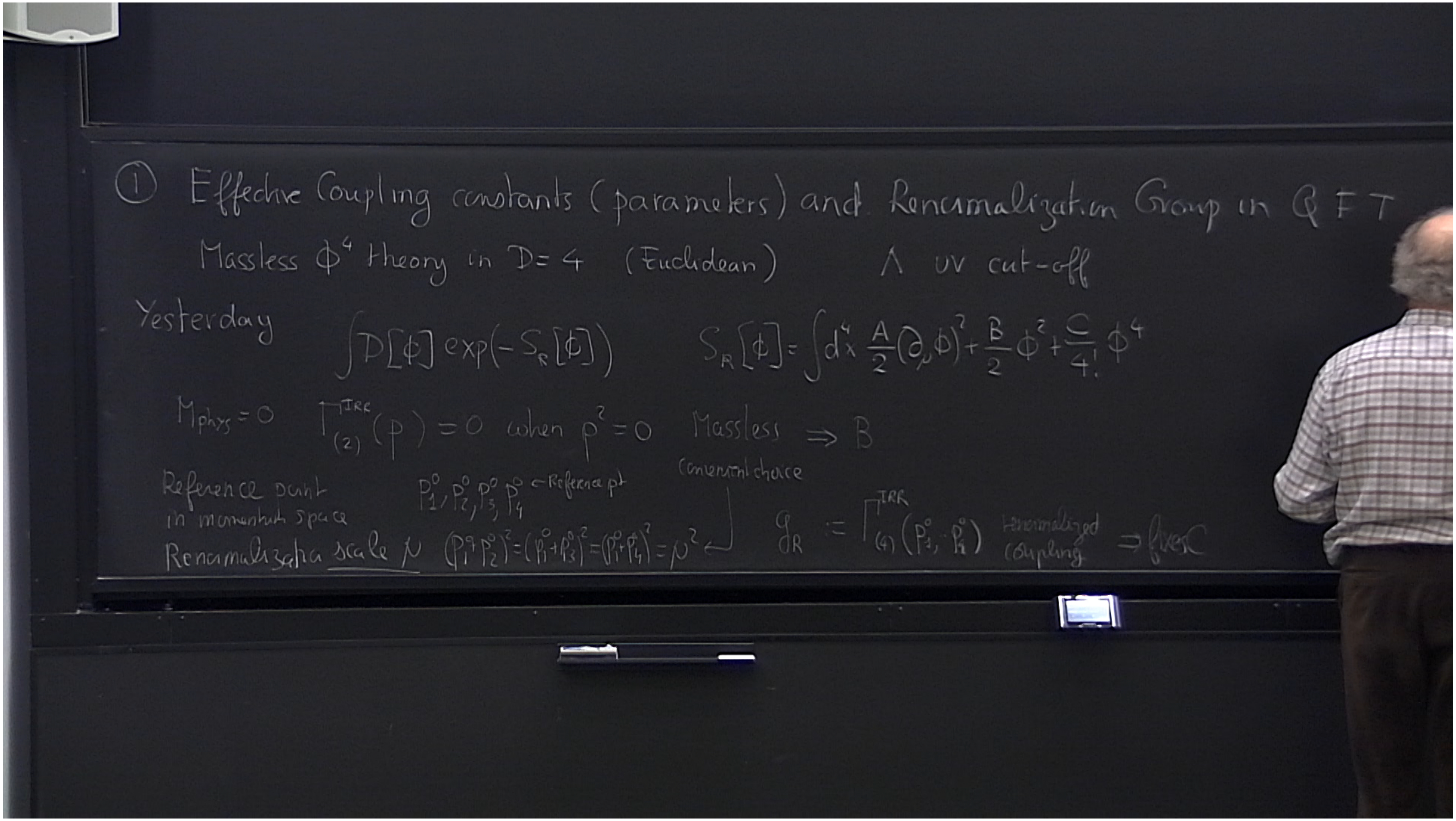
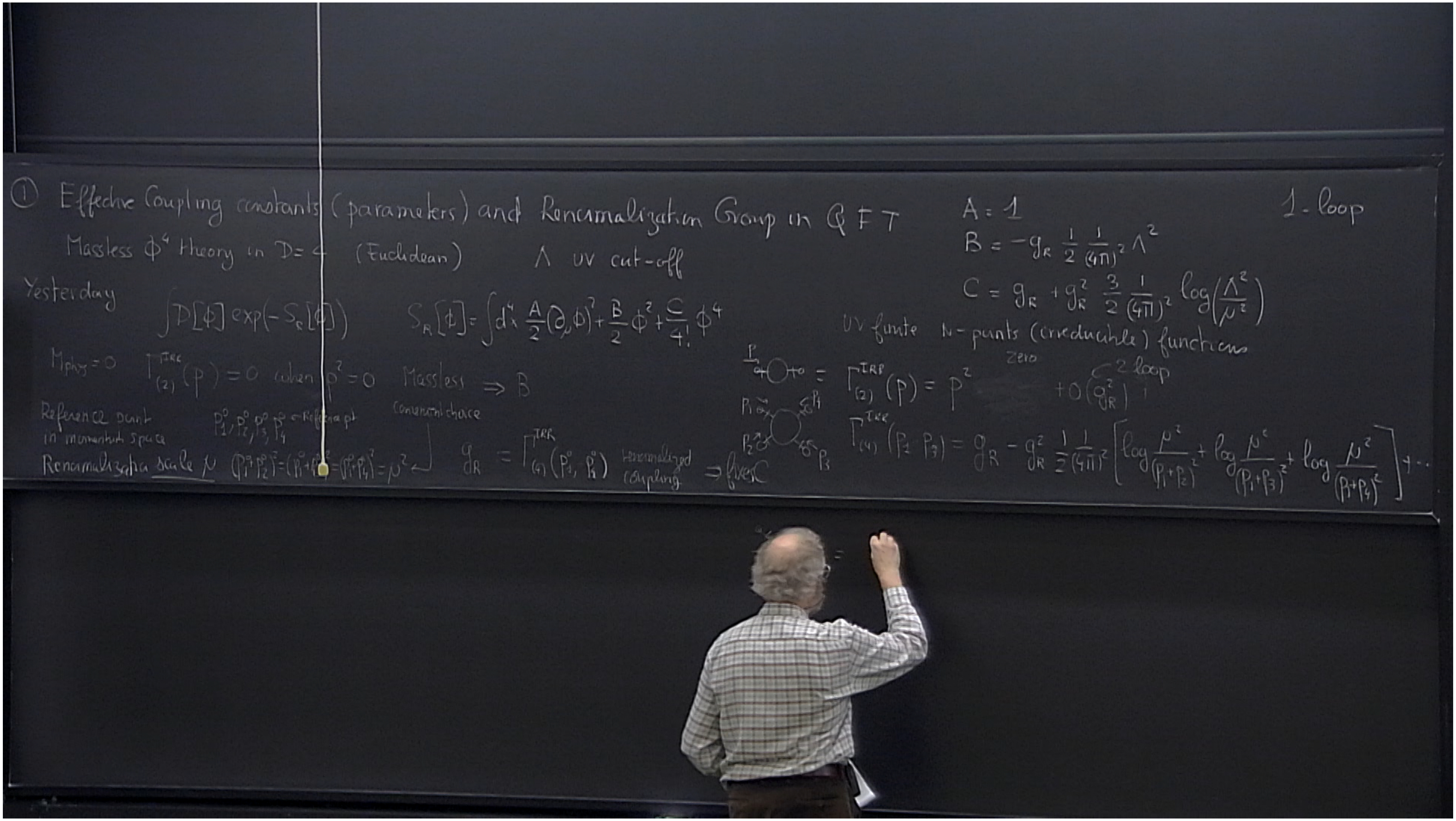
Title: PSI 2016/2017 Quantum Field Theory II - Lecture 9

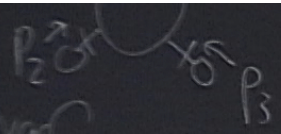
Date: Nov 17, 2016 09:00 AM

URL: http://pirsa.org/16110009

Abstract:


① Effective Coupling constants (parameters) and Renormalization Group in QFT
Massless ϕ^4 theory in $D=4$ (Euclidean) Λ UV cut-off
Yesterday $\int D[\phi] \exp(-S_R[\phi])$ $S_R[\phi] = \int d^4x \frac{A}{2}(\partial_\mu\phi)^2 + \frac{B}{2}\phi^2 + \frac{C}{4!}\phi^4$
$M_{phys} = 0$ $\Gamma^{IRR}_{(2)}(p) = 0$ when $p^2 = 0$ Massless $\Rightarrow B$
Reference point in momentum space
$p_1^0, p_2^0, p_3^0, p_4^0$ ← Reference pt
Convenient choice
Renormalization scale μ $(p_1^0+p_2^0)^2 = (p_1^0+p_3^0)^2 = (p_1^0+p_4^0)^2 = \mu^2$
$g_R := \Gamma^{IRR}_{(4)}(p_1^0, \cdots p_4^0)$ Renormalized coupling $\Rightarrow$ fixes C


