Title: PSI 2016/2017 Quantum Field Theory II - Lecture 10

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Abstract:

Perturbative Renormalization in QFT $\Longleftrightarrow$

g_R^0 $\quad$ $g_R(\mu)$ $\quad$ $g_R(\Lambda)$ $\leftarrow$ coupling at the "cut-off" scale

$\mu_0 \longrightarrow \mu \longrightarrow \Lambda$ $\quad$ Energy / Momentum Scale

renormalization Scale (physics) $\quad$ effective coupling (running) $\quad$ cut-off scale

$g_{eff} \longleftarrow g$ $\quad$ physical microscopic couplings

large scale $l \gg a$ $\quad$ microscopic scale a

$\mu \sim 1/l \ll \Lambda = 1/a$ $\quad$ $\Lambda = 1/a$

Real Space Wilson Renormalization in Stat. Mech.

RG calculation: Local Potential Approximation

$$\mathcal{A}[\phi] = \int d^D x \, \frac{1}{2} (\partial_\mu \phi)^2 + V(\phi) \qquad V(\phi) =$$

⇒ Real Space Wilson Renormalization in Stat. Mech.

RG calculation; Local Potential Approximation even: $V(\phi) = V(-\phi)$ not necessary

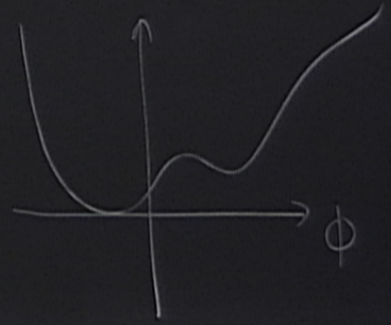
$$\mathcal{A}[\phi] = \int d^D x \, \frac{1}{2}(\partial_\mu \phi)^2 + V(\phi)$$

$$V(\phi) = v_0 + \frac{v_2}{2}\phi^2 + \frac{v_4}{4!}\phi^4 + \frac{v_6}{6!}\phi^6 + \frac{v_8}{8!}\phi^8 + \cdots$$

Sharp cutoff $|k| < \Lambda$

$$\int D[\phi] \exp(-\mathcal{A}[\phi])$$

Euclidean



⇒ Real Space Wilson Renormalization in Stat. Mech.

RG calculation: Local Potential Approximation even:

$$\mathcal{A}[\phi] = \int d^D x \, \frac{1}{2}(\partial_\mu \phi)^2 + V(\phi) \qquad V(\phi) = v_0 + \frac{v_2}{2}$$

Sharp cutoff $|k| < \Lambda$

$$\int D[\phi] \; \exp(-\mathcal{A}[\phi])$$

Euclidean $\phi(x) = \int \frac{d^D k}{(2\pi)^D} e^{i k \cdot x} \hat{\phi}(k) \qquad \hat{\phi}(k) = 0 \quad |k| > \Lambda$



pace Wilson Renormalization in Stat. Mech.

culation: Local Potential Approximation even. $V(\phi) = V(-\phi)$ not

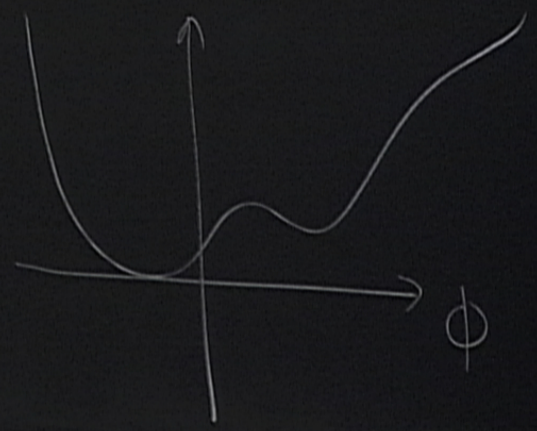
$S] = \int d^D x \, \frac{1}{2} (\partial_\mu \phi)^2 + V(\phi)$ $\quad V(\phi) = v_0 + \frac{v_2}{2}\phi^2 + \frac{v_4}{4!}\phi^4 + \frac{v_6}{6!}\phi^6 + \frac{v}{8}$

cutoff $|k| < \Lambda$

Euclidean Invariance

$\phi] = \exp(-A[\phi])$

dean $\phi(x) = \int \frac{d^D k}{(2\pi)^D} e^{ik\cdot x} \hat{\phi}(k)$ $\quad \hat{\phi}(k) = 0 \quad |k| > \Lambda$



pace Wilson Renormalization in Stat. Mech.

culation: Local Potential Approximation even. $V(\phi) = V(-\phi)$ not

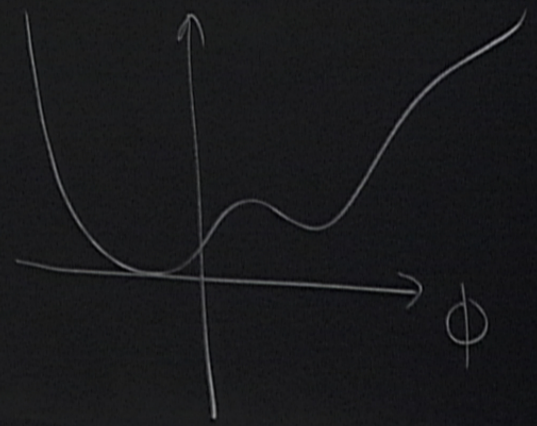
$S] = \int d^D x \, \frac{1}{2} (\partial_\mu \phi)^2 + V(\phi)$ $\quad V(\phi) = v_0 + \frac{v_2}{2}\phi^2 + \frac{v_4}{4!}\phi^4 + \frac{v_6}{6!}\phi^6 + \frac{v}{8}$

