ℝ)

Defined by its generators, + and × rules

dim. of the algebra G_N (over ℂ)

enerators $\theta_1 \cdots \theta_N$, $\bar\theta_1 \cdots \bar\theta_N$ ("conjugates")

ciative $(AB)\cdot C = A\cdot(BC)$

$\theta_a\theta_b + \theta_b\theta_a = 0$ whenever $\theta_{a,b}$ are θ or $\bar\theta$

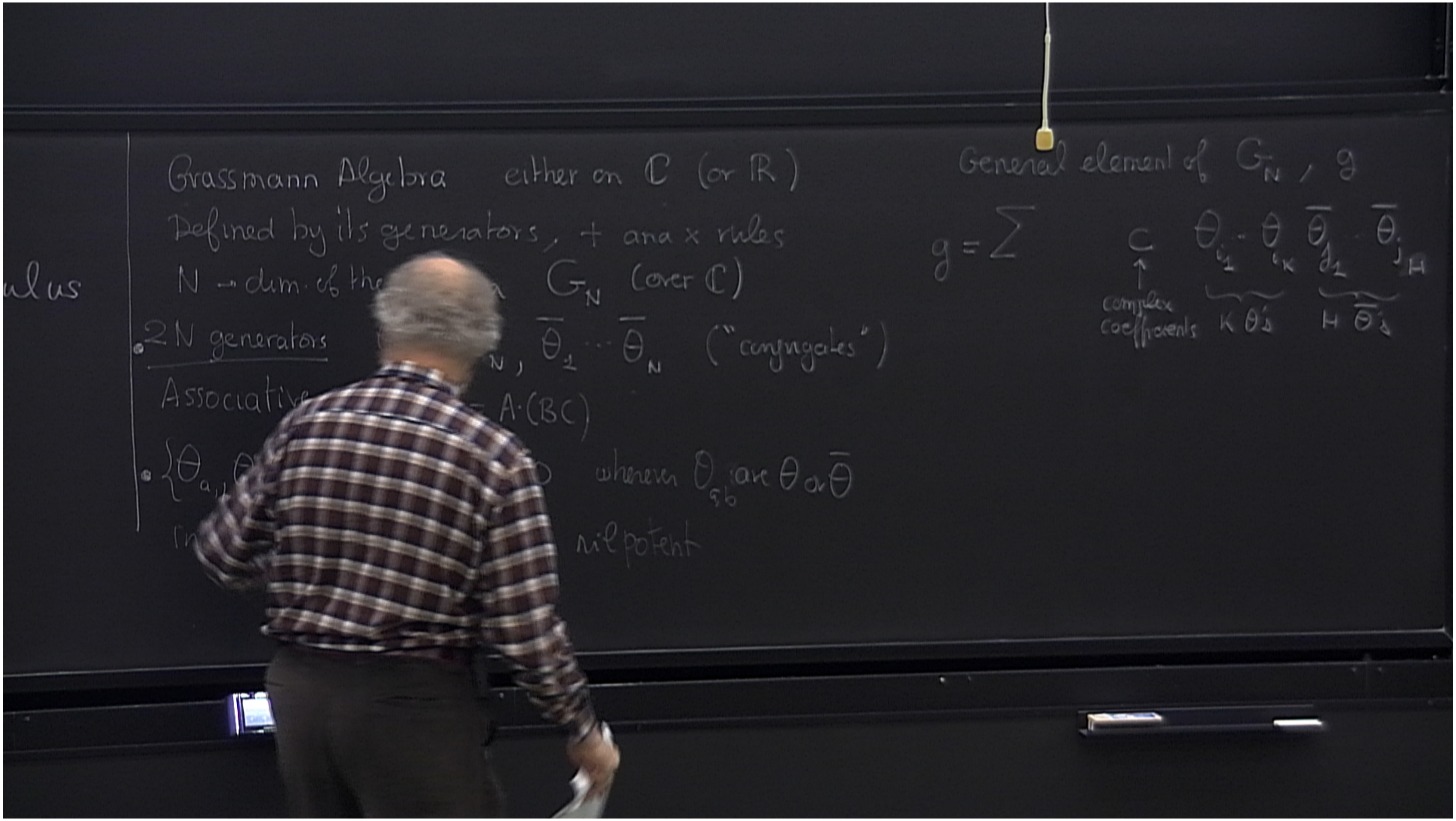
ticular θ_a = $\bar\theta_a^2 = 0$ nilpotent

General element of G_N, g

$g = \sum$ C

↑

complex coefficients



ulus

Grassmann Algebra either on $\mathbb{C}$ (or $\mathbb{R}$)

Defined by its generators, + and × rules

N → dim. of the … G_N (over $\mathbb{C}$)

- 2N generators … $_N$, $\bar\theta_1 \cdots \bar\theta_N$ ("conjugates")

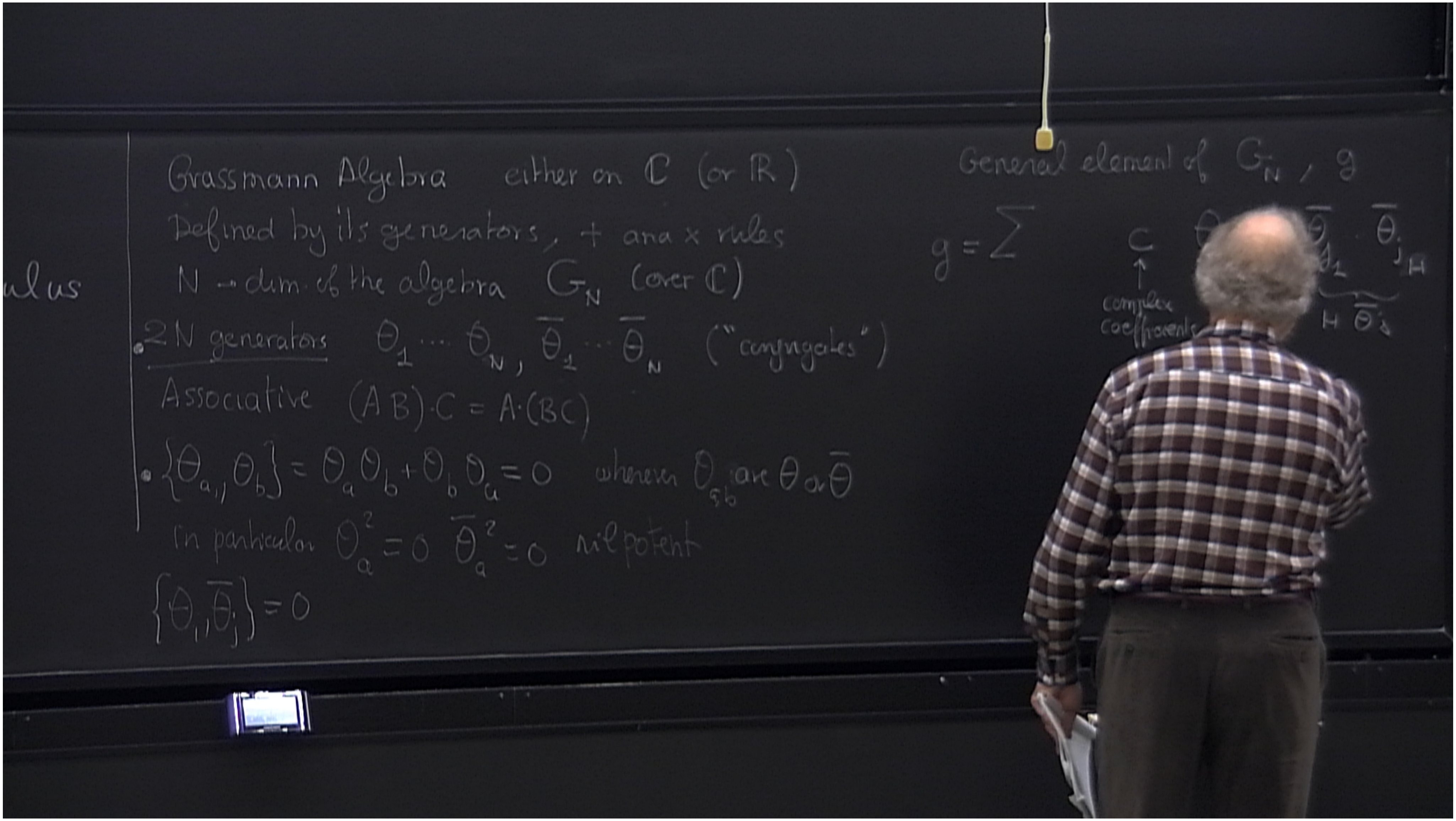
Associative … $= A\cdot(BC)$

- $\{\theta_{a}, \theta$ … 0 whenever $\theta_{a,b}$ are θ or $\bar\theta$

in … nilpotent

General element of G_N, g

$$g = \sum \underset{\substack{\uparrow \\ \text{complex} \\ \text{coefficients}}}{C} \underbrace{\theta_{i_1} \cdots \theta_{i_K}}_{K\ \theta\text{'s}} \underbrace{\bar\theta_{j_1} \cdots \bar\theta_{j_H}}_{H\ \bar\theta\text{'s}}$$



ulus

Grassmann Algebra either on ℂ (or ℝ)

Defined by its generators, + and × rules

N → dim. of the algebra G_N (over ℂ)

- 2N generators $\theta_1 \cdots \theta_N, \bar{\theta}_1 \cdots \bar{\theta}_N$ ("conjugates")

Associative $(AB)\cdot C = A\cdot(BC)$

- $\{\theta_{a}, \theta_b\} = \theta_a \theta_b + \theta_b \theta_a = 0$ whenever $\theta_{a,b}$ are θ or $\bar{\theta}$

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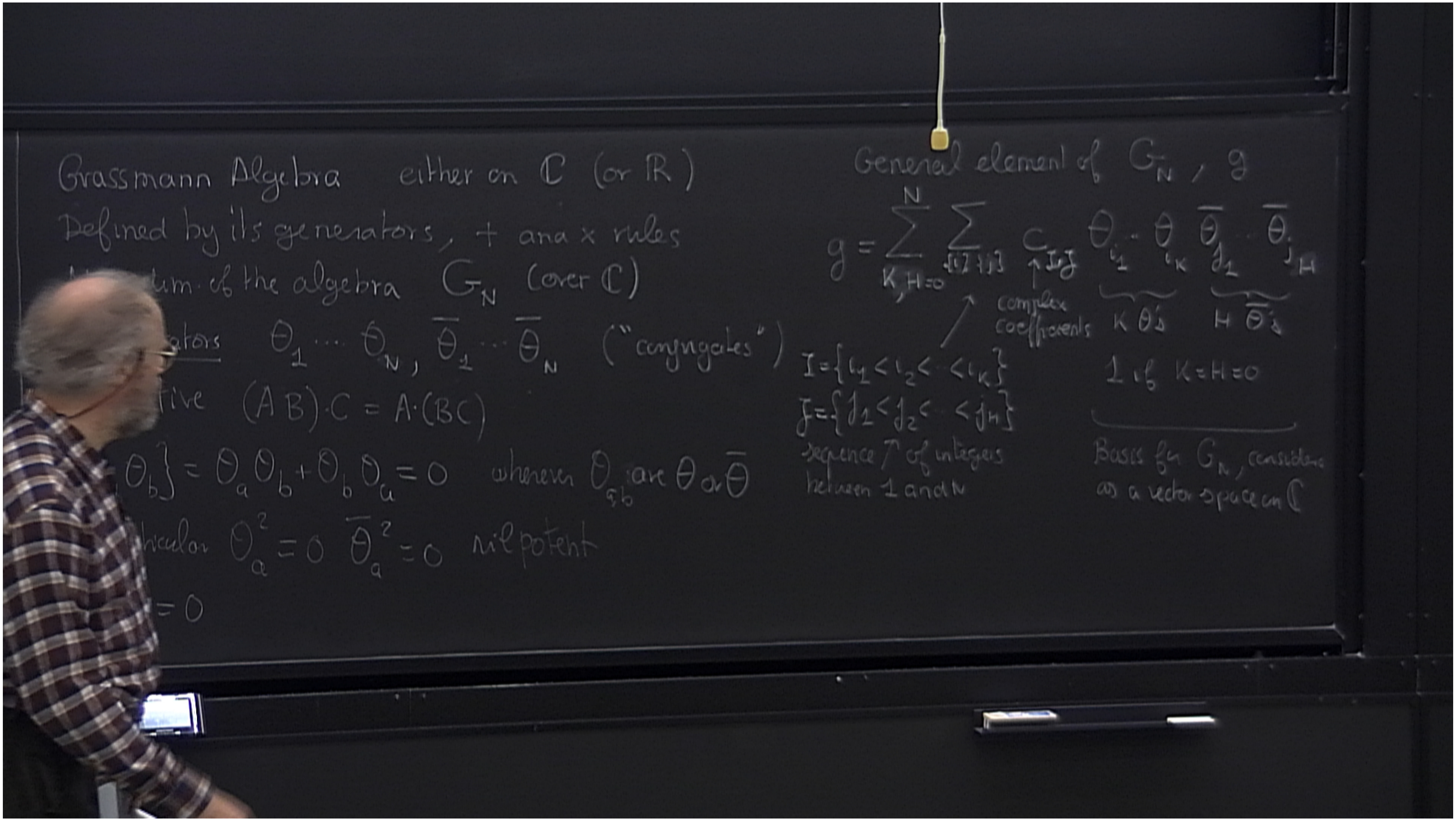
$\{\theta_i, \bar{\theta}_j\} = 0$

General element of G_N, g

$$g = \sum \underset{\substack{\uparrow \\ \text{complex} \\ \text{coefficients}}}{C} \; \theta \cdots \underbrace{\bar{\theta}_{j_1} \cdot \bar{\theta}_{j_H}}_{H\ \bar{\theta}\text{'s}}$$



Grassmann Algebra either on $\mathbb{C}$ (or $\mathbb{R}$)

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$= 0$

General element of G_N, g

$$g = \sum_{K,H=0}^{N} \sum_{\{i\}\{j\}} \underset{\uparrow}{C_{IJ}} \underbrace{\theta_{i_1} \cdots \theta_{i_K}}_{K\ \theta's} \underbrace{\bar{\theta}_{j_1} \cdots \bar{\theta}_{j_H}}_{H\ \bar{\theta}'s}$$

complex coefficients

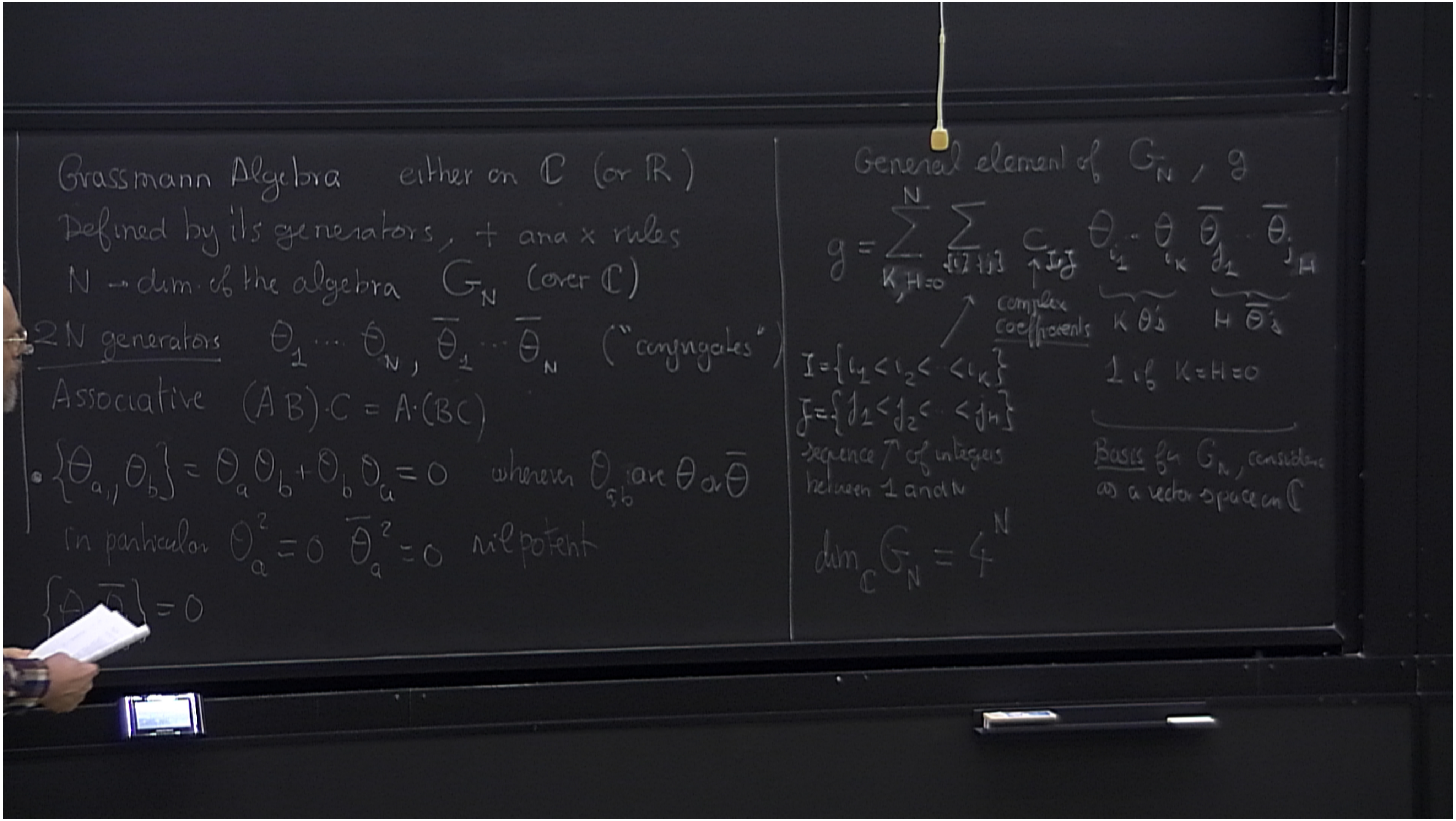
$I = \{i_1 < i_2 < \cdots < i_K\}$

$J = \{j_1 < j_2 < \cdots < j_H\}$

sequence ↗ of integers between 1 and N

1 if $K = H = 0$

Basis for G_N, considered as a vector space on $\mathbb{C}$



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N → dim. of the algebra G_N (over $\mathbb{C}$)

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- $\{\theta_{a,}, \theta_b\} = \theta_a \theta_b + \theta_b \theta_a = 0$ whenever $\theta_{a,b}$ are θ or $\bar{\theta}$

in particular $\theta_a^2 = 0$ $\bar{\theta}_a^2 = 0$ nilpotent

$\{\theta, \bar{\theta}\} = 0$

General element of G_N, g

$$g = \sum_{K,H=0}^{N} \sum_{\{i\}\{j\}} \underset{\uparrow}{C_{I,J}} \underbrace{\theta_{i_1} \cdots \theta_{i_K}}_{K\ \theta's} \underbrace{\bar{\theta}_{j_1} \cdots \bar{\theta}_{j_H}}_{H\ \bar{\theta}'s}$$

complex coefficients

1 if K = H = 0

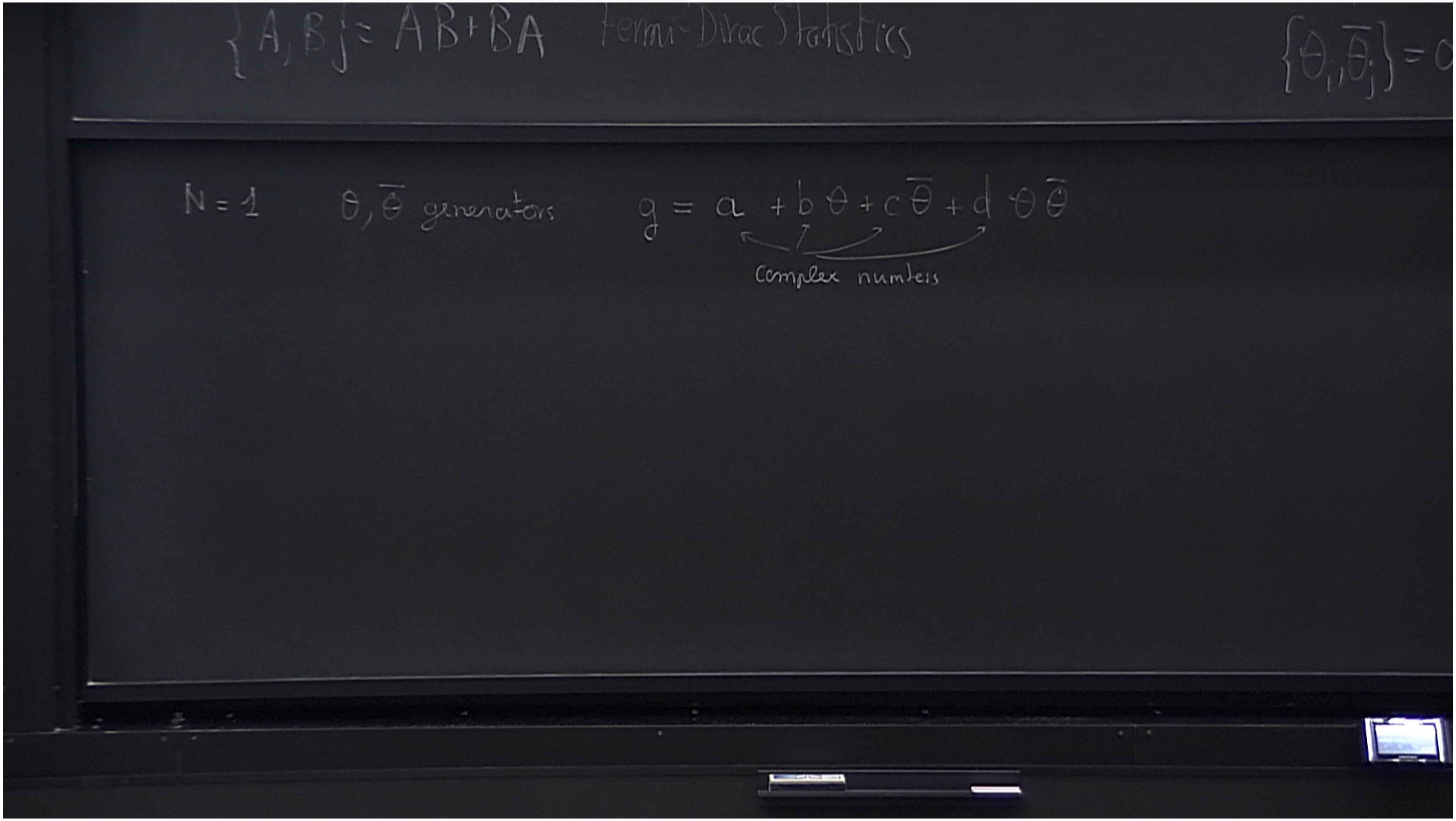
$I = \{i_1 < i_2 < \cdots < i_K\}$

$J = \{j_1 < j_2 < \cdots < j_H\}$

sequence ↗ of integers between 1 and N

Basis for G_N, considered as a vector space on $\mathbb{C}$

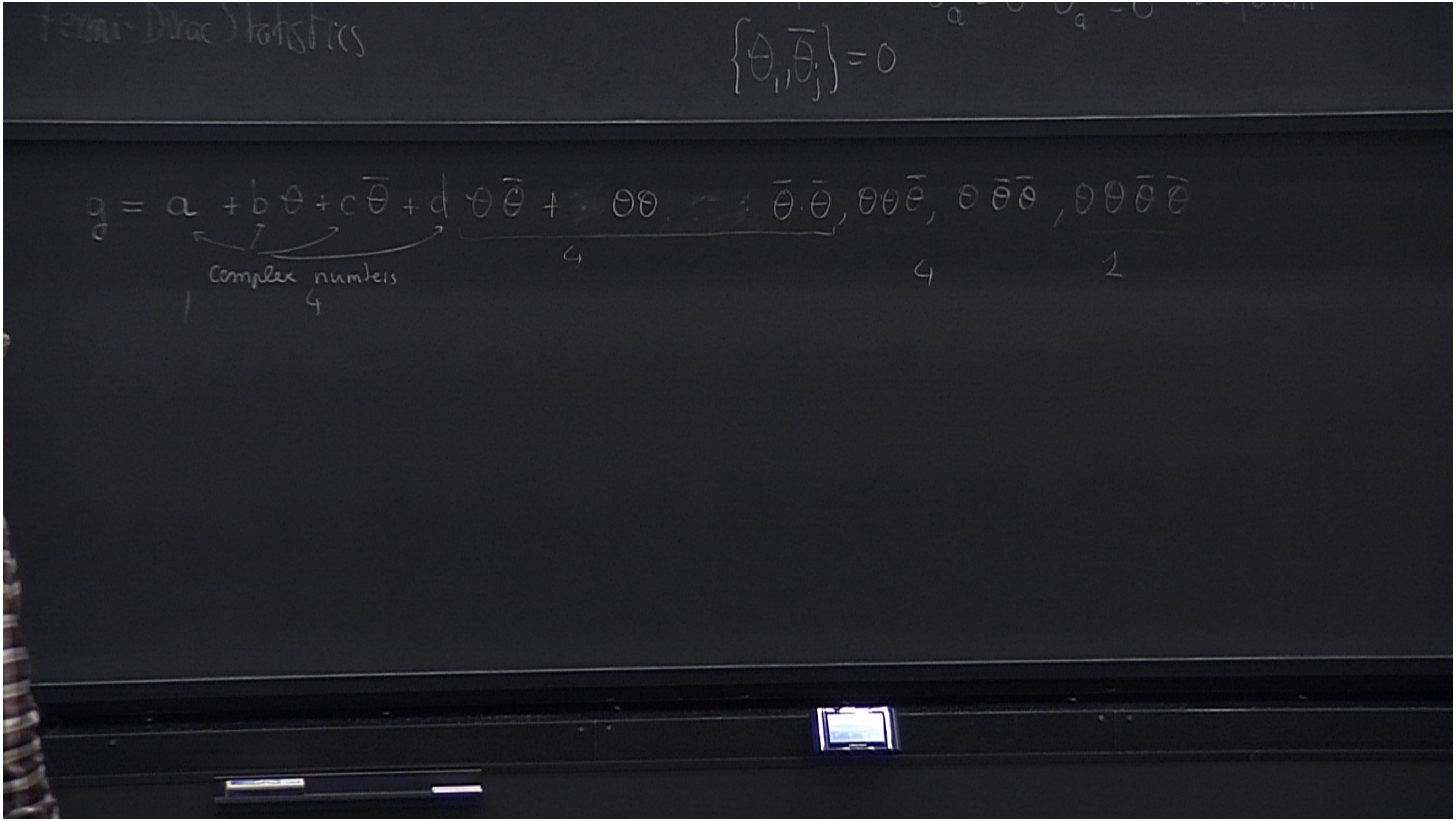
$$\dim_{\mathbb{C}} G_N = 4^N$$



$\{A,B\} = AB + BA$ Fermi-Dirac Statistics $\{\theta_i, \bar{\theta}_j\} =$

$N = 1$ $\theta, \bar{\theta}$ generators $g = a + b\theta + c\bar{\theta} + d\theta\bar{\theta}$

complex numbers



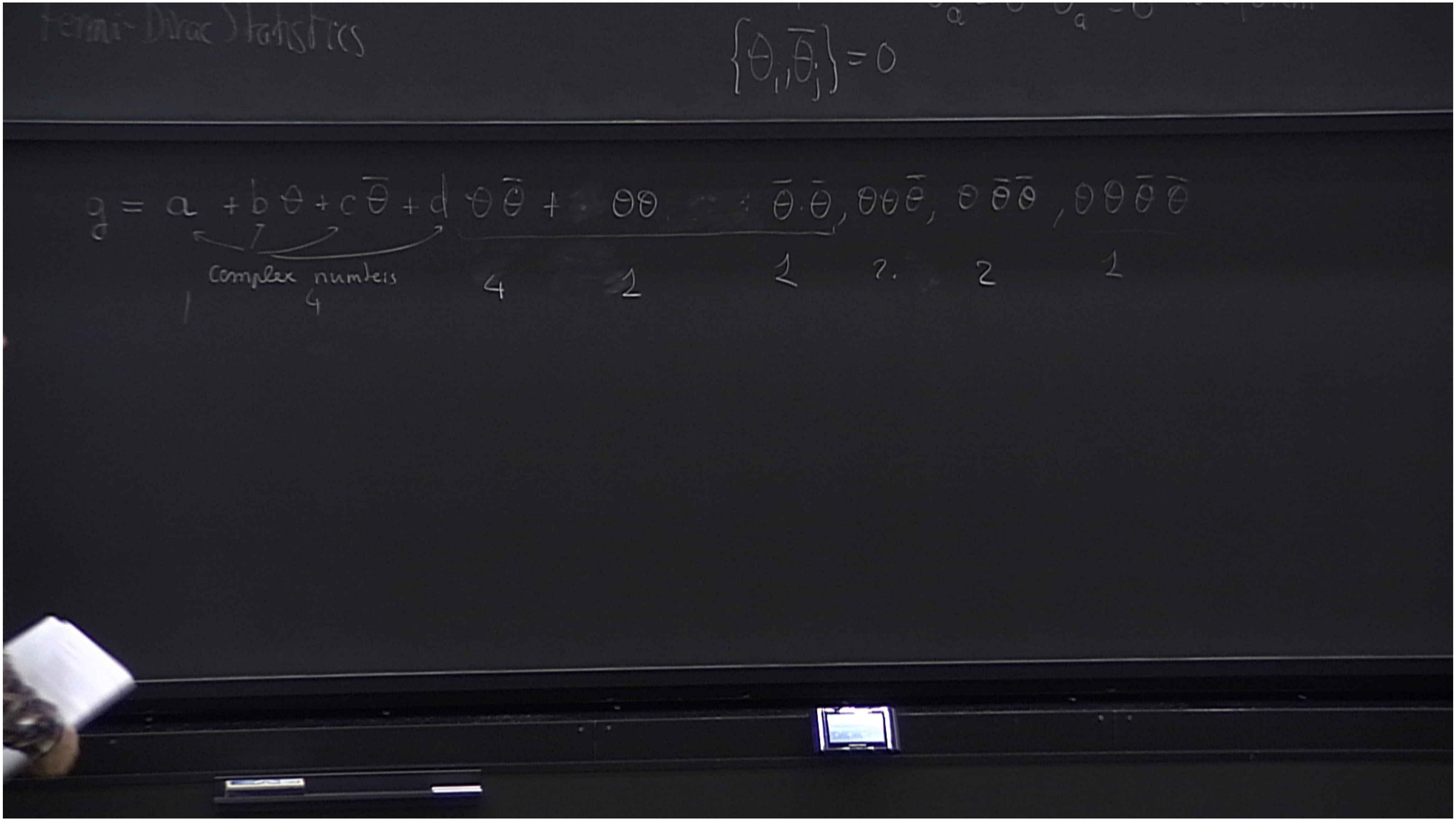
Fermi-Dirac Statistics

$$\{\theta_i, \bar\theta_j\} = 0$$

$$g = a + b\theta + c\bar\theta + d\,\underbrace{\theta\bar\theta + \theta\theta \quad \bar\theta\cdot\bar\theta}_{4},\ \underbrace{\theta\theta\bar\theta,\ \theta\bar\theta\bar\theta}_{4},\ \underbrace{\theta\theta\bar\theta\bar\theta}_{1}$$

complex numbers

1 4



Fermi-Dirac Statistics

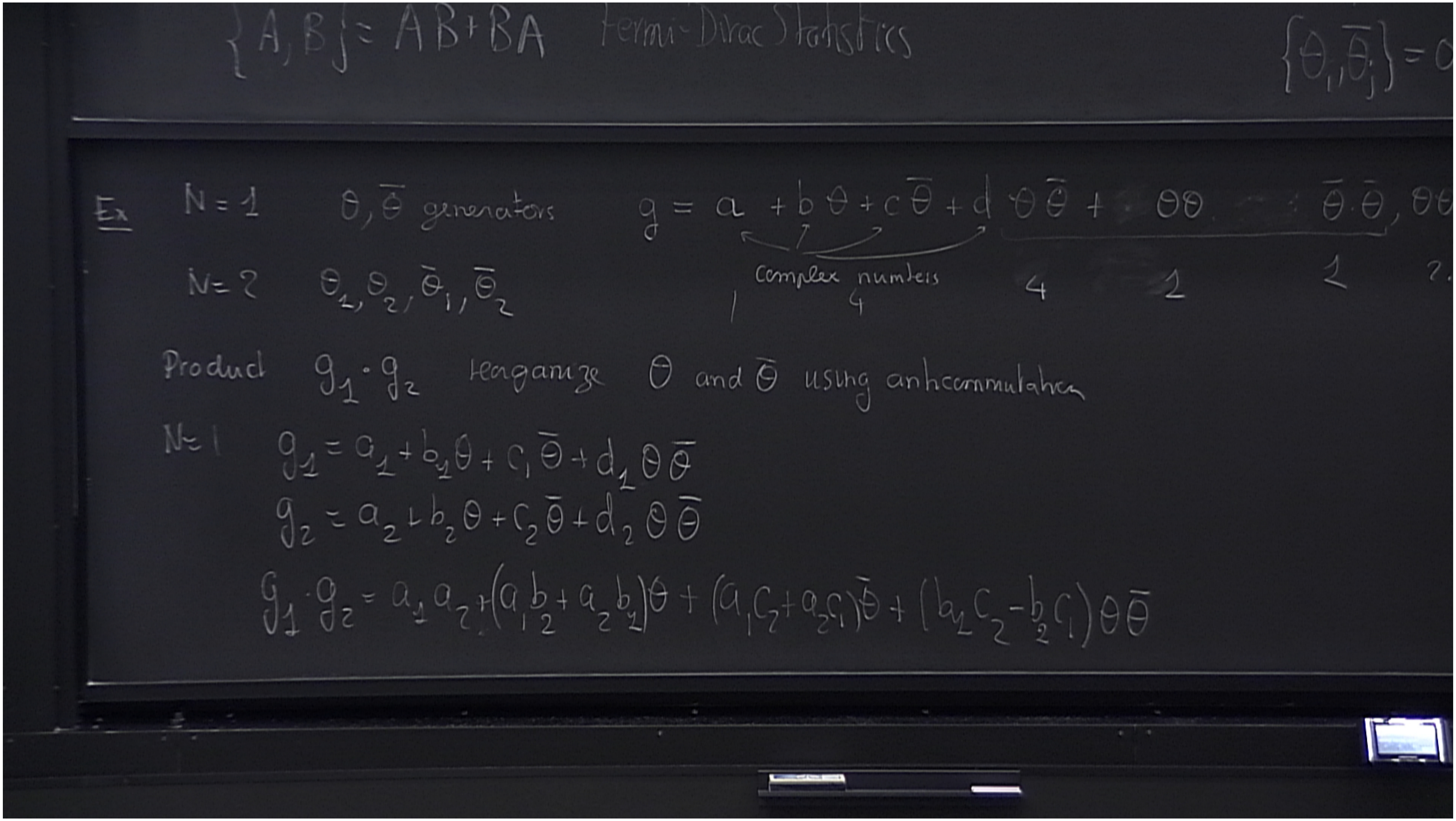
$\{\theta_i, \bar{\theta}_j\} = 0$

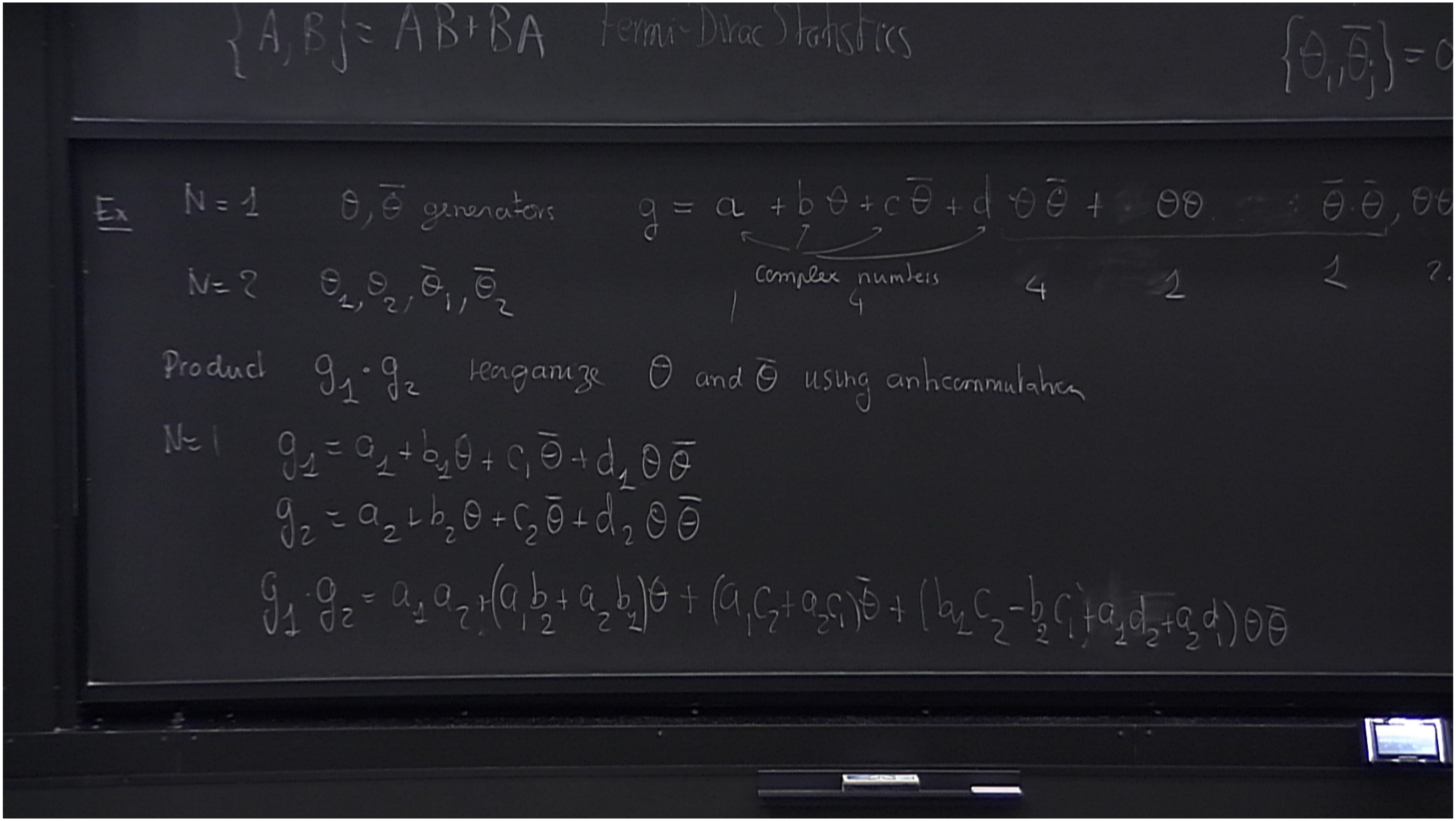
$g = a + b\theta + c\bar{\theta} + d\,\theta\bar{\theta} + \theta\theta \quad \bar{\theta}\cdot\bar{\theta}, \theta\theta\bar{\theta}, \theta\bar{\theta}\bar{\theta}, \theta\theta\bar{\theta}\bar{\theta}$

complex numbers

1 4

4 1 1 ?. 2 1





$\{A,B\} = AB + BA$ Fermi-Dirac Statistics $\{\theta_i,\bar{\theta}_j\} = 0$

Ex $N = 1$ $\theta, \bar{\theta}$ generators $g = a + b\,\theta + c\,\bar{\theta} + d\,\underbrace{\theta\bar{\theta} + \theta\theta \quad \bar{\theta}\cdot\bar{\theta}}, \theta\theta$

complex numbers

$N = 2$ $\theta_1, \theta_2, \bar{\theta}_1, \bar{\theta}_2$ 1 4 4 2 1 2.