① Effective Coupling constants (parameters) and Renormalization Group in QFT
Massless φ⁴ theory in D=4 (Euclidean)
Λ UV cut-off
Yesterday
∫D[φ] exp(−S_E[φ])
S_R[φ] = ∫d⁴x A/2 (∂_μφ)² + B/2 φ² + C/4! φ⁴
A = 1
1-loop
B = −g_R 1/2 1/(4π)² Λ²
C = g_R + g_R² 3/2 1/(4π)² log(Λ²/μ²)
UV finite N-points (irreducible) functions
M_phys = 0
Γ^IRR_(2)(p) = 0 when p² = 0
Massless ⇒ B
Convenient choice
zero
2 loop
Γ^IRR_(2)(p) = p² + O(g_R²)
Reference point in momentum space
Renormalization scale μ
g_R = Γ^IRR_(4)(p₁⁰, … p₄⁰)
Renormalized coupling
⇒ fixes C
Γ^IRR_(4)(p₁ … p₄) = g_R − g_R² 1/2 1/(4π)² [log μ²/(p₁+p₂)² + log μ²/(p₁+p₃)² + log μ²/(p₁+p₄)²] + …



p_2 p_3

...ized ...ng $\Rightarrow$ fixes C

$$\Gamma_{(4)}(p_1 \cdots p_3) = g_R - g_R^2 \frac{1}{2}\frac{1}{(4\pi)^2}\left[\log\frac{\mu^2}{(p_1+p_2)^2} + \log\frac{\mu^2}{(p_1+p_3)^2} + \log\frac{\mu^2}{(p_1+p_4)^2}\right]$$

g_R^3

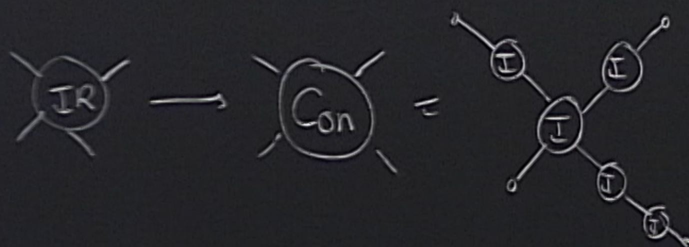
UV finite

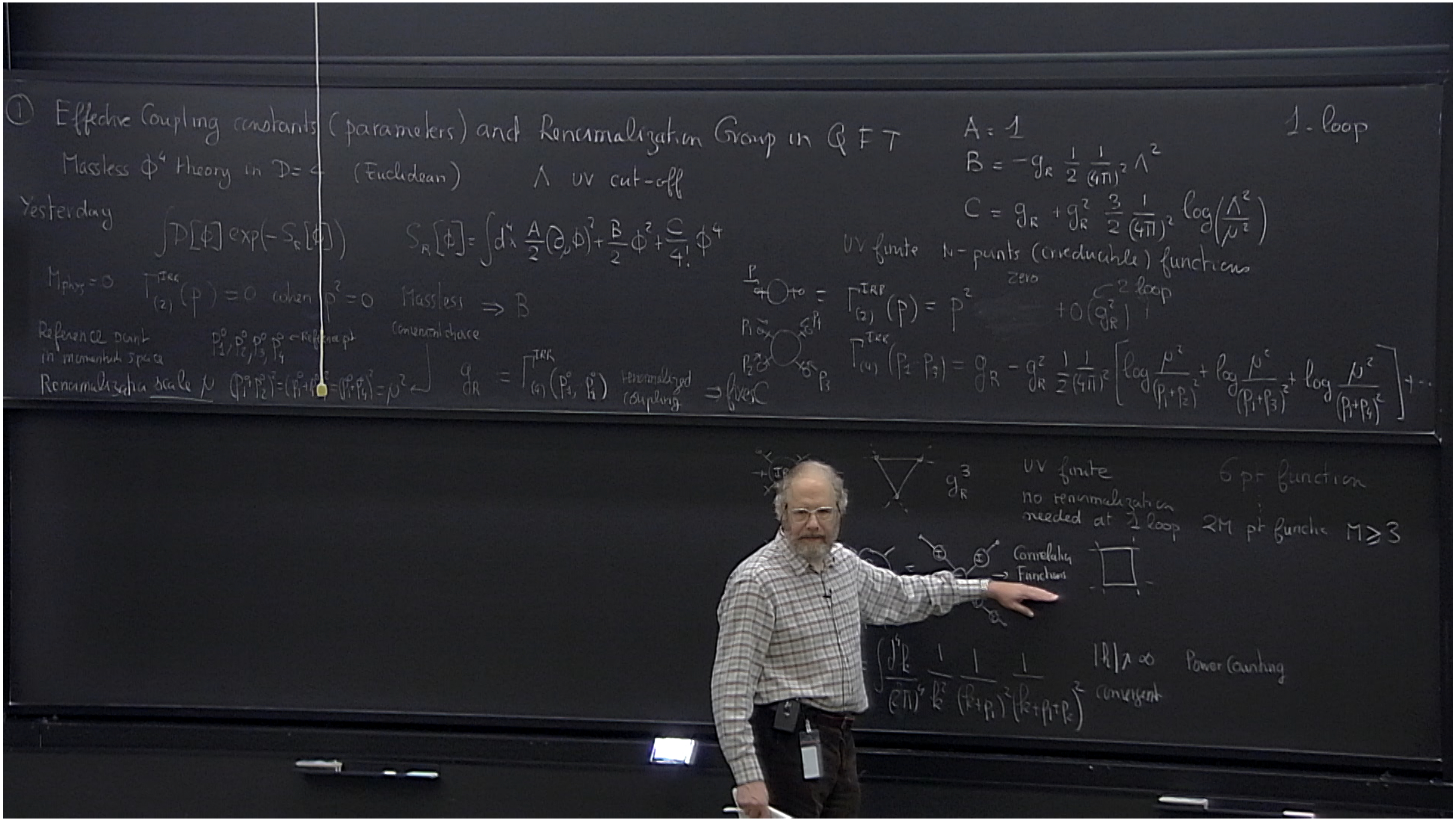
no renormalization needed at 1 loop

6 pt function

⋮

2M pt functio $M \geq 3$

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Γ^IRR_(4)(p₁ … p₄) = g_R − g_R² 1/2 1/(4π)² [log μ²/(p₁+p₂)² + log μ²/(p₁+p₃)² + log μ²/(p₁+p₄)²] + …
g_R³
UV finite
no renormalization needed at 1 loop
6 pt function
2M pt function M ≥ 3
Correlation Functions
|k| → ∞
Power Counting
convergent
∫ d⁴k/(2π)⁴ 1/k² 1/(k+p₁)² 1/(k+p₁+p₂)²

...lized
...ling $\Rightarrow$ fixes C

p_2 p_3

$$\Gamma_{(4)}(p_1 \cdot p_3) = g_R - g_R^2 \frac{1}{2}\frac{1}{(4\pi)^2}\left[\log\frac{\mu}{(p_1+p_2)^2} + \log\frac{\mu}{(p_1+p_3)^2} + \log\frac{\mu}{(p_1+p_4)^2}\right.$$

g_R^3

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no renormalization
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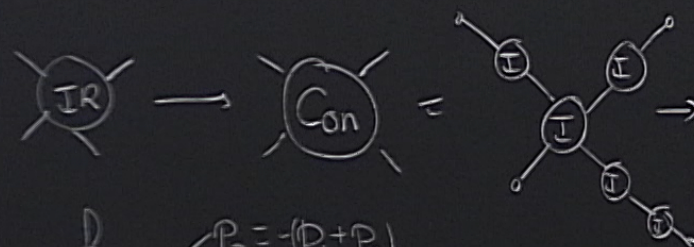
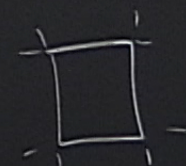
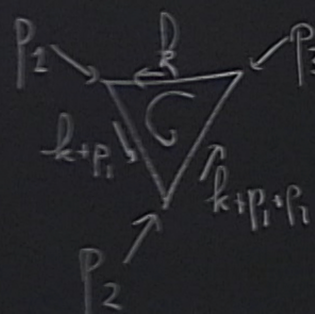


$\rightarrow$ Correlation Functions





$$= \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\frac{1}{(k+p_1)^2}\frac{1}{(k+p_1+p_2)^2}$$