cutoff $|k| < \Lambda$

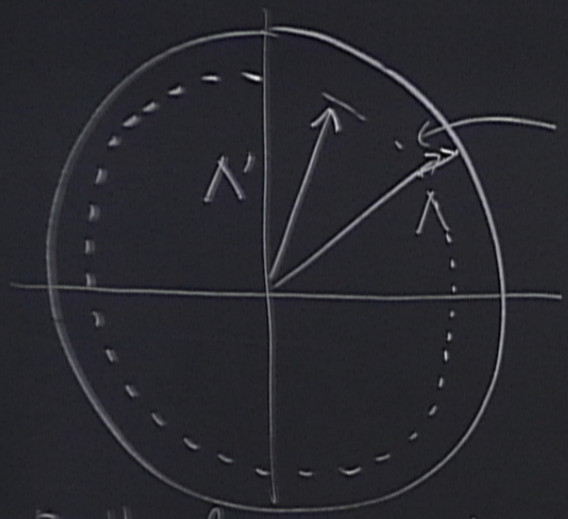
Euclidean Invariance

$\phi] : \exp(-A[\phi])$ "Unitarity" is problematic

dean $\phi(x) = \int \frac{d^D k}{(2\pi)^D} e^{ik\cdot x} \hat{\phi}(k)$ $\quad \hat{\phi}(k) = 0 \quad |k| > \Lambda$



② RG Wilsonian



$\Lambda' < \Lambda \qquad \Lambda' = \Lambda(1 - ds)$

momentum shell

$$\hat{\phi}(k) = \hat{\phi}_{Fast}(k) + \hat{\phi}_{slow}(k)$$

$\Lambda' < |k| < \Lambda \qquad |k| < \Lambda'$

Ball of radius Λ where the $\hat{\phi}(k)$ are defined

Integrate over the fast modes

Step 1: Integrate over the fast modes / effective theory

$$\int \mathcal{D}[\phi_{fast}] \exp(-\mathcal{A}[\phi]) = \exp(-\mathcal{A}_{eff}[\phi_{slow}])$$

Step 2: Rescale positions/momenta, and the ϕ_{slow} so that we recover a theory with cut-off Λ

Exact Renormalizing Group (ERGE)

LPA $\Leftarrow$ Renormalized Effective Potential

- Keep only the 1-loop terms in the funct.
- $\Gamma_{eff}[\phi_{slow}]$ is projected onto the same fu

$(\partial\phi_{slow})^2$, $V_{eff}(\phi_{slow})$ | $U(\phi_{slow})(\partial\phi_{s}$ $(\partial\phi_{slow})^4$, $(\partial^2$

Larger Space of theories than just Landau-Ginsburg-Wilson ϕ^4

alized Effective Potential Method

1-loop terms in the funct. integral calculation

is projected onto the same form we started from

$V(\phi_{slow})$ | $V(\phi_{slow})(\partial\phi_{slow})^2$ are neglected

than

ers-Wilson ϕ^4 | $(\partial\phi_{slow})^4, (\partial^2\phi_{slow})^2$ ← to be checked

Full QFT $\Gamma[\varphi] = \mathcal{A}[\varphi] + \frac{1}{2} \mathrm{Tr}\left[\mathrm{Log}\left(\mathcal{A}''[\varphi] \right) \right]$

$\mathcal{A}''[\varphi]$: Hessian

$$\mathcal{A}''[\varphi] = -\Delta + V''(\varphi)$$

$$\Delta = \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\mu}$$

$$\mathcal{A}''[\varphi] \cdot \psi(x) = -\Delta_x \psi(x) + V''(\varphi(x)) \psi(x)$$

$$V''(\varphi) = v_2 + \frac{v_4}{2} \varphi^2 + \frac{v_6}{4!} \varphi^4 + \cdots$$

$$\Gamma[\varphi] = A[\varphi] + \frac{1}{2}\,\mathrm{Tr}\left[\mathrm{Log}\left(A''[\varphi]\right)\right] + \cdots$$

$+ V''(\varphi)$

↑ Hessian

Σ higher loops contributions

$$-\Delta_x \psi(x) + V''(\varphi(x))\,\psi(x)$$

$$+ \frac{v_6}{4!}\varphi^4 + \cdots$$

③ If $\varphi(x) = \varphi$ does not depend on x

$$\mathrm{Tr}\left(\mathrm{Log}\left[\mathcal{A}''[\varphi]\right]\right) = \int d^D x \, \langle x | \log\left(\mathcal{A}''[\varphi]\right) | x \rangle =$$

$\varphi = \mathrm{cst}$ $\quad \mathcal{A}''[\varphi] = -\Delta_x + V''(\varphi)$ operator commutes with the translations

$V''(\varphi)$ cst

Fourier Transform $\quad \mathcal{A}''[\varphi] = k^2 + V''(\varphi) \leftarrow$ e.values

$\mathcal{A}''[\varphi]$ acts of $\hat{\varphi}(k)$

φ does not depend on x

$$\Gamma[\varphi] = \int d^D x \, \langle x | \log\left(A''[\varphi]\right) | x \rangle = \int d^D x \cdot \int \frac{d^D k}{(2\pi)^D} \log\left[k^2 + V''(\varphi)\right]$$