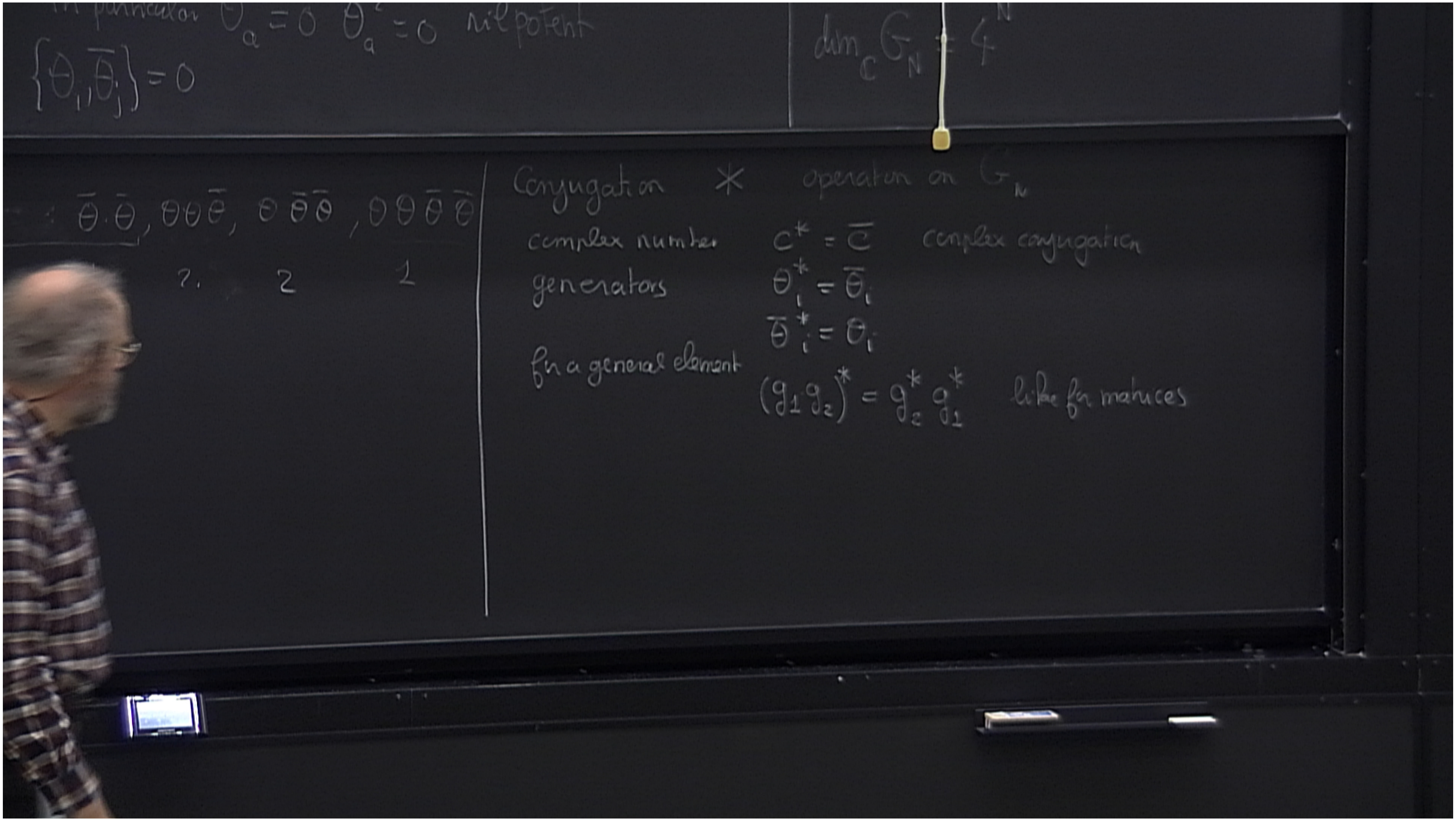
Product $g_1 \cdot g_2$ reorganize θ and $\bar{\theta}$ using anticommutation

$N = 1$

$$g_1 = a_1 + b_1\theta + c_1\bar{\theta} + d_1\,\theta\bar{\theta}$$

$$g_2 = a_2 + b_2\theta + c_2\bar{\theta} + d_2\,\theta\bar{\theta}$$

$$g_1 \cdot g_2 = a_1 a_2 + (a_1 b_2 + a_2 b_1)\theta + (a_1 c_2 + a_2 c_1)\bar{\theta} + (b_1 c_2 - b_2 c_1 + a_1 d_2 + a_2 d_1)\,\theta\bar{\theta}$$



in particular $\Theta_a = 0$ $\Theta_a^2 = 0$ nilpotent

$\{\Theta_i, \bar{\Theta}_j\} = 0$

$\dim_{\mathbb{C}} G_N = 4^N$

$\bar{\Theta}\cdot\bar{\Theta}, \Theta\Theta\bar{\Theta}, \Theta\bar{\Theta}\bar{\Theta}, \Theta\Theta\bar{\Theta}\bar{\Theta}$

?. 2 1

Conjugation $*$ operation on G_N

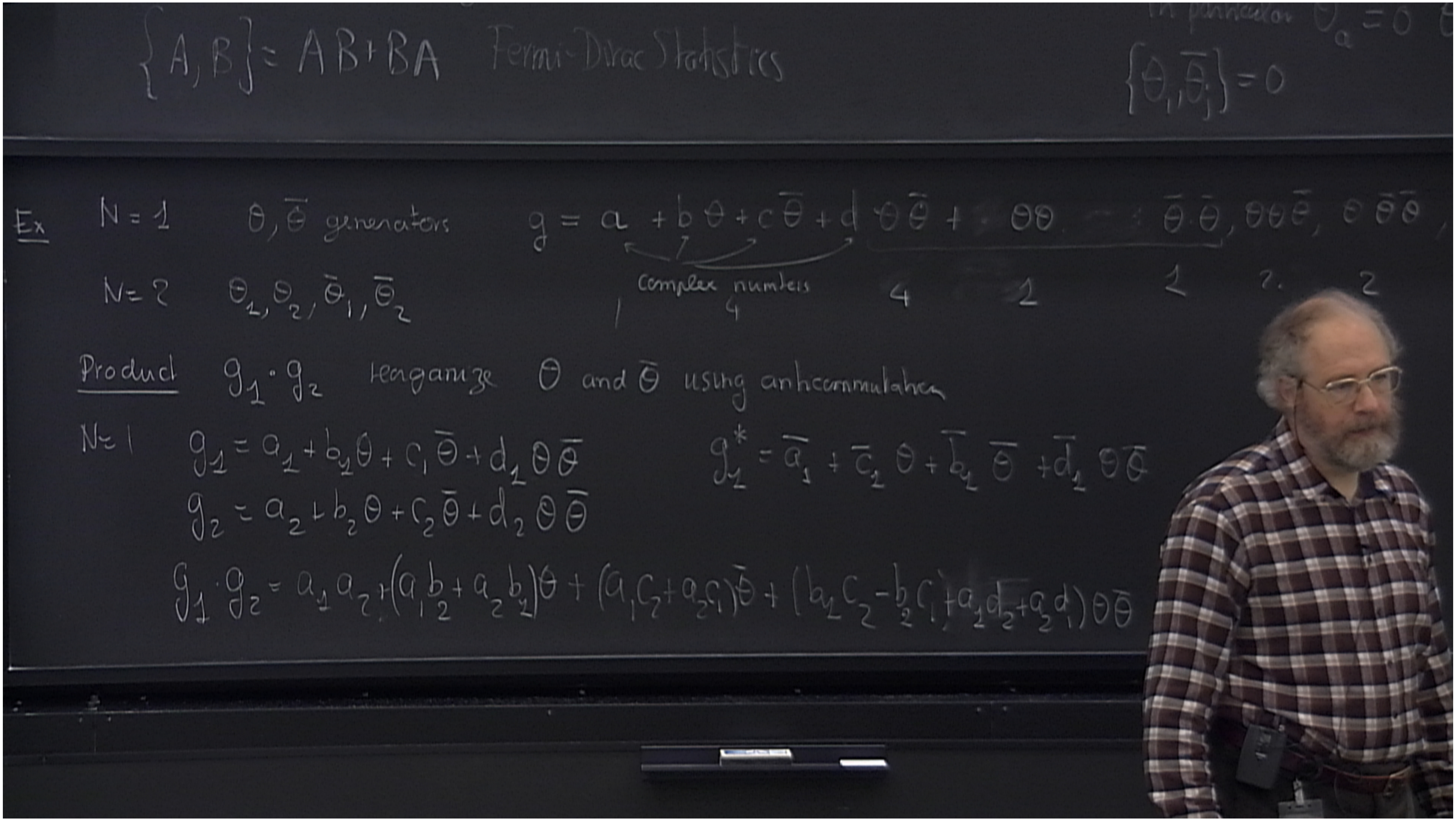
complex number $c^* = \bar{c}$ complex conjugation

generators $\Theta_i^* = \bar{\Theta}_i$

$\bar{\Theta}_i^* = \Theta_i$

for a general element

$(g_1 g_2)^* = g_2^* g_1^*$ like for matrices



$\{A,B\} = AB + BA$ Fermi-Dirac Statistics

In particular $\theta_a = 0$ θ

$\{\theta_i, \bar{\theta}_j\} = 0$

Ex $N = 1$ $\theta, \bar{\theta}$ generators $g = a + b\theta + c\bar{\theta} + d\,\theta\bar{\theta} + \theta\theta$ $\bar{\theta}\cdot\bar{\theta}, \theta\theta\bar{\theta}, \theta\bar{\theta}\bar{\theta},$

complex numbers

$N = 2$ $\theta_1, \theta_2, \bar{\theta}_1, \bar{\theta}_2$ 1 4 4 2 1 2. 2

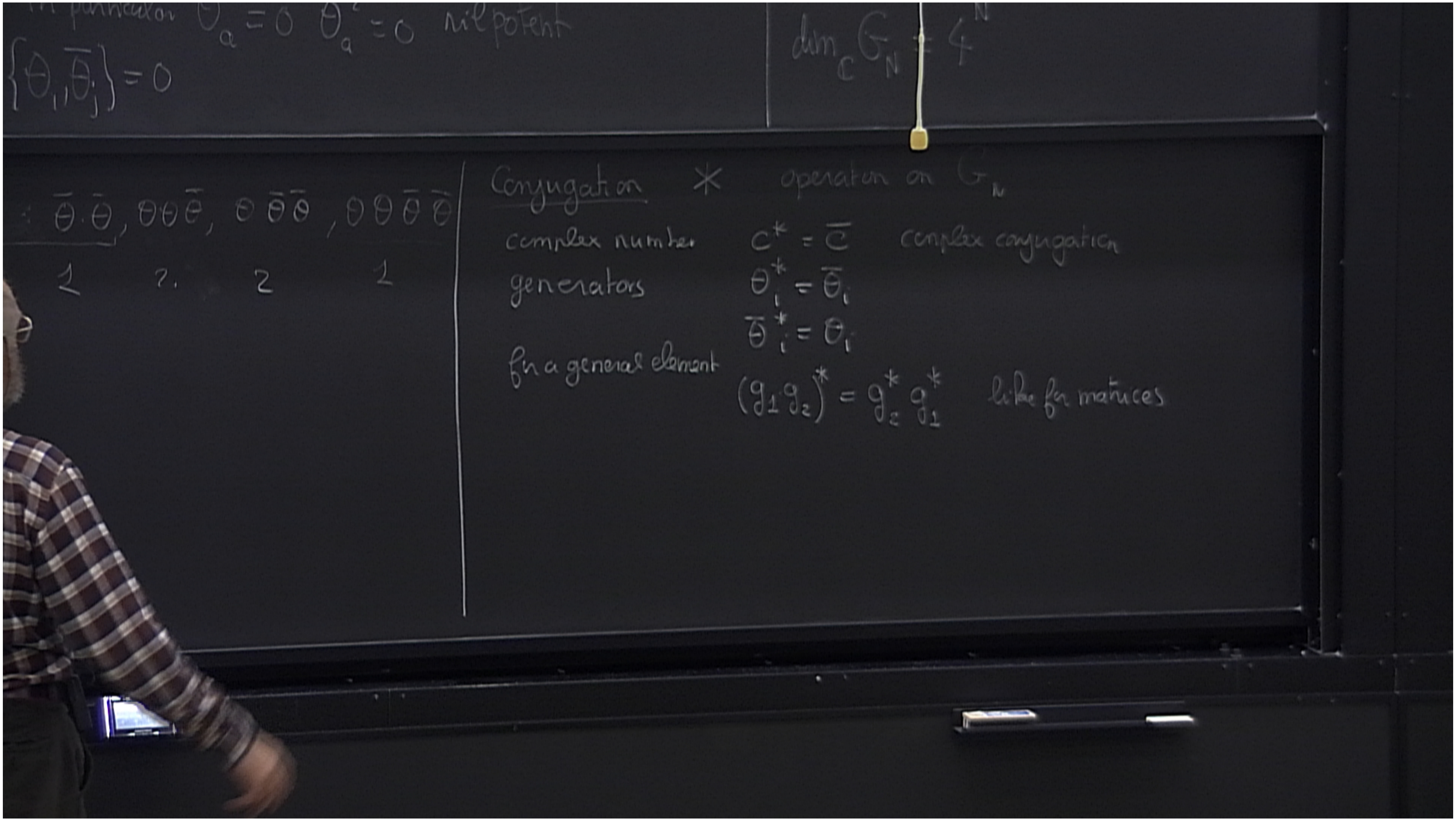
Product $g_1 \cdot g_2$ reorganize θ and $\bar{\theta}$ using anticommutation

$N = 1$

$$g_1 = a_1 + b_1\theta + c_1\bar{\theta} + d_1\theta\bar{\theta} \qquad g_1^* = \bar{a}_1 + \bar{c}_2\theta + \bar{b}_2\bar{\theta} + \bar{d}_1\theta\bar{\theta}$$

$$g_2 = a_2 + b_2\theta + c_2\bar{\theta} + d_2\theta\bar{\theta}$$

$$g_1 \cdot g_2 = a_1 a_2 + (a_1 b_2 + a_2 b_1)\theta + (a_1 c_2 + a_2 c_1)\bar{\theta} + (b_1 c_2 - b_2 c_1 + a_1 d_2 + a_2 d_1)\theta\bar{\theta}$$



particular $\theta_a = 0$ $\theta_a^2 = 0$ nilpotent

$\{\theta_i, \bar\theta_j\} = 0$

$\dim_{\mathbb{C}} G_N = 4^N$

$\bar\theta \cdot \bar\theta, \theta\theta\bar\theta, \theta\bar\theta\bar\theta, \theta\theta\bar\theta\bar\theta$

1 2. 2 1

Conjugation $*$ operation on G_N

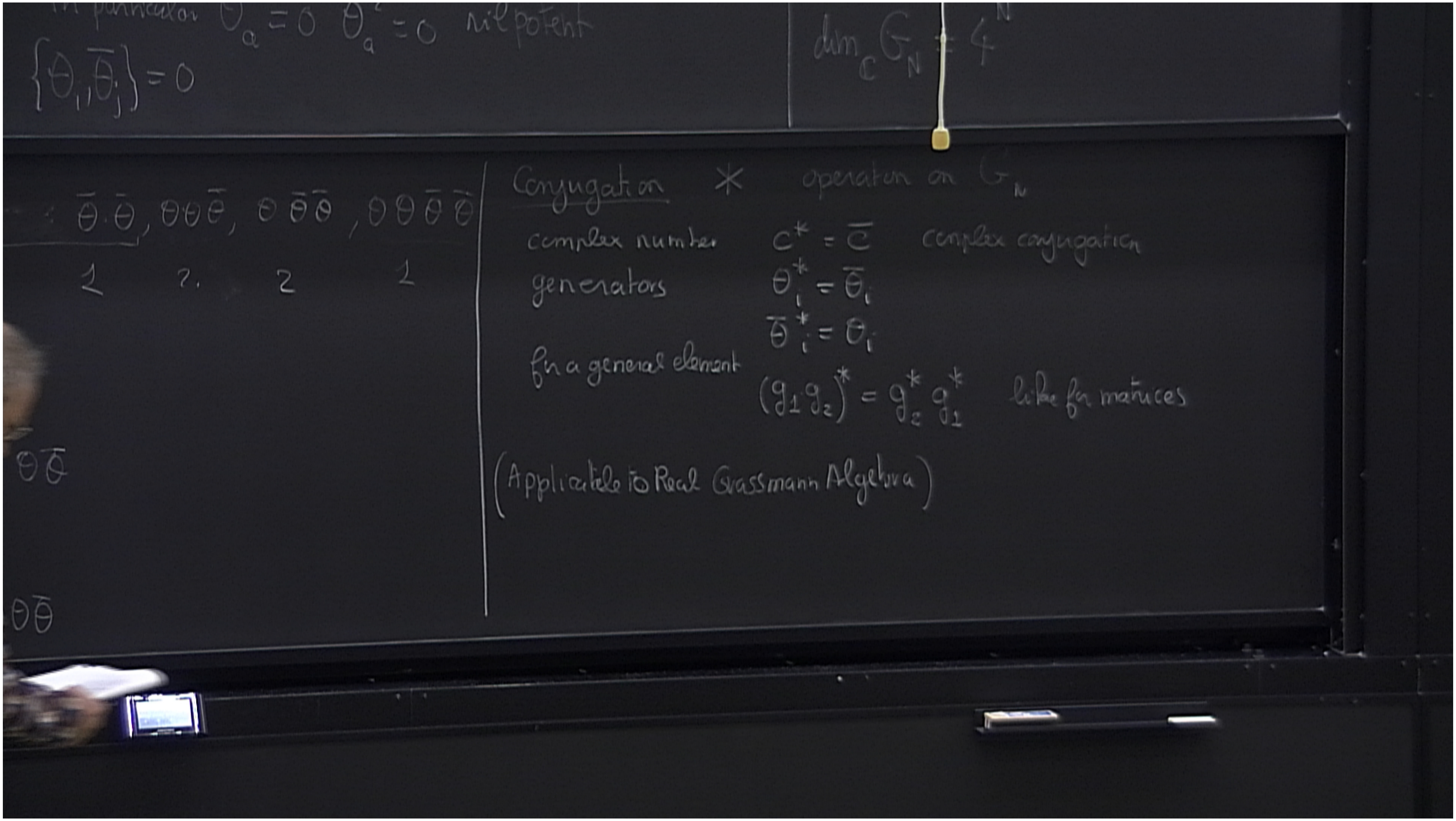
complex number $c^* = \bar{c}$ complex conjugation

generators $\theta_i^* = \bar\theta_i$

$\bar\theta_i^* = \theta_i$

for a general element

$(g_1 \cdot g_2)^* = g_2^* g_1^*$ like for matrices



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$\{\theta_i, \bar{\theta}_j\} = 0$

$\dim_{\mathbb{C}} G_N = 4^N$

$\bar{\theta}\cdot\bar{\theta}, \theta\theta\bar{\theta}, \theta\bar{\theta}\bar{\theta}, \theta\theta\bar{\theta}\bar{\theta}$

1 2. 2 1

$\theta\bar{\theta}$

$\theta\bar{\theta}$

Conjugation $*$ operation on G_N

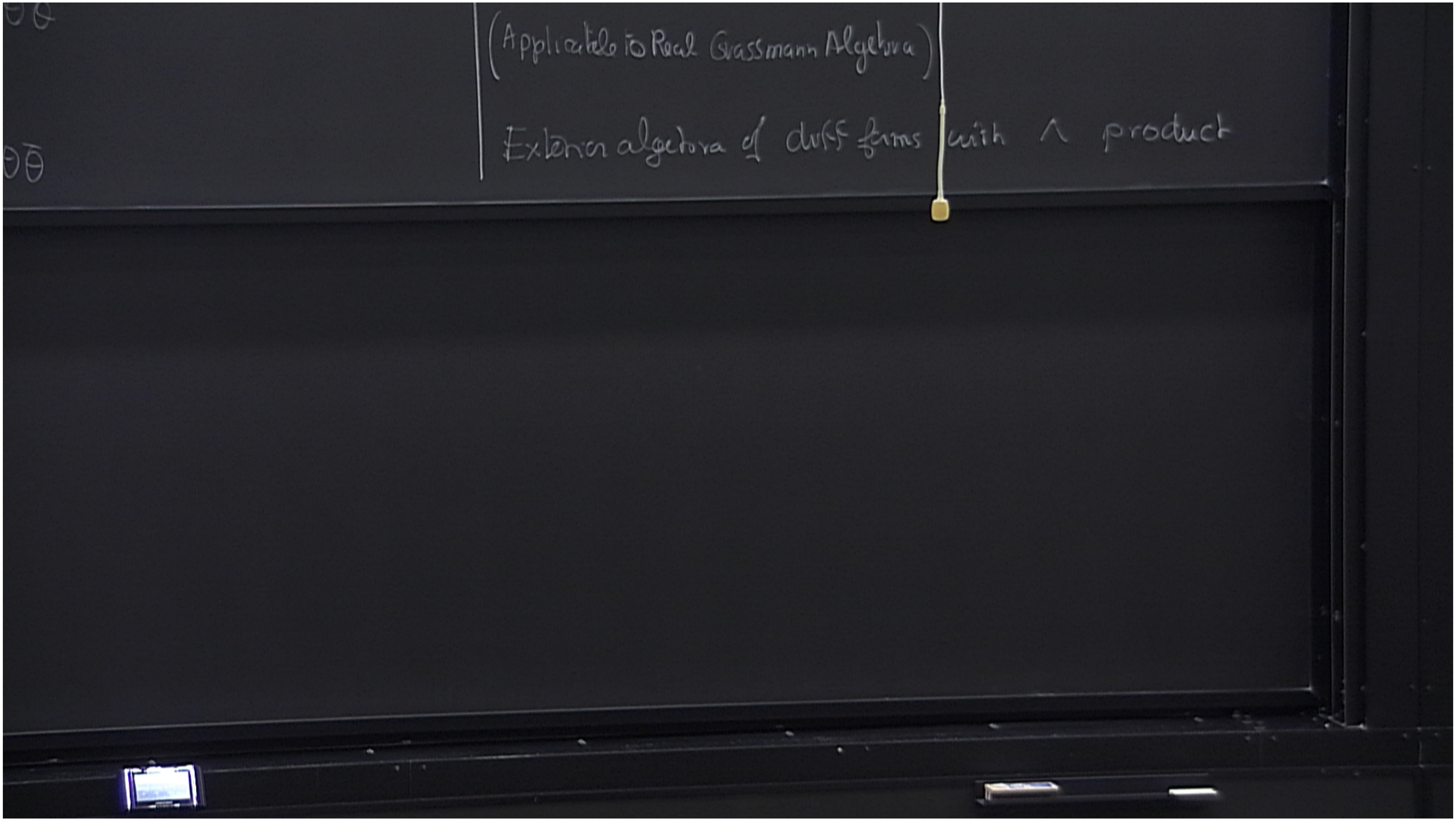
complex number $c^* = \bar{c}$ complex conjugation

generators $\theta_i^* = \bar{\theta}_i$

$\bar{\theta}_i^* = \theta_i$

for a general element $(g_1 g_2)^* = g_2^* g_1^*$ like for matrices

(Applicable to Real Grassmann Algebra)

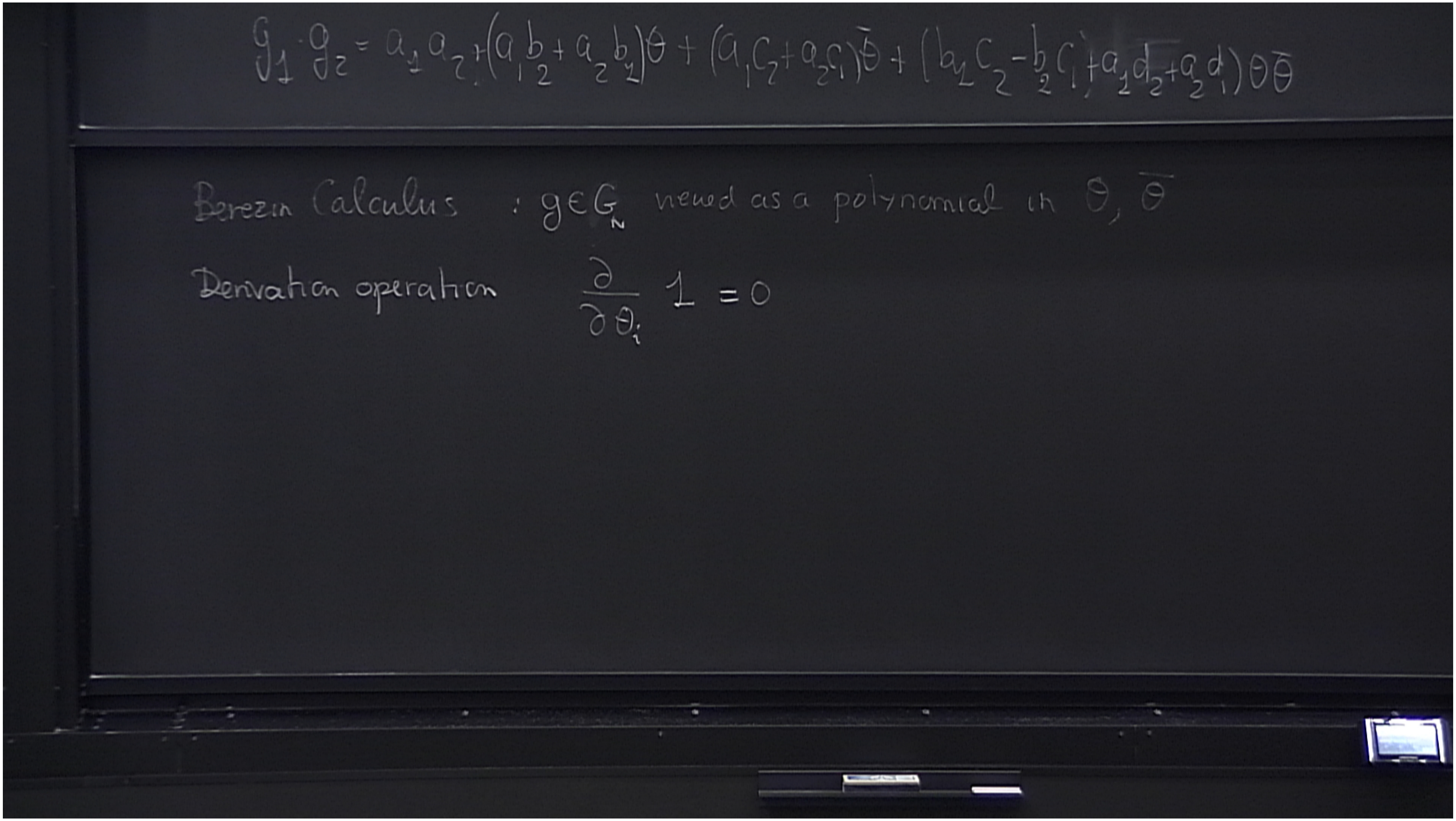


$\theta\theta$

$\theta\bar{\theta}$

(Applicable to Real Grassmann Algebra)

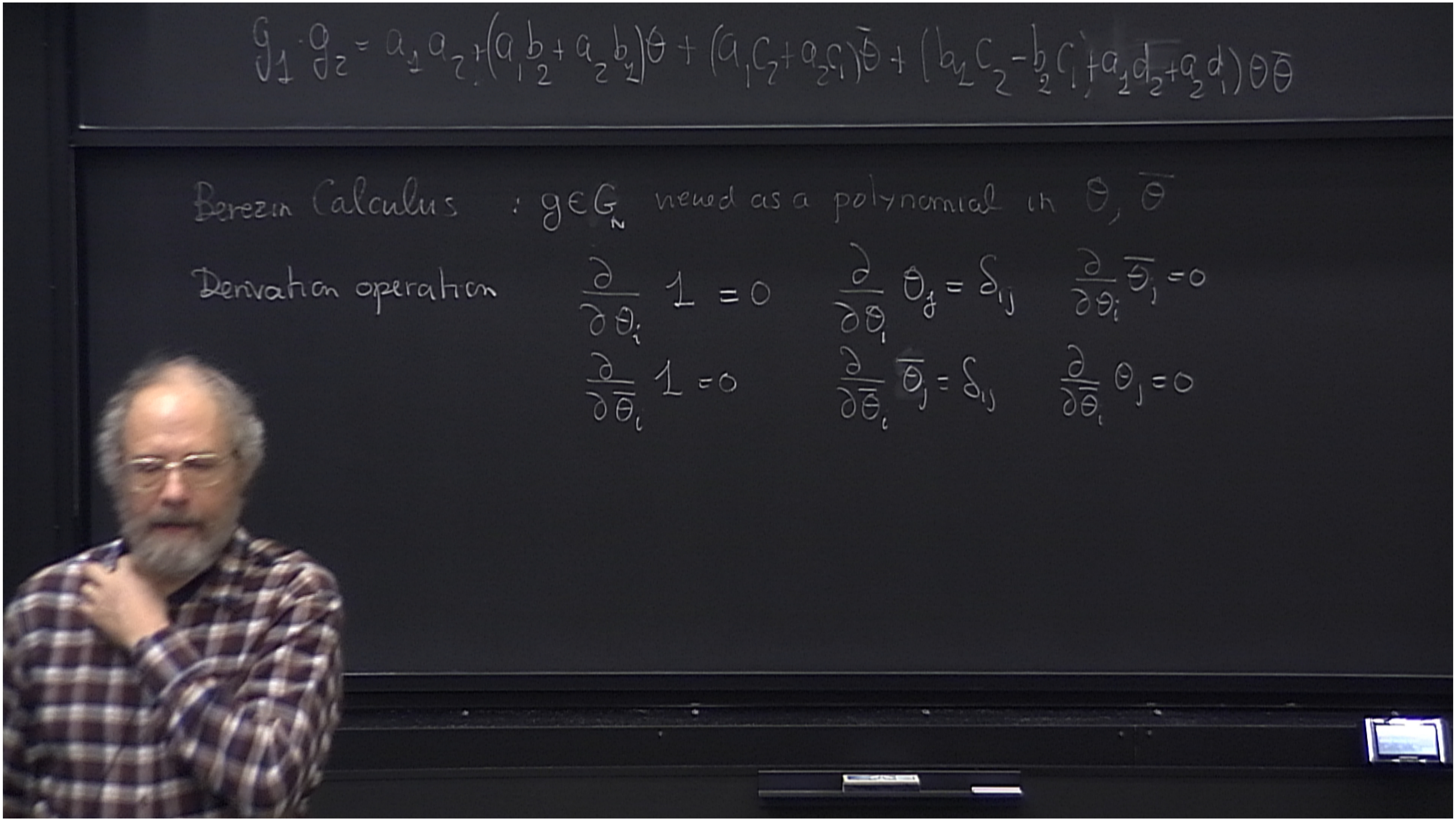
Exterior algebra of diff forms with ∧ product



$$g_1 \cdot g_2 = a_1 a_2 + (a_1 b_2 + a_2 b_1)\theta + (a_1 c_2 + a_2 c_1)\bar{\theta} + (b_1 c_2 - b_2 c_1 + a_1 d_2 + a_2 d_1)\theta\bar{\theta}$$

Berezin Calculus : $g \in G_N$ viewed as a polynomial in $\theta, \bar{\theta}$

Derivation operation $\frac{\partial}{\partial \theta_i} 1 = 0$



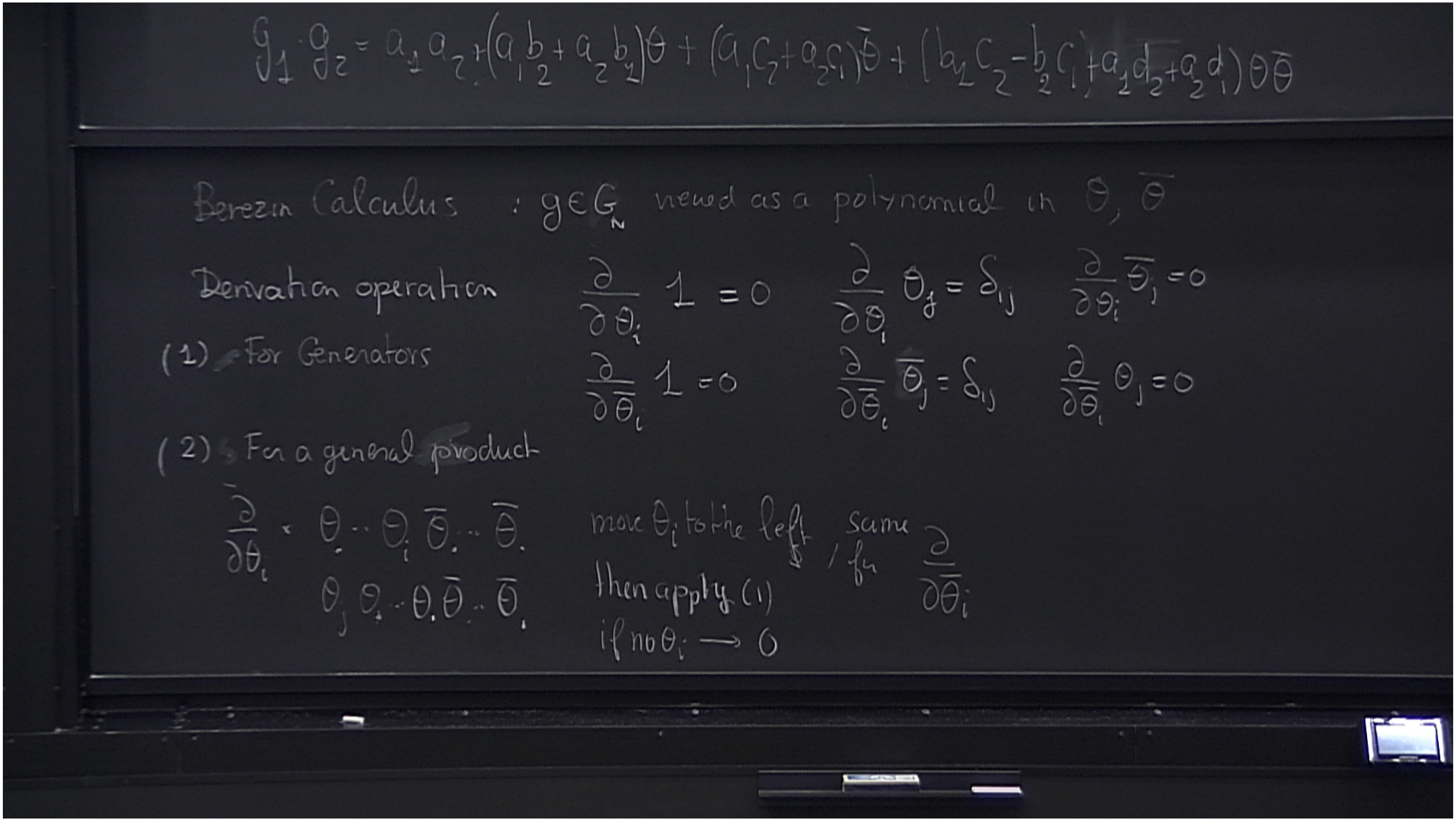
$$g_1 \cdot g_2 = a_1 a_2 + (a_1 b_2 + a_2 b_1)\theta + (a_1 c_2 + a_2 c_1)\bar{\theta} + (b_1 c_2 - b_2 c_1 + a_1 d_2 + a_2 d_1)\theta\bar{\theta}$$