$|k| \nearrow \infty$ Power Counting

convergent

Reference point in momentum space $p_1^0, p_2^0, p_3^0, p_4^0$ ← Reference pt

Renormalization scale μ $\quad (p_1^0+p_2^0)^2=(p_1^0+p_3^0)^2=(p_1^0+p_4^0)^2=\mu^2$

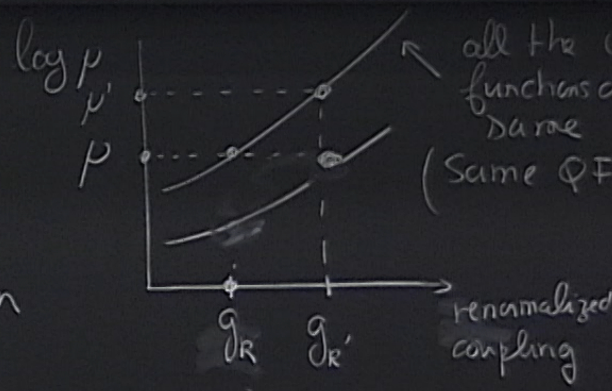
$g_R := \Gamma^{IRR}_{(4)}(p_1^0, \cdot\, p_4^0)$ renormalized coupling

② μ versus g_R

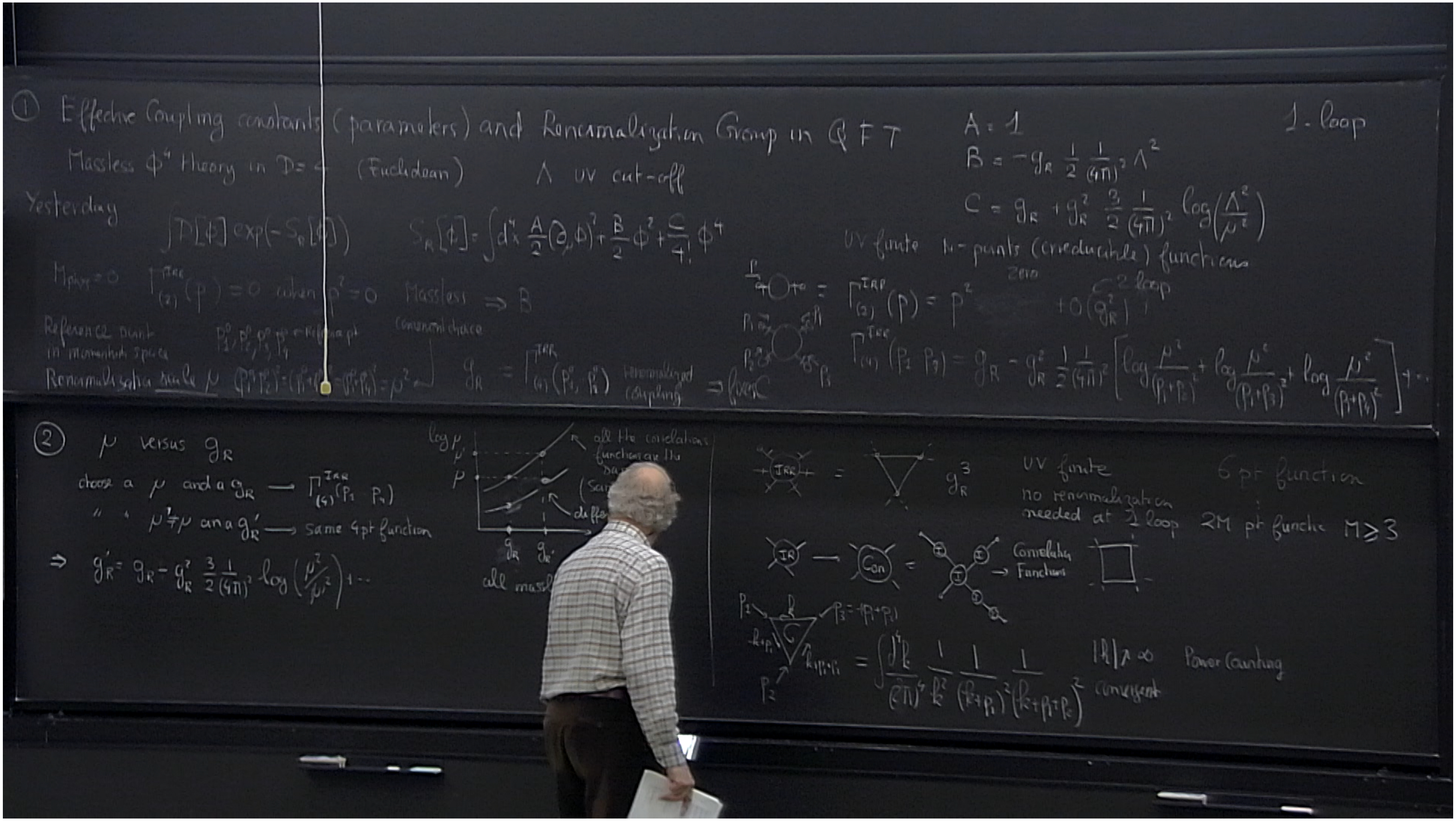
choose a μ and a g_R $\longrightarrow$ $\Gamma^{IRR}_{(4)}(p_1 \cdot p_4)$

" " $\mu' \neq \mu$ an a g_R' $\longrightarrow$ same 4pt function

$$\Rightarrow \quad g_R' = g_R - g_R^2\,\frac{3}{2}\,\frac{1}{(4\pi)^2}\log\left(\frac{\mu^2}{\mu'^2}\right)+\dots$$



all the correlations functions are the same (Same QFT)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Λ UV cut-off
Yesterday
1-loop
A = 1
UV finite n-point (irreducible) functions
Massless ⇒ B
Reference point in momentum space
canonical choice
Renormalization scale μ
Renormalized coupling
② μ versus g_R
choose a μ and a g_R
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6 pt function
2M pt function M ≥ 3
Correlation Functions
Power Counting
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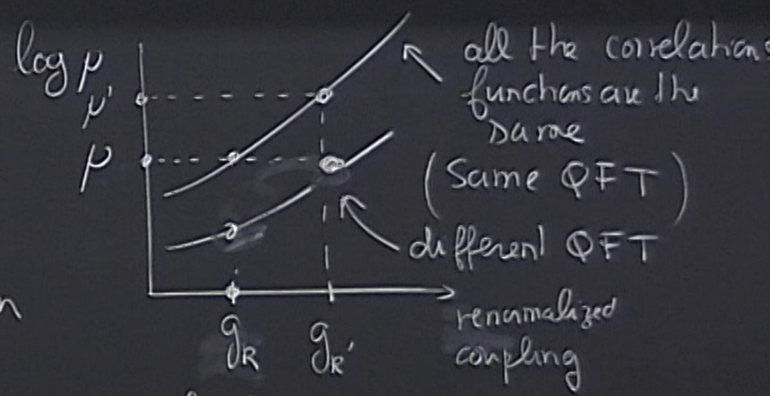
Renormalization scale μ $\quad (p_1^0+p_2^0)^2=(p_1^0+p_3^0)^2=(p_1^0+p_4^0)^2=\mu^2$ $\quad g_R := \Gamma_{(4)}(p_1^0,\dots,p_4^0)$ renormalized coupling $\Rightarrow$

② μ versus g_R

choose a μ and a $g_R \longrightarrow \Gamma^{IRR}_{(4)}(p_1 \dots p_4)$

" " $\mu' \neq \mu$ and a $g_R' \longrightarrow$ same 4pt function

$$\Rightarrow \quad g_R' = g_R - g_R^2 \, \frac{3}{2} \frac{1}{(4\pi)^2} \log\left(\frac{\mu^2}{\mu'^2}\right) + \dots$$



all massless QFT

① Effective Coupling constants (parameters) and Renormalization Group in QFT

Massless ϕ^4 theory in $D=4$ (Euclidean) $\quad \Lambda$ UV cut-off

Yesterday

$$\int \mathcal{D}[\phi]\exp(-S_R[\phi]) \qquad S_R[\phi]=\int d^4x\, \frac{A}{2}(\partial_\mu\phi)^2+\frac{B}{2}\phi^2+\frac{C}{4!}\phi^4$$

$M_{phys}=0 \quad \Gamma^{IRR}_{(2)}(p)=0$ when $p^2=0$ Massless $\Rightarrow B$

Reference point in momentum space $\quad p_1^0,p_2^0,p_3^0,p_4^0 \leftarrow$ Reference pt $\quad$ Convenient choice

Renormalization scale μ $\quad (p_1^0+p_2^0)^2=(p_1^0+p_3^0)^2=(p_1^0+p_4^0)^2=\mu^2$