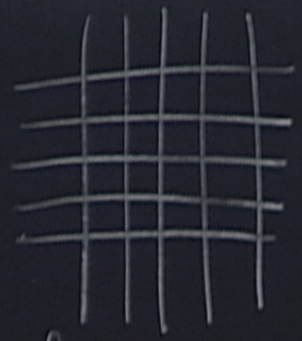
x extensivity; k sum over eigenvalues

$A''[\varphi] = -\Delta_x + V''(\varphi)$ operator commutes with the translations

cst

$A''[\varphi] = k^2 + V''(\varphi)$ ← e.values

$|k\rangle$



lattice $\leftrightarrow$ mesh a

$|k\rangle = \frac{1}{\sqrt{V}} \exp[i k x]$

$x = i a \quad i \in \mathbb{Z}^D / (N\mathbb{Z})^D$

$N = L/a$

size L periodic b.c $\quad k = \frac{2\pi}{a} \cdot j \quad j \in \mathbb{Z}^D / (N\mathbb{Z})^D$

$\langle k | k' \rangle = \delta(k - k')$

$$\lambda_k = \frac{1}{a^2} \sum_{\mu=1}^{D} \sin^2(k_\mu a) + V''(\varphi)$$

$\varphi(x) = \varphi$ does not depend on x

$$\log(A''[\varphi]) = \int d^D x \, \langle x | \log(A''[\varphi]) | x \rangle = \int d^D x \cdot \int \frac{d^D k}{(2\pi)^D}$$

extensively

$A''[\varphi] = -\Delta_x + V''(\varphi)$ operator commutes with the translations

csf

from $A''[\varphi] = k^2 + V''(\varphi)$ ← e.values

$\tilde{\varphi}(k)$

$$[\varphi] + \frac{1}{2}\,\mathrm{Tr}\left[\mathrm{Log}\left(\mathcal{A}''[\varphi]\right)\right] + \cdots$$

↑ Hessian

↑ higher loops contributions

Integrate over all the momenta

$\varphi(x)\,\psi(x)$

$$|x\rangle = \int d^D x \cdot \int \frac{d^D k}{(2\pi)^D} \log\left[k^2 + V''(\varphi)\right]$$

↑ extensivity

↑ Σ Sum over eigenvalues

$|k| < \Lambda$

utes

ations

e-values

$\mathrm{Tr}\left(\mathrm{Log}\left[\mathcal{A}''[\varphi]\right]\right) = \int dx \langle x| \log(\mathcal{A}[\varphi])|x\rangle$ = ...

extensi

$\varphi = \mathrm{cst}$ $\quad \mathcal{A}''[\varphi] = -\Delta_x + V''(\varphi)$ operator commutes with the translations

cst

Fourier Transform $\quad \mathcal{A}''[\varphi] = k^2 + V''(\varphi)$ ← e.values

$\mathcal{A}''[\varphi]$ acts of $\hat{\varphi}(k)$

When Integrating only on the Fast Modes $\quad \varphi_{\mathrm{fast}}(k) \quad \Lambda' < |k| < \Lambda$

$< \Lambda$

$$\frac{1}{2}\int d^D x \int_{\Lambda' < |k| < \Lambda} \frac{d^D k}{(2\pi)^D} \log\left[\frac{k^2 + V''[\varphi(x)]}{\Lambda^2}\right] + V[\phi_{slow}] = V_{eff}[\phi_{slow}]$$

$$\phi_{slow})^2 + V_{eff}[\phi_{slow}(x)]$$

Result in the LPA.

$$\Gamma_{k-s}[\phi_{slow}] = S[\phi_{slow}] + \frac{1}{2}\int d^D x \int_{\Lambda'<|k|<\Lambda} \frac{d^D k}{(2\pi)^D}$$

$$= \int dx \, \frac{1}{2}(\partial_\mu \phi_{slow})^2 + V_{eff}[\phi_{slow}]$$

Small momentum shell

$$\Lambda(1-ds) < |k| < \Lambda \quad \Rightarrow \quad V_{eff}(\phi_{slow}) = V(\phi_{slow}) + \frac{1}{2} ds \int$$