Berezin Calculus : $g \in G_N$ viewed as a polynomial in $\theta, \bar{\theta}$

Derivation operation

$$\frac{\partial}{\partial \theta_i} 1 = 0 \qquad \frac{\partial}{\partial \theta_i} \theta_j = \delta_{ij} \qquad \frac{\partial}{\partial \theta_i} \bar{\theta}_j = 0$$

$$\frac{\partial}{\partial \bar{\theta}_i} 1 = 0 \qquad \frac{\partial}{\partial \bar{\theta}_i} \bar{\theta}_j = \delta_{ij} \qquad \frac{\partial}{\partial \bar{\theta}_i} \theta_j = 0$$



$$g_1 \cdot g_2 = a_1 a_2 + (a_1 b_2 + a_2 b_1)\theta + (a_1 c_2 + a_2 c_1)\bar\theta + (b_1 c_2 - b_2 c_1 + a_1 d_2 + a_2 d_1)\theta\bar\theta$$

Berezin Calculus : $g \in G_N$ viewed as a polynomial in $\theta, \bar\theta$

Derivation operation

(1) For Generators

$$\frac{\partial}{\partial\theta_i} 1 = 0 \qquad \frac{\partial}{\partial\theta_i} \theta_j = \delta_{ij} \qquad \frac{\partial}{\partial\theta_i} \bar\theta_j = 0$$

$$\frac{\partial}{\partial\bar\theta_i} 1 = 0 \qquad \frac{\partial}{\partial\bar\theta_i} \bar\theta_j = \delta_{ij} \qquad \frac{\partial}{\partial\bar\theta_i} \theta_j = 0$$

(2) For a general product

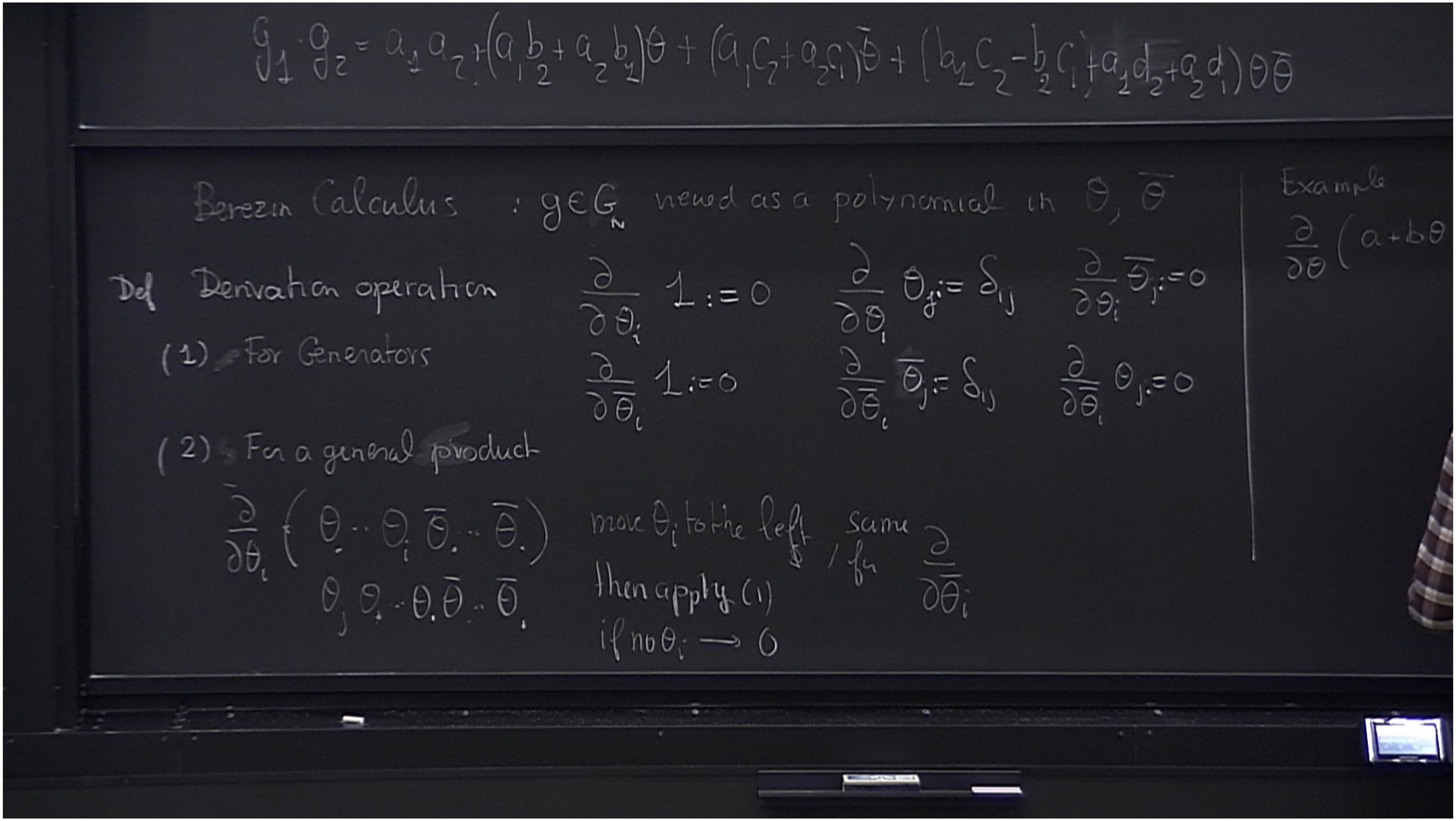
$$\frac{\partial}{\partial\theta_i} \times \theta_{\cdot} \cdots \theta_i \bar\theta_{\cdot} \cdots \bar\theta_{\cdot}$$

$$\theta_j \theta_{\cdot} \cdots \theta_{\cdot} \bar\theta_{\cdot} \cdots \bar\theta_{\cdot}$$

move θ_i to the left, same / for $\frac{\partial}{\partial\bar\theta_i}$

then apply (1)

if no $\theta_i \longrightarrow 0$



$$g_1 \cdot g_2 = a_1 a_2 + (a_1 b_2 + a_2 b_1)\theta + (a_1 c_2 + a_2 c_1)\bar\theta + (b_1 c_2 - b_2 c_1 + a_1 d_2 + a_2 d_1)\theta\bar\theta$$

Berezin Calculus : $g \in G_N$ viewed as a polynomial in $\theta, \bar\theta$

Def Derivation operation

(1) For Generators

$$\frac{\partial}{\partial\theta_i} 1 := 0 \qquad \frac{\partial}{\partial\theta_i}\theta_j := \delta_{ij} \qquad \frac{\partial}{\partial\theta_i}\bar\theta_j := 0$$

$$\frac{\partial}{\partial\bar\theta_i} 1 := 0 \qquad \frac{\partial}{\partial\bar\theta_i}\bar\theta_j := \delta_{ij} \qquad \frac{\partial}{\partial\bar\theta_i}\theta_j := 0$$

(2) For a general product

$$\frac{\partial}{\partial\theta_i}\left(\theta_\cdot \cdots \theta_i \bar\theta_\cdot \cdots \bar\theta_\cdot\right)$$

$$\theta_j \theta_\cdot \cdots \theta_\cdot \bar\theta \cdots \bar\theta_\cdot$$

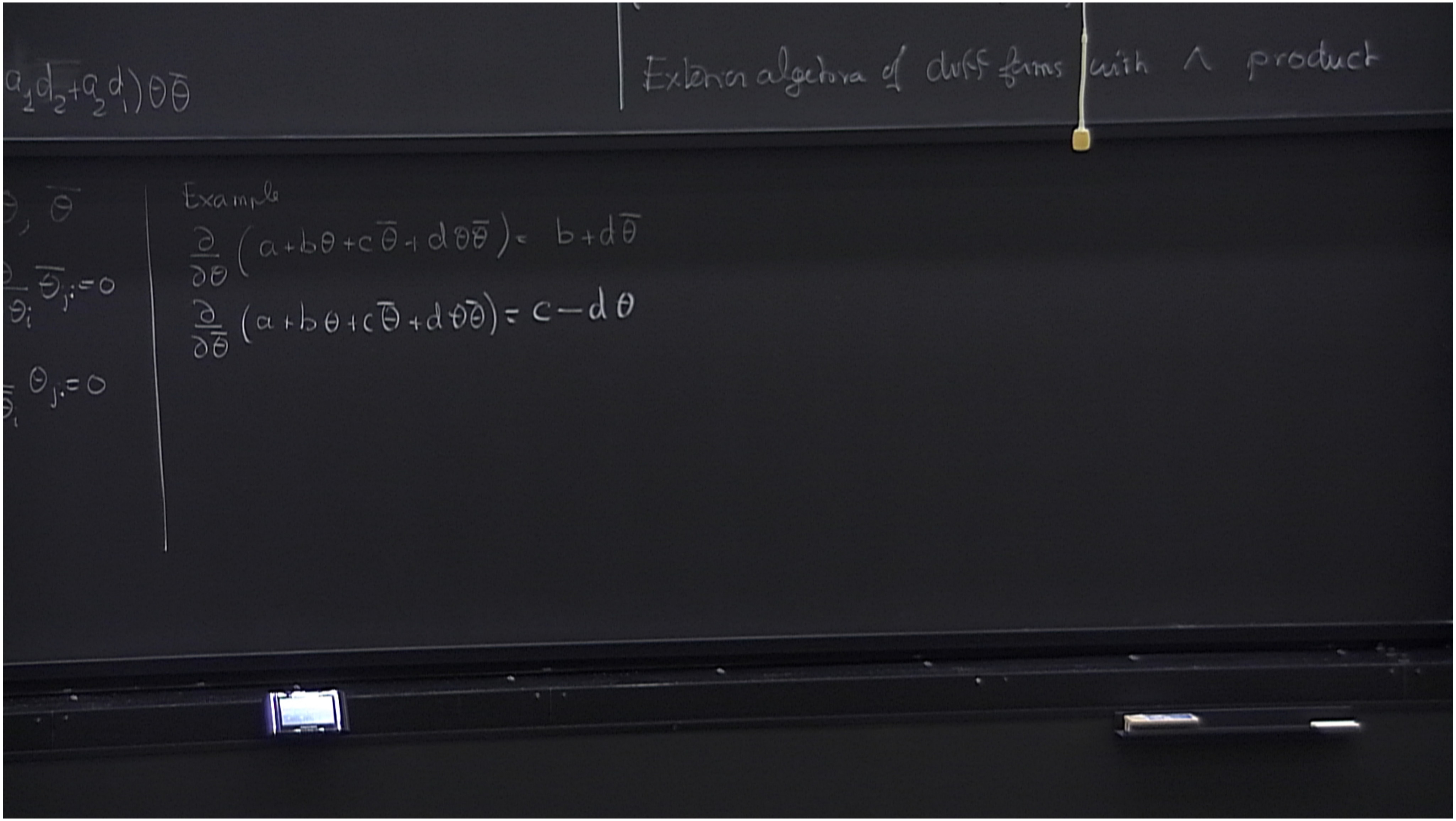
move θ_i to the left, same for $\frac{\partial}{\partial\bar\theta_i}$

then apply (1)

if no $\theta_i \longrightarrow 0$

Example

$\frac{\partial}{\partial\theta}\left(a + b\theta\right.$



$a_1 d_2 + a_2 d_1) \theta \bar{\theta}$

Exterior algebra of diff forms with $\wedge$ product

$\theta, \bar{\theta}$

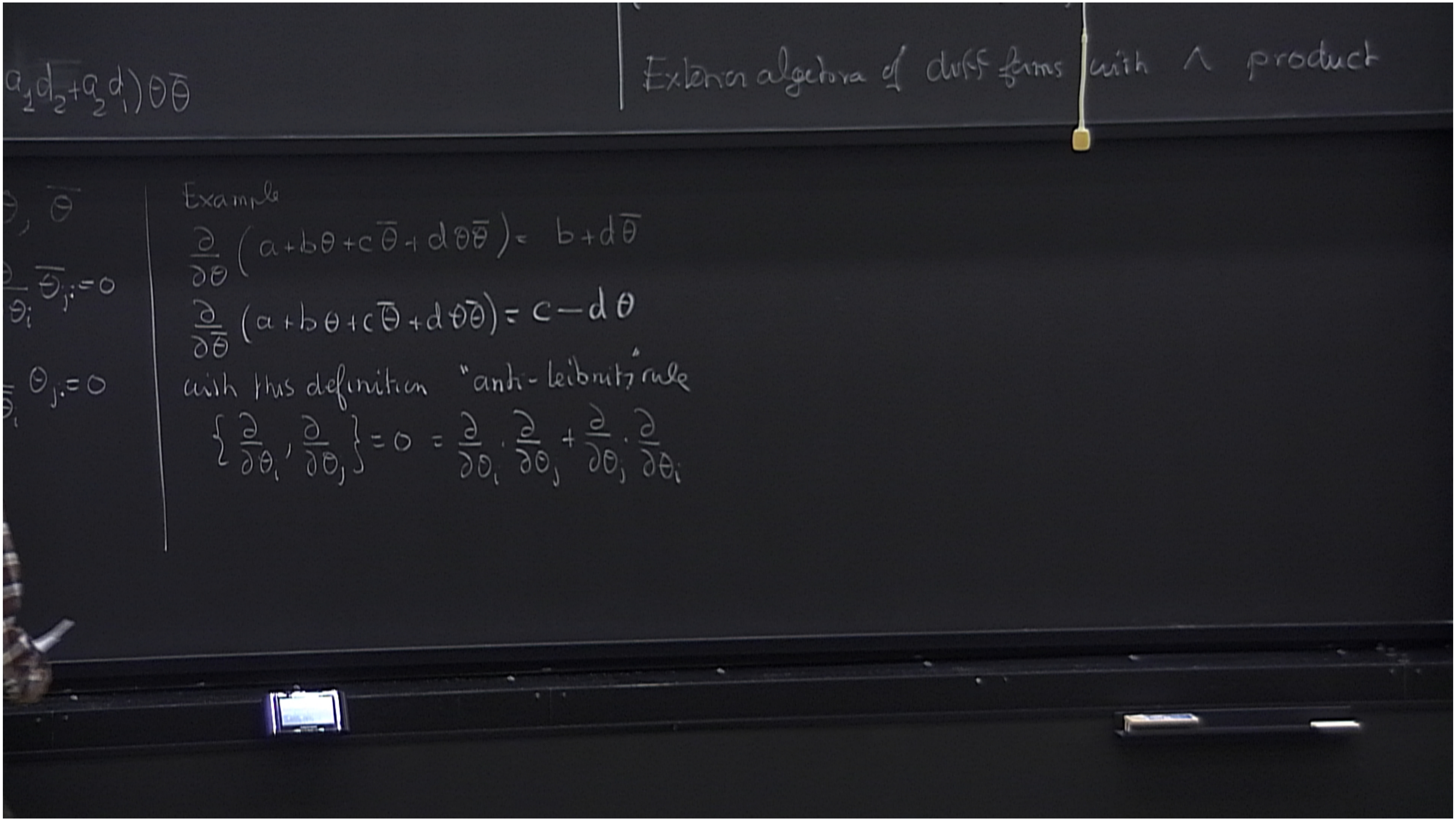
$\theta_i \quad \bar{\theta}_j := 0$

$\bar{\theta}_i \quad \theta_j := 0$

Example

$$\frac{\partial}{\partial \theta}\left(a + b\theta + c\bar{\theta} + d\theta\bar{\theta}\right) = b + d\bar{\theta}$$

$$\frac{\partial}{\partial \bar{\theta}}\left(a + b\theta + c\bar{\theta} + d\theta\bar{\theta}\right) = c - d\theta$$



$a_1 d_2 + a_2 d_1)\theta\bar{\theta}$

Exterior algebra of diff forms with $\wedge$ product

$\theta, \bar{\theta}$

$\bar{\theta}_j := 0$

$\theta_j := 0$

Example

$$\frac{\partial}{\partial\theta}\left(a + b\theta + c\bar{\theta} + d\theta\bar{\theta}\right) = b + d\bar{\theta}$$

$$\frac{\partial}{\partial\bar{\theta}}\left(a + b\theta + c\bar{\theta} + d\theta\bar{\theta}\right) = c - d\theta$$

with this definition "anti-leibnitz" rule

$$\left\{\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j}\right\} = 0 = \frac{\partial}{\partial\theta_i}\cdot\frac{\partial}{\partial\theta_j} + \frac{\partial}{\partial\theta_j}\cdot\frac{\partial}{\partial\theta_i}$$

$a_1 d_2 + a_2 d_1)\theta\bar{\theta}$

Exterior algebra of diff forms with $\wedge$ product

$\theta, \bar{\theta}$

$\frac{}{\theta_i}\bar{\theta}_j = 0$

$\overline{\theta}_i \; \theta_j = 0$

Example

$$\frac{\partial}{\partial\theta}\left(a + b\theta + c\bar{\theta} + d\theta\bar{\theta}\right) = b + d\bar{\theta}$$

$$\frac{\partial}{\partial\bar{\theta}}\left(a + b\theta + c\bar{\theta} + d\theta\bar{\theta}\right) = c - d\theta$$

with this definition "anti-leibnitz" rule

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$$\left\{\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\bar{\theta}_j}\right\} = 0, \left\{\frac{\partial}{\partial\bar{\theta}_i}, \frac{\partial}{\partial\bar{\theta}_j}\right\} = 0$$

$d_1)\theta\bar{\theta}$

$=0$

Exterior algebra of diff forms with $\wedge$ product

Example

$$\frac{\partial}{\partial\theta}\left(a+b\theta+c\bar{\theta}+d\theta\bar{\theta}\right)= b+d\bar{\theta}$$

$$\frac{\partial}{\partial\bar{\theta}}\left(a+b\theta+c\bar{\theta}+d\theta\bar{\theta}\right)= c-d\theta$$

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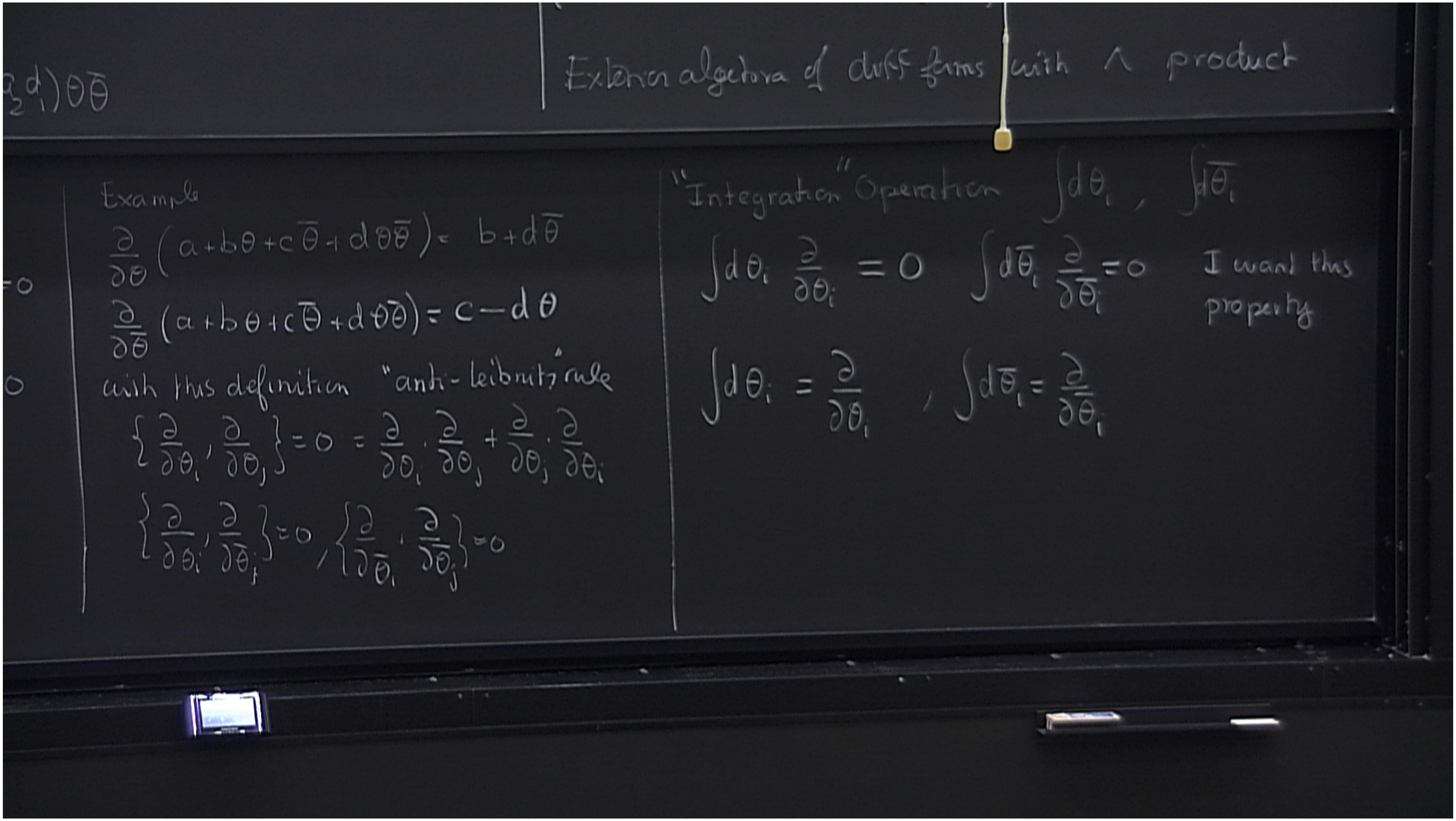
$$\left\{\frac{\partial}{\partial\theta_i},\frac{\partial}{\partial\theta_j}\right\}=0=\frac{\partial}{\partial\theta_i}\cdot\frac{\partial}{\partial\theta_j}+\frac{\partial}{\partial\theta_j}\cdot\frac{\partial}{\partial\theta_i}$$

$$\left\{\frac{\partial}{\partial\theta_i},\frac{\partial}{\partial\bar{\theta}_j}\right\}=0,\ \left\{\frac{\partial}{\partial\bar{\theta}_i},\frac{\partial}{\partial\bar{\theta}_j}\right\}=0$$

Integration Operation $\int d\theta_i$, $\int d\bar{\theta}_i$

$\int d\theta_i\frac{\partial}{\partial\theta_i}=0$ $\int d\bar{\theta}_i\frac{\partial}{\partial\bar{\theta}_i}=0$ I want this



$q_2 d_1)\theta\bar{\theta}$

Exterior algebra of diff forms with $\wedge$ product

$=0$

0

Example

$$\frac{\partial}{\partial\theta}\left(a+b\theta+c\bar{\theta}+d\theta\bar{\theta}\right)=b+d\bar{\theta}$$

$$\frac{\partial}{\partial\bar{\theta}}\left(a+b\theta+c\bar{\theta}+d\theta\bar{\theta}\right)=c-d\theta$$

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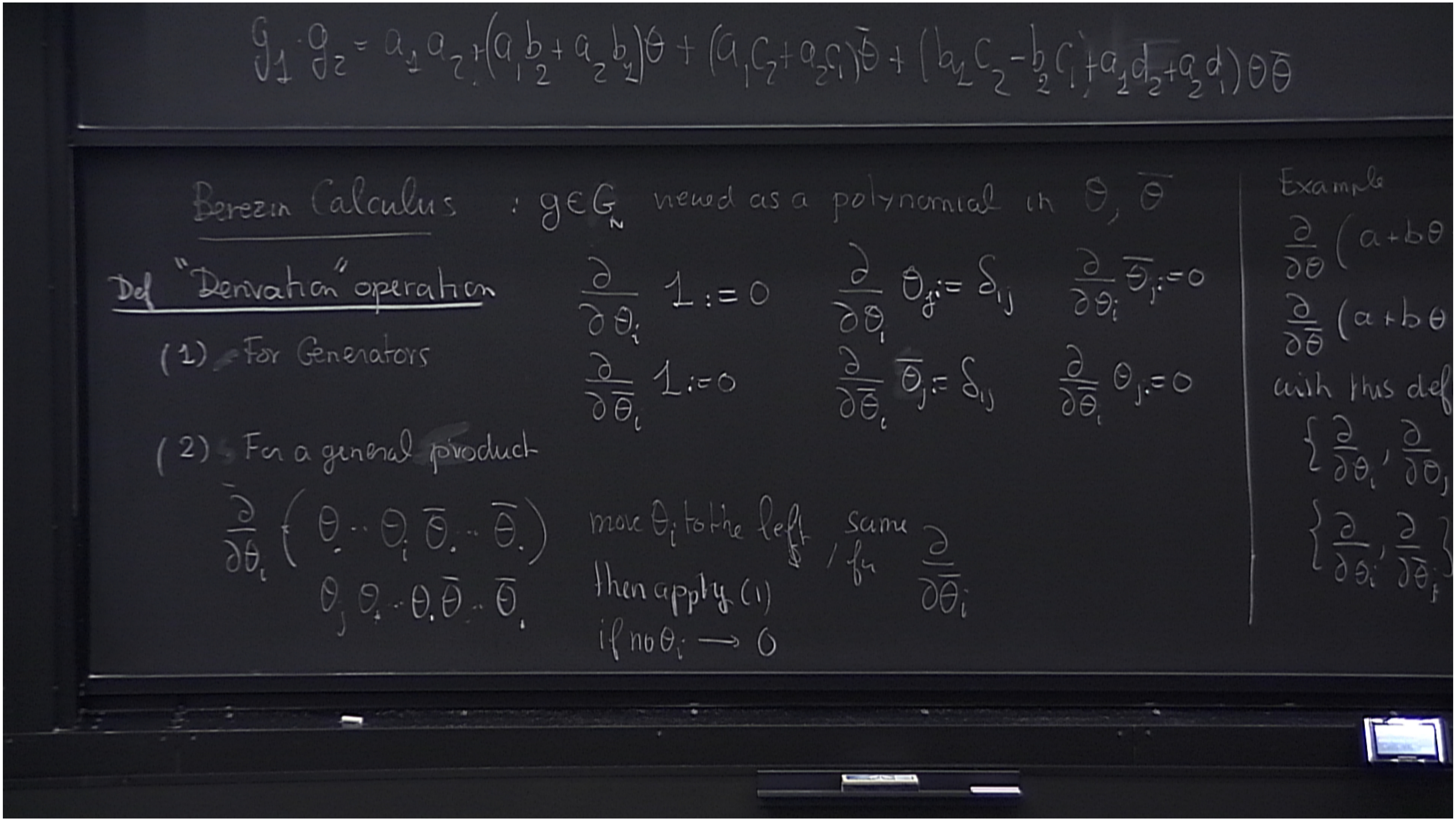
$$\left\{\frac{\partial}{\partial\theta_i},\frac{\partial}{\partial\bar{\theta}_j}\right\}=0,\left\{\frac{\partial}{\partial\bar{\theta}_i},\frac{\partial}{\partial\bar{\theta}_j}\right\}=0$$

"Integration" Operation $\int d\theta_i$, $\int d\bar{\theta}_i$

$$\int d\theta_i\,\frac{\partial}{\partial\theta_i}=0\qquad\int d\bar{\theta}_i\,\frac{\partial}{\partial\bar{\theta}_i}=0$$

I want this property

$$\int d\theta_i=\frac{\partial}{\partial\theta_i},\quad\int d\bar{\theta}_i=\frac{\partial}{\partial\bar{\theta}_i}$$



$$g_1 \cdot g_2 = a_1 a_2 + (a_1 b_2 + a_2 b_1)\theta + (a_1 c_2 + a_2 c_1)\bar\theta + (b_1 c_2 - b_2 c_1 + a_1 d_2 + a_2 d_1)\theta\bar\theta$$

Berezin Calculus : $g \in G_N$ viewed as a polynomial in $\theta, \bar\theta$

Def "Derivation" operation

(1) For Generators

$$\frac{\partial}{\partial\theta_i} 1 := 0 \qquad \frac{\partial}{\partial\theta_i}\theta_j := \delta_{ij} \qquad \frac{\partial}{\partial\theta_i}\bar\theta_j := 0$$

$$\frac{\partial}{\partial\bar\theta_i} 1 := 0 \qquad \frac{\partial}{\partial\bar\theta_i}\bar\theta_j := \delta_{ij} \qquad \frac{\partial}{\partial\bar\theta_i}\theta_j := 0$$

(2) For a general product

$$\frac{\partial}{\partial\theta_i}\left(\theta_\cdot \cdots \theta_i \bar\theta_\cdot \cdots \bar\theta_\cdot\right)$$

$$\theta_j \theta_\cdot \cdots \theta_\cdot \bar\theta \cdots \bar\theta_\cdot$$

move θ_i to the left

then apply (1)

if no $\theta_i \longrightarrow 0$

same / for $\frac{\partial}{\partial\bar\theta_i}$

Example

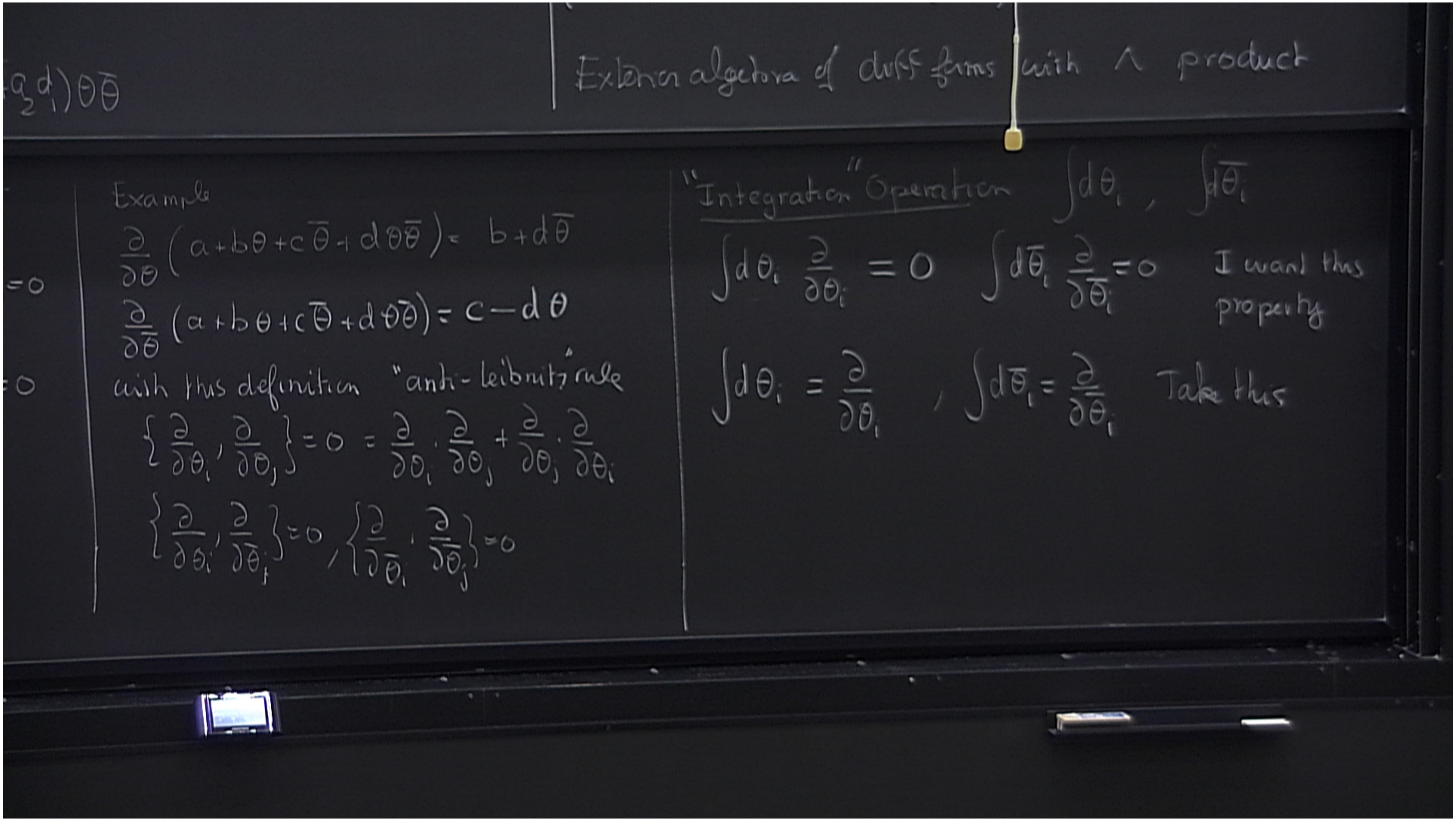
$\frac{\partial}{\partial\theta}\left(a + b\theta\right.$

$\frac{\partial}{\partial\bar\theta}\left(a + b\theta\right.$

with this def

$\left\{\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j}\right.$

$\left\{\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\bar\theta_j}\right\}$



$a_2 d_1)\theta\bar{\theta}$

$=0$

$=0$

Exterior algebra of diff forms with $\wedge$ product

Example

$$\frac{\partial}{\partial\theta}\left(a+b\theta+c\bar{\theta}+d\theta\bar{\theta}\right)= b+d\bar{\theta}$$

$$\frac{\partial}{\partial\bar{\theta}}\left(a+b\theta+c\bar{\theta}+d\theta\bar{\theta}\right)= c-d\theta$$

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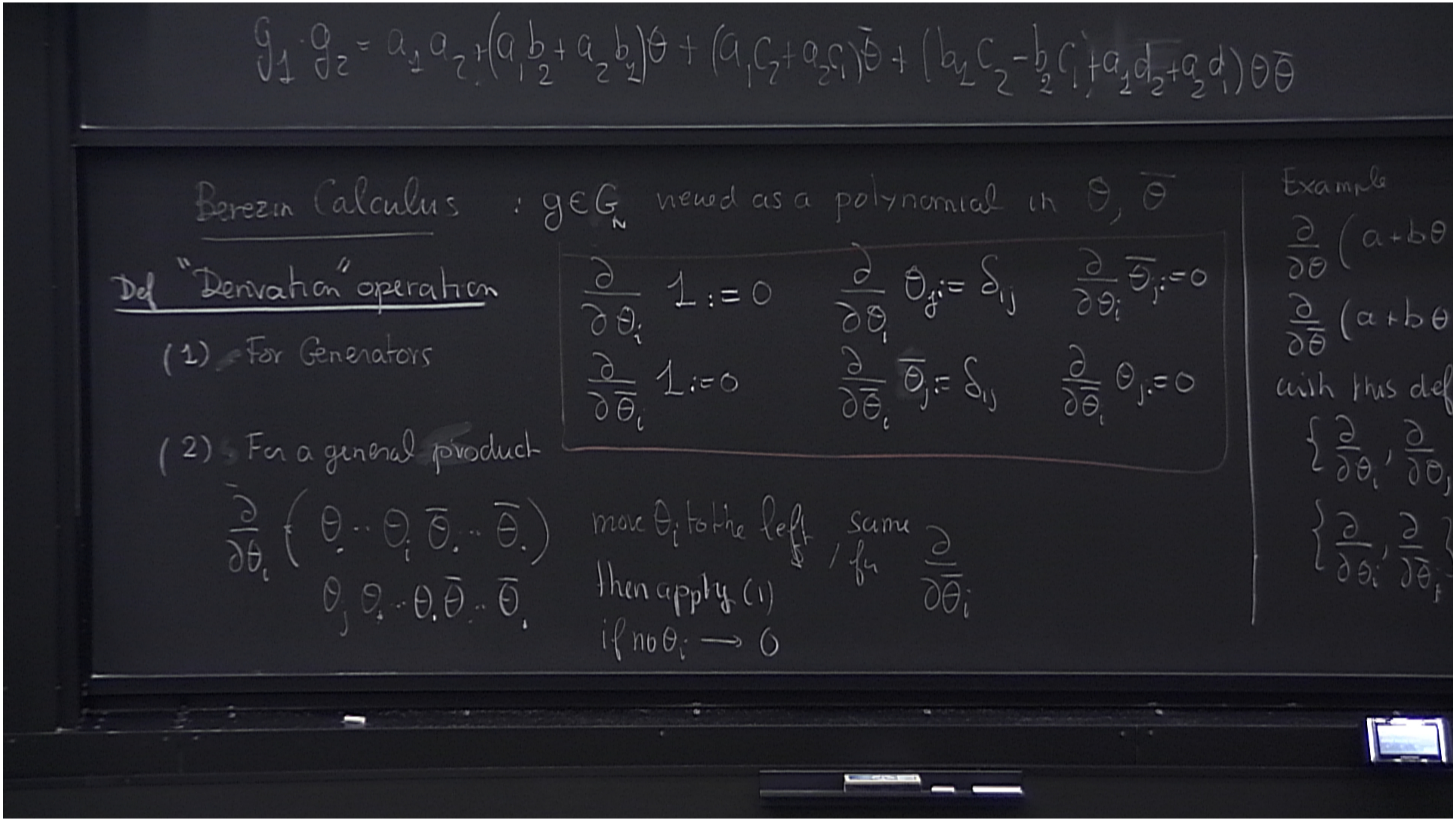
$$\int d\theta_i \frac{\partial}{\partial\theta_i}=0 \qquad \int d\bar{\theta}_i\frac{\partial}{\partial\bar{\theta}_i}=0$$