$g_R = \Gamma^{IRR}_{(4)}(p_1^0,\ldots p_4^0)$ Renormalized Coupling $\Rightarrow$ fixes C

1-loop

$A=1$

$B=-g_R\,\frac{1}{2}\frac{1}{(4\pi)^2}\Lambda^2$

$C=g_R+g_R^2\,\frac{3}{2}\frac{1}{(4\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)$

UV finite n-points (irreducible) functions

$\Gamma^{IRR}_{(2)}(p)=p^2 + \underset{\text{zero}}{0} + O(g_R^2)$ (2 loop)

$\Gamma^{IRR}_{(4)}(p_1\ldots p_3)=g_R-g_R^2\,\frac{1}{2}\frac{1}{(4\pi)^2}\left[\log\frac{\mu^2}{(p_1+p_2)^2}+\log\frac{\mu^2}{(p_1+p_3)^2}+\log\frac{\mu^2}{(p_1+p_4)^2}\right]+\cdots$

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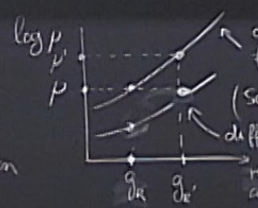


all massless QFT

$=g_R^3$ UV finite, no renormalization needed at 1 loop

6pt function ⋮ 2M pt function $M\geq 3$

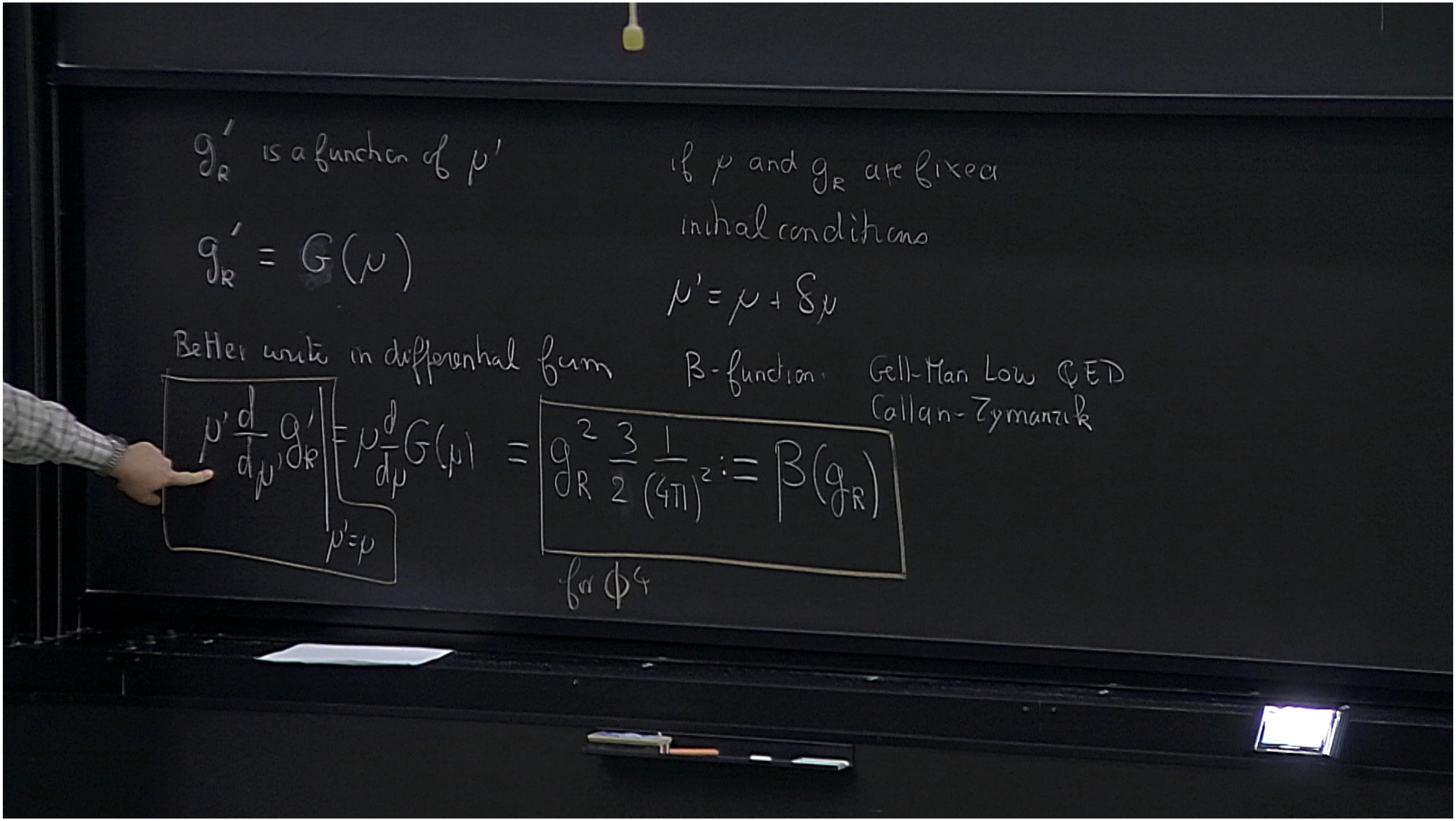
IRR $\longrightarrow$ Con $=$ $\to$ Correlation Functions

$p_1,\ p_2,\ p_3=-(p_1+p_2)$, k, $k+p_1$, $k+p_1+p_2$

$$=\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\frac{1}{(k+p_1)^2}\frac{1}{(k+p_1+p_2)^2}$$

$|k|\to\infty$ convergent $\quad$ Power Counting

g_R' is a function of μ'


g_R' is a function of μ'
$g_R' = G(\mu)$
Better write in differential form
$\mu' \frac{d}{d\mu'} g_R' \Big|_{\mu'=\mu} = \mu \frac{d}{d\mu} G(\mu) = g_R^2 \frac{3}{2} \frac{1}{(4\pi)^2} := \beta(g_R)$
for ϕ^4
if μ and g_R are fixed
initial conditions
$\mu' = \mu + \delta\mu$
β-function: Gell-Man Low QED
Callan-Zymanzik

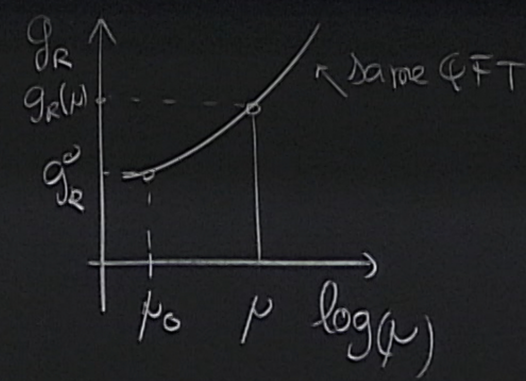
p_2 $\frac{\quad}{(2\pi)^4 k^2 (k+p_1)^2 (k+p_1+p_2)^2}$ convergent

ixed

How does g_R depends on μ?

For a given QFT (specific choice of μ_0, g_R^0)

Integrate the Flow Equation



Man Low QED

an-Zymanzik

$$\mu \frac{d}{d\mu} g_R(\mu) = \beta(g_R(\mu))$$

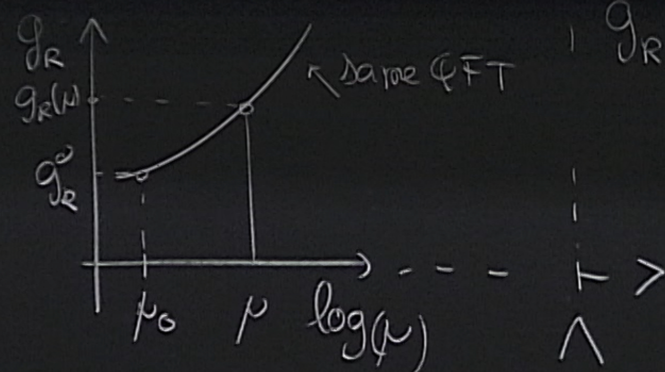
p_2 $\quad \frac{}{(2\pi)^4\, k^2\, (k+p_1)^2\, (k+p_1+p_2)^2}$ convergent

...pends on μ ?