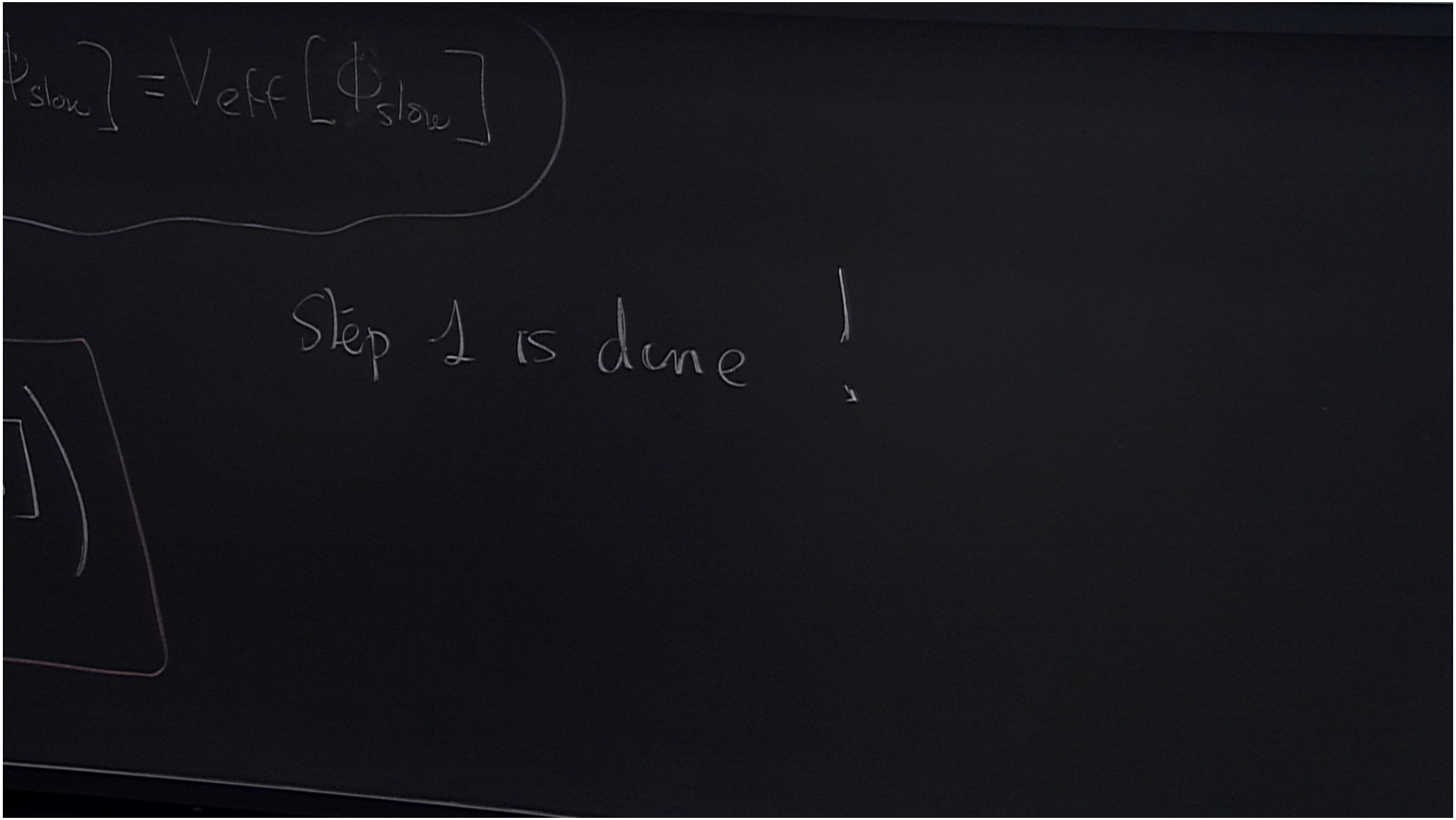
volume

$$\int d^D x \int_{\Lambda' < |k| < \Lambda} \frac{d^D k}{(2\pi)^D} \log\left[\frac{k^2 + V''[\varphi(x)]}{\Lambda^2}\right] + V[\phi_{slow}] = V_{eff}[\phi_{slow}]$$

$$\left(\phi_{slow}\right)^2 + V_{eff}[\phi_{slow}(x)]$$

$$\left(\phi_{slow}\right) + \frac{1}{2} ds \frac{\Lambda^D}{(2\pi)^D} \frac{2\pi^{D/2}}{\Gamma(D/2)} \log\left(1 + \frac{1}{\Lambda^2} V''[\phi_{slow}]\right)$$

volume of $D-1$ dim unit sphere


$\phi_{slow}] = V_{eff}[\phi_{slow}]$
Step 1 is done !

Now step 2 rescaling

rescaling of space

$k \longrightarrow \bar{k} = k(1+ds)$ so that new cut-off $\bar{\Lambda} = \Lambda'(1+ds) = \Lambda$

$x \longrightarrow \bar{x} = x(1-ds)$

$d^D x \longrightarrow d^D \bar{x}\,(1+Dds)$

$\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial \bar{x}^\mu}\cdot(1-ds)$

rescaling of field ϕ

$\phi_{slow}(x) = \bar{\phi}(\bar{x})\left(1-\frac{(D-2)}{2}ds\right)$

$$\mathcal{A}_{\text{eff}}[\phi_{\text{slow}}] = \mathcal{A}_{\text{REN}}[\bar{\phi}] = \int d^D\bar{x}\ \frac{1}{2}\left(\frac{\partial}{\partial\bar{x}^\mu}\bar{\phi}\right)^2 + V_{\text{REN}}(\bar{\phi})$$

$$V_{\text{REN}}(\phi) = V(\phi) + d\delta\left[\mathcal{D}\,V(\phi) - \frac{D-2}{2}\,\phi\cdot V'(\phi) + \mathcal{A}_D \log\left(1+\frac{1}{\Lambda^2}V''(\phi)\right)\right]$$

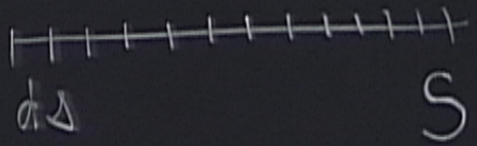
↑ x-rescaling ↑ ϕ rescaling ↑ step 1

That new cut-off $\Lambda = \Lambda(1+ds) \equiv \Lambda$

$$V_{REN}(\phi) = V(\phi) + d$$

Integrate

Thus



$V_S(\phi)$ renormalized potential after a scale transformation by a factor S in position space

$$V_{REN}(\phi) = V(\phi) + d\Delta\left[D\,V(\phi) - \frac{D-2}{2}\,\phi \cdot V'(\phi) + A_D \log(1\right.$$