I want this property

$$\int d\theta_i = \frac{\partial}{\partial\theta_i}, \quad \int d\bar{\theta}_i=\frac{\partial}{\partial\bar{\theta}_i}$$

Take this



$$g_1 \cdot g_2 = a_1 a_2 + (a_1 b_2 + a_2 b_1)\theta + (a_1 c_2 + a_2 c_1)\bar{\theta} + (b_1 c_2 - b_2 c_1 + a_1 d_2 + a_2 d_1)\theta\bar{\theta}$$

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Def "Derivation" operation

(1) For Generators

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(2) For a general product

$$\frac{\partial}{\partial \theta_i} \left(\theta_{\cdot} \cdots \theta_i \, \bar{\theta}_{\cdot} \cdots \bar{\theta}_{\cdot} \right)$$

$$\theta_j \, \theta_{\cdot} \cdots \theta_{\cdot} \bar{\theta}_{\cdot} \cdots \bar{\theta}_{\cdot}$$

move θ_i to the left, same for $\frac{\partial}{\partial \bar{\theta}_i}$

then apply (1)

if no $\theta_i \longrightarrow 0$

Example

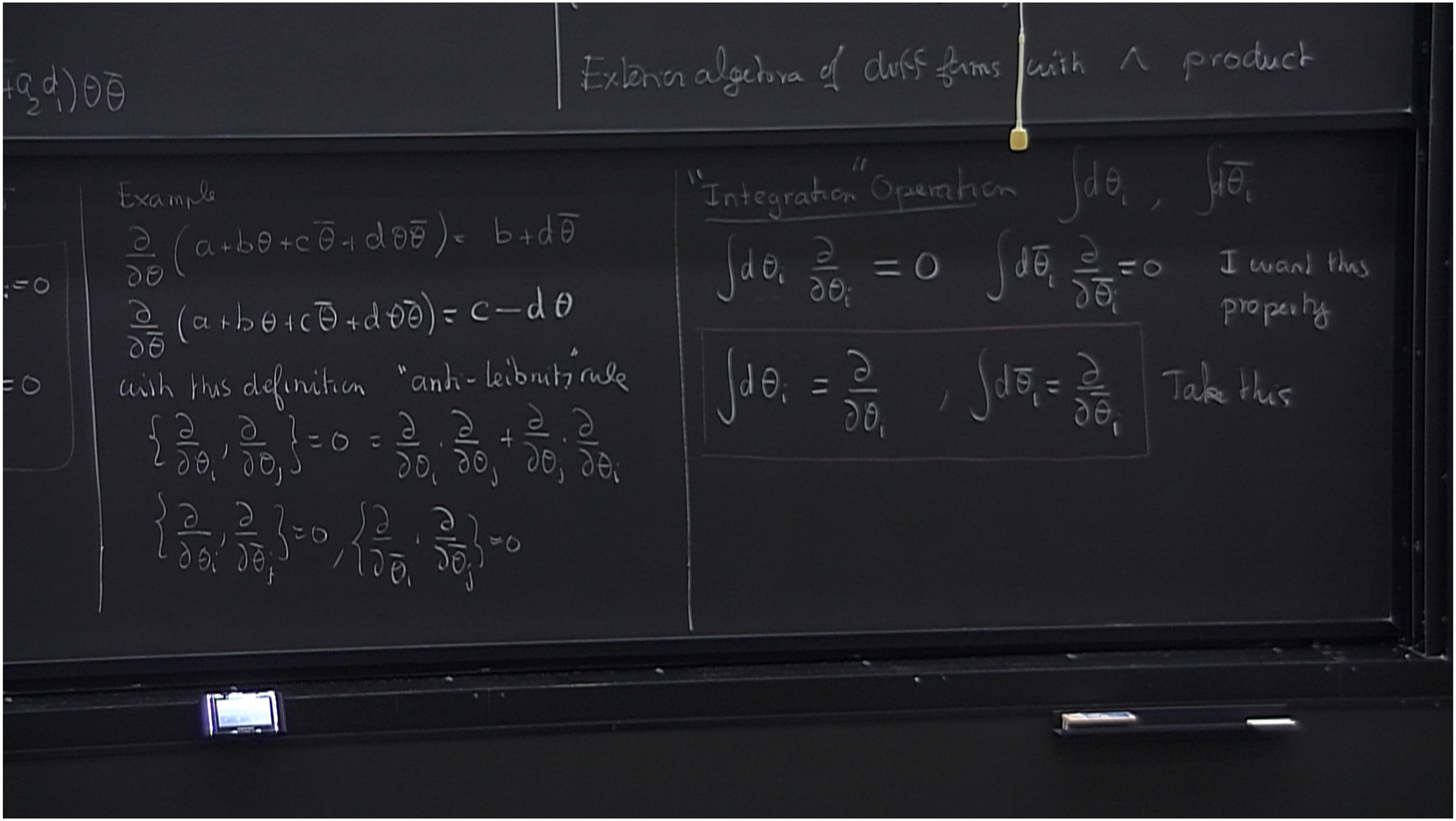
$\frac{\partial}{\partial \theta} (a + b\theta$

$\frac{\partial}{\partial \bar{\theta}} (a + b\theta$

with this def

$\{ \frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j}$

$\{ \frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \bar{\theta}_j} \}$



$+a_2 d_1)\theta\bar{\theta}$

Exterior algebra of diff forms with $\wedge$ product

$=0$

$=0$

Example

$$\frac{\partial}{\partial\theta}\left(a+b\theta+c\bar{\theta}+d\theta\bar{\theta}\right)=b+d\bar{\theta}$$

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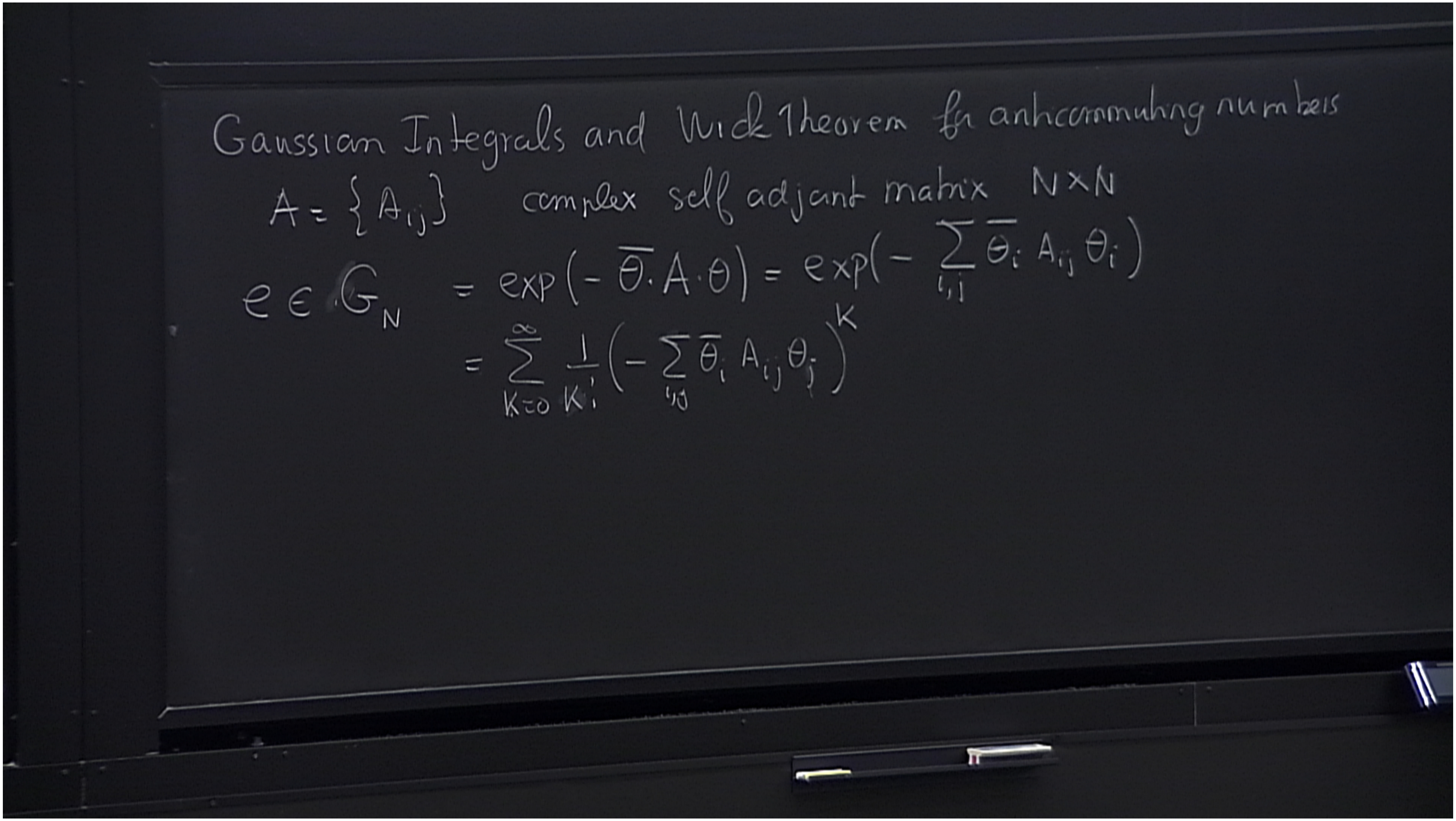
I want this property

$$\boxed{\int d\theta_i=\frac{\partial}{\partial\theta_i}\ ,\ \int d\bar{\theta}_i=\frac{\partial}{\partial\bar{\theta}_i}}$$

Take this

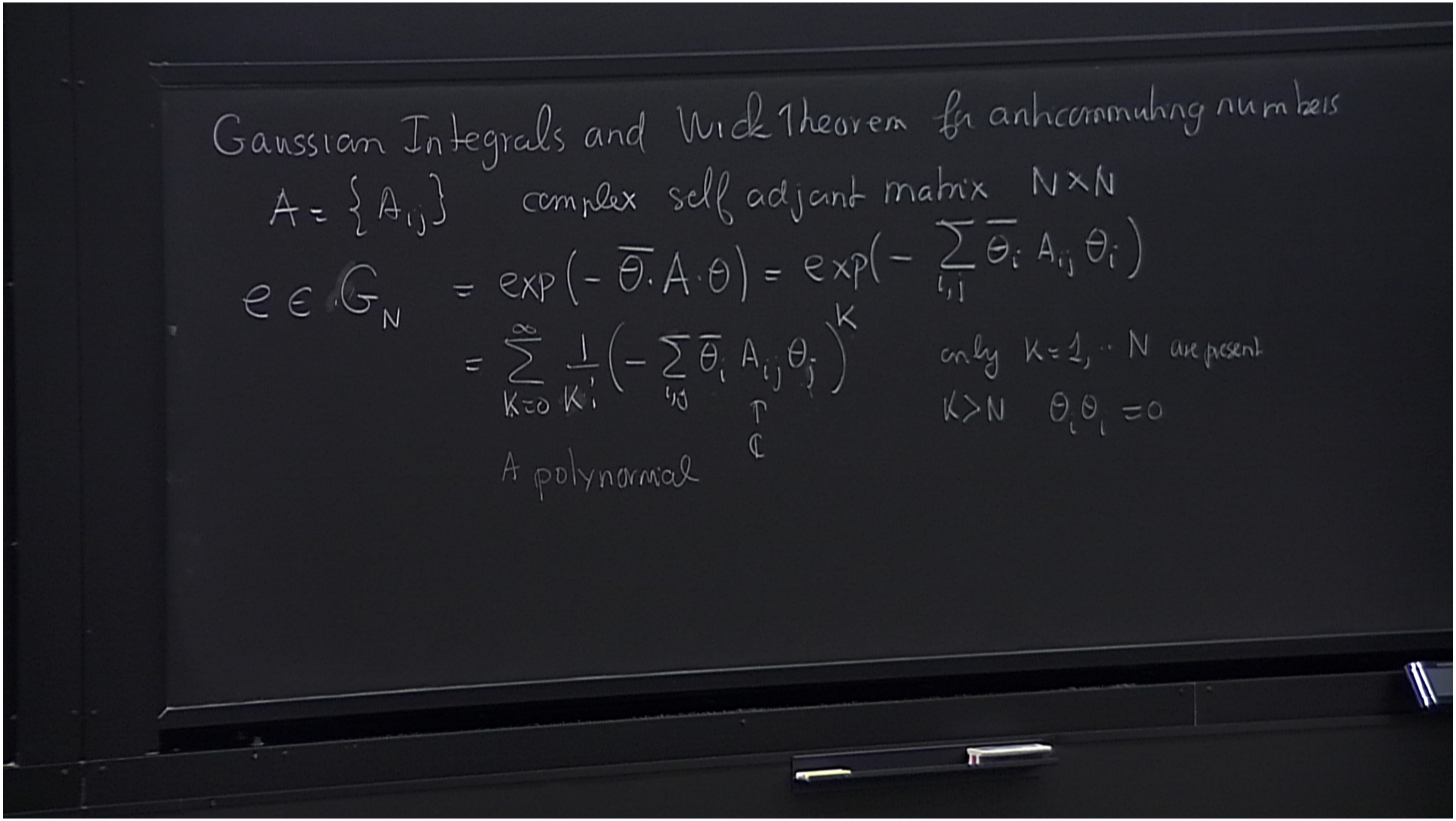


Gaussian Integrals and Wick Theorem fr anticommuting numbers

$A = \{A_{ij}\}$ complex self adjant matrix $N \times N$

$$e \in G_N = \exp(-\bar{\theta} \cdot A \cdot \theta) = \exp(-\sum_{i,j} \bar{\theta}_i A_{ij} \theta_i)$$

$$= \sum_{K=0}^{\infty} \frac{1}{K!} \left(-\sum_{ij} \bar{\theta}_i A_{ij} \theta_j\right)^K$$



Gaussian Integrals and Wick Theorem for anticommuting numbers

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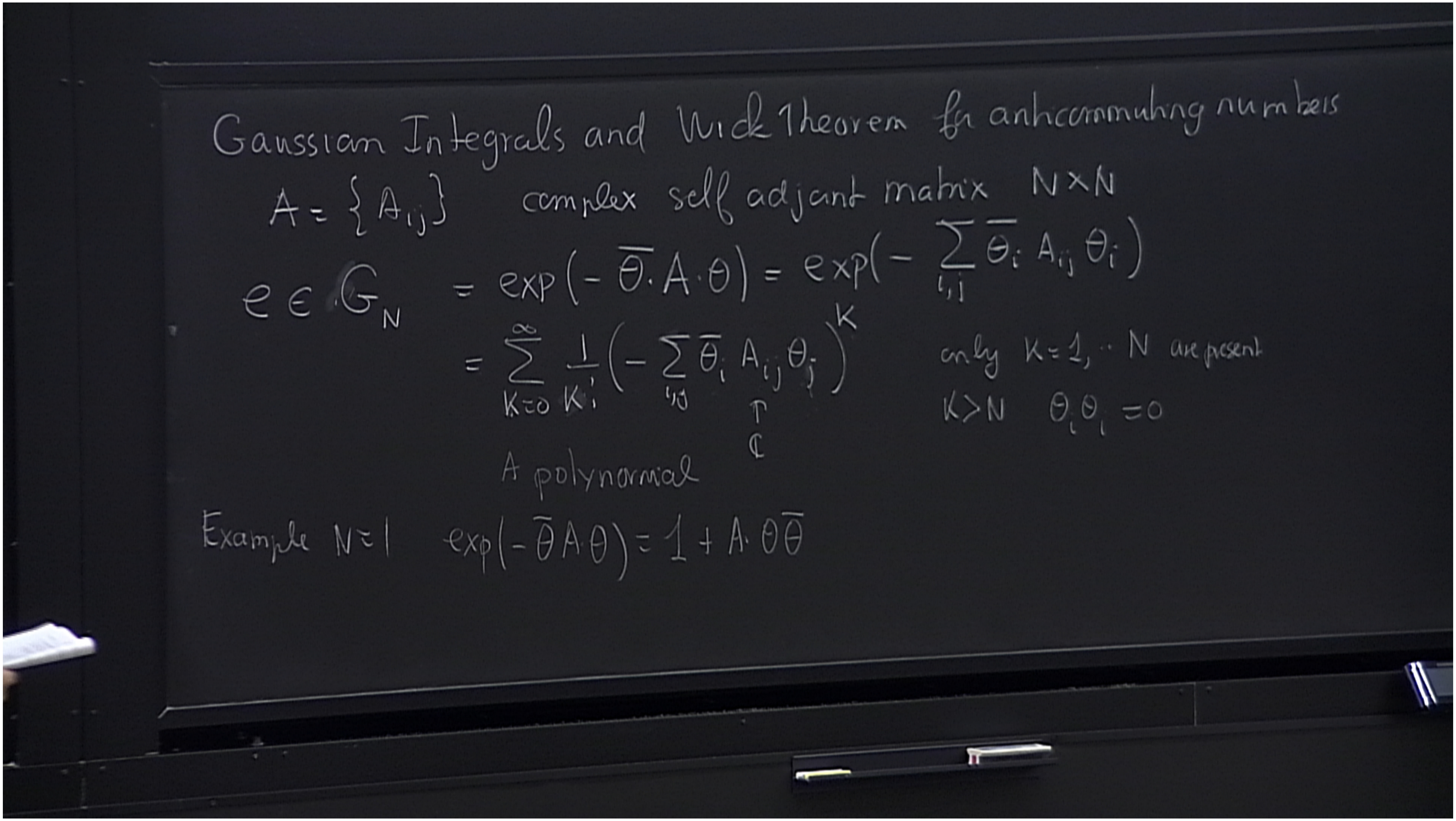
only $K = 1, \cdots N$ are present

$K > N \quad \theta_i \theta_i = 0$

A polynormial



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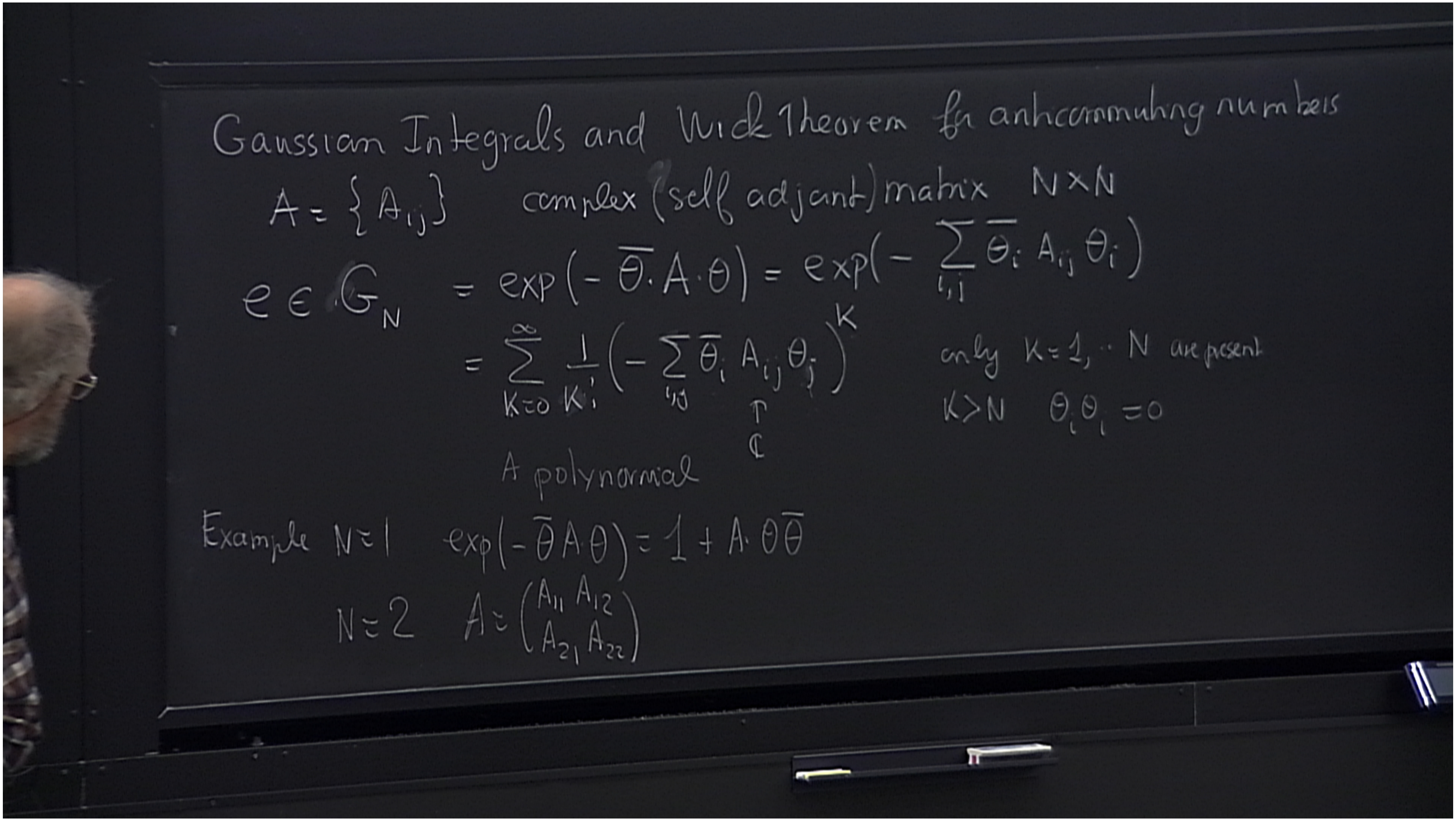
A polynormial

Example $N = 1 \quad \exp(-\bar{\theta} A \theta) = 1 + A \cdot \theta \bar{\theta}$



Gaussian Integrals and Wick Theorem fr anticommuting numbers

$A = \{A_{ij}\}$ complex (self adjant) matrix $N \times N$

$$e \in G_N = \exp(-\bar{\theta} \cdot A \cdot \theta) = \exp(-\sum_{i,j} \bar{\theta}_i A_{ij} \theta_i)$$

$$= \sum_{K=0}^{\infty} \frac{1}{K!} \Big(-\sum_{ij} \bar{\theta}_i \underset{\underset{\mathbb{C}}{\uparrow}}{A_{ij}} \theta_j\Big)^K$$

only $K = 1, \cdot\cdot N$ are present

$K > N \quad \theta_i \theta_i = 0$

A polynormial

Example $N=1$ $\exp(-\bar{\theta} A \theta) = 1 + A \cdot \theta \bar{\theta}$

$N = 2$ $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$

Gaussian Integrals and Wick Theorem fr anticommuting numbers

$A = \{A_{ij}\}$ complex (self adjoint) matrix $N \times N$

$$e \in G_N \quad = \exp(-\bar{\theta}\cdot A\cdot\theta) = \exp\Big(-\sum_{i,j}\bar{\theta}_i A_{ij}\theta_i\Big)$$

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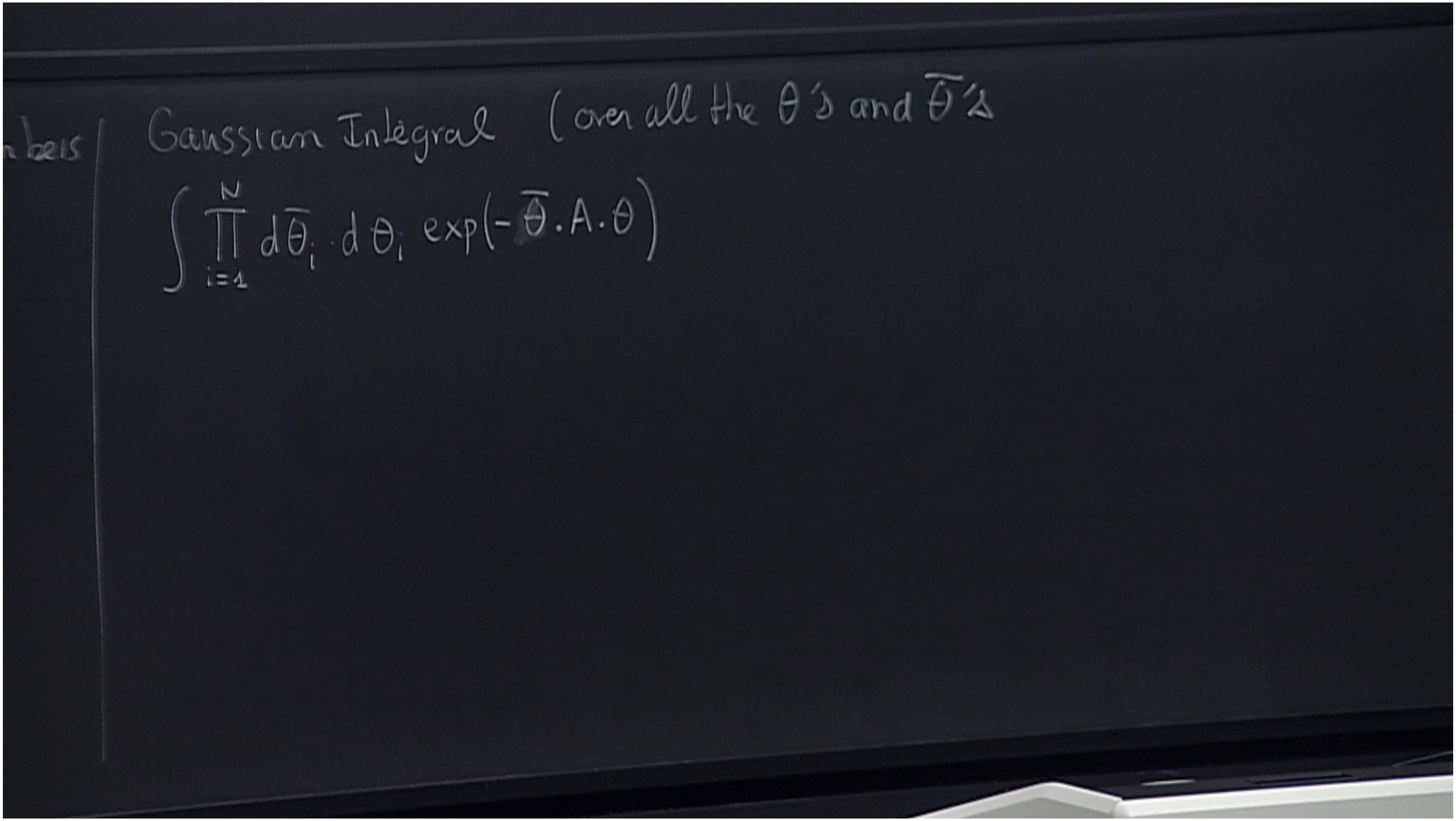
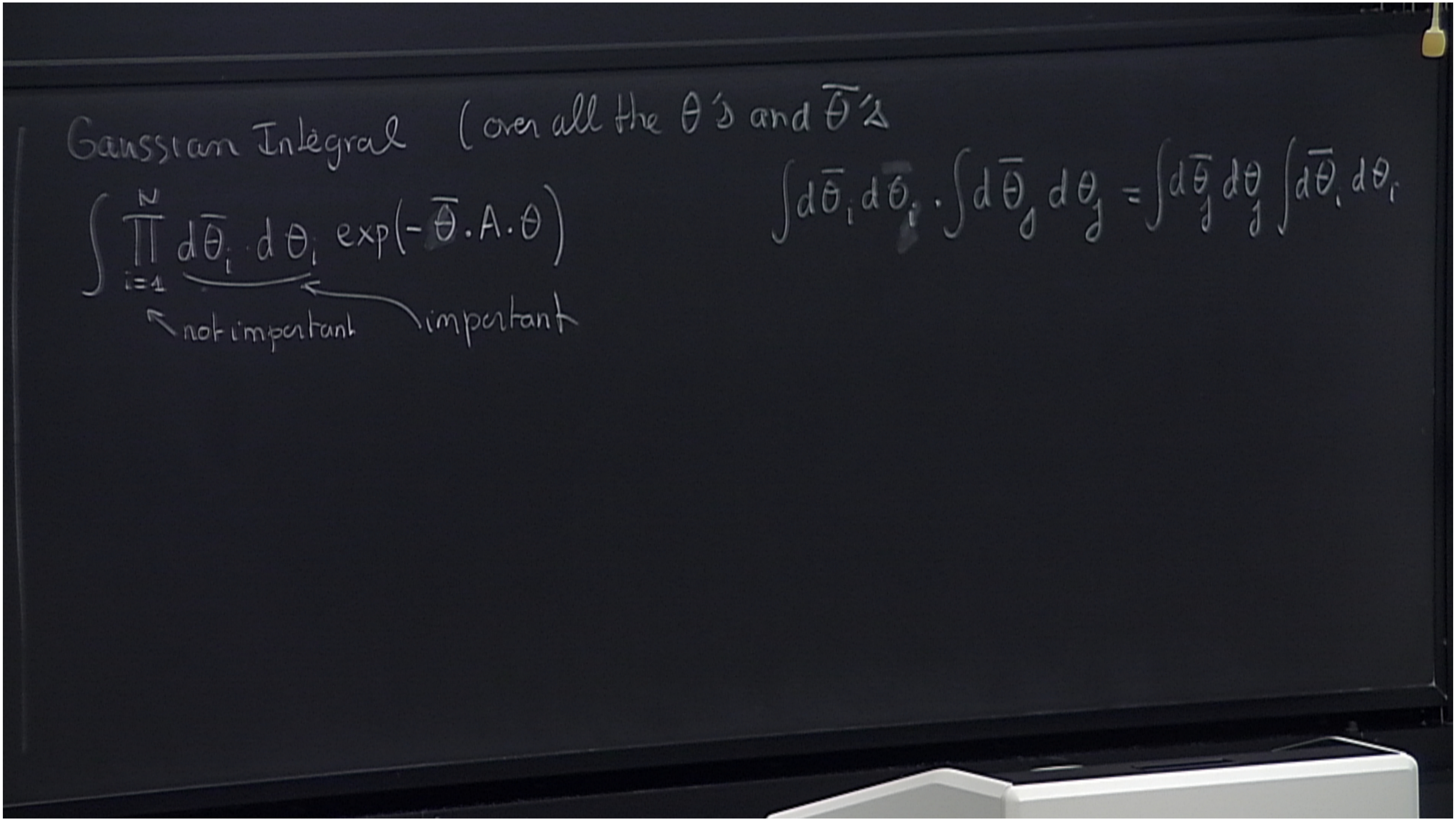


nbers

Gaussian Integral (over all the θ's and $\bar{\theta}$'s

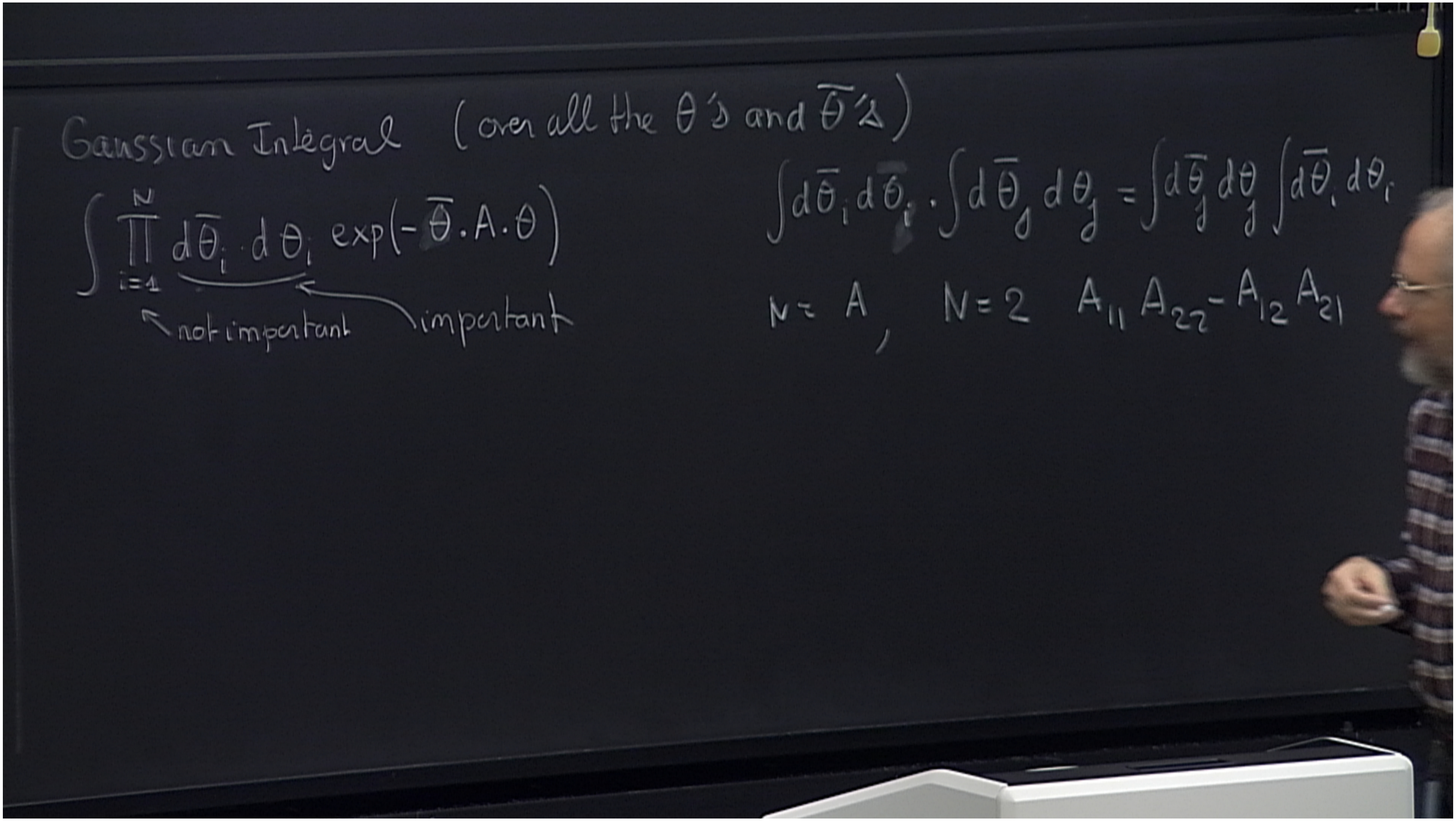
$$\int \prod_{i=1}^{N} d\bar{\theta}_i \cdot d\theta_i \, \exp(-\bar{\theta} . A . \theta)$$



Gaussian Integral (over all the θ's and $\bar\theta$'s

$$\int \underset{\text{not important}}{\prod_{i=1}^{N}} \underbrace{d\bar\theta_i \cdot d\theta_i}_{\text{important}} \exp(-\bar\theta . A . \theta)$$

$$\int d\bar\theta_i \, d\theta_i \cdot \int d\bar\theta_j \, d\theta_j = \int d\bar\theta_j \, d\theta_j \int d\bar\theta_i \, d\theta_i$$