(specific choice of μ_0, g_R^0)

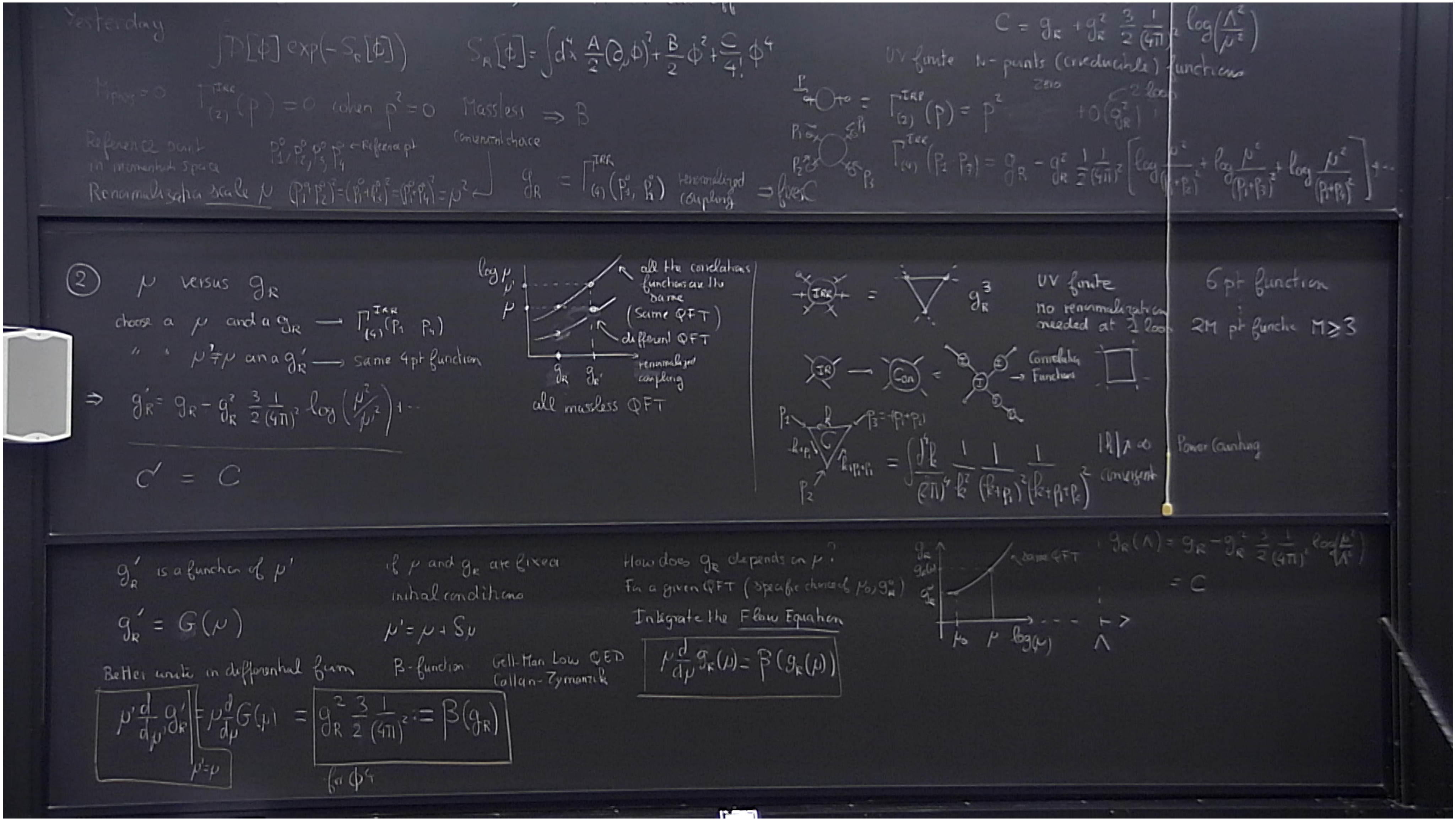
Flow Equation

$$= \boxed{\beta(g_R(\mu))}$$



$$g_R(\Lambda) = g_R - g_R^2\, \frac{3}{2}\, \frac{1}{(4\pi)^2} \log\left(\frac{\mu^2}{\Lambda^2}\right)$$

$$= C$$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S_R[φ] = ∫d⁴x A/2 (∂_μφ)² + B/2 φ² + C/4! φ⁴
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UV finite N-points (irreducible) functions
Massless ⇒ B
Conventional choice
Reference point in momentum space
Renormalization scale μ
g_R = Γ^IRR_(4)(p₁⁰, … p₄⁰)
renormalized coupling ⇒ fixes C
Γ^IRR_(2)(p) = p² + O(g_R²)
② μ versus g_R
choose a μ and a g_R ⟶ Γ^IRR_(4)(p₁ … p₄)
same 4pt function
g_R′ = g_R − g_R² 3/2 1/(4π)² log(μ′²/μ²) + …
C′ = C
all the correlations functions are the same
(Same QFT)
different QFT
renormalized coupling
all massless QFT
UV finite
no renormalization needed at 1 loop
6 pt function
2M pt function M ≥ 3
Correlation Functions
Power Counting
Convergent
g_R′ is a function of μ′
g_R′ = G(μ′)
if μ and g_R are fixed initial conditions
μ′ = μ + δμ
How does g_R depends on μ?
For a given QFT (specific choice of μ₀, g_R⁰)
Integrate the Flow Equation
same QFT
= C
Better write in differential form
β-function
Gell-Man Low QED
Callan-Zymanzik
μ d/dμ g_R(μ) = β(g_R(μ))
μ′ d/dμ′ g_R′ |_{μ′=μ} = μ d/dμ G(μ) = g_R² 3/2 1/(4π)² := β(g_R)
for φ⁴

p_2

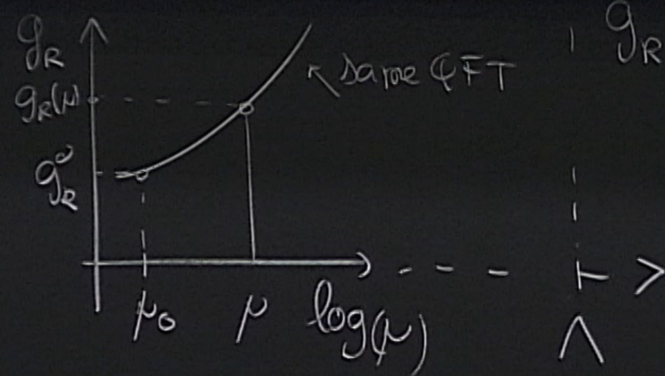
$$\frac{}{(2\pi)^4\, k^2\, (k+p_1)^2\, (k+p_1+p_2)^2} \quad \text{convergent}$$

-pends on μ?

(specific choice of μ_0, g_R^0)

Flow Equation

$$\boxed{= \beta(g_R(\mu))}$$



$$g_R(\Lambda) = g_R - g_R^2\, \frac{3}{2}\, \frac{1}{(4\pi)^2} \log\left(\frac{\mu^2}{\Lambda^2}\right)$$

$$= C$$

$$\boxed{g_R(\Lambda) = C}$$

④ Running Coupling Constant ---> Wilsonian RG

$\Gamma^{IRR}_{(4)}(\{p_i\}, g_R; \mu)$

$\Gamma^{IRR}_{(4)}(\{p_i/s\}, g_R; \mu) = \Gamma^{IRR}_{(4)}(\{p_i\}, g_R; \mu s)$

Exact - Dimensional Analysis

$\Gamma_{(4)}$ and g_R dimensionless

it can depend only on (p_i/μ)

Scale transformation on the momenta

$\{p_i\}$, $\{p_i' = \frac{1}{s} p_i\}$

$s \ll 1$ High Momentas

$s \gg 1$ Low Momenta

④ Running Coupling Constant ---> Wilsonian RG

$\Gamma^{IRR}_{(4)}(\{p_i\}, g_R; \mu)$

$\Gamma^{IRR}_{(4)}(\{p_i/s\}, g_R; \mu) = \Gamma^{IRR}_{(4)}(\{p_i\}, g_R; \mu s)$

Exact - Dimensional Analysis

RG Beta function $= \Gamma^{IRR}_{(4)}(\{p_i\}, g_R(s); \mu)$

$$\boxed{s\frac{d}{ds}g_R(s) = -\beta[g_R(s)]}$$

minus sign !

Scale transformation on the momenta

$\{p_i\}$, $\{p_i' = \frac{1}{s}p_i\}$

$s \ll 1$ High Momenta

$s \gg 1$ Low Momenta

$\Gamma_{(4)}$ and g_R dimensionless

it can depend only on (p_i/μ)

$\mu \longrightarrow \mu' = \mu/s$

so that

$\mu' s = \mu$

④ Running Coupling Constant ---> Wilsonian RG

$\Gamma^{IRR}_{(4)}(\{p_i\}, g_R; \mu)$

$\Gamma^{IRR}_{(4)}(\{p_i/s\}, g_R; \mu) = \Gamma^{IRR}_{(4)}(\{p_i\}, g_R; \mu s)$

Exact
Dimensional
Analysis

RG Beta function $= \Gamma^{IRR}_{(4)}(\{p_i\}, g_R(s); \mu)$

$$\boxed{s\frac{d}{ds} g_R(s) = -\beta\left[g_R(s)\right]}$$

minus!
sign!