↑ x- rescaling ↑ ϕ rescaling ↑ step

RG Flow Equation For $V_s(\phi)$

$$s\frac{\partial}{\partial s} V_s[\phi]$$

spac

$$ds\left[D\,V(\phi) - \frac{D-2}{2}\,\phi\cdot V'(\phi) + A_D \log\left(1+\frac{1}{\Lambda^2}V''(\phi)\right)\right]$$

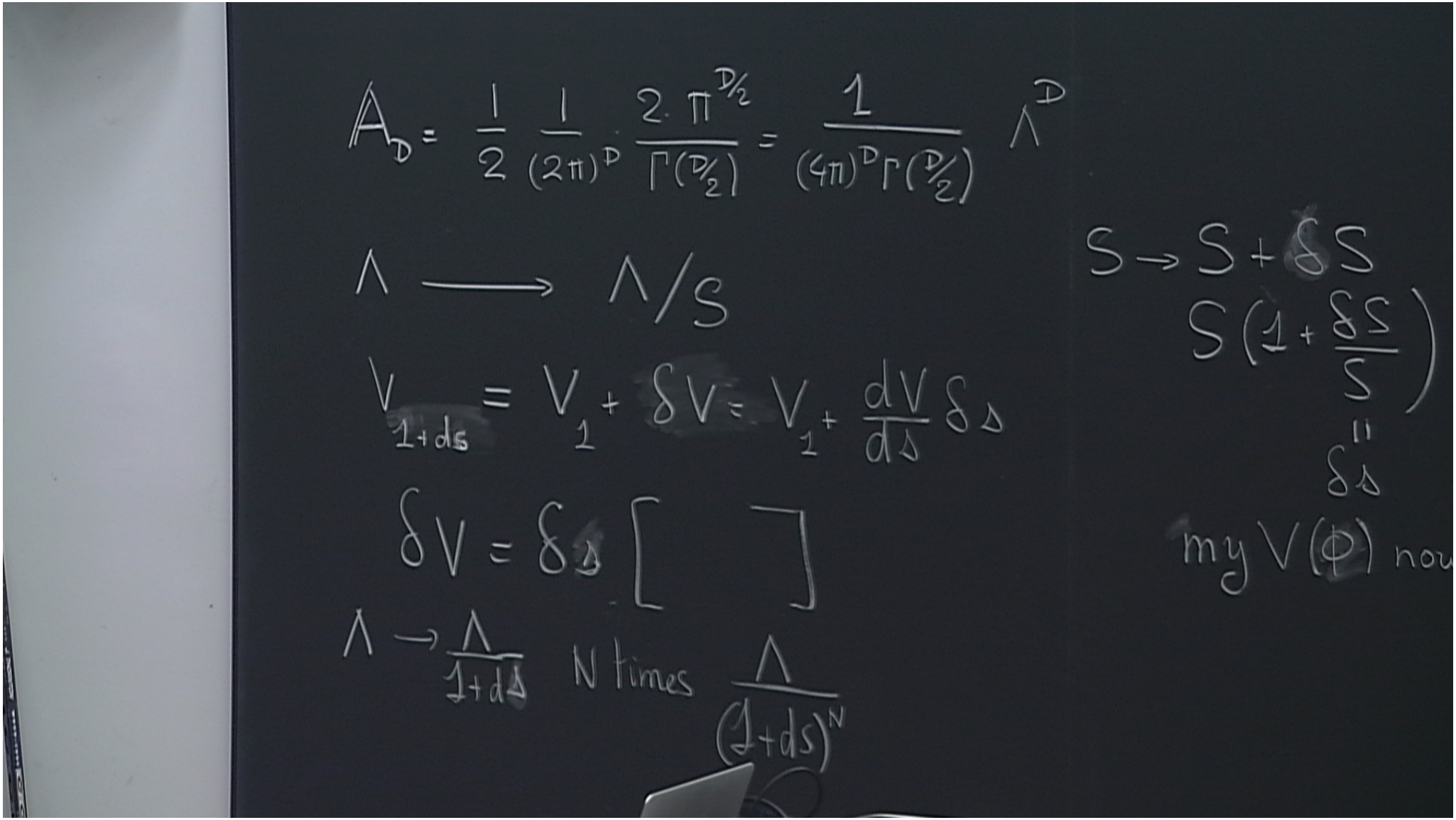
↑ X-rescaling ↑ ϕ rescaling ↑ step 1

$$s\frac{\partial}{\partial s}V_s[\phi] = D\,V_s[\phi] - \frac{D-2}{2}\phi\,V_s'(\phi) + A_D \log\left(1+\frac{1}{\Lambda^2}V_s''(\phi)\right)$$

$$-d\Delta\left[D\,V(\phi)-\frac{D-2}{2}\,\phi\cdot V'(\phi)+\mathbb{A}_D\log\left(1+\frac{1}{\Lambda^2}V''(\phi)\right)\right]$$

↑ χ-rescaling ↑ ϕ rescaling ↑ step 1

$$s\frac{\partial}{\partial s}V_s[\phi]=D\,V_s[\phi]-\frac{D-2}{2}\,\phi\,V_s'(\phi)+\mathbb{A}_D\log\left(1+\frac{1}{\Lambda^2}V_s''(\phi)\right)$$