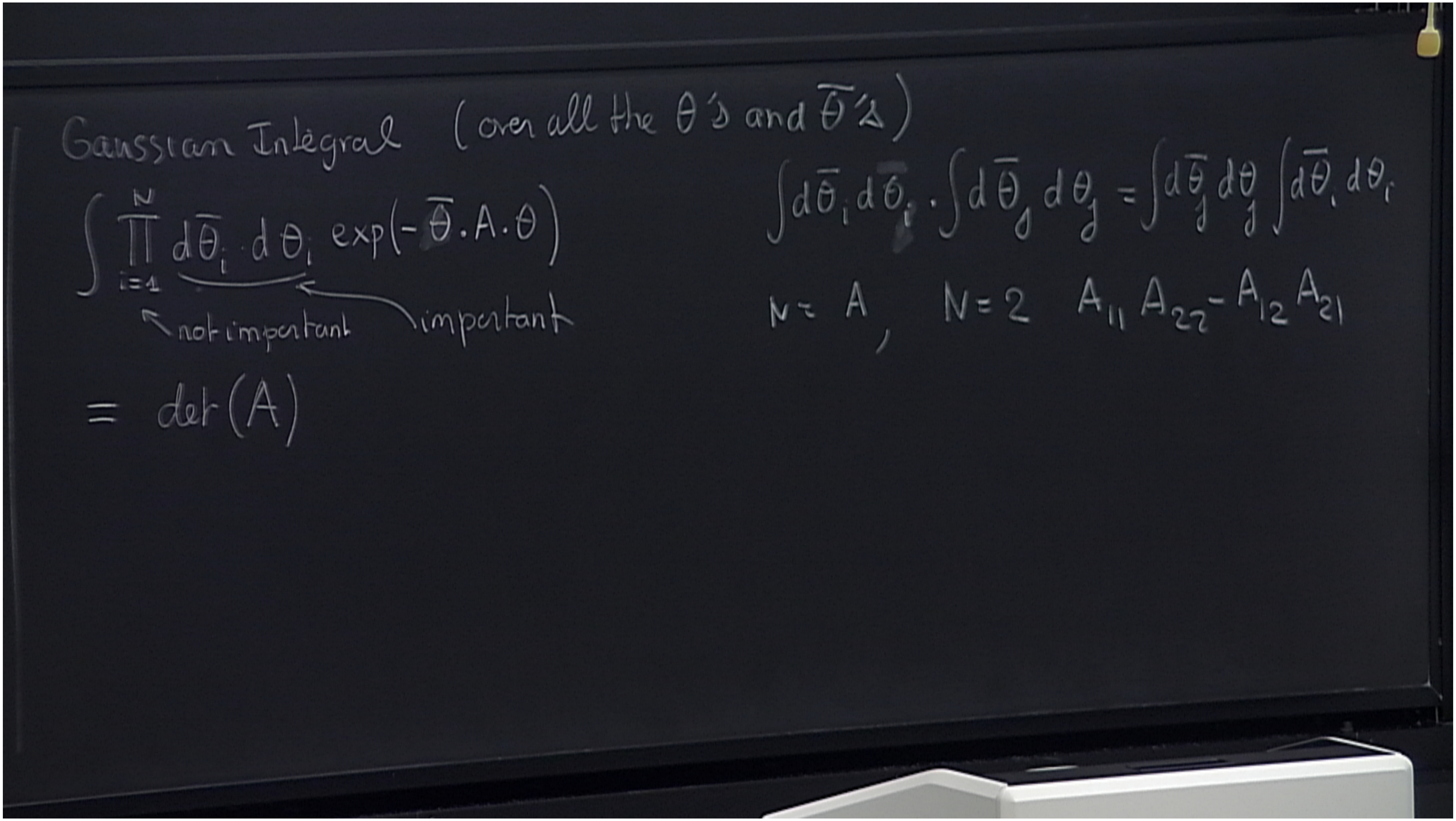
Gaussian Integral (over all the θ's and $\bar\theta$'s)

$$\int \prod_{i=1}^{N} \underbrace{d\bar\theta_i \cdot d\theta_i} \exp(-\bar\theta . A . \theta)$$

not important   important

$$\int d\bar\theta_i \, d\bar\theta_i \cdot \int d\bar\theta_j \, d\theta_j = \int d\bar\theta_j \, d\theta_j \int d\bar\theta_i \, d\theta_i$$

$N = A$, $N = 2$ $A_{11} A_{22} - A_{12} A_{21}$



Gaussian Integral (over all the θ's and $\bar{\theta}$'s)

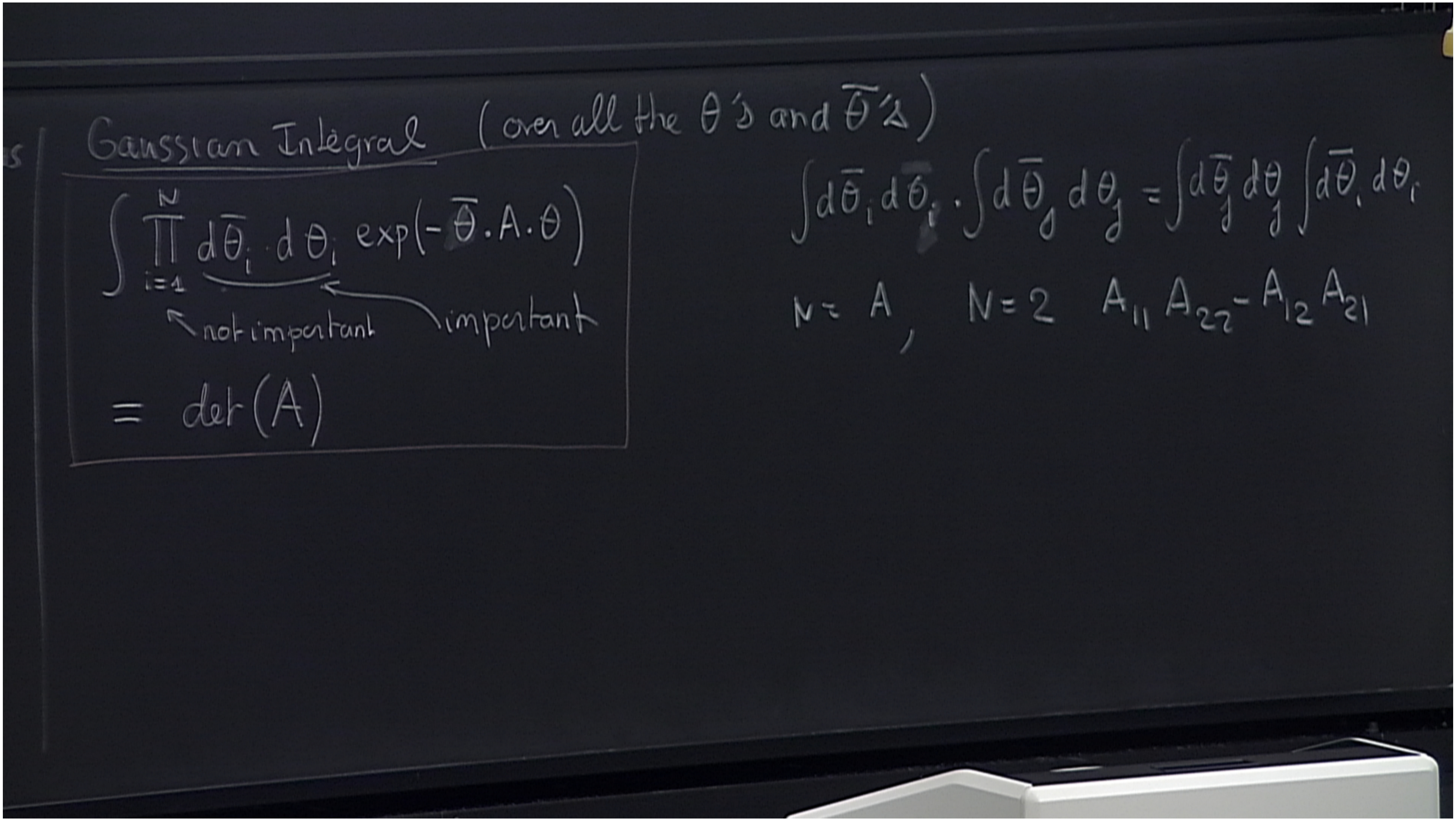
$$\int \prod_{i=1}^{N} \underbrace{d\bar{\theta}_i \cdot d\theta_i} \exp(-\bar{\theta} . A . \theta)$$

not important important

$$= \det(A)$$

$$\int d\bar{\theta}_i \, d\theta_i \, . \int d\bar{\theta}_j \, d\theta_j = \int d\bar{\theta}_j \, d\theta_j \int d\bar{\theta}_i \, d\theta_i$$

$N = A$, $N = 2$ $A_{11} A_{22} - A_{12} A_{21}$

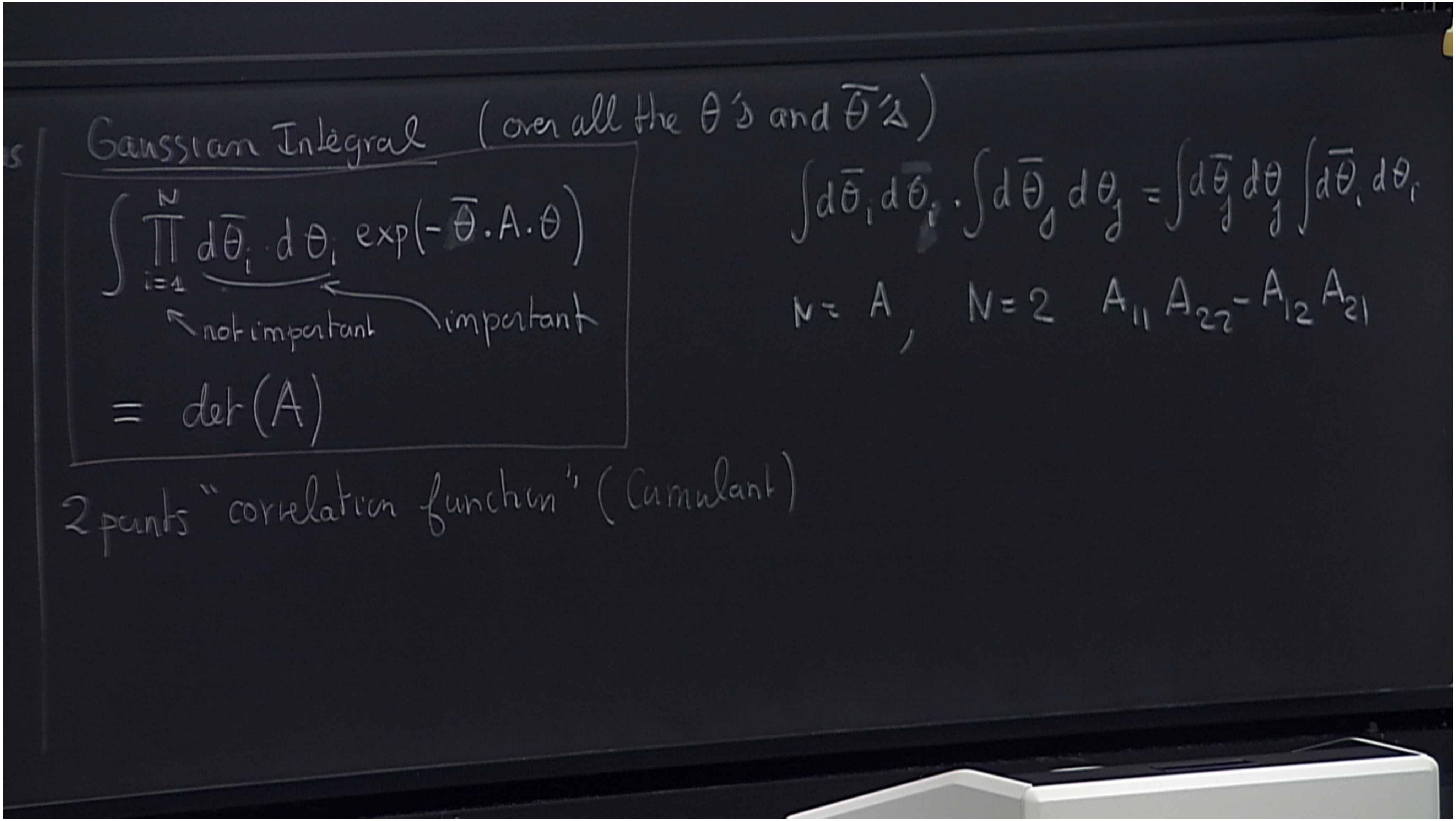


Gaussian Integral (over all the θ's and $\bar\theta$'s)

$$\boxed{\begin{aligned} &\int \underbrace{\prod_{i=1}^{N}}_{\text{not important}} \underbrace{d\bar\theta_i \cdot d\theta_i}_{\text{important}} \exp(-\bar\theta . A . \theta) \\ &= \det(A) \end{aligned}}$$

$$\int d\bar\theta_i \, d\theta_i \cdot \int d\bar\theta_j \, d\theta_j = \int d\bar\theta_j \, d\theta_j \int d\bar\theta_i \, d\theta_i$$

$N = A$, $\quad N = 2 \quad A_{11} A_{22} - A_{12} A_{21}$



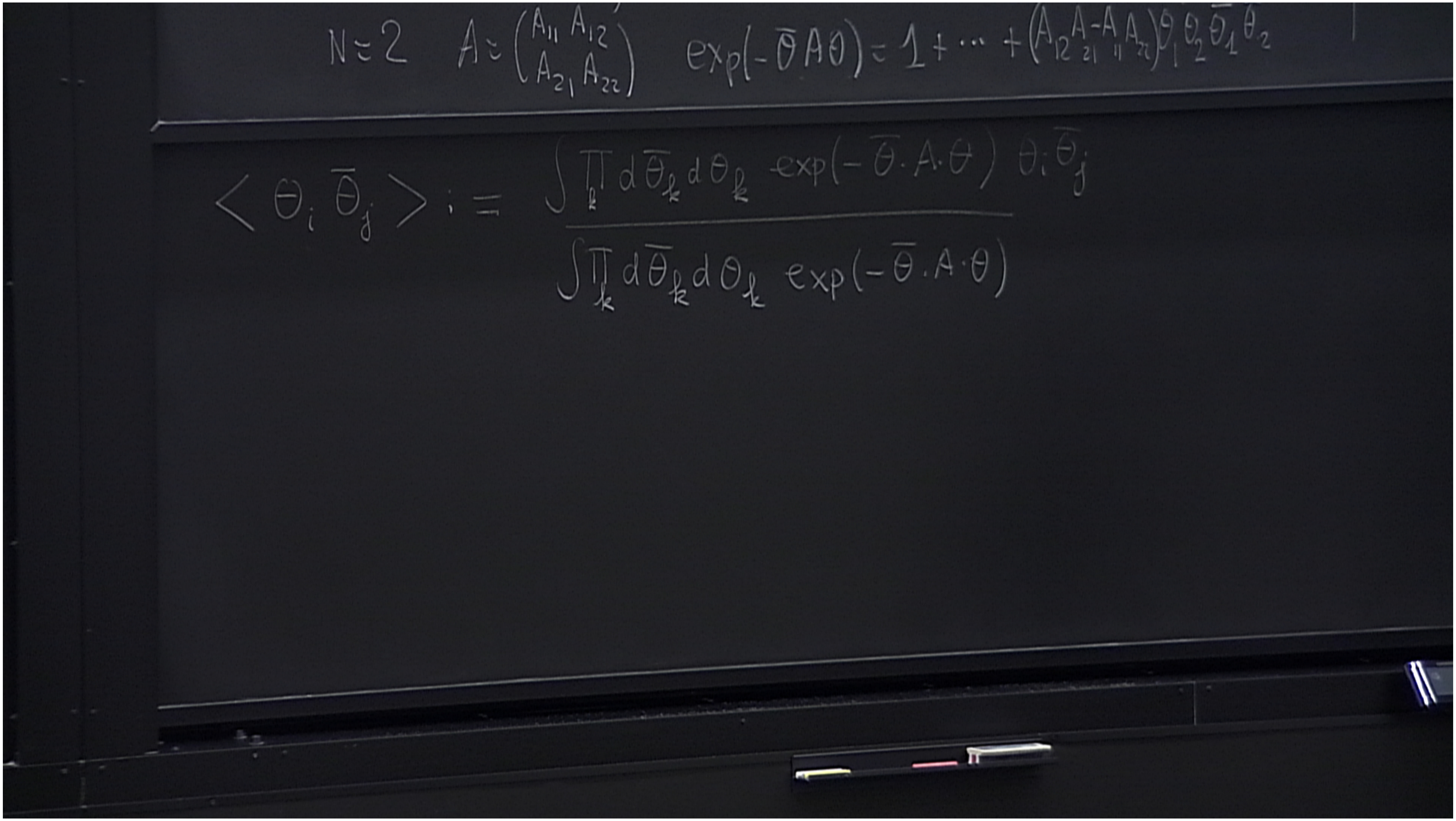
Gaussian Integral (over all the θ's and $\bar\theta$'s)

$$\boxed{\int \underset{\nwarrow \text{not important}}{\prod_{i=1}^{N}} \underset{\nwarrow \text{important}}{\underbrace{d\bar\theta_i \cdot d\theta_i}} \exp(-\bar\theta . A . \theta) = \det(A)}$$

$$\int d\bar\theta_i \, d\theta_i \cdot \int d\bar\theta_j \, d\theta_j = \int d\bar\theta_j \, d\theta_j \int d\bar\theta_i \, d\theta_i$$

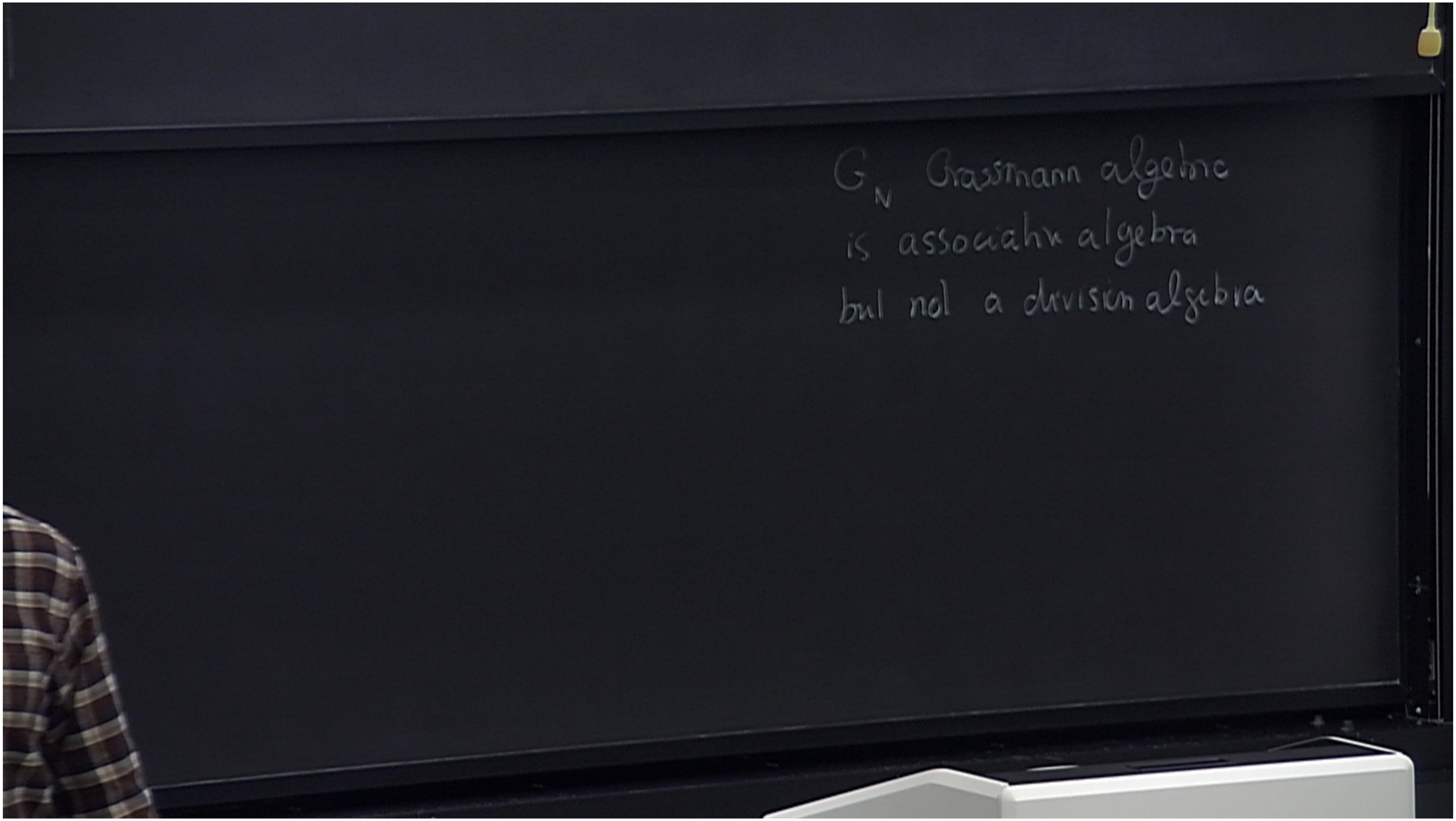
$N = A$, $N = 2$ $A_{11} A_{22} - A_{12} A_{21}$

2 points "correlation function" (Cumulant)



$$N = 2 \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \exp(-\bar{\theta} A \theta) = 1 + \cdots + (A_{12}A_{21} - A_{11}A_{22})\theta_1 \theta_2 \bar{\theta}_1 \theta_2$$

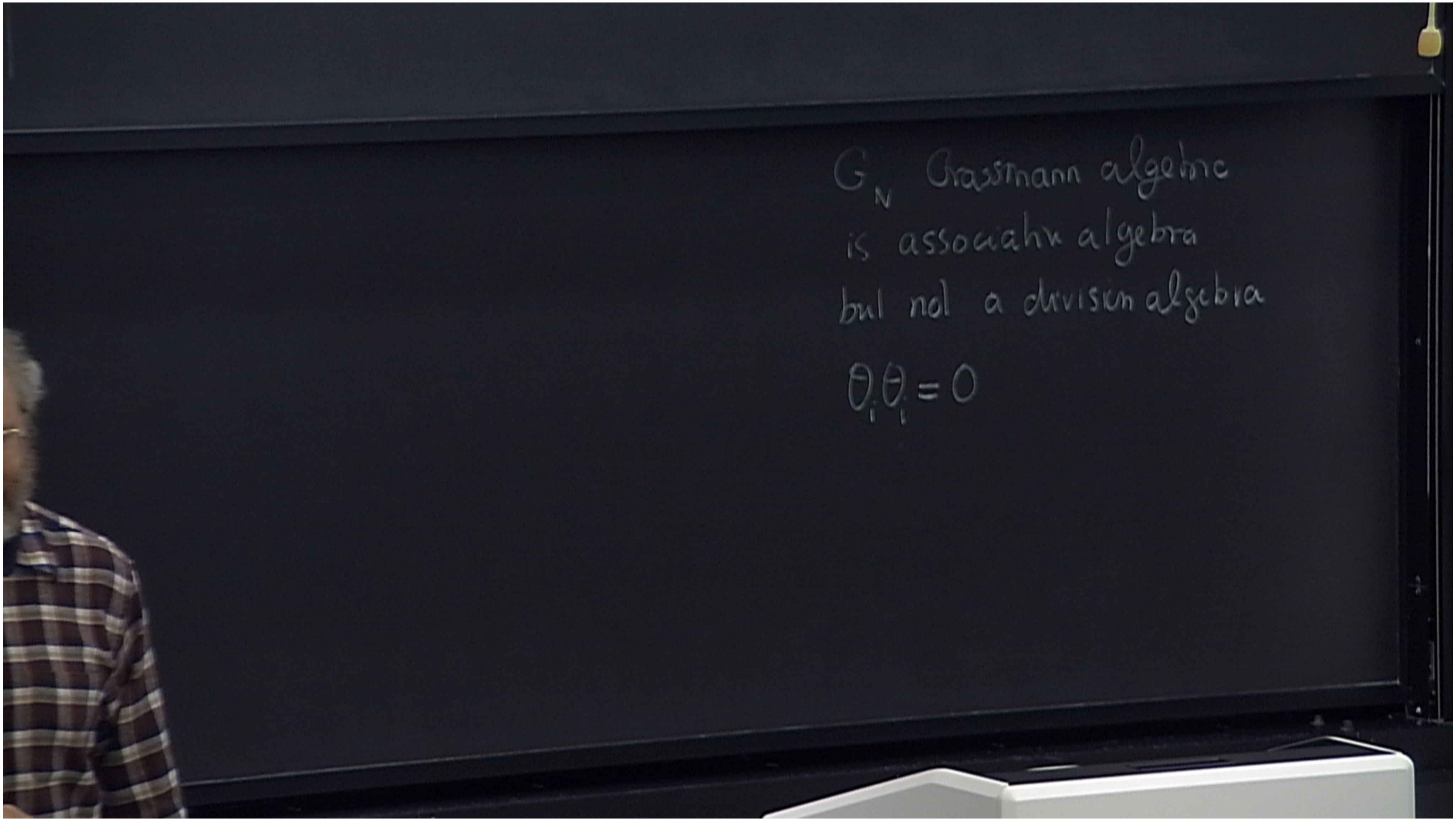
$$\langle \theta_i \bar{\theta}_j \rangle := \frac{\int \prod_k d\bar{\theta}_k d\theta_k \exp(-\bar{\theta} \cdot A \cdot \theta)\, \theta_i \bar{\theta}_j}{\int \prod_k d\bar{\theta}_k d\theta_k \exp(-\bar{\theta} \cdot A \cdot \theta)}$$



G_N Grassmann algebra

is associative algebra

but not a division algebra



G_N Grassmann algebra

is associative algebra

but not a division algebra

$\theta_i\theta_i = 0$

$$\exp(-\bar{\theta} A \theta) = 1 + A \cdot \bar{\theta}\theta$$

$$N = 2 \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \exp(-\bar{\theta} A \theta) = 1 + \cdots + (A_{12}A_{21} - A_{11}A_{22})\theta_1 \theta_2 \bar{\theta}_1 \bar{\theta}_2$$

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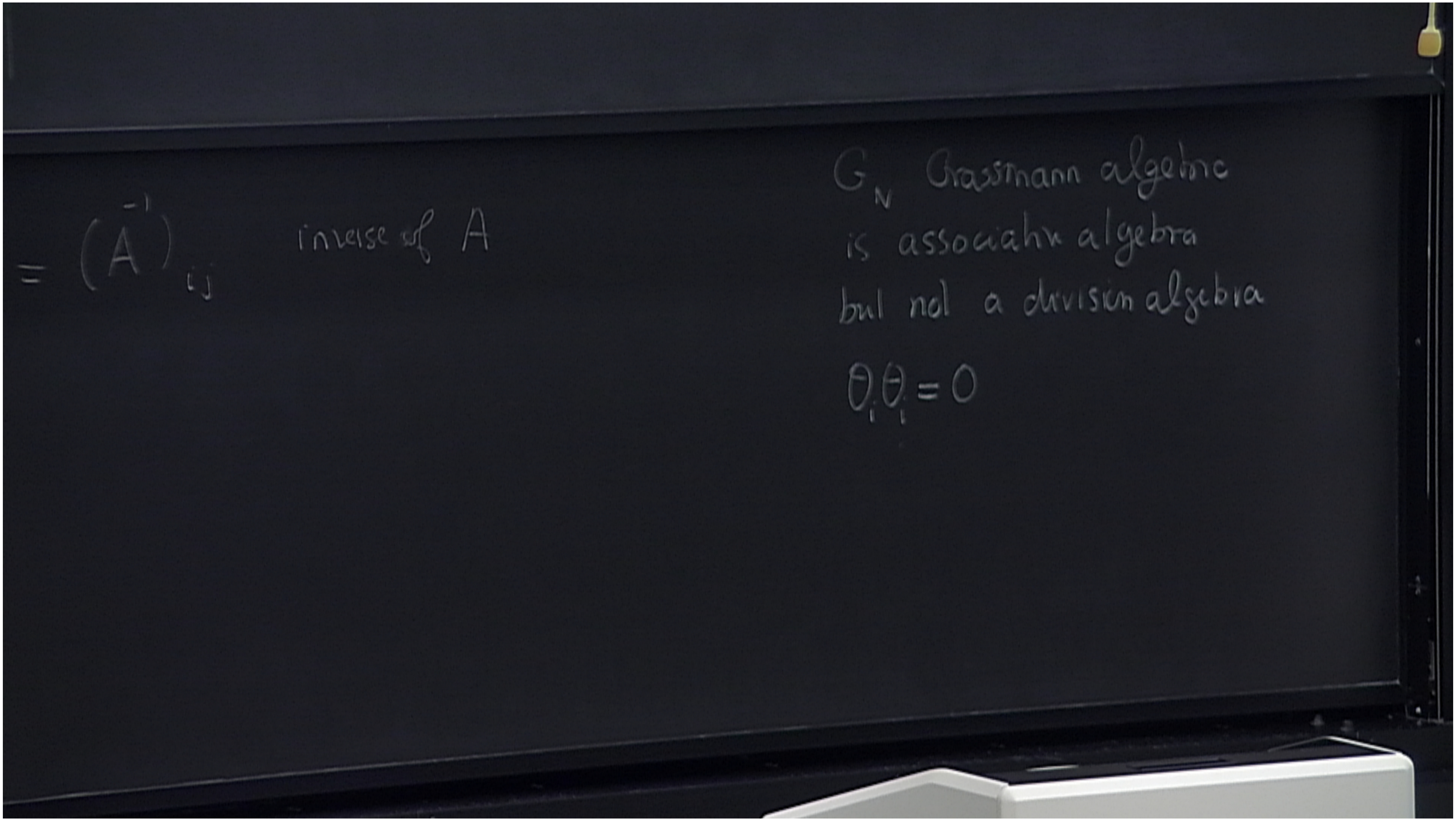
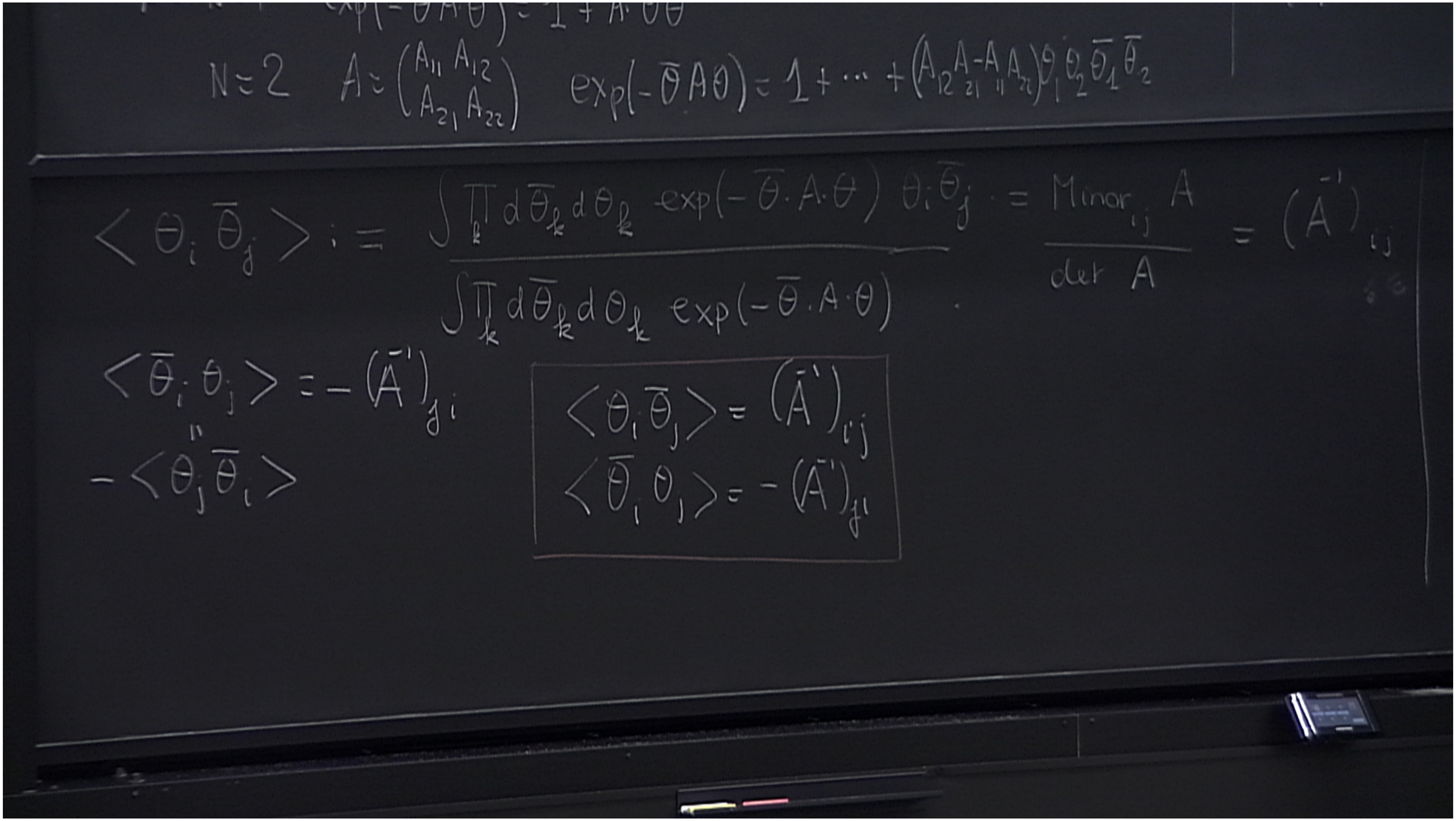


$= (A^{-1})_{ij}$ inverse of A

G_N Grassmann algebra
is associative algebra
but not a division algebra

$\theta_i \theta_i = 0$



$\exp(-\theta A\cdot\theta) = 1 + A\cdot\theta\theta$

$N=2 \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \exp(-\bar\theta A\theta) = 1 + \cdots + (A_{12}A_{21} - A_{11}A_{22})\theta_1\theta_2\bar\theta_1\bar\theta_2$

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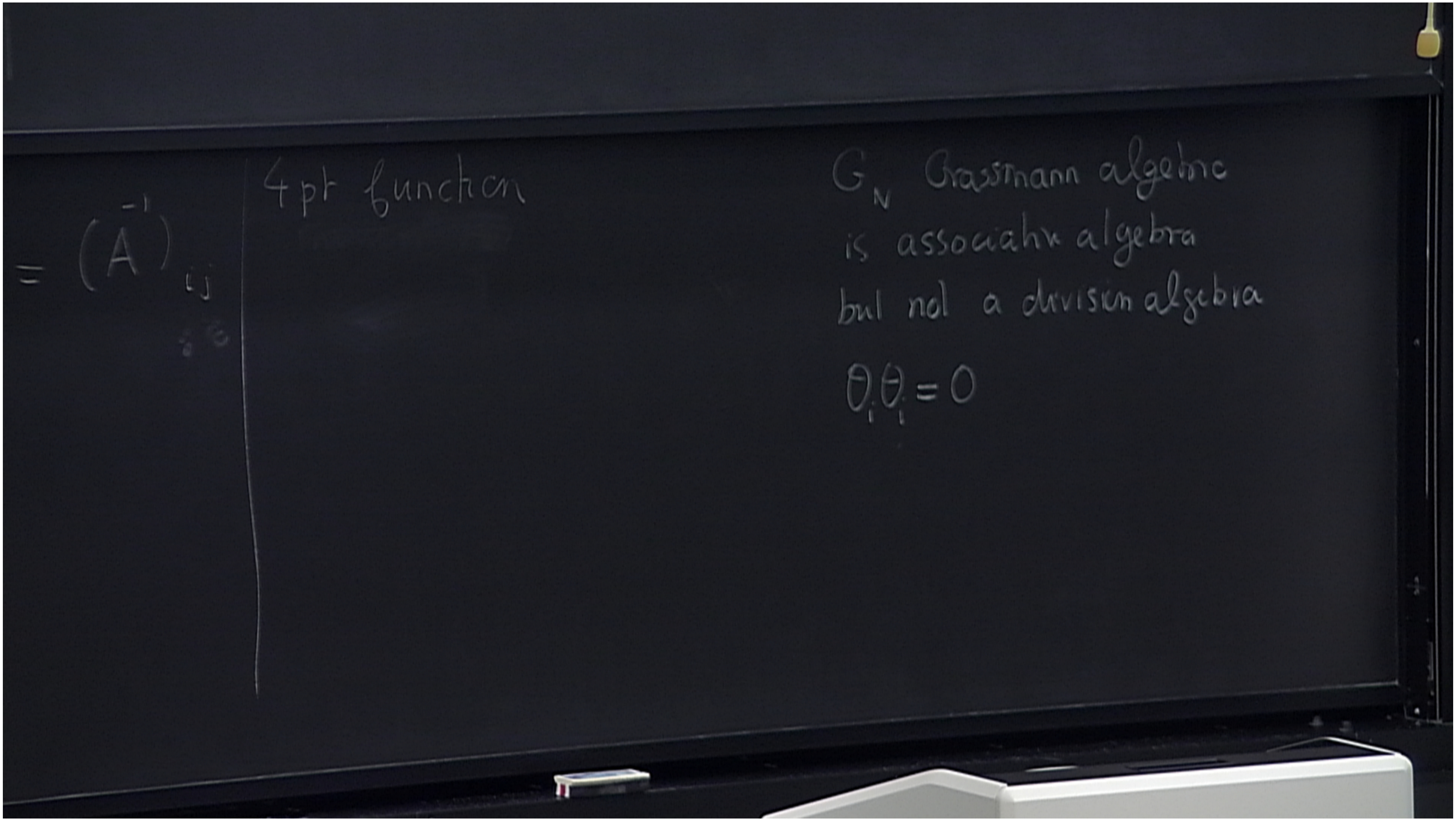
$$\langle \bar\theta_i \theta_j \rangle = -(A^{-1})_{ji}$$

$$\overset{\shortparallel}{} -\langle \theta_j \bar\theta_i \rangle$$

$$\boxed{\begin{aligned} \langle \theta_i \bar\theta_j \rangle &= (A^{-1})_{ij} \\ \langle \bar\theta_i \theta_j \rangle &= -(A^{-1})_{ji} \end{aligned}}$$