$\Gamma_{(4)}$ and g_R dimensionless
it can depend only on (p_i/μ)

Scale transformation on the momenta

$\{p_i\}$, $\{p_i' = \frac{1}{s} p_i\}$

$s \ll 1$ High Momenta,
$s \gg 1$ Low Momenta

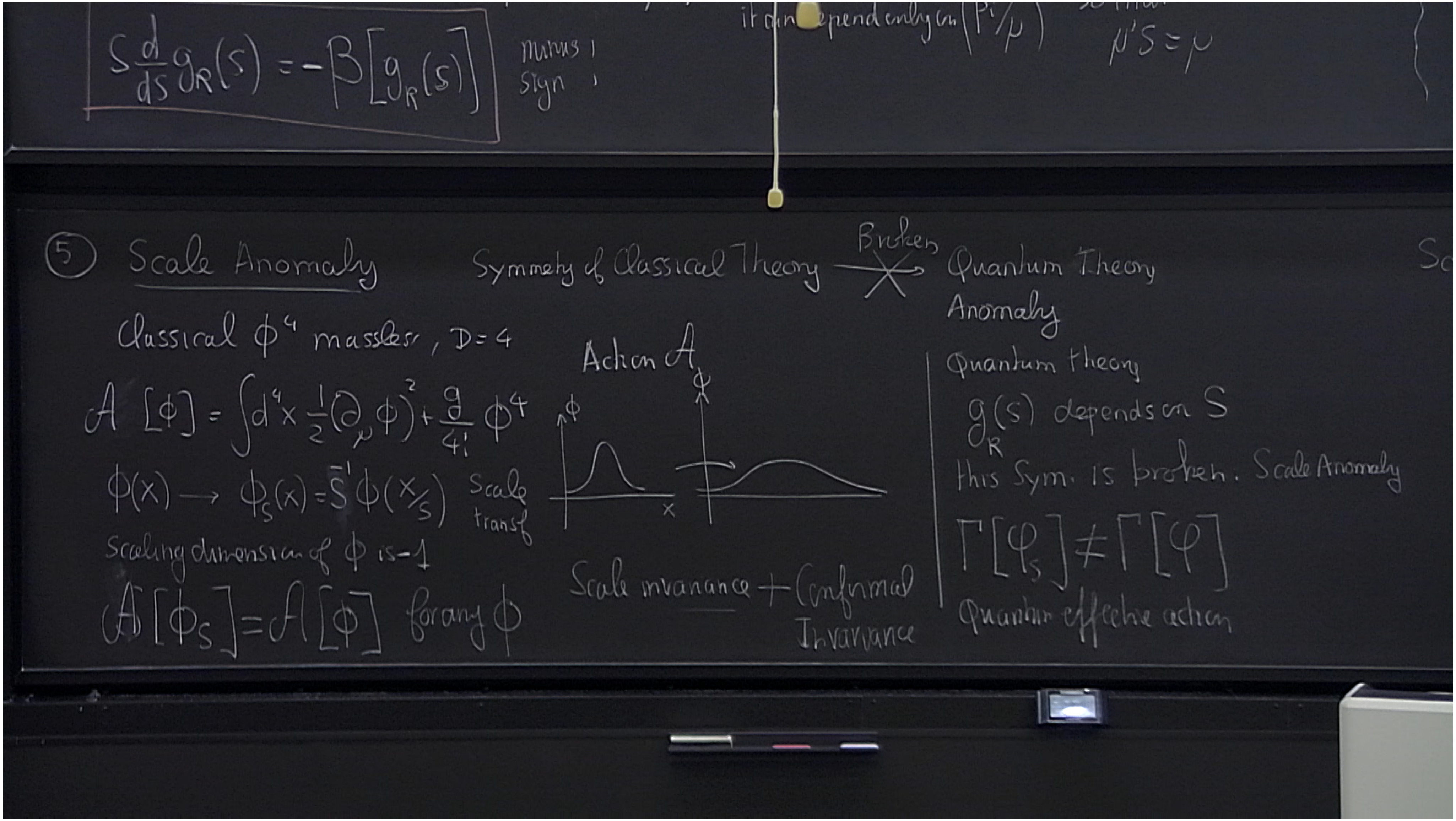
$\mu \longrightarrow \mu' = \mu/s$
so that
$\mu' s = \mu$

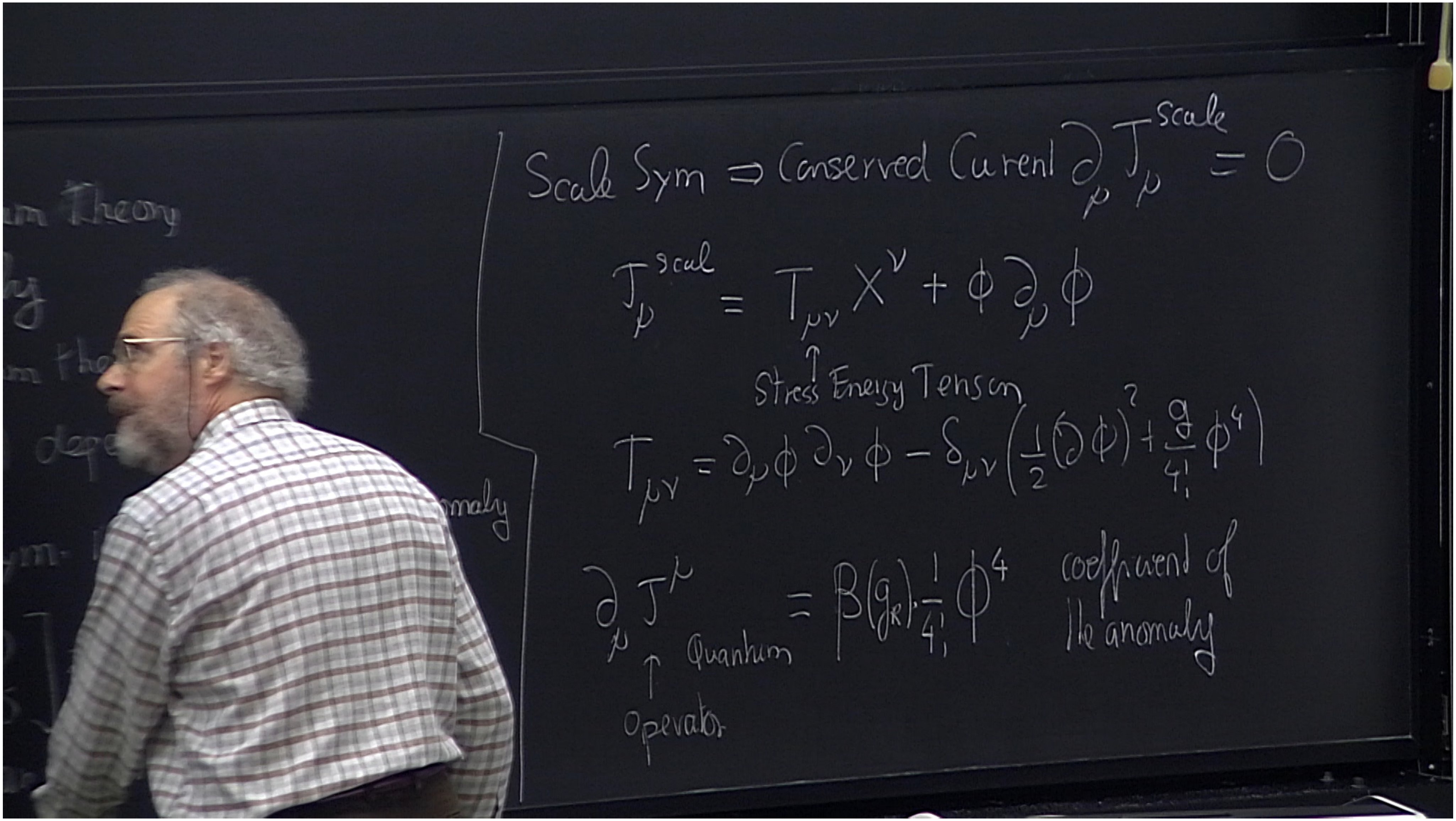
$g_R(s)$ = Running Coupling Constant
with the "length scale factor" s

s = inverse of a momenta/energy scale factor

$p_i \longrightarrow u\, p_i$ $\quad u \nearrow \quad g(u)$ Momentum rescalin

$u\frac{d}{du} g_R = \beta(g_R(u))$ + sign.