$A_D = \frac{1}{2}\frac{1}{(2\pi)^D}\frac{2\pi^{D/2}}{\Gamma(D/2)} = \frac{1}{(4\pi)^D\Gamma(D/2)}\Lambda^D$
$\Lambda \longrightarrow \Lambda/s$
$V_{1+ds} = V_1 + \delta V = V_1 + \frac{dV}{d\Lambda}\delta\Lambda$
$\delta V = \delta\Lambda\,[\ \ ]$
$\Lambda \to \frac{\Lambda}{1+d\Lambda}$ N times $\frac{\Lambda}{(1+ds)^N}$
$S \to S + \delta S$
$S\left(1 + \frac{\delta S}{S}\right)$
$= \delta\Lambda$
my $V(\phi)$ nou

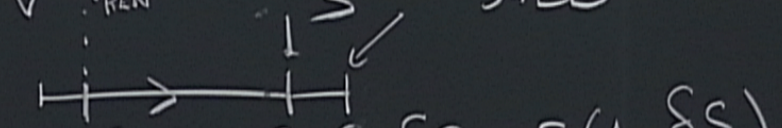
$S \to S + \delta S$

$S\left(1 + \frac{\delta S}{S}\right)$

$= \delta s$

$\Lambda \qquad \Lambda \qquad \Lambda$

$V \; V_{REN} \qquad V_S \qquad V_{S+\delta S}$



my $V(\phi)$ now is $V_S(\phi) \longrightarrow V_{S_{REN}}(\phi)$

conespondind $d s = \frac{dS}{S}$

$S \to S + \delta S$

$S\left(1 + \frac{\delta S}{S}\right)$

$\delta\Delta$

$V \; V_{REN} \; V_S \; V_{S+\delta S}$

$1 \; 1+\delta\Delta \; S \; S+\delta S = S\left(1+\frac{\delta S}{S}\right)$

$\delta V = V_1 + \frac{dV}{d\Delta}\delta\Delta$

my $V(\phi)$ now is $V_S(\phi) \longrightarrow V_{S\,REN}(\phi)$

Scale transformation corresponding $d\Delta = \frac{dS}{S}$

$1 \to 2 \to 4 \to 8$

group of scale transformation is additive in log (scale factor)

$\frac{\Lambda}{(1+ds)^N}$

$S+\delta S$

$= S\left(1+\frac{\delta S}{S}\right)$

$V_{S\ \mathrm{REN}}(\phi)$

$\Delta = \frac{dS}{S}$

ddd

R_S scale by S

$R_{S_2} \circ R_{S_1} = R_{S_2 \cdot S_1}$

S_{new}

$\log S_{new} = \log S_1 + \log S_2$

$$V''(\phi) = v_2 + \frac{v_4}{2}\phi^4 \rightarrow \log\left(1 + V''(\phi)\right) = \log\left(1 + v_2 + \frac{1}{2}v_4\phi^2\right)$$

$$\log(1+v_2) + \log\left(1 + \frac{v_4}{2(1+v_2)}\phi^2\right)$$

low

$\phi^2, \phi^4 \xrightarrow{RG}$ generate $\phi^2 + \phi^4 + \phi^6 \ldots$

Explicit RG flows

$$V(\phi) = v_0 + \frac{v_2}{2}\phi^2 + \frac{v_4}{4!}\phi^4$$

Further approx: truncate to ϕ^4

Approximate flow, truncated at order ϕ^4, ϕ^6, ...

Local potential approximation

RG flow equations at 1 loop and order ϕ^6

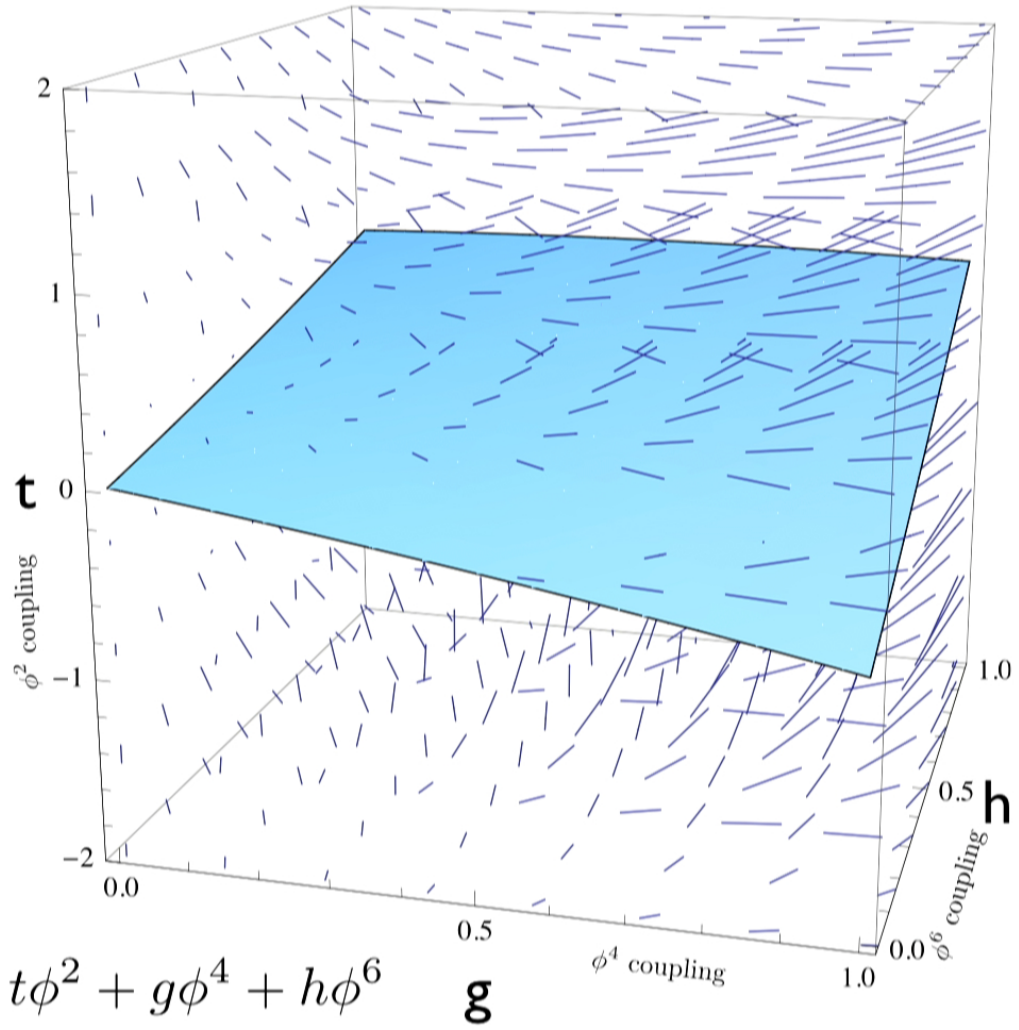
$$V(\phi) = \frac{t}{2}\phi^2 + \frac{g}{4!}\phi^4 + \frac{h}{6!}\phi^6$$