$= (A^{-1})_{ij}$

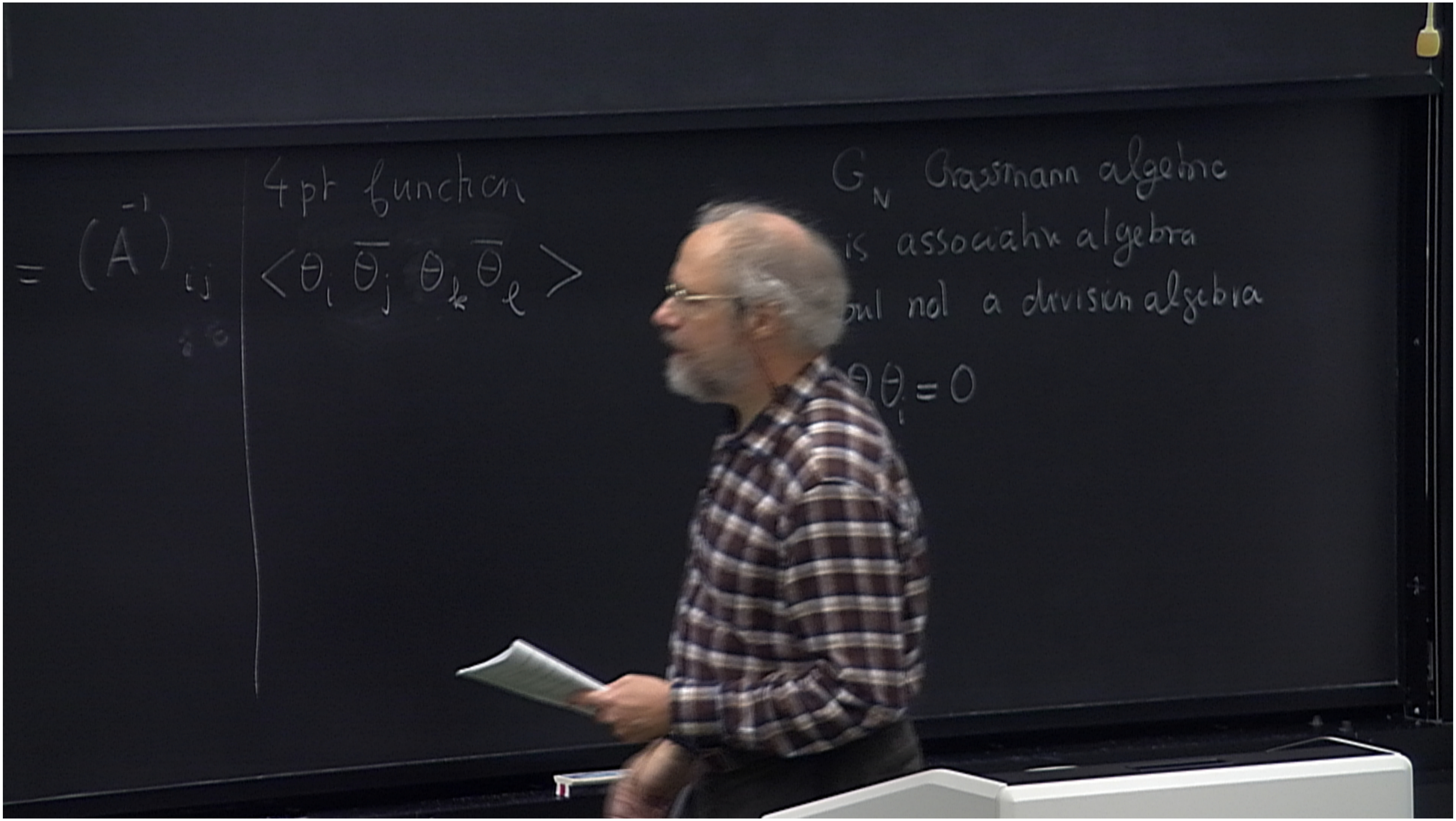
4pt function

G_N Grassmann algebra

is associative algebra

but not a division algebra

$\theta_i \theta_i = 0$



$= (A^{-1})_{ij}$

4pt function

$\langle \theta_i \bar{\theta}_j \theta_k \bar{\theta}_\ell \rangle$

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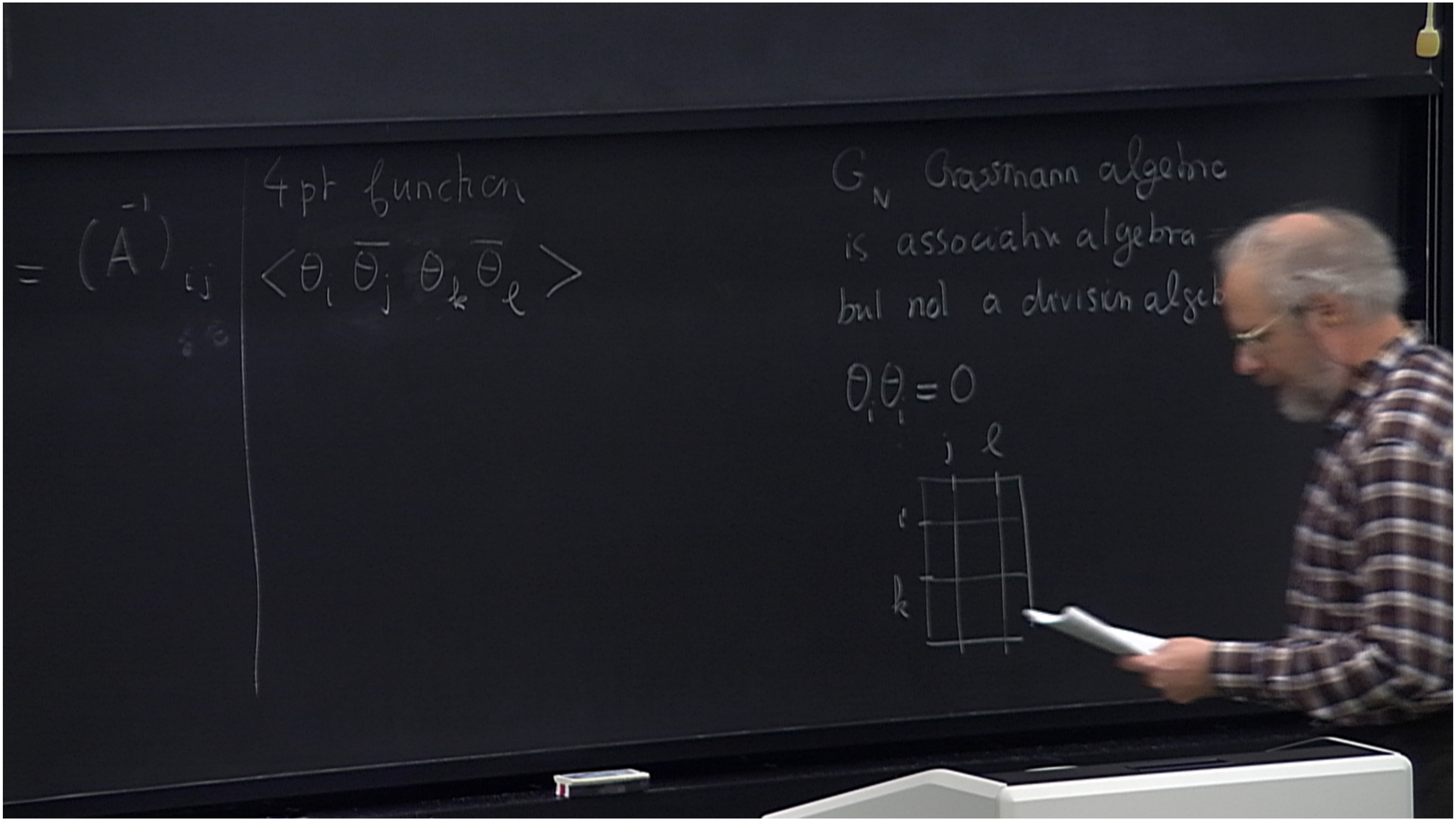

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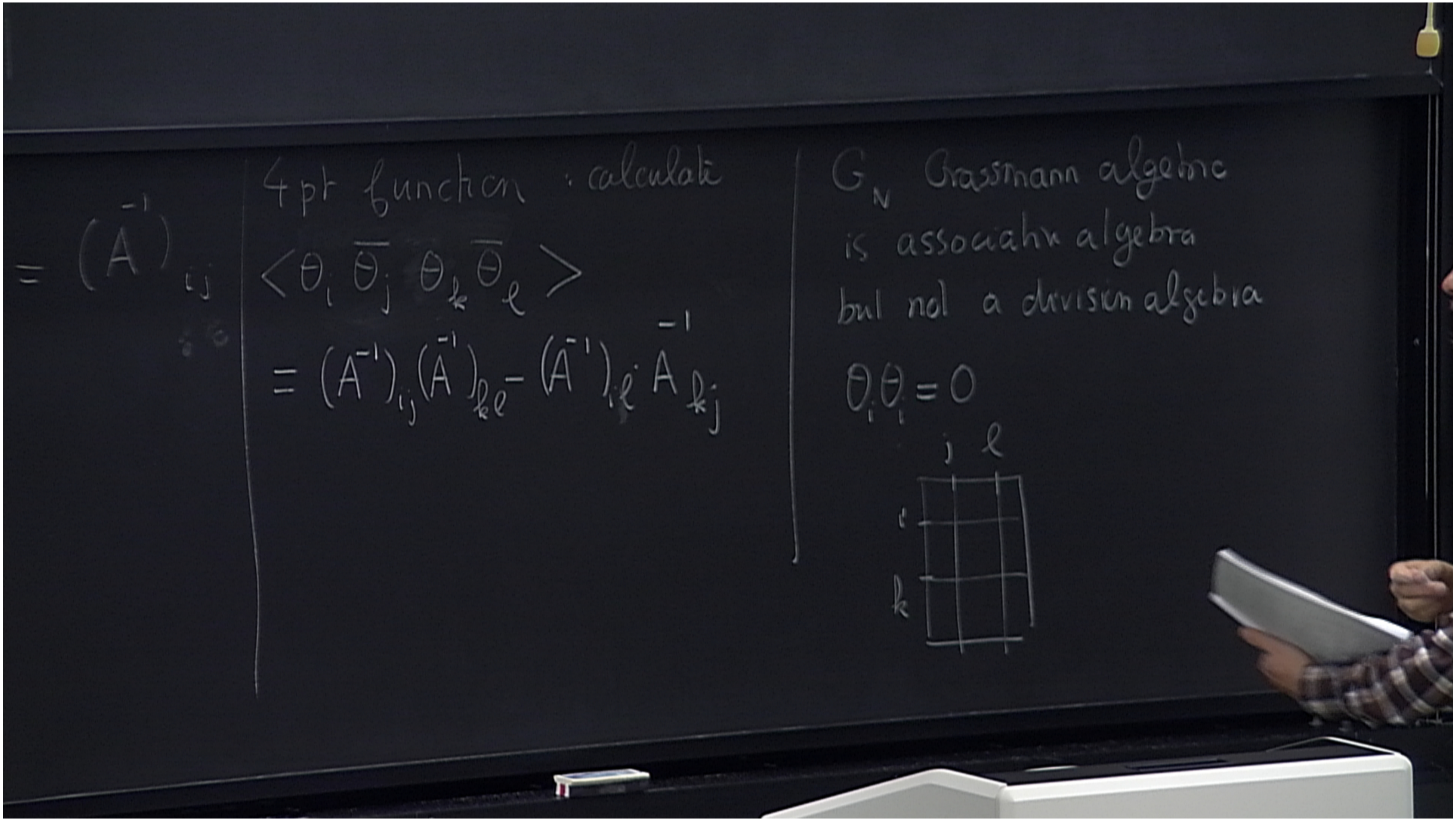

$= (A^{-1})_{ij}$

4pt function : calculate

$\langle \theta_i \bar{\theta}_j \theta_k \bar{\theta}_\ell \rangle$

$= (A^{-1})_{ij} (A^{-1})_{k\ell} - (A^{-1})_{i\ell} A^{-1}_{kj}$

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$= \langle \theta_i \bar{\theta}_j \rangle \langle \theta_k \bar{\theta}_l \rangle$

$- \langle \theta_i \bar{\theta}_l \rangle \langle \theta_k \bar{\theta}_j \rangle$

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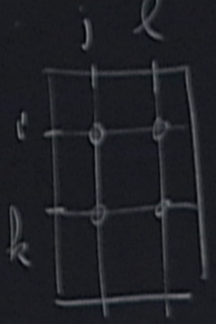
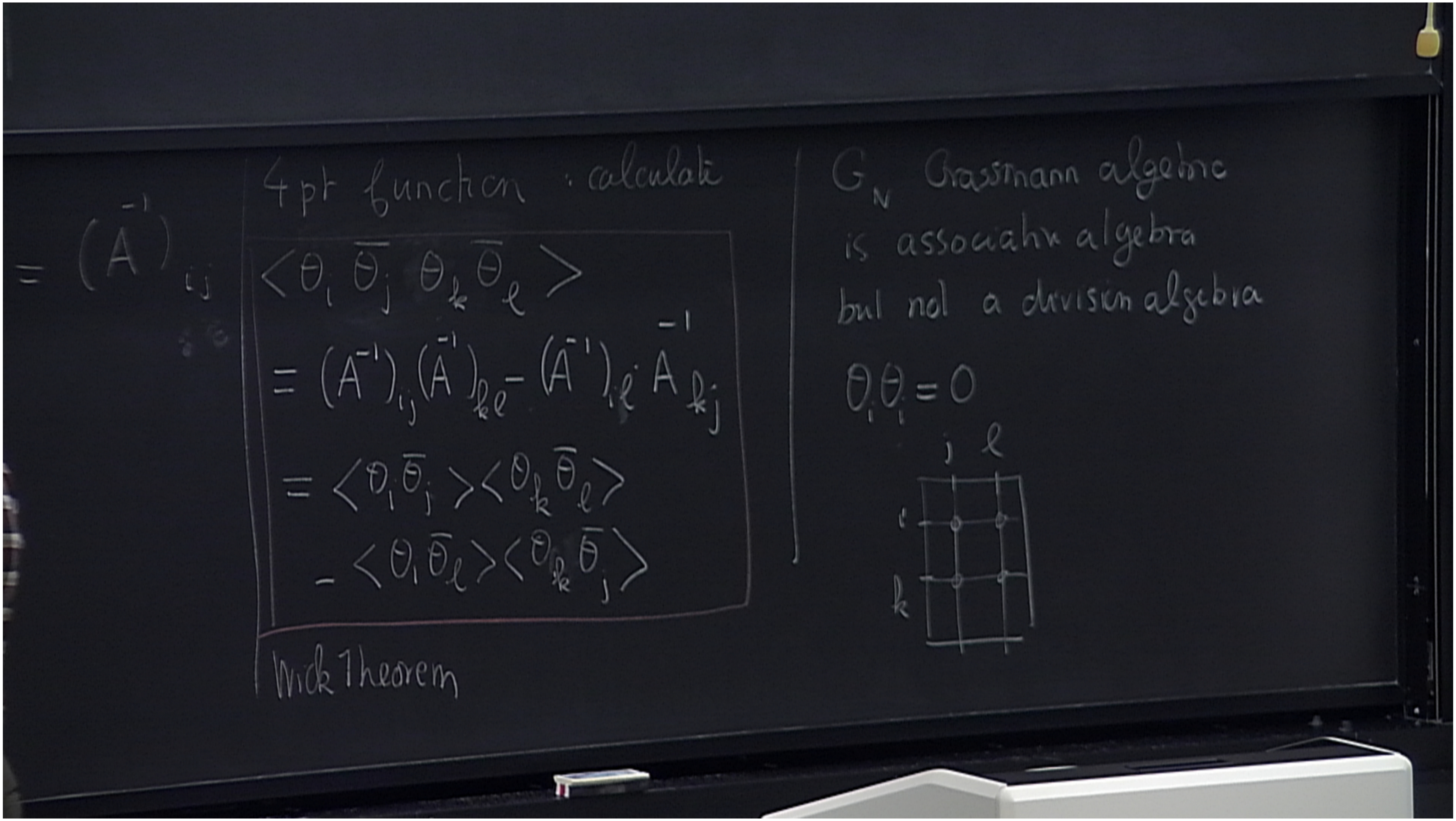




$= (A^{-1})_{ij}$

4pt function : calculate

$$\langle \theta_i \bar\theta_j \theta_k \bar\theta_\ell \rangle$$

$$= (A^{-1})_{ij} (A^{-1})_{k\ell} - (A^{-1})_{i\ell} A^{-1}_{kj}$$

$$= \langle \theta_i \bar\theta_j \rangle \langle \theta_k \bar\theta_\ell \rangle - \langle \theta_i \bar\theta_\ell \rangle \langle \theta_k \bar\theta_j \rangle$$

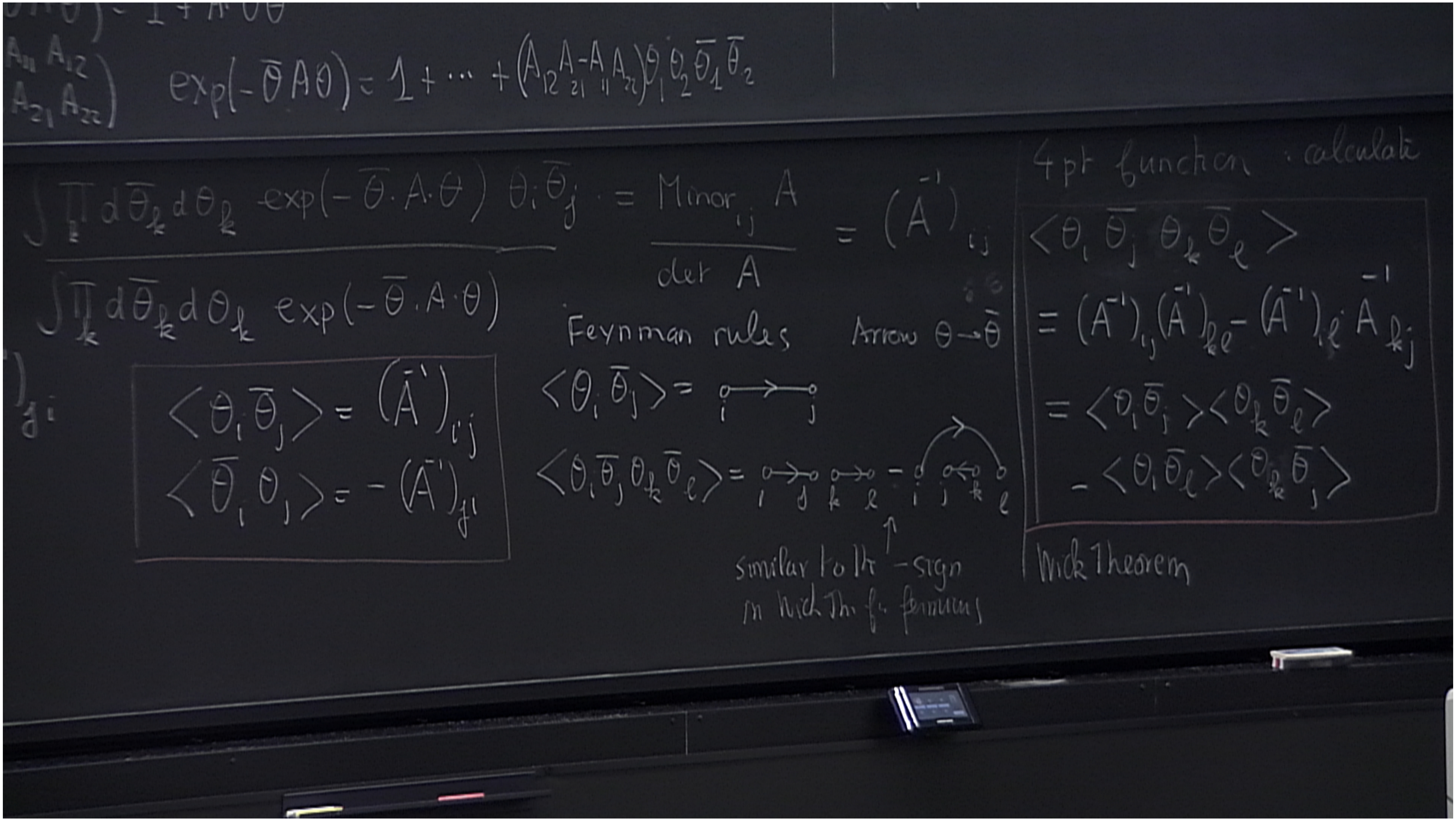
Wick Theorem

G_N Grassmann algebra

is associative algebra

but not a division algebra

$\theta_i \theta_i = 0$



$$\exp(-\bar\theta A\theta) = 1 + \cdots + (A_{12}A_{21} - A_{11}A_{22})\theta_1\theta_2\bar\theta_1\bar\theta_2$$

$$\frac{\int \prod_k d\bar\theta_k d\theta_k \exp(-\bar\theta\cdot A\cdot\theta)\,\theta_i\bar\theta_j}{\int \prod_k d\bar\theta_k d\theta_k \exp(-\bar\theta\cdot A\cdot\theta)} = \frac{\text{Minor}_{ij} A}{\det A} = (A^{-1})_{ij}$$

$$\boxed{\begin{aligned}\langle\theta_i\bar\theta_j\rangle &= (A^{-1})_{ij}\\ \langle\bar\theta_i\theta_j\rangle &= -(A^{-1})_{ji}\end{aligned}}$$

Feynman rules Arrow $\theta \to \bar\theta$

$$\langle\theta_i\bar\theta_j\rangle = \underset{i}{\circ}\!\longrightarrow\!\underset{j}{\circ}$$

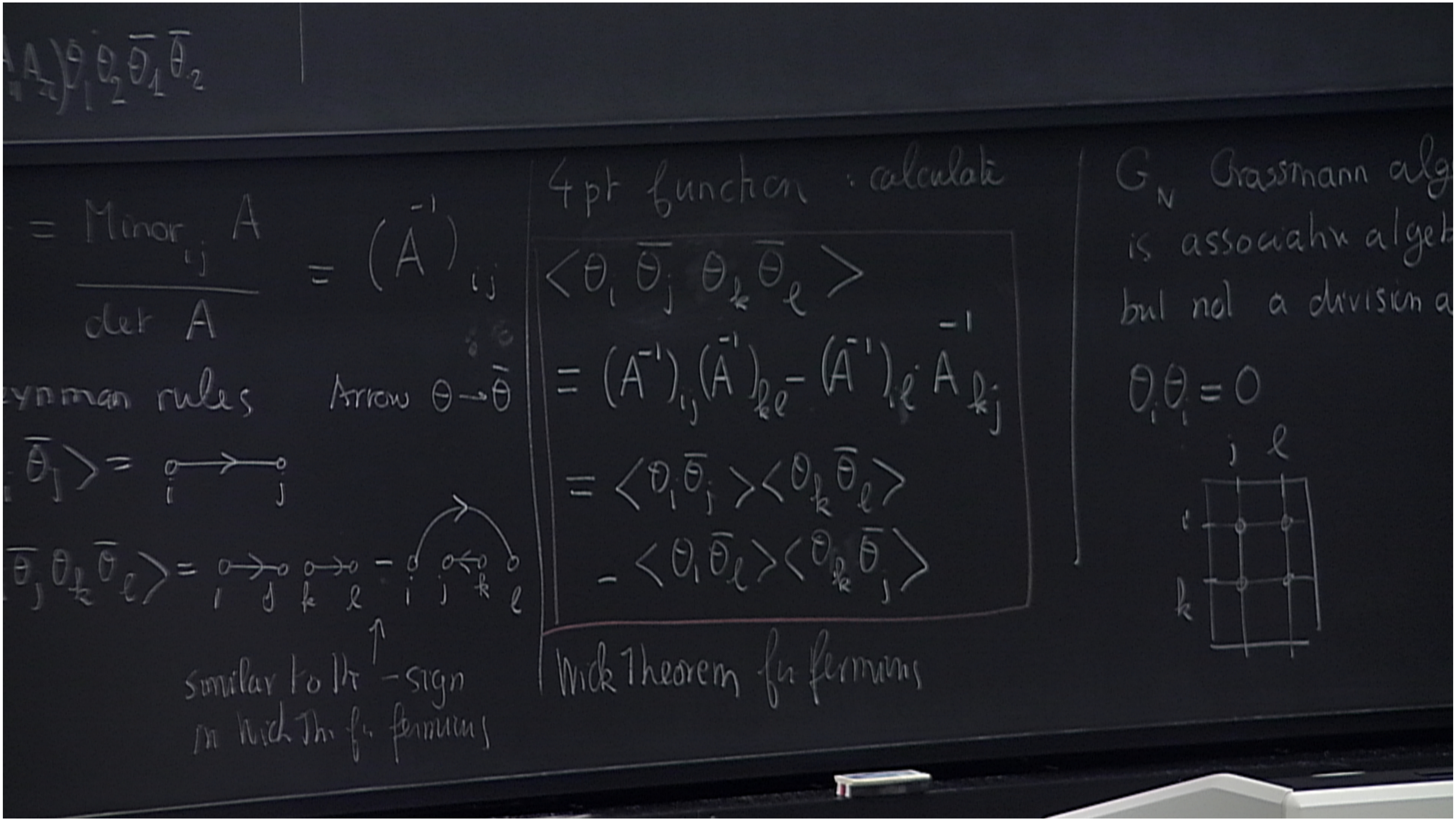
$$\langle\theta_i\bar\theta_j\theta_k\bar\theta_\ell\rangle = \underset{i}{\circ}\!\rightarrow\!\underset{j}{\circ}\;\underset{k}{\circ}\!\rightarrow\!\underset{\ell}{\circ} - \underset{i}{\circ}\;\underset{j}{\circ}\!\leftarrow\!\underset{k}{\circ}\;\underset{\ell}{\circ}$$

↑ similar to the − sign in Wick Th. for fermions

4pt function: calculate

$$\boxed{\begin{aligned}&\langle\theta_i\bar\theta_j\theta_k\bar\theta_\ell\rangle\\ &= (A^{-1})_{ij}(A^{-1})_{k\ell} - (A^{-1})_{i\ell}A^{-1}_{kj}\\ &= \langle\theta_i\bar\theta_j\rangle\langle\theta_k\bar\theta_\ell\rangle\\ &\quad - \langle\theta_i\bar\theta_\ell\rangle\langle\theta_k\bar\theta_j\rangle\end{aligned}}$$

Wick Theorem



$A A_{21})\theta_1 \theta_2 \bar{\theta}_1 \bar{\theta}_2$

$$= \frac{\text{Minor}_{ij} A}{\det A} = (A^{-1})_{ij}$$

eynman rules

Arrow $\theta \to \bar{\theta}$

$\bar{\theta}_j \rangle = \underset{i}{\circ} \longrightarrow \underset{j}{\circ}$

$\bar{\theta}_j \theta_k \bar{\theta}_l \rangle = \underset{i}{\circ}\!\rightarrow\!\underset{j}{\circ} \;\; \underset{k}{\circ}\!\rightarrow\!\underset{l}{\circ} \; - \; \underset{i}{\circ} \;\; \underset{j}{\circ}\!\leftarrow\!\underset{k}{\circ} \;\; \underset{l}{\circ}$

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$$\boxed{\begin{aligned} &\langle \theta_i \bar{\theta}_j \theta_k \bar{\theta}_l \rangle \\ &= (A^{-1})_{ij}(A^{-1})_{kl} - (A^{-1})_{il} A^{-1}_{kj} \\ &= \langle \theta_i \bar{\theta}_j \rangle \langle \theta_k \bar{\theta}_l \rangle \\ &\quad - \langle \theta_i \bar{\theta}_l \rangle \langle \theta_k \bar{\theta}_j \rangle \end{aligned}}$$

Wick Theorem for fermions

G_N Grassmann alg

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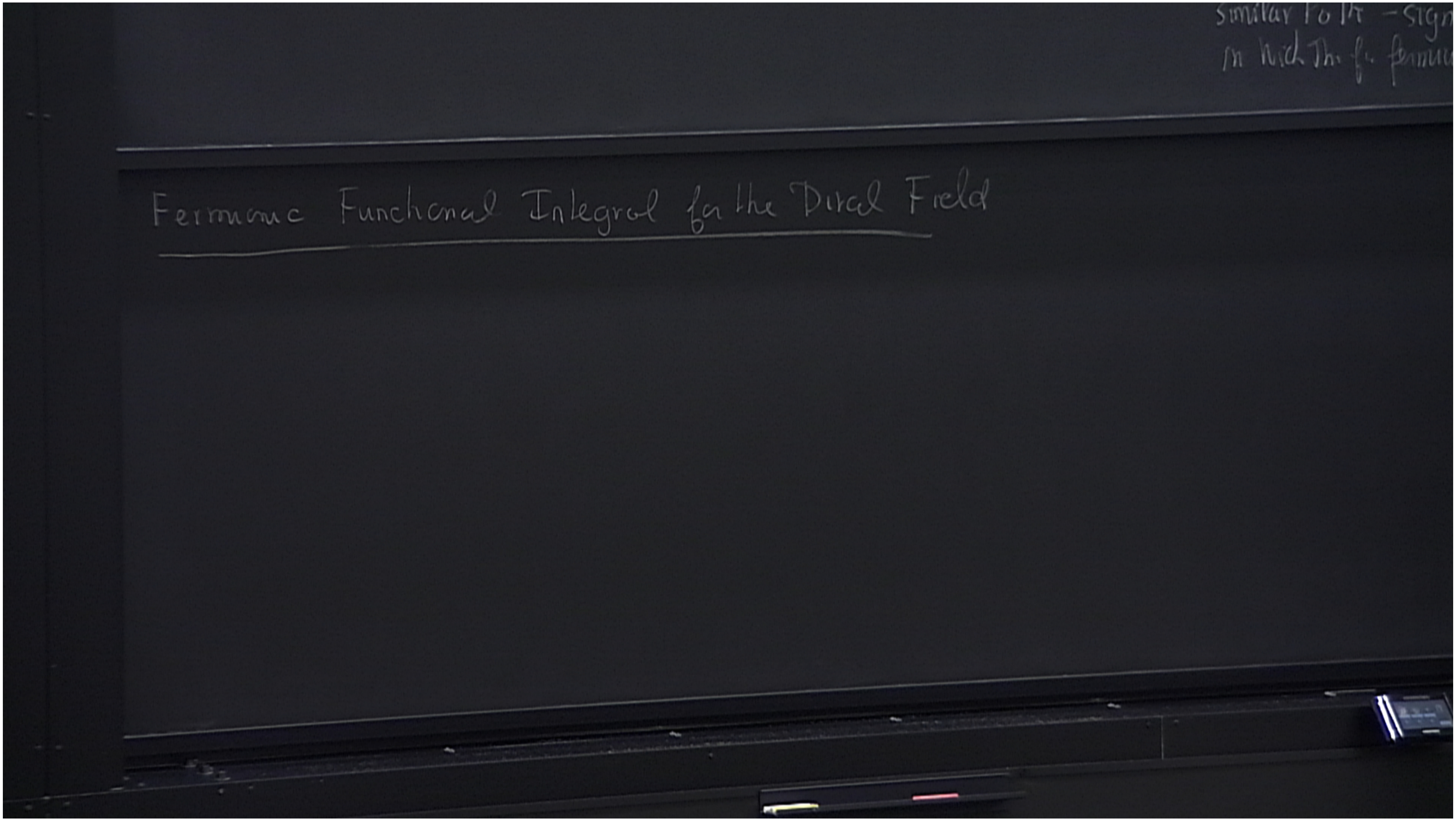
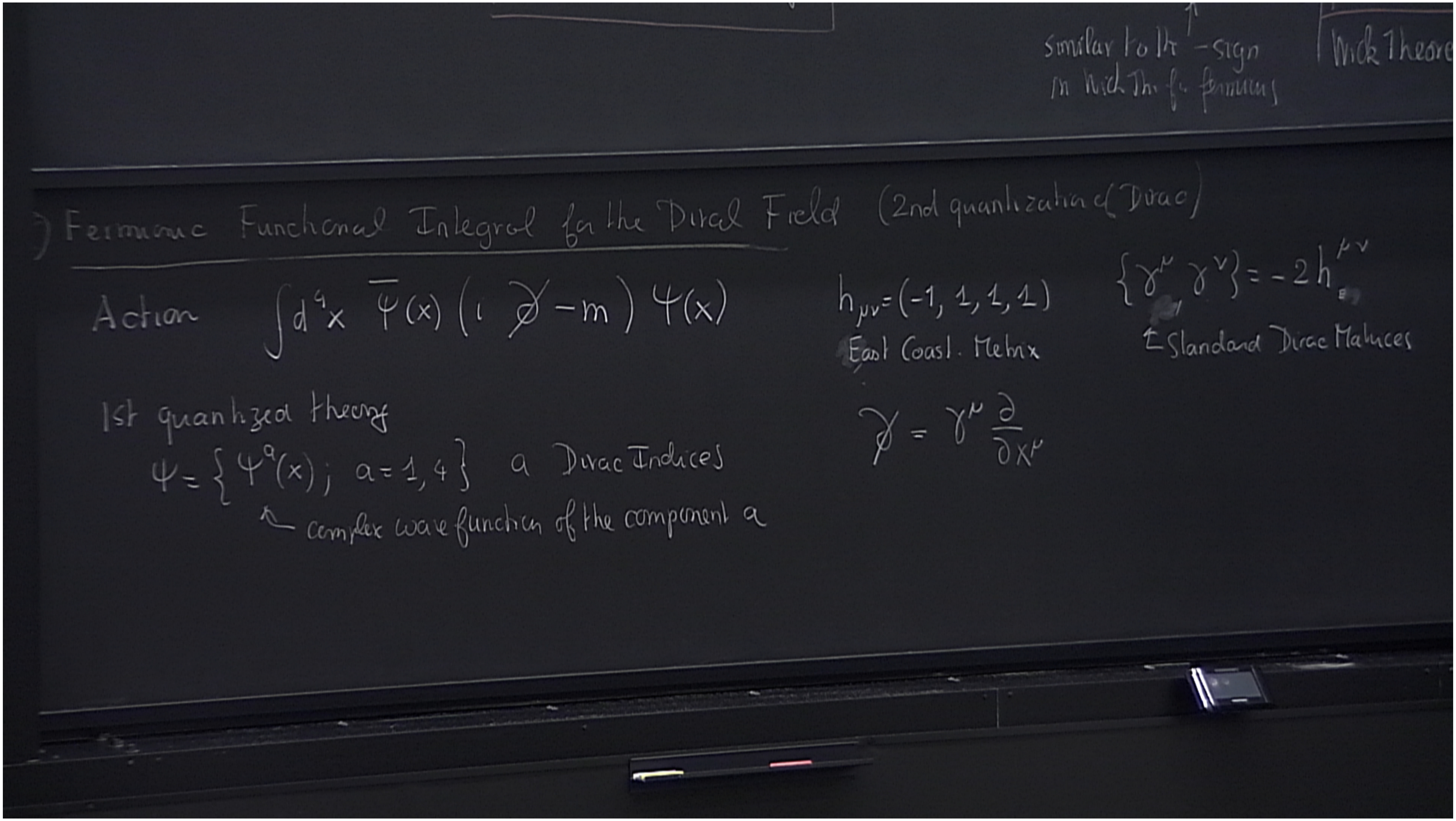


similar to PI – sign

in which the for fermi

Fermionic Functional Integral for the Dirac Field



similar to 1st – sign
in Wick Thm for fermions

Wick Theore

7) Fermionic Functional Integral for the Dirac Field (2nd quantization of Dirac)