$s\frac{d}{ds}g_R(s) = -\beta[g_R(s)]$
minus sign
it can depend only on (p_i/μ)
$\mu' s = \mu$
5 Scale Anomaly
Symmetry of Classical Theory $\xrightarrow{\text{Broken}}$ Quantum Theory
Anomaly
Classical ϕ^4 massless, D = 4
$A[\phi] = \int d^4x \frac{1}{2}(\partial_\mu \phi)^2 + \frac{g}{4!}\phi^4$
$\phi(x) \to \phi_s(x) = s^{-1}\phi(x/s)$ Scale transf
Scaling dimension of ϕ is -1
$A[\phi_s] = A[\phi]$ for any ϕ
Action A
Scale invariance + Conformal Invariance
Quantum theory
$g_R(s)$ depends on S
This Sym. is broken. Scale Anomaly
$\Gamma[\varphi_s] \neq \Gamma[\varphi]$
Quantum effective action



Scale Sym ⇒ Conserved Curent $\partial_\mu J_\mu^{scale} = 0$
$J_\mu^{scal} = T_{\mu\nu} X^\nu + \phi \partial_\mu \phi$
Stress Energy Tensor
$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \delta_{\mu\nu}\left(\frac{1}{2}(\partial\phi)^2 + \frac{g}{4!}\phi^4\right)$
$\partial_\mu J^\mu_{\text{Quantum}} = \beta(g_R) \cdot \frac{1}{4!}\phi^4$
coefficient of
the anomaly
Operator
Theory
anomaly

(5) ∘ Massive Theory ← Tutorial. It can be renormalized when renormalized mass m_R

B has to be adjusted

- Running Coupling Constants :

μ_0 scale, coupling g_R^0 ⟶ scale μ, equivalent coupling $g_R(\mu)$

$$\mu \frac{d}{d\mu} g_R(\mu) = \beta(g_R(\mu)) = \frac{3}{2}\frac{1}{(4\pi)^2} g_R^2(\mu) \Rightarrow \boxed{g_R(\mu) = \frac{g_R^0}{1 - \frac{3}{(4\pi)^2} g_R^0 \log(\mu/\mu_0)}}$$

Quantum effective action

Operator

when $\mu \to 0$ low energy $g_R(\mu) \searrow 0$ as $\frac{1}{\log \mu}$

$\mu \nearrow$ high energy at a finite energy

$$\mu = \mu_0 \exp\left(\frac{(4\pi)^2}{3}\frac{1}{g_R^0}\right), \quad g_R(\mu) \to \infty$$

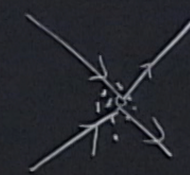
at this scale: the theory breaks down!

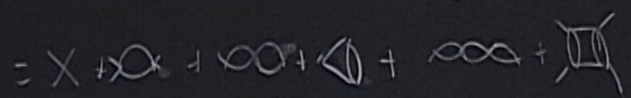
very large ; non-perturbative phenomenon

effective coupling at low energies $\searrow 0$

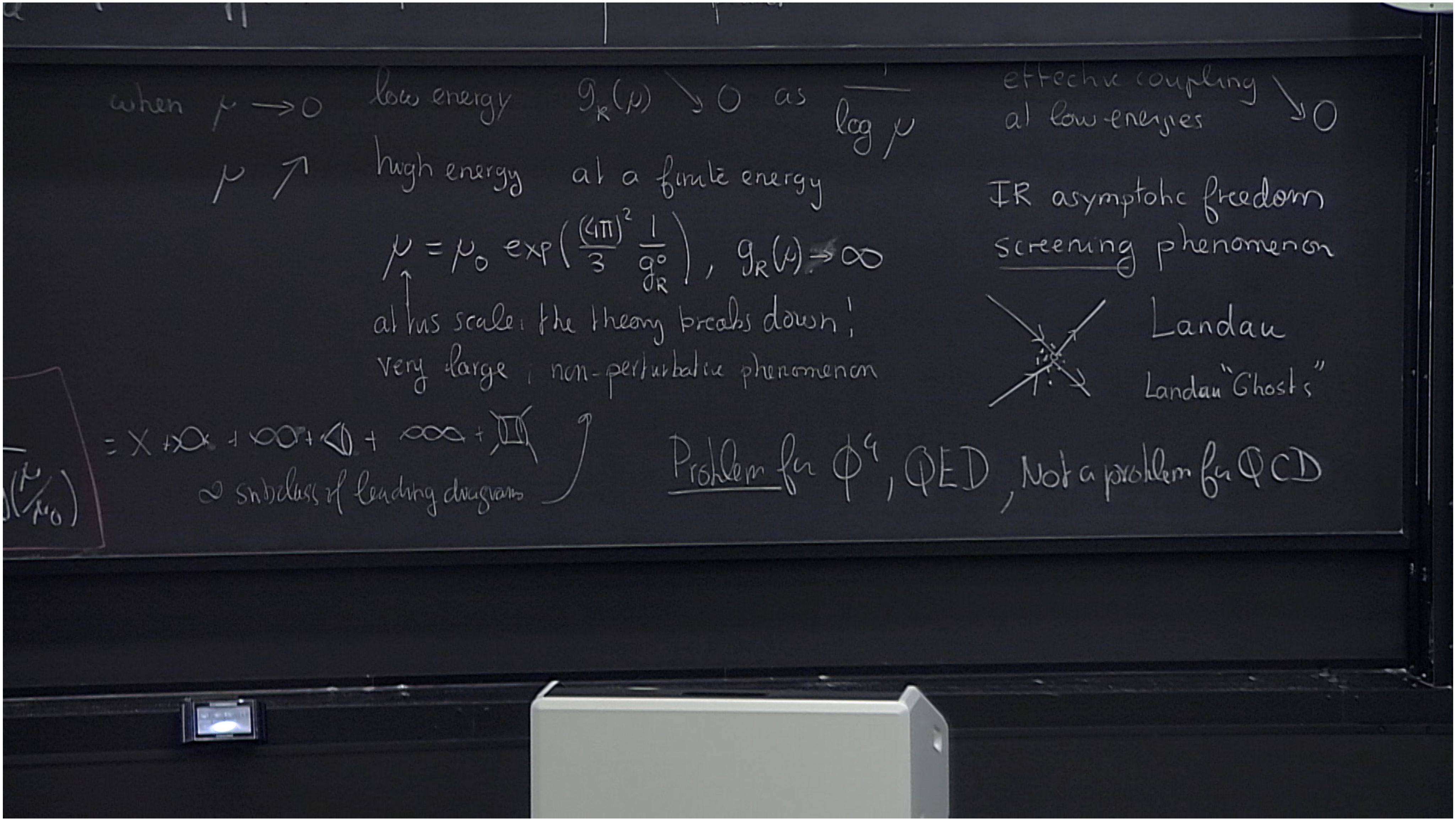
IR asymptotic freedom

screening phenomenon





∞ subclass of leading diagrams


when $\mu \to 0$ low energy $g_R(\mu) \searrow 0$ as $\frac{1}{\log \mu}$
effective coupling $\searrow 0$ at low energies
$\mu \nearrow$ high energy at a finite energy
IR asymptotic freedom
screening phenomenon
$\mu = \mu_0 \exp\left(\frac{(4\pi)^2}{3} \frac{1}{g_R^0}\right)$, $g_R(\mu) \to \infty$
at this scale: the theory breaks down!
very large ; non-perturbative phenomenon
Landau
Landau "Ghosts"
∞ subclass of leading diagrams
Problem for ϕ^4, QED, Not a problem for QCD

...n be renormalized

...malized mass m_R

...has to be adjusted

...valent coupling $g_R(\mu)$

$$\Rightarrow g_R(\mu) = \frac{g_R^0}{1-\frac{3}{(4\pi)^2} g_R^0 \log(\mu/\mu_0)}$$

∞ subclass of leading diagrams

when $\mu \to 0$ low energy $g_R(\mu) \searrow 0$ as $\frac{1}{\log \mu}$

$\mu \nearrow$ high energy at a finite energy

$$\mu = \mu_0 \exp\left(\frac{(4\pi)^2}{3}\frac{1}{g_R^0}\right), \quad g_R(\mu) \to \infty$$

at this scale: the theory breaks down!