$$S\frac{\partial}{\partial S}t = -\beta_t = 2t + \frac{g}{16\pi^2(t+1)}$$

$$S\frac{\partial}{\partial S}g = -\beta_g = -\frac{3g^2}{16\pi^2(t+1)^2} + \frac{h}{16\pi^2(t+1)^2} + \frac{ht}{16\pi^2(t+1)^2}$$

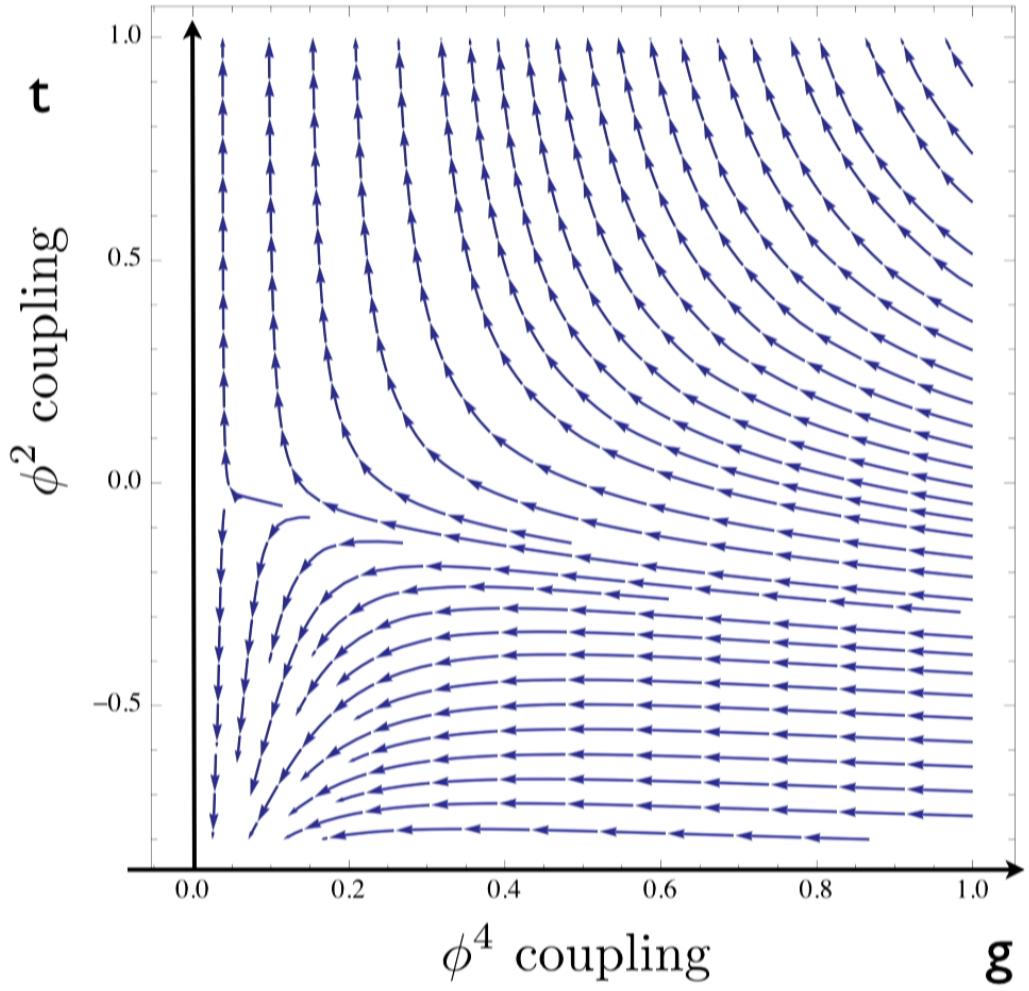
$$S\frac{\partial}{\partial S}h = -\beta_h = -2h + \frac{15g^3}{8\pi^2(t+1)^3} - \frac{15ght}{16\pi^2(t+1)^3} - \frac{15gh}{16\pi^2(t+1)^3}$$