Action $\int d^4x \; \bar{\Psi}(x) \left(i \not{\partial} - m \right) \Psi(x)$

$h_{\mu\nu} = (-1, 1, 1, 1)$
East Coast. Metric

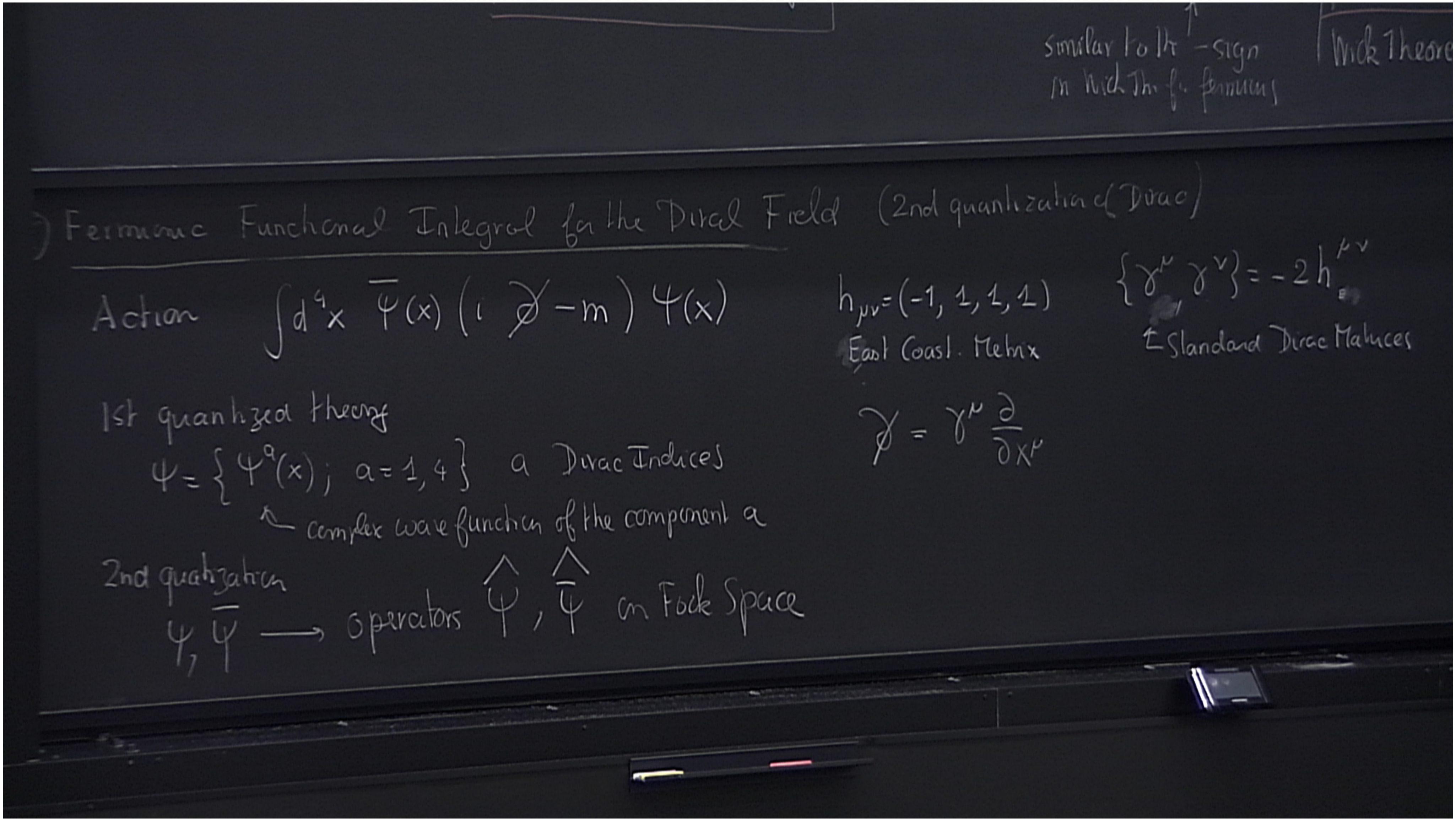
$\{ \gamma^\mu , \gamma^\nu \} = -2 h^{\mu\nu}$
← Standard Dirac Matrices

$\not{\partial} = \gamma^\mu \frac{\partial}{\partial x^\mu}$

1st quantized theory

$\Psi = \{ \Psi^a(x) ; \; a = 1, 4 \}$ a Dirac Indices

↖ complex wave function of the component a



similar to the − sign in Wick Thm for fermions

Wick Theore

Fermionic Functional Integral for the Dirac Field (2nd quantization of Dirac)

Action $\int d^4x \, \overline{\Psi}(x) \left(i \not{\partial} - m \right) \Psi(x)$

$h_{\mu\nu} = (-1, 1, 1, 1)$

East Coast. Metrix

$\{ \gamma^\mu , \gamma^\nu \} = -2 h^{\mu\nu}$

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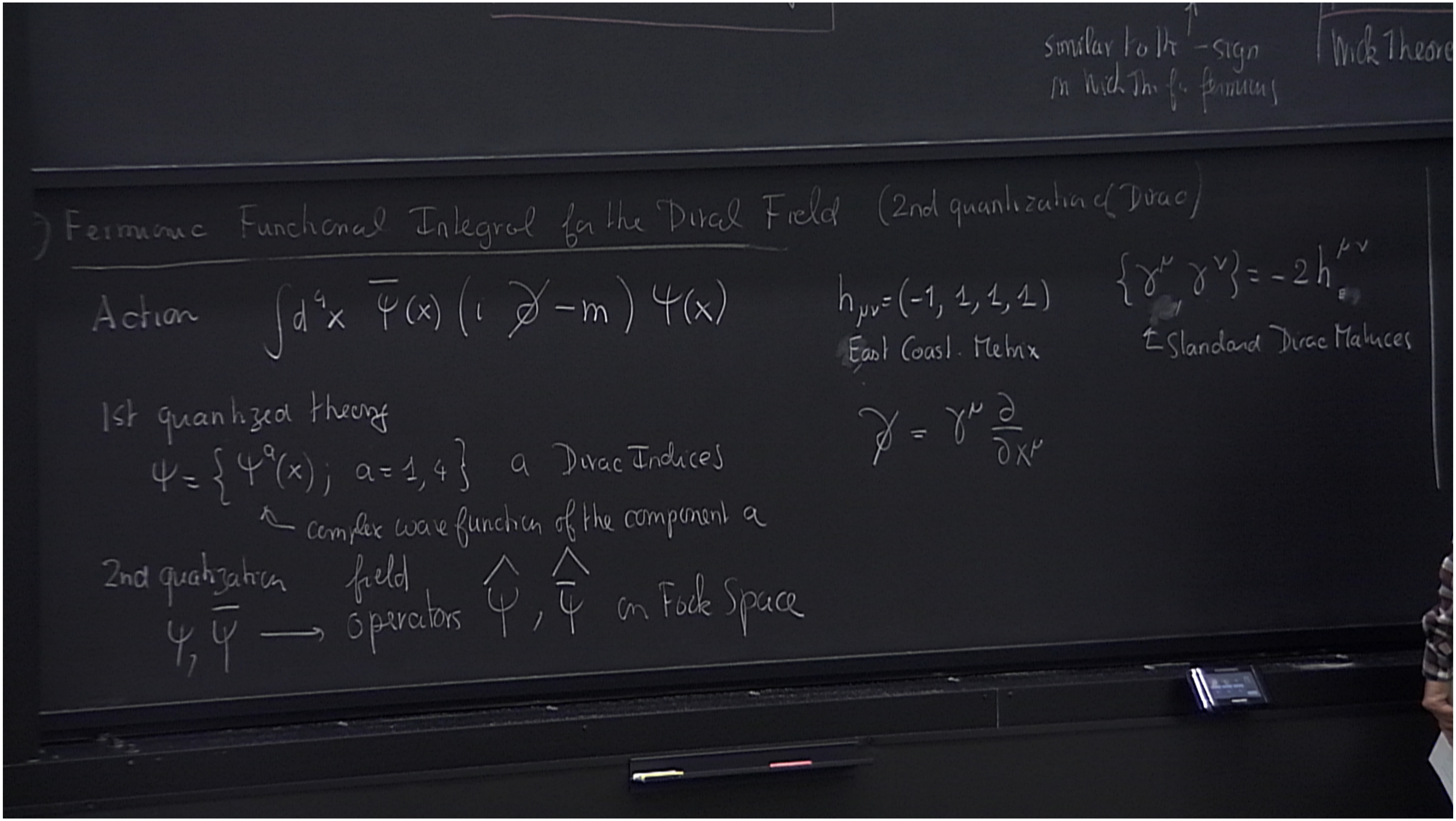
2nd quantization

$\Psi , \overline{\Psi} \longrightarrow$ operators $\hat{\Psi} , \hat{\overline{\Psi}}$ on Fock Space



similar to the - sign
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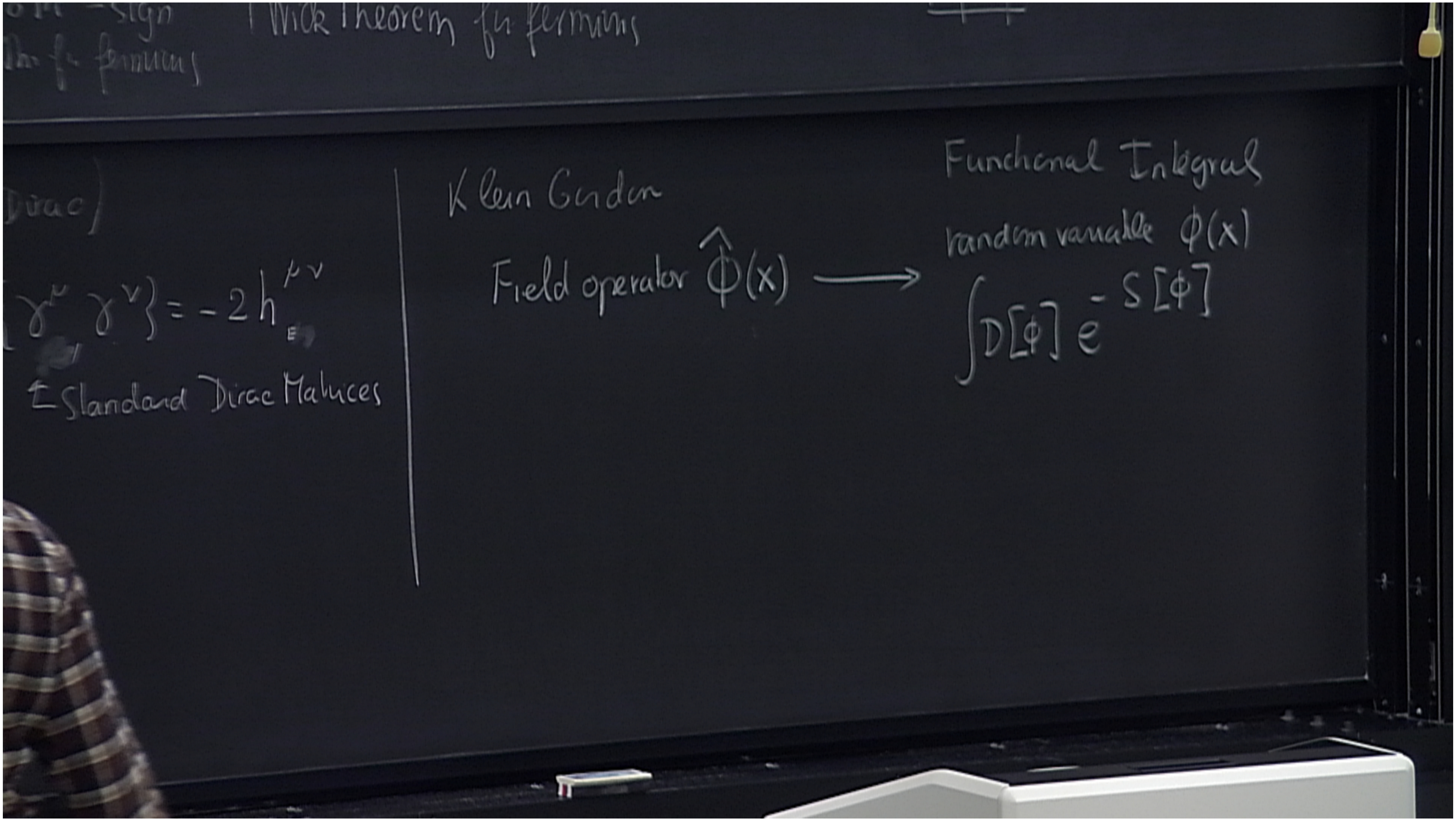


sign Wick Theorem for fermions

Th. for fermions

Dirac)

$\{\gamma^\mu, \gamma^\nu\} = -2\, h^{\mu\nu}_{E}$

Standard Dirac Matrices

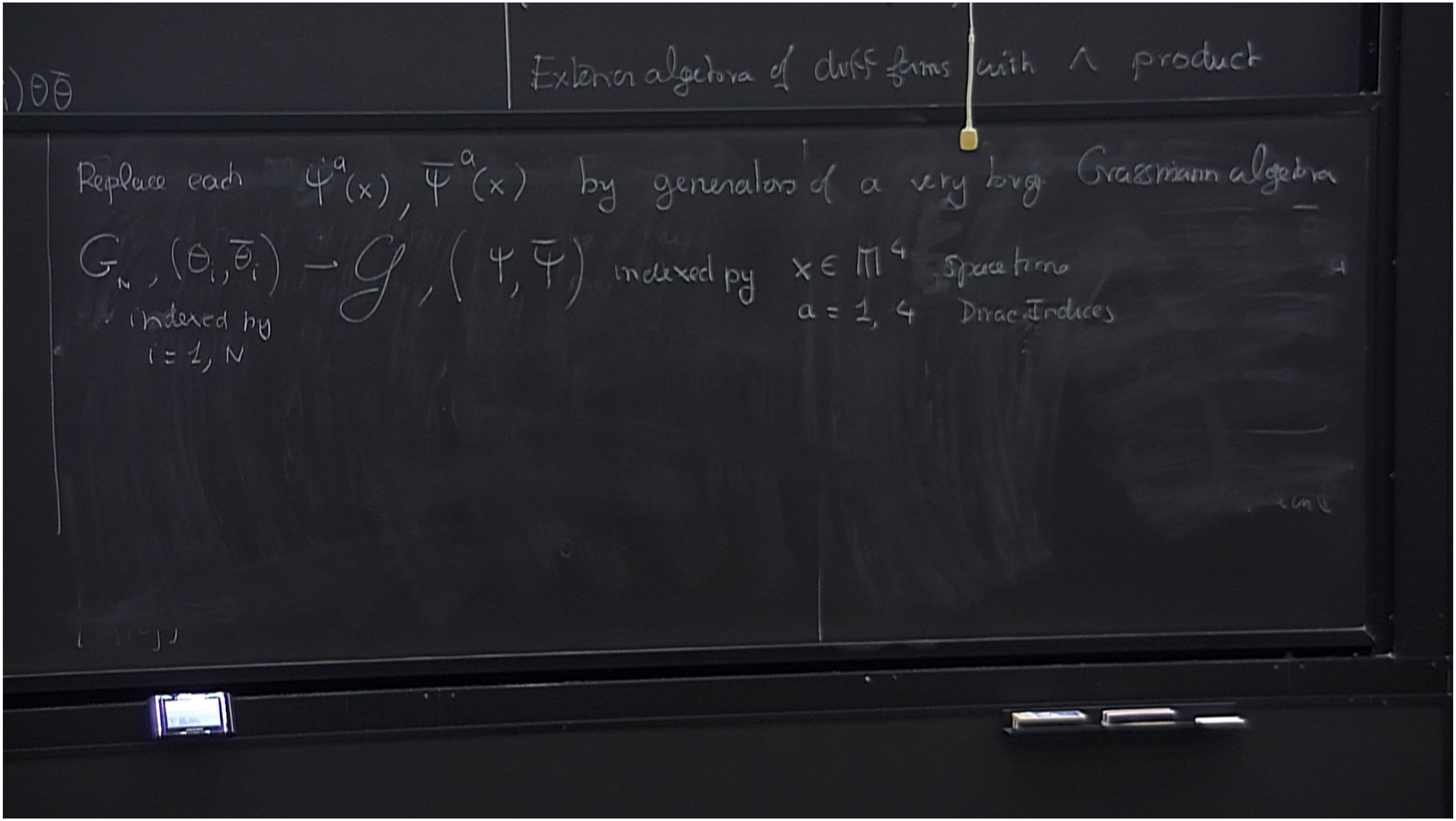
Klein Gordon

Field operator $\hat{\phi}(x) \longrightarrow$

Functional Integral

random variable $\phi(x)$

$\int D[\phi]\, e^{-S[\phi]}$



$)\theta\bar{\theta}$

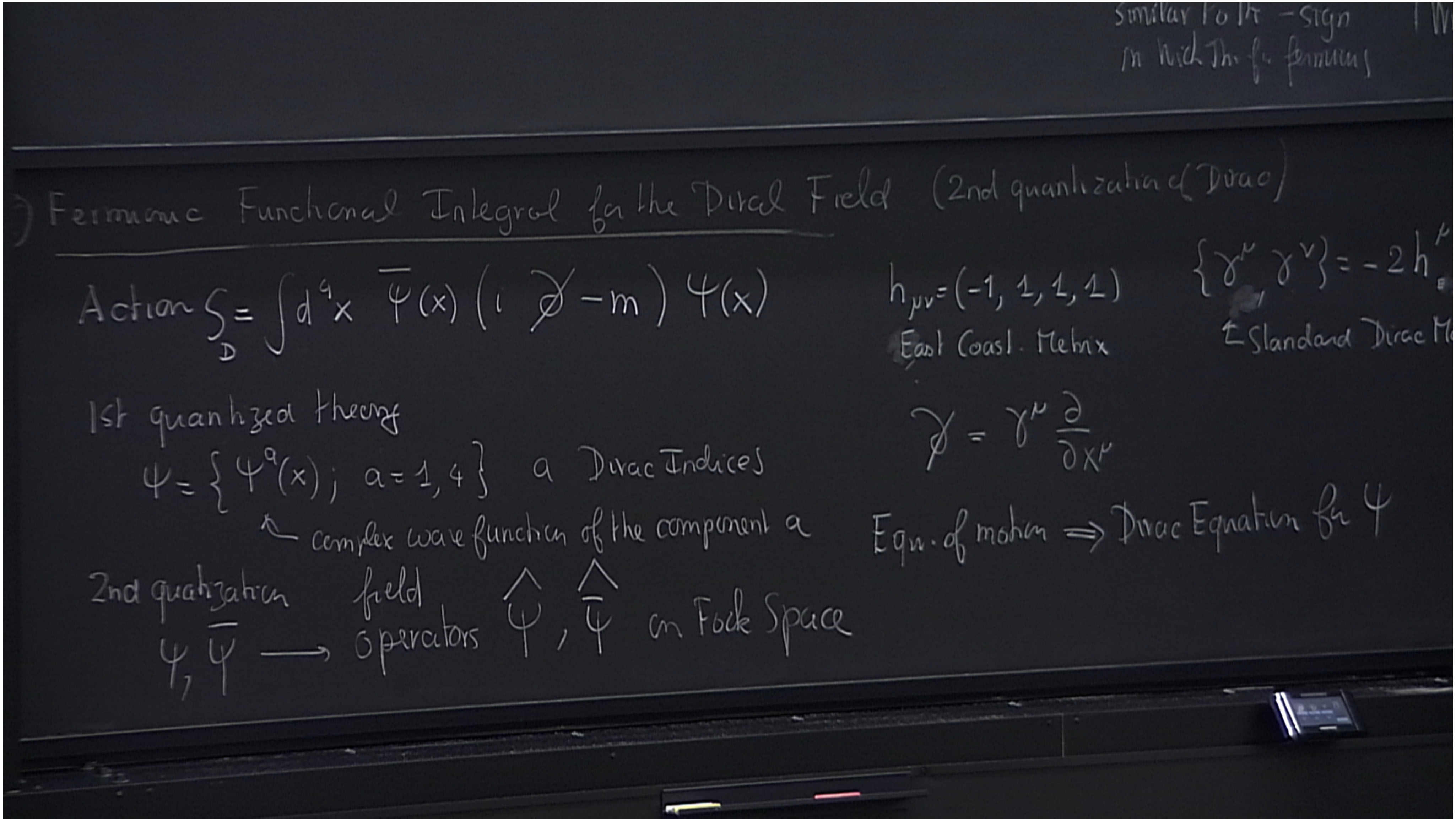
Exterior algebra of diff forms with $\wedge$ product

Replace each $\psi^a(x)$, $\bar{\psi}^a(x)$ by generators of a very large Grassmann algebra

G_N, $(\theta_i, \bar{\theta}_i) \to \mathcal{G}$, $(\psi, \bar{\psi})$ indexed by $x \in M^4$ space time

indexed by $i = 1, N$

$a = 1, 4$ Dirac Indices



Similar to th – sign
in Wick Th. for fermions

Fermionic Functional Integral for the Dirac Field (2nd quantization of Dirac)

Action $S_D = \int d^4x \; \bar{\psi}(x) \left(i \not{\partial} - m \right) \psi(x)$

$h_{\mu\nu} = (-1, 1, 1, 1)$

East Coast. Metric

$\{\gamma^\mu, \gamma^\nu\} = -2h^{\mu}$

← Standard Dirac M

1st quantized theory

$\psi = \{ \psi^a(x) ; \; a = 1, 4 \}$ a Dirac Indices

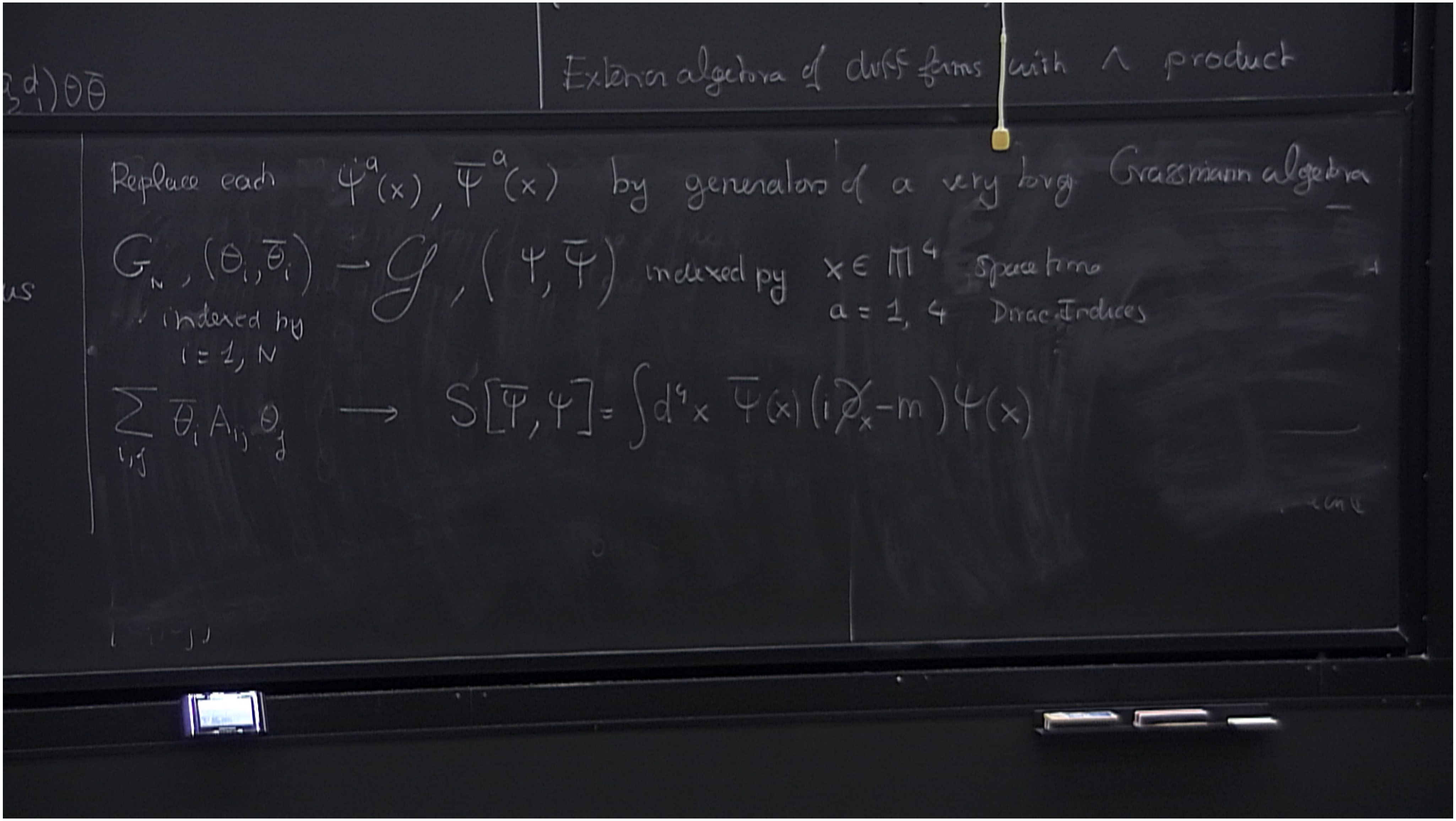
↖ complex wave function of the component a

$\not{\partial} = \gamma^\mu \frac{\partial}{\partial x^\mu}$

Eqn. of motion ⟹ Dirac Equation for ψ

2nd quantization field

$\psi, \bar{\psi} \longrightarrow$ operators $\hat{\psi}, \hat{\bar{\psi}}$ on Fock Space



$d_1)\theta\bar{\theta}$

Exterior algebra of diff forms with $\wedge$ product

us

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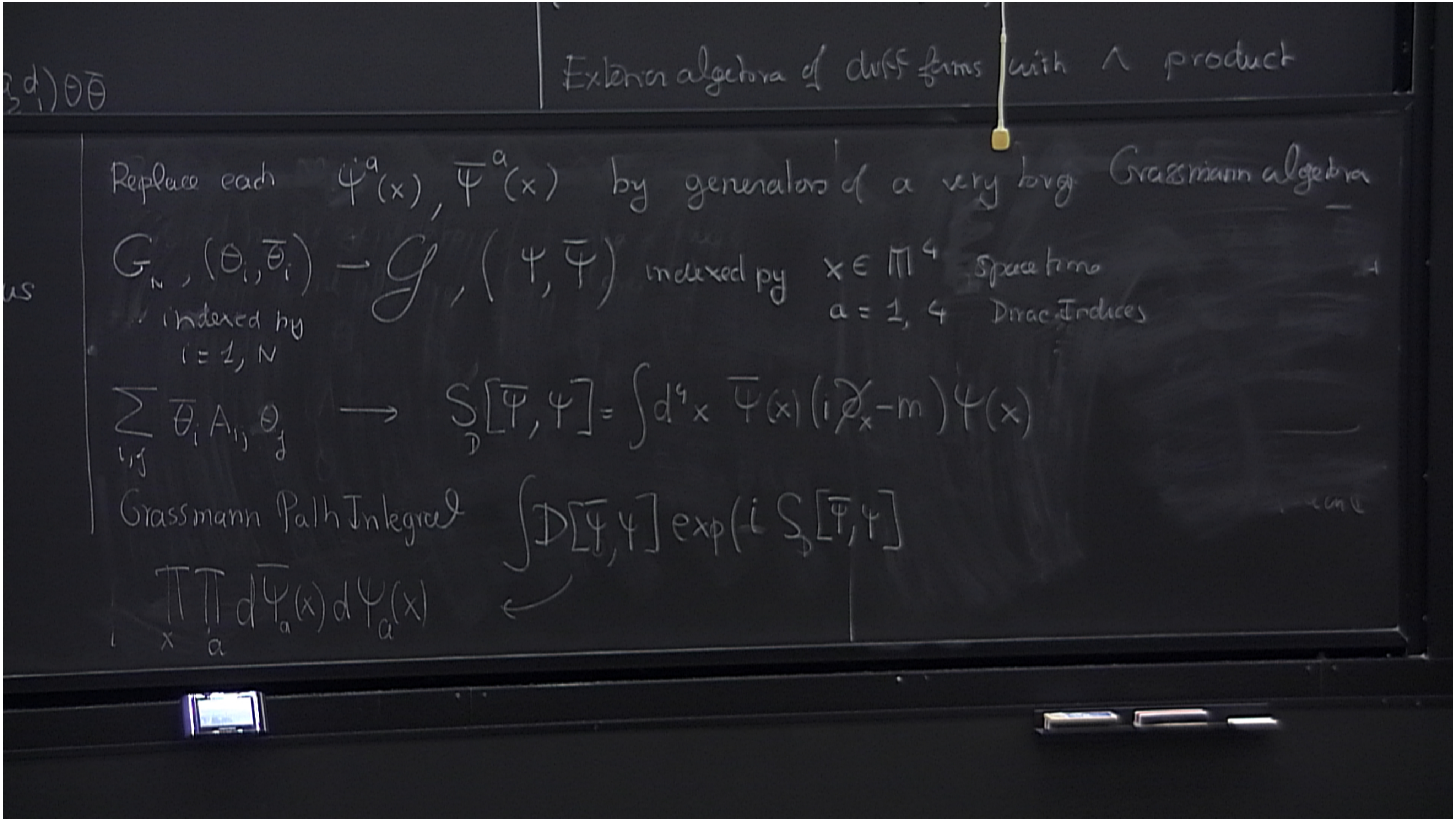
G_N, $(\theta_i, \bar{\theta}_i) \rightarrow \mathcal{G}$, $(\psi, \bar{\psi})$ indexed by $x \in M^4$ space time

$a = 1, 4$ Dirac Indices

indexed by
$i = 1, N$

$$\sum_{i,j} \bar{\theta}_i A_{ij} \theta_j \longrightarrow S[\bar{\psi}, \psi] = \int d^4x \, \bar{\psi}(x)(i\cancel{\partial}_x - m)\psi(x)$$



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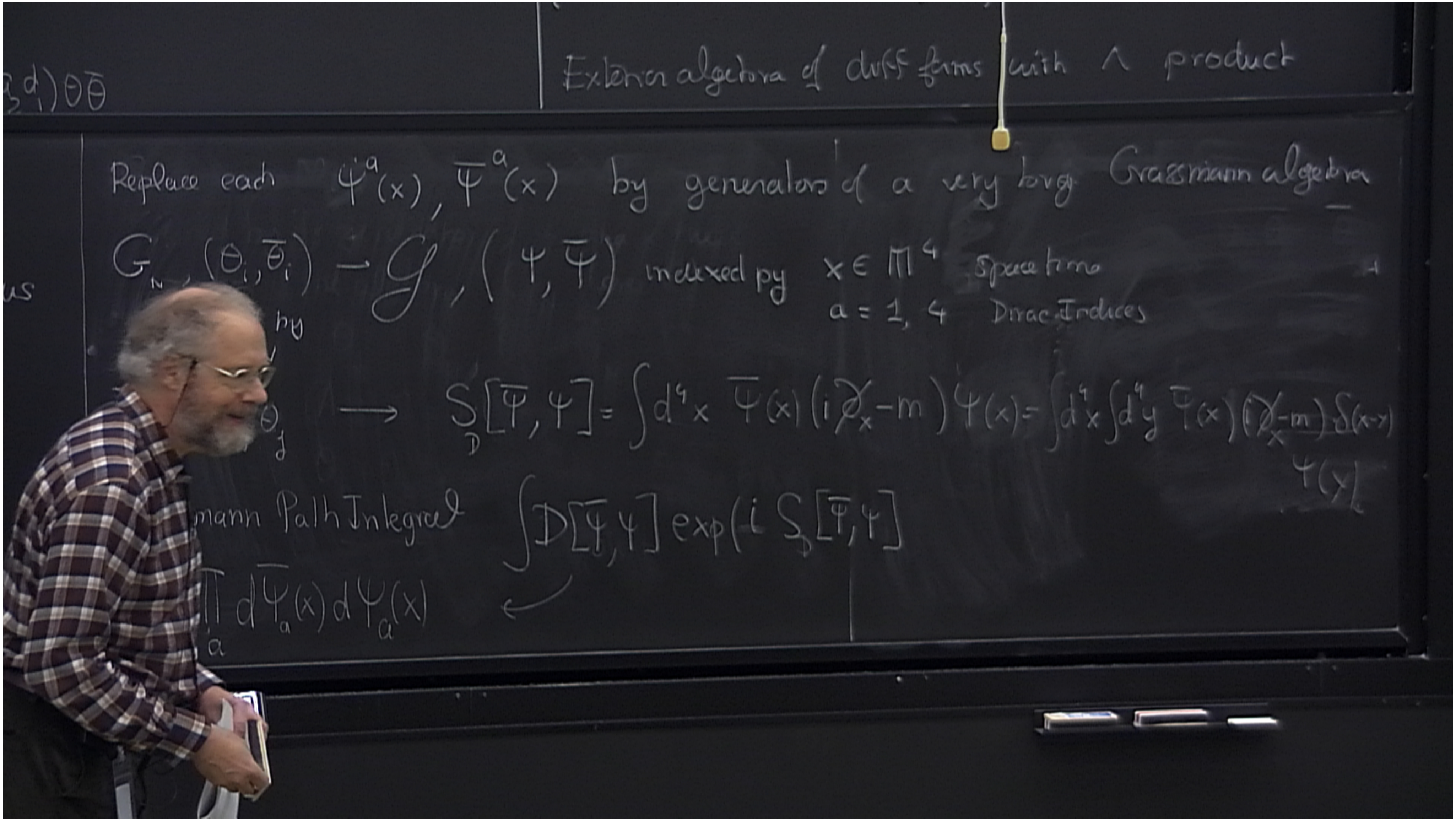
$$\sum_{i,j} \bar{\theta}_i A_{ij} \theta_j \longrightarrow S_D[\bar{\psi}, \psi] = \int d^4x \, \bar{\psi}(x)(i\not{\partial}_x - m)\psi(x)$$

Grassmann Path Integral $\int D[\bar{\psi}, \psi] \exp(i S_D[\bar{\psi}, \psi]$

$$\prod_x \prod_a d\bar{\psi}_a(x) \, d\psi_a(x) \longleftarrow$$



$d_1)\theta\bar{\theta}$

Exterior algebra of diff forms with $\wedge$ product

Replace each $\psi^a(x), \overline{\psi}^a(x)$ by generators of a very large Grassmann algebra

$G_N, (\theta_i, \bar{\theta}_i) \rightarrow \mathcal{G}, (\psi, \overline{\psi})$ indexed by $x \in M^4$ Space time

$a = 1, 4$ Dirac Indices

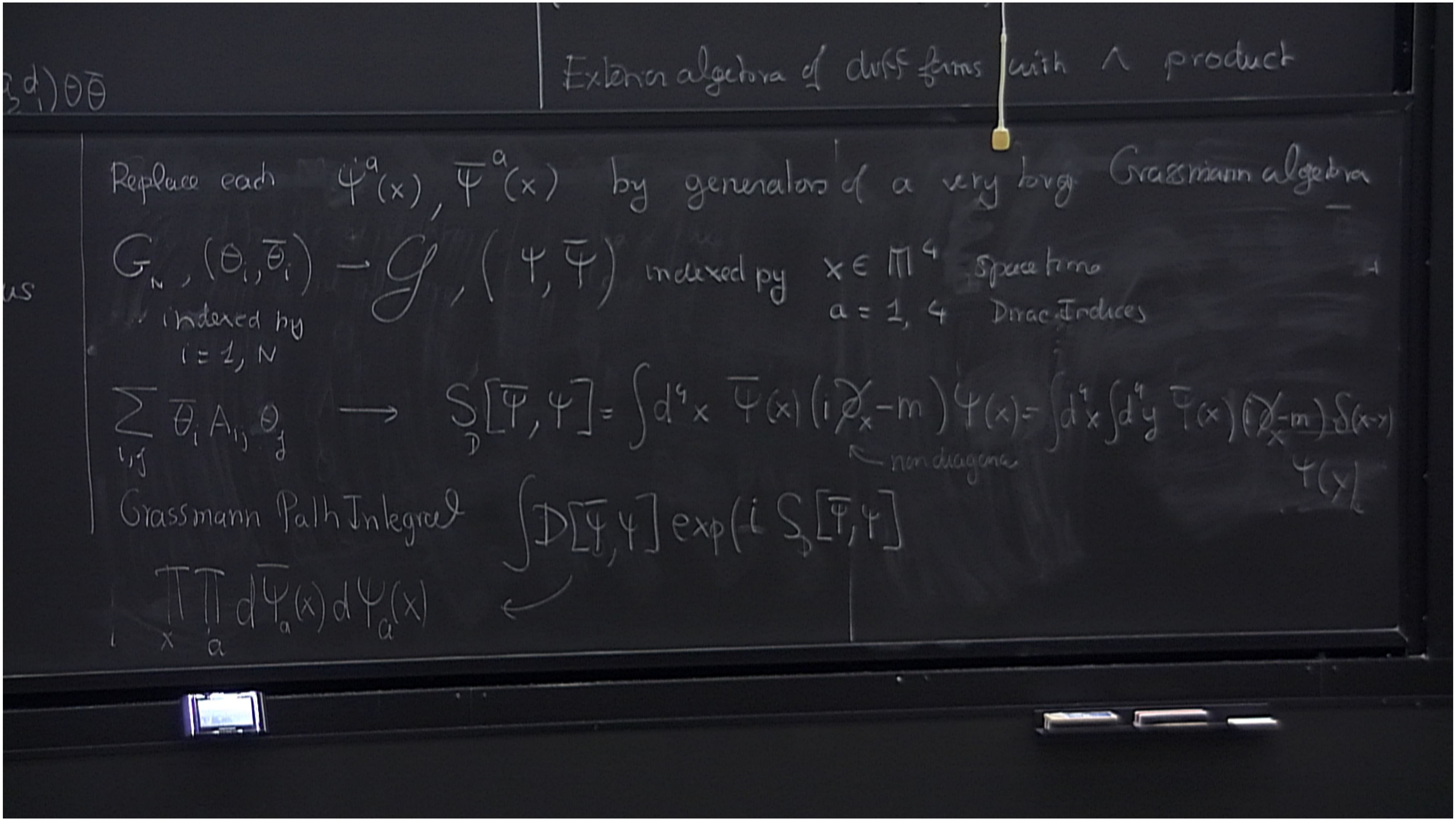
us

by

$\theta_j \longrightarrow S_D[\overline{\psi}, \psi] = \int d^4x\, \overline{\psi}(x)(i\not{\partial}_x - m)\psi(x) = \int d^4x \int d^4y\, \overline{\psi}(x)(i\not{\partial}_x - m)\delta(x-y)\psi(y)$

mann Path Integral $\int \mathcal{D}[\overline{\psi}, \psi] \exp(i S_D[\overline{\psi}, \psi]$

$\prod_a d\overline{\psi}_a(x) d\psi_a(x) \leftarrow$



$d_1)\theta\bar{\theta}$

Exterior algebra of diff forms with $\wedge$ product

us

Replace each $\psi^a(x), \bar{\psi}^a(x)$ by generators of a very large Grassmann algebra