RG flows in the (t,g,h) space

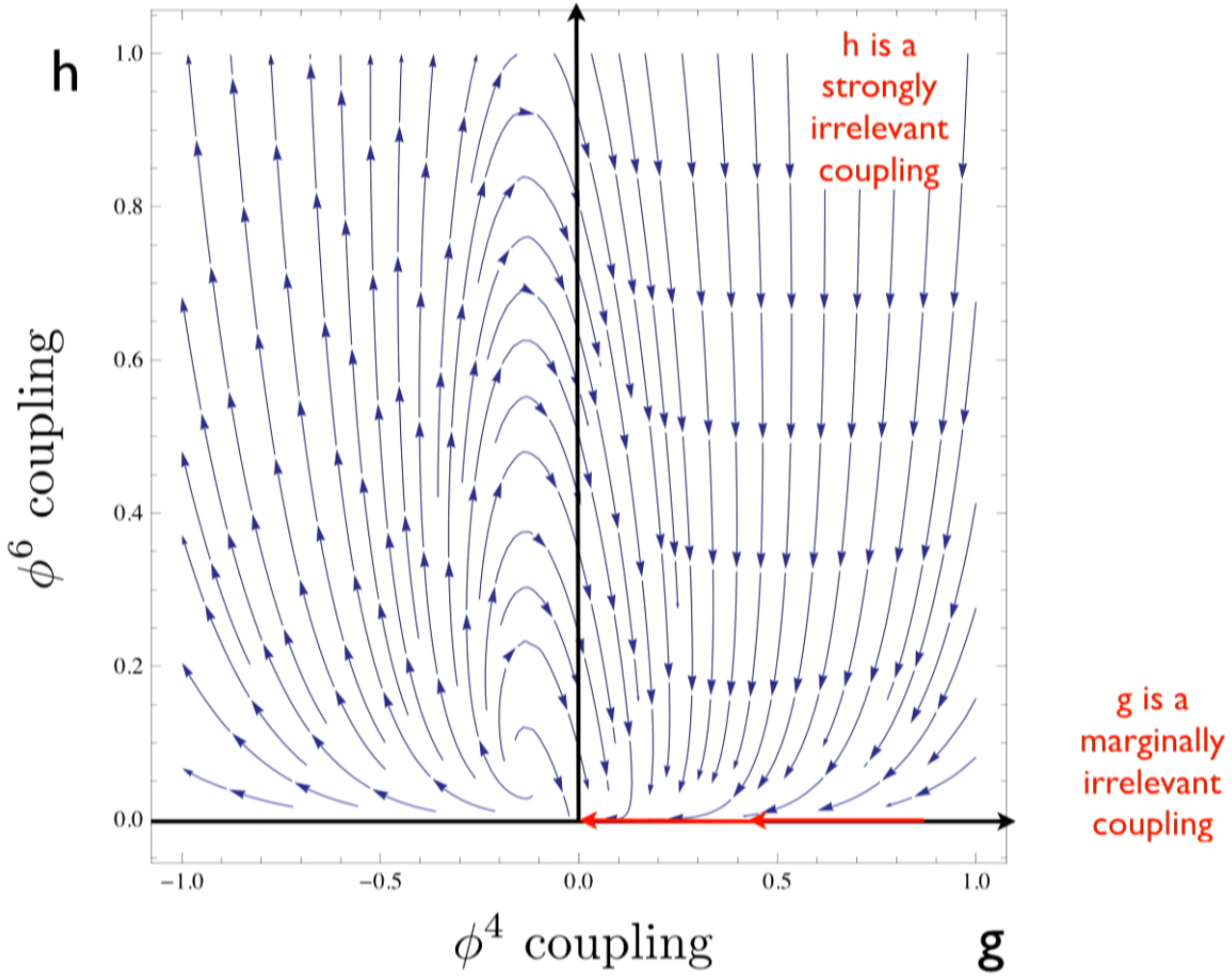


$$V(\phi) = t\phi^2 + g\phi^4 + h\phi^6$$

RG flows in the (t,g) plane



RG flows on the critical surface



Explicit RG flows

$$V(\phi) = v_0 + \frac{v_2}{2}\phi^2 + \frac{v_4}{4!}\phi^4 \quad \phi^6$$

Further approx: truncate to ϕ^4 — t, g, h

Approximate flows; truncated at order ϕ^4, ϕ^6; $D = 4$ case

$$V_s(\phi) = v_0(s) + \frac{v_2(s)}{2}\phi^2 + \cdots$$

Wilson Function

$$W_g(g(s)) = s\frac{d}{ds}g(s) = -\beta_g(g(s)) \Leftarrow \begin{matrix}\text{QFT} \\ \text{Callan-Symanzik} \\ \text{RG Functions}\end{matrix}$$

$$V(\phi) = v_0 + \frac{v_2}{2}\phi^2 + \frac{v_4}{4!}\phi^4 \quad \phi^6 \quad V''(\phi) =$$

truncate to ϕ^4 t. g h

truncated at order ϕ^4, ϕ^6; ... $D = 4$ case

$$= v_0(s) + \frac{v_2(s)}{2}\phi^2 + \dots$$

$$\frac{d}{ds} g(s) = -\beta_g(g(s)) \Leftarrow \begin{matrix} \phi \text{ FT} \\ \text{Callan-Symanzik} \\ \text{RG Functions} \end{matrix}$$