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$a = 1, 4$ Dirac Indices

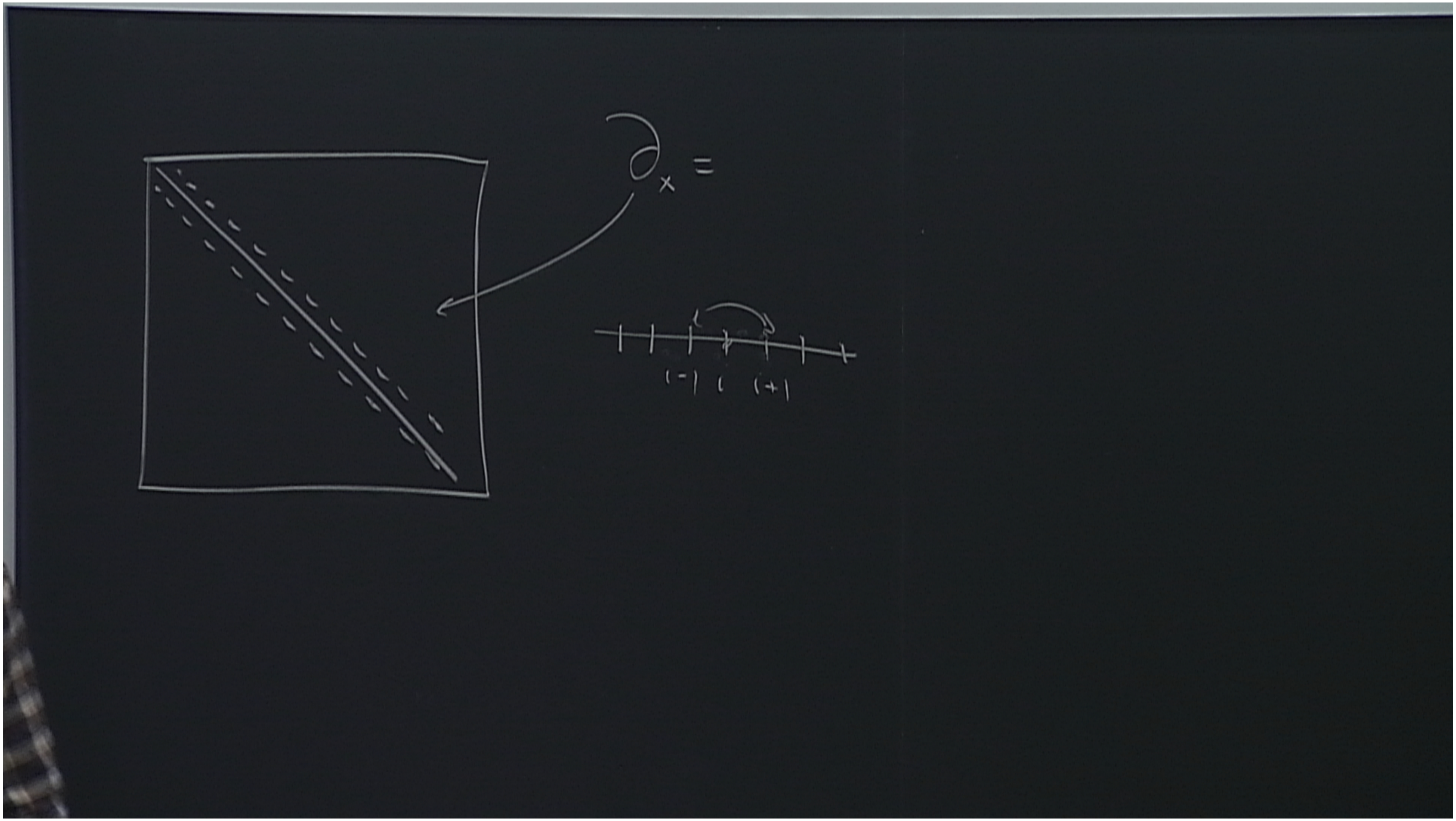
indexed by
$i = 1, N$

$$\sum_{i,j} \bar{\theta}_i A_{ij} \theta_j \longrightarrow S_D[\bar{\psi}, \psi] = \int d^4x \, \bar{\psi}(x)(i \not{\partial}_x - m)\psi(x) = \int d^4x \int d^4y \, \bar{\psi}(x)(i\not{\partial}_x - m)\delta(x-y)\psi(y)$$

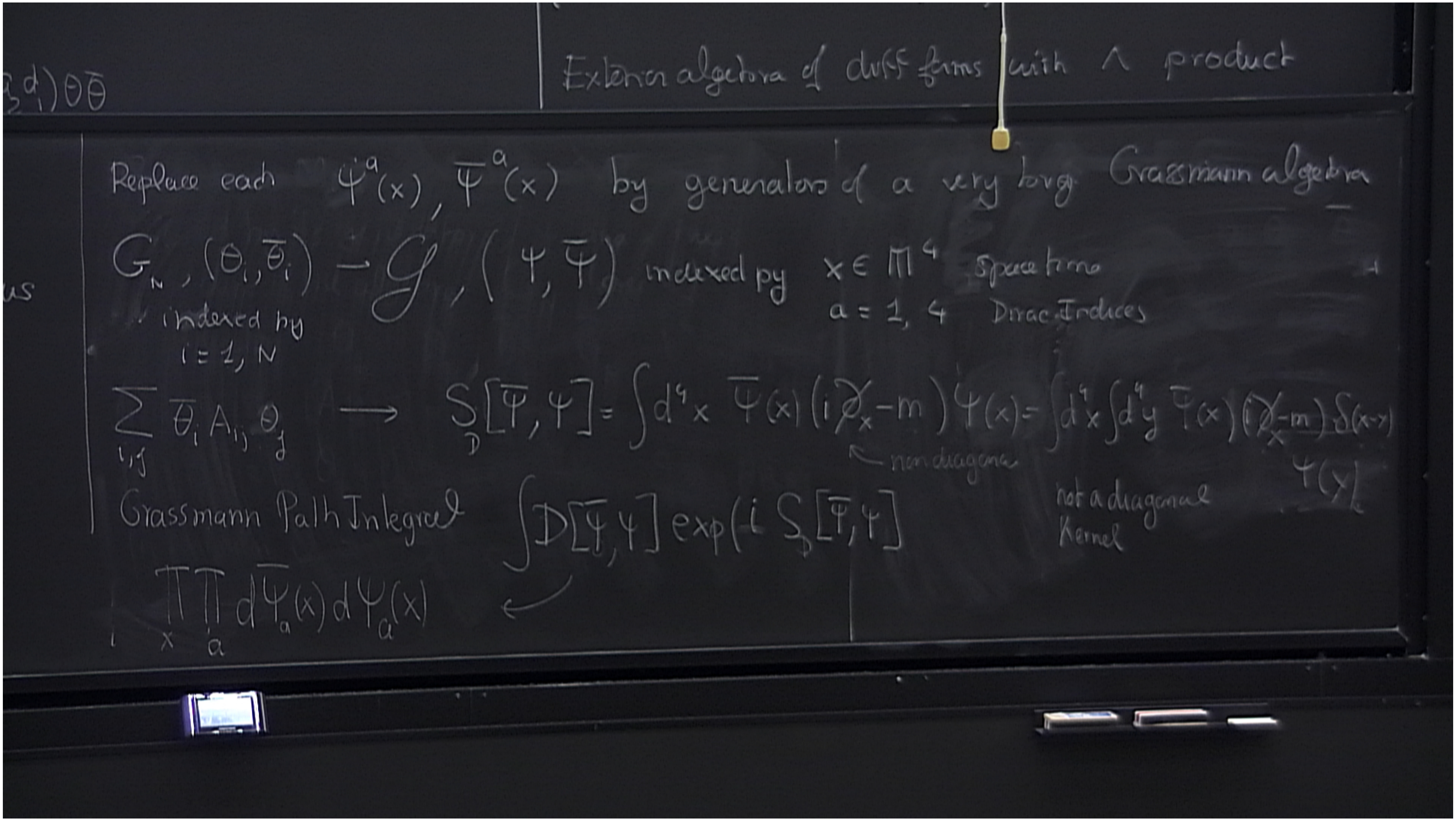
← non diagonal

Grassmann Path Integral $\int \mathcal{D}[\bar{\psi}, \psi] \exp(i S_D[\bar{\psi}, \psi]$

$$\prod_{x} \prod_{a} d\bar{\psi}_a(x) \, d\psi_a(x)$$



$\partial_x =$



$d_1)\theta\bar\theta$

Exterior algebra of diff forms with $\wedge$ product

us

Replace each $\psi^a(x)$, $\bar\psi^a(x)$ by generators of a very large Grassmann algebra

$G_N, (\theta_i, \bar\theta_i) \to \mathcal{G}, (\psi, \bar\psi)$ indexed by $x \in M^4$ Space time

$a = 1, 4$ Dirac Indices

indexed by
$i = 1, N$

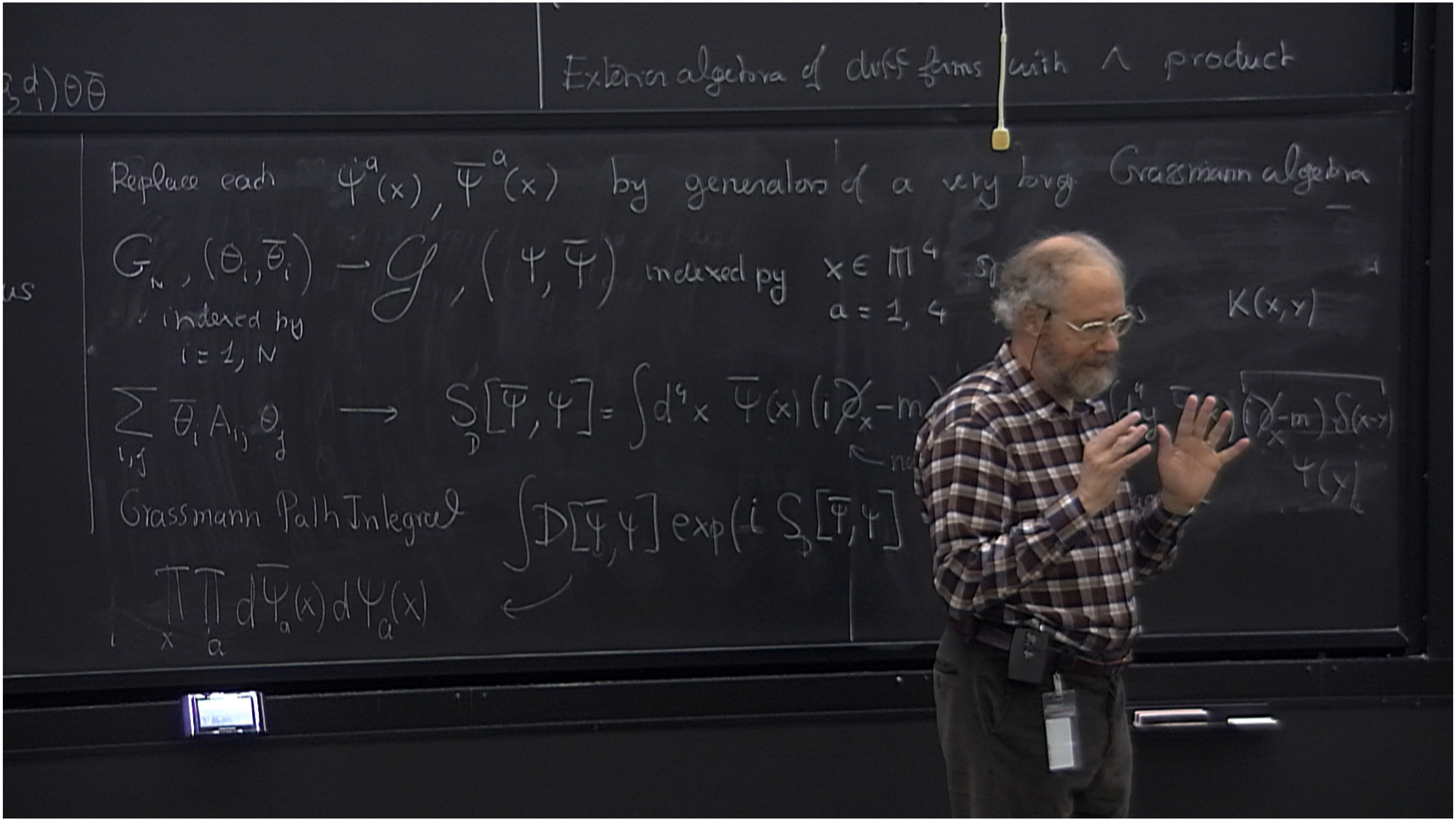
$$\sum_{i,j} \bar\theta_i A_{ij} \theta_j \longrightarrow S_D[\bar\psi, \psi] = \int d^4x\, \bar\psi(x)(i\not\partial_x - m)\psi(x) = \int d^4x \int d^4y\, \bar\psi(x)(i\not\partial_x - m)\delta(x-y)\psi(y)$$

$\leftarrow$ non diagonal

not a diagonal Kernel

Grassmann Path Integral $\int D[\bar\psi, \psi] \exp(i S_D[\bar\psi, \psi]$

$$\prod_x \prod_a d\bar\psi_a(x)\, d\psi_a(x) \leftarrow$$



$d_1)\theta\bar\theta$

Exterior algebra of diff forms with $\wedge$ product

us

Replace each $\psi^a(x)$, $\bar\psi^a(x)$ by generators of a very large Grassmann algebra

$G_N, (\theta_i, \bar\theta_i) \to \mathcal{G}, (\psi, \bar\psi)$ indexed by $x \in M^4$
$a = 1, 4$

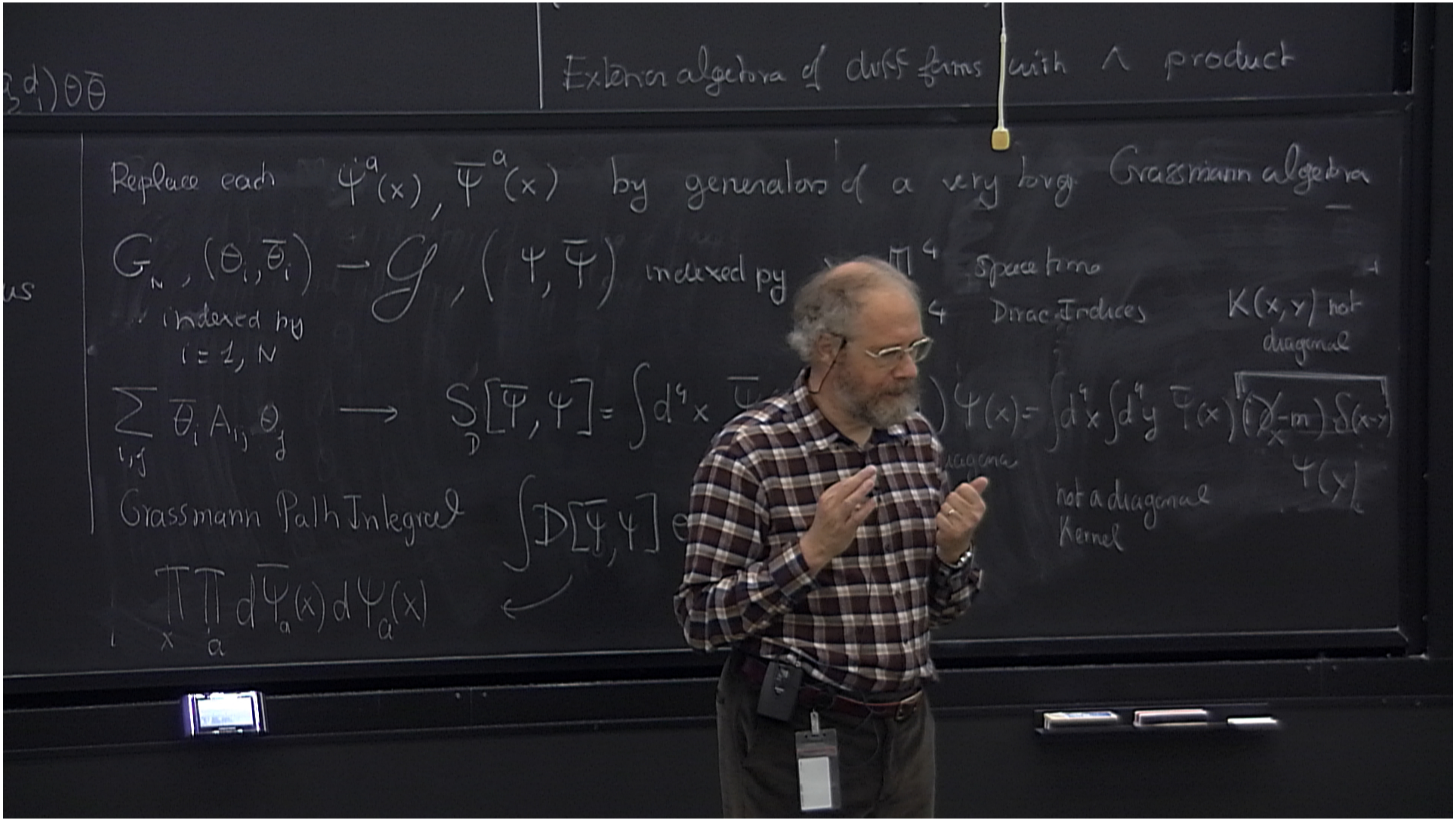
indexed by
$i = 1, N$

$K(x,y)$

$$\sum_{i,j} \bar\theta_i A_{ij} \theta_j \longrightarrow S_D[\bar\psi, \psi] = \int d^4x\, \bar\psi(x)(i\not\partial_x - m) \quad \int d^4y \quad (i\not\partial_x - m)\delta(x-y)\,\psi(y)$$

Grassmann Path Integral $\int \mathcal{D}[\bar\psi, \psi] \exp(i S_D[\bar\psi, \psi]$

$$\prod_x \prod_a d\bar\psi_a(x)\, d\psi_a(x)$$



$d_1)\theta\bar\theta$

Exterior algebra of diff forms with ∧ product

us

Replace each $\psi^a(x)$, $\bar\psi^a(x)$ by generators of a very large Grassmann algebra

$G_N, (\theta_i, \bar\theta_i) \to \mathcal{G}, (\psi, \bar\psi)$ indexed by $\mathbb{M}^4$ Space time

indexed by $i = 1, N$ 4 Dirac Indices

$K(x,y)$ not diagonal

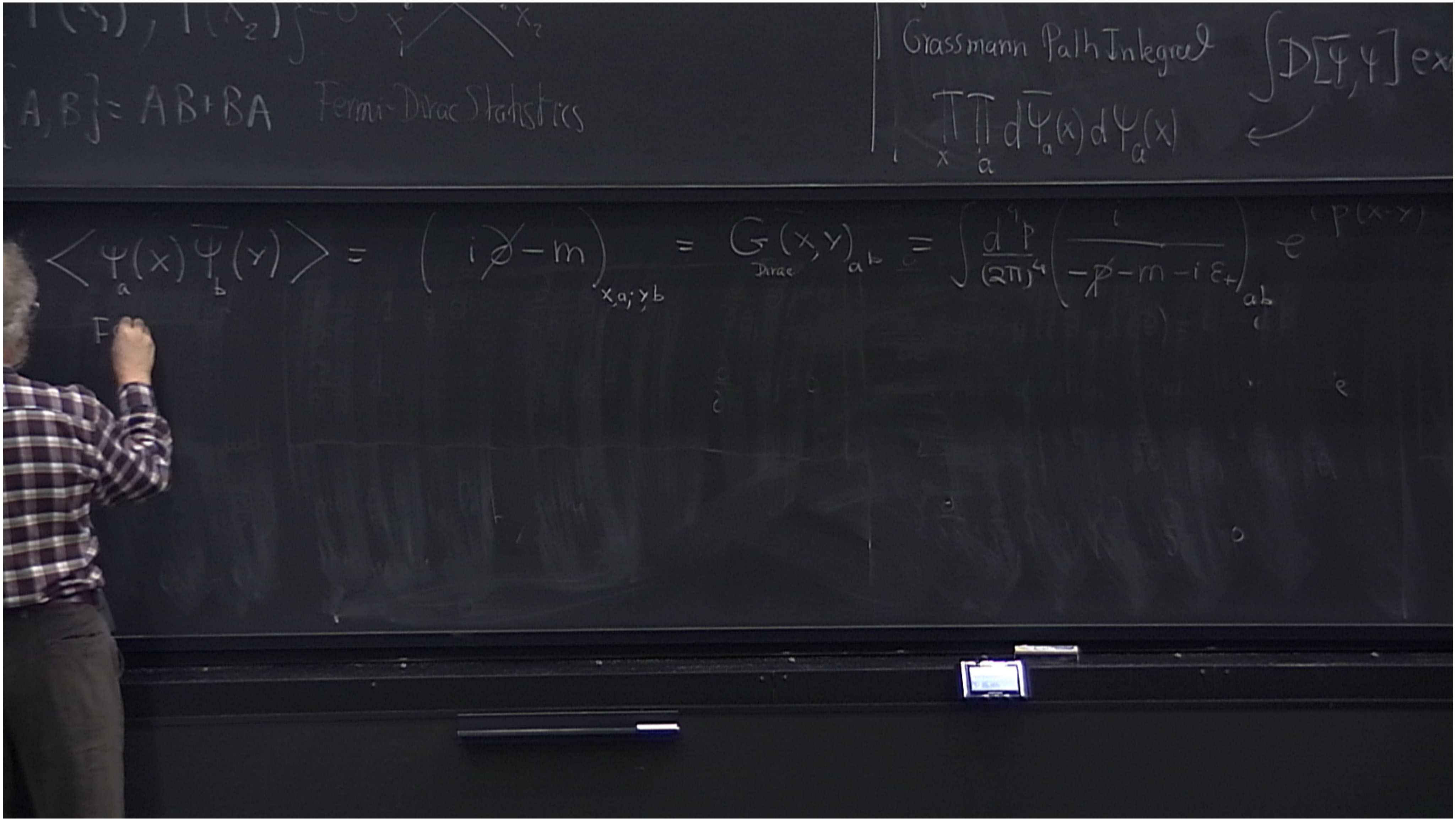
$$\sum_{i,j} \bar\theta_i A_{ij} \theta_j \longrightarrow S_D[\bar\psi, \psi] = \int d^4x\, \bar\psi \quad)\psi(x) = \int d^4x \int d^4y\, \bar\psi(x) \boxed{(i\not\partial_x - m)\delta(x-y)} \psi(y)$$

diagonal

not a diagonal Kernel

Grassmann Path Integral $\int D[\bar\psi, \psi]\, e$

$$\prod_x \prod_a d\bar\psi_a(x)\, d\psi_a(x) \longleftarrow$$



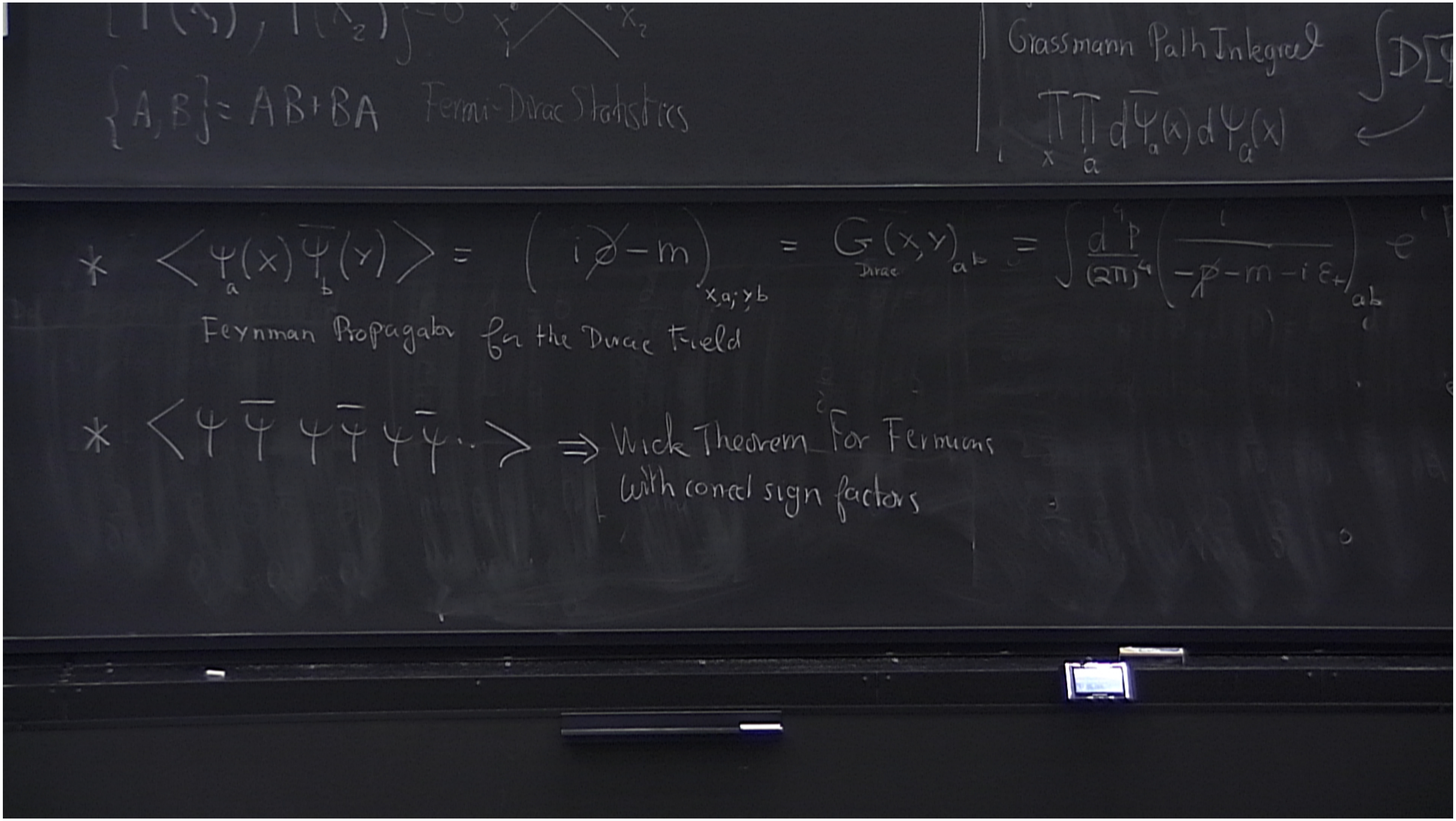
$\{A,B\} = AB + BA$ Fermi-Dirac Statistics

Grassmann Path Integral $\int D[\bar{\psi},\psi]$ ex

$\prod_x \prod_a d\bar{\psi}_a(x)\, d\psi_a(x)$

$$\langle \psi_a(x)\bar{\psi}_b(y)\rangle = \left(i\cancel{\partial} - m \right)_{xa;yb} = G_{\text{Dirac}}(x,y)_{ab} = \int \frac{d^4p}{(2\pi)^4}\left(\frac{i}{-\cancel{p}-m-i\varepsilon_+}\right)_{ab} e^{ip(x-y)}$$

F



$\{T(x_1), T(x_2)\} = 0$

$\{A, B\} = AB + BA$ Fermi-Dirac Statistics

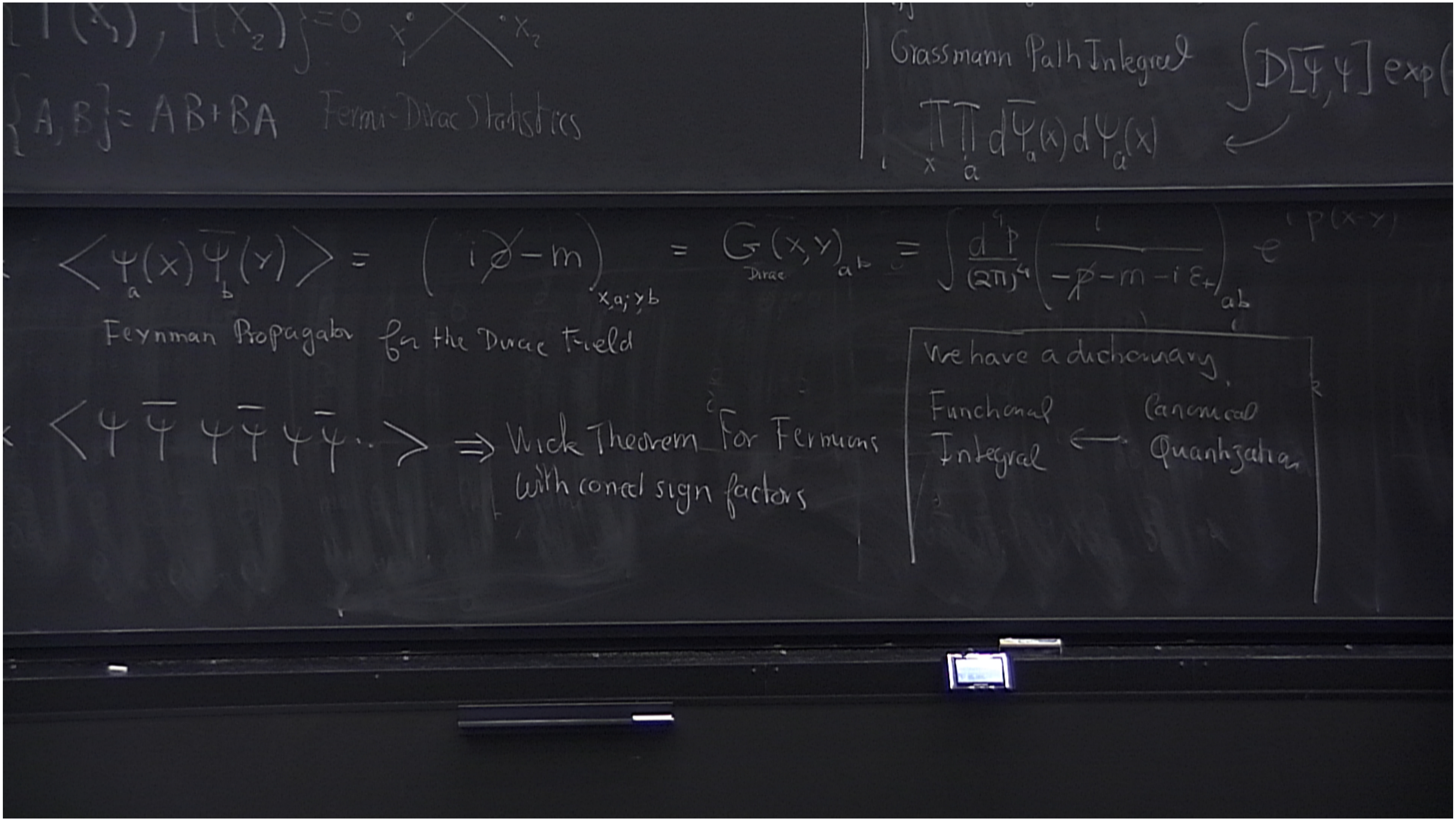
Grassmann Path Integral $\int D[\Psi$

$\prod_x \prod_a d\bar{\Psi}_a(x)\, d\Psi_a(x)$

$$* \quad \langle \Psi_a(x) \bar{\Psi}_b(y) \rangle = \left(i\partial\!\!\!/ - m \right)_{x,a;\, y,b} = G_{\text{Dirac}}(x,y)_{ab} = \int \frac{d^4 p}{(2\pi)^4} \left(\frac{i}{-p\!\!\!/ - m - i\varepsilon_+} \right)_{ab} e$$

Feynman Propagator for the Dirac Field

$$* \quad \langle \Psi \bar{\Psi} \Psi \bar{\Psi} \Psi \bar{\Psi} \cdots \rangle$$ $\Rightarrow$ Wick Theorem For Fermions with correct sign factors



$\{\Psi(x_1), \Psi(x_2)\} = 0$

$\{A, B\} = AB + BA$ Fermi-Dirac Statistics

Grassmann Path Integral $\int D[\bar\Psi, \Psi] \exp($

$\prod_x \prod_a d\bar\Psi_a(x)\, d\Psi_a(x)$

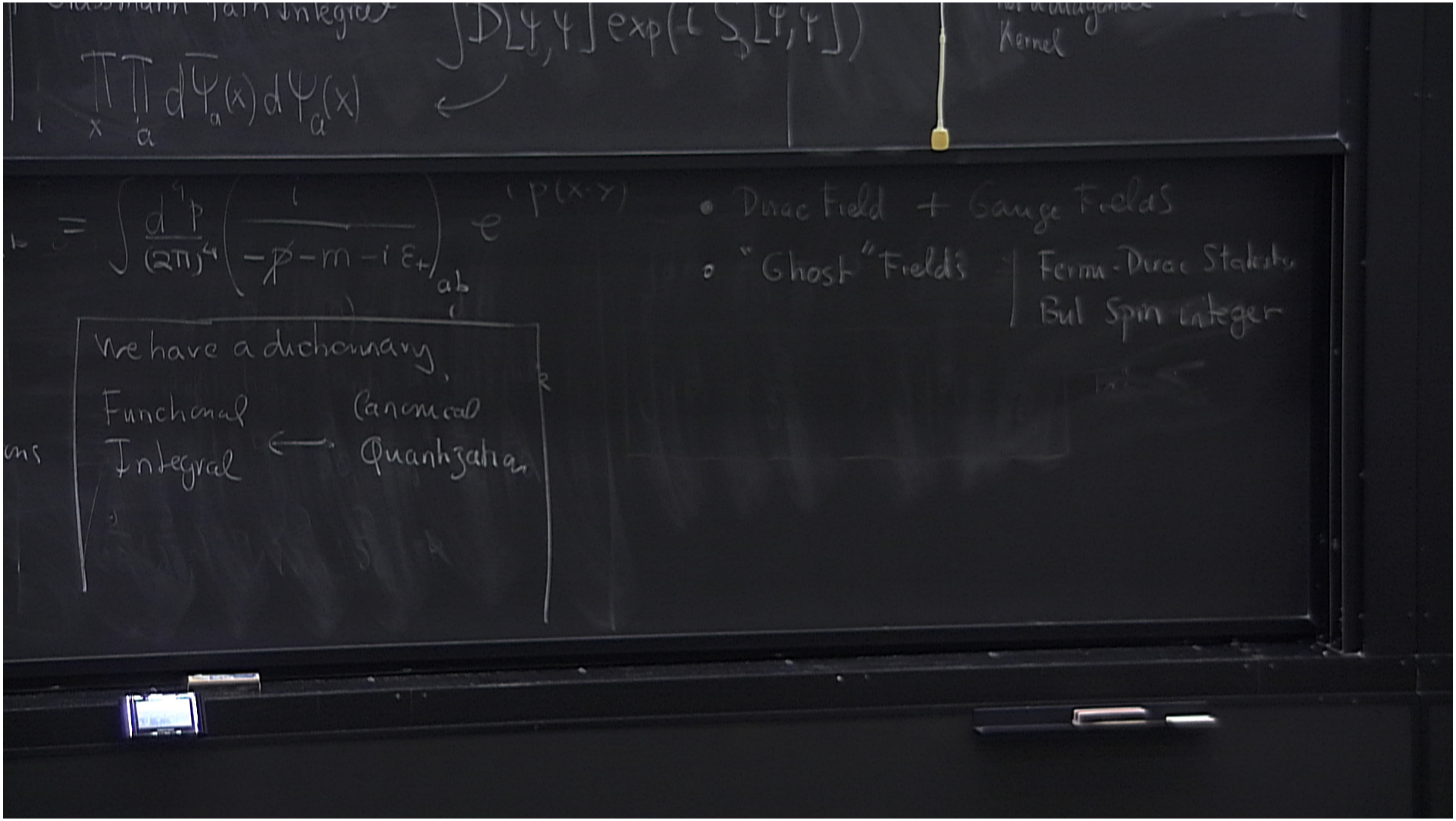
$$\langle \Psi_a(x) \bar\Psi_b(y) \rangle = \left(i\not{\partial} - m \right)_{x,a;y,b} = G_{\text{Dirac}}(x,y)_{ab} = \int \frac{d^4p}{(2\pi)^4} \left(\frac{i}{-\not{p} - m - i\varepsilon_+} \right)_{ab} e^{ip(x-y)}$$

Feynman Propagator for the Dirac Field

$\langle \Psi \bar\Psi \Psi \bar\Psi \Psi \bar\Psi \cdots \rangle \Rightarrow$ Wick Theorem for Fermions with correct sign factors

We have a dictionary.

Functional Integral $\longleftrightarrow$ Canonical Quantization



$$\int D[\bar{\psi},\psi]\,\exp(-i\,S[\bar{\psi},\psi])$$

$$\prod_x \prod_a d\bar{\psi}_a(x)\,d\psi_a(x)$$

Kernel

$$= \int \frac{d^4p}{(2\pi)^4}\left(\frac{i}{-\not{p}-m-i\varepsilon_+}\right)_{ab} e^{ip(x-y)}$$

- Dirac Field + Gauge Fields
- "Ghost" Fields | Fermi-Dirac Statistics
  But Spin integer

We have a dichotomy.

Functional Integral ⟷ Canonical Quantization