very large : non-perturbative phenomenon

effective coupling at low energies $\searrow 0$

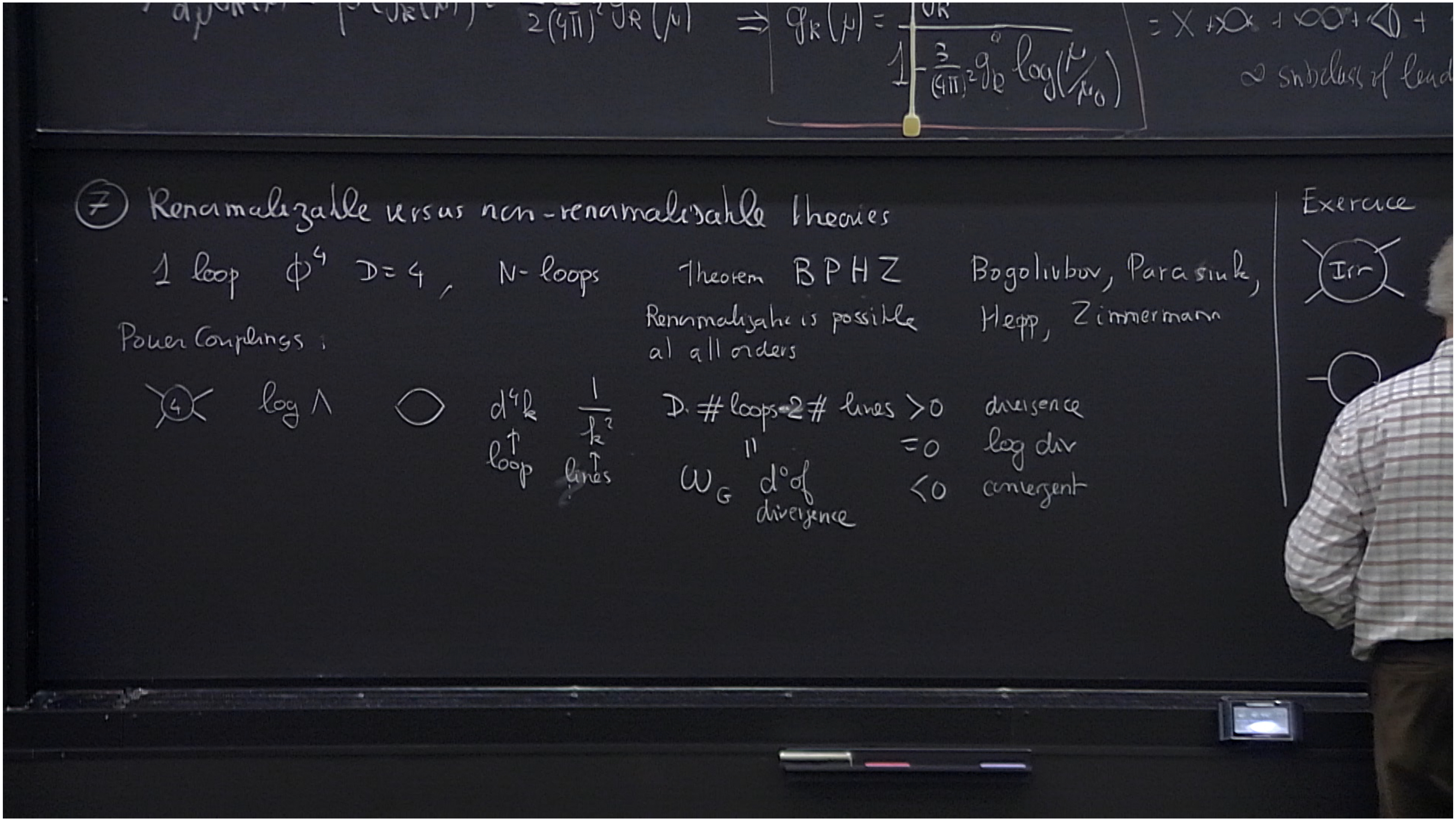
IR asymptotic freedom/UV

screening phenomenon

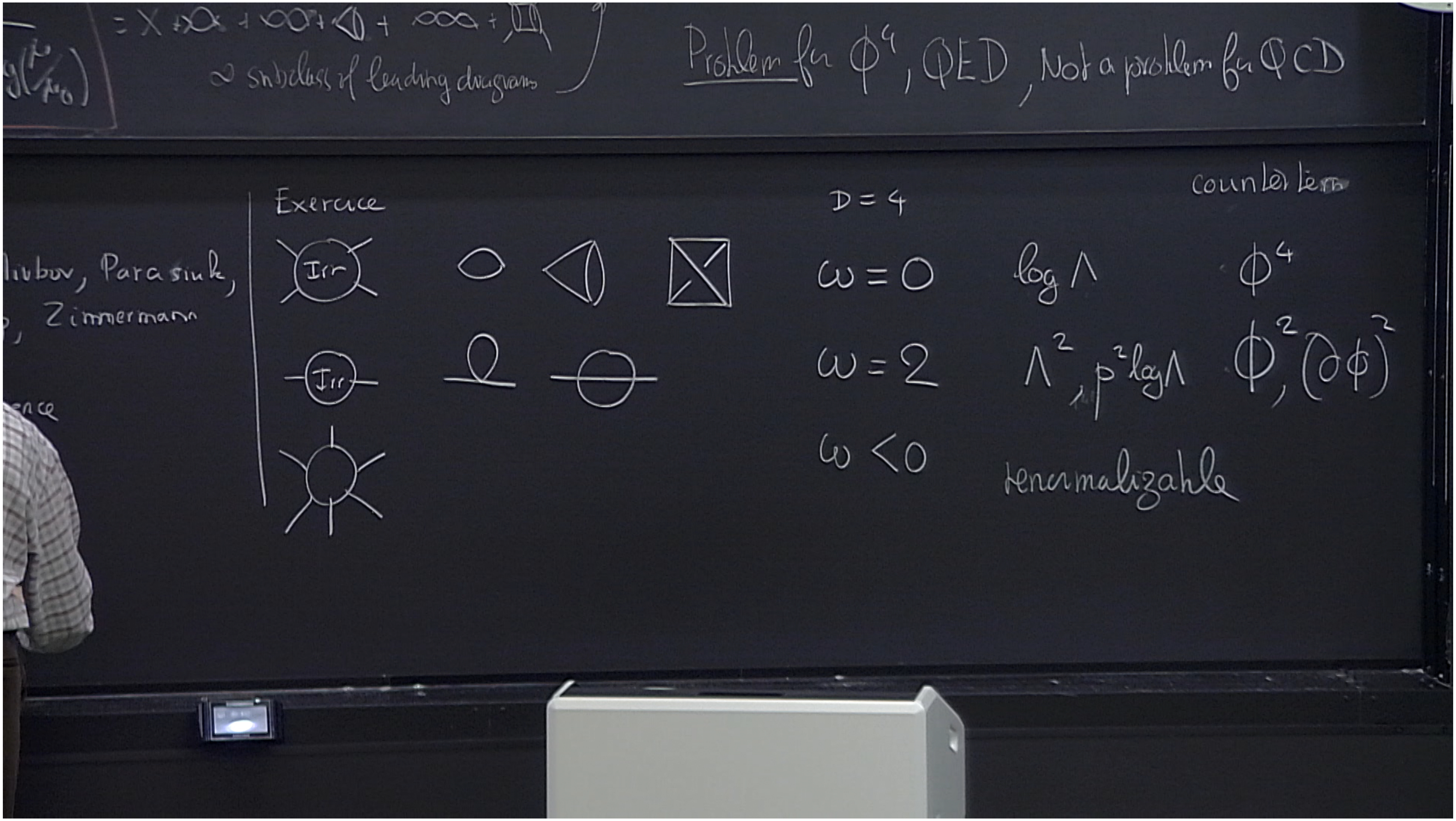
Landau

Landau "Ghosts"

Problem for ϕ^4, QED, Not a problem for QCD



⇒ g_R(μ) = g_R / (1 − 3/(4π)^2 g_R log(μ/μ_0))
∞ subclass of lead
7 Renormalizable versus non-renormalisable theories
1 loop φ^4 D = 4, N-loops
Theorem BPHZ
Bogolivbov, Parasiuk, Hepp, Zimmermann
Renormalisation is possible at all orders
Power Couplings :
4
log Λ
d^4k
1/k^2
loop
lines
D. # loops − 2 # lines > 0 divergence
= 0 log div
< 0 convergent
ω_G d° of divergence
Exercice
Irr


∝ subclass of leading diagrams
Problem for φ⁴, QED, Not a problem for QCD
Exercice
Irr
Irr
D = 4
ω = 0
ω = 2
ω < 0
log Λ
Λ², p² log Λ
renormalizable
counterterm
φ⁴
φ², (∂φ)²
Zimmermann

μ_0 scale, coupling $g_R^0 \longrightarrow$ scale μ, equivalent coupling $g_R(\mu)$

$$\mu \frac{d}{d\mu} g_R(\mu) = \beta(g_R(\mu)) = \frac{3}{2}\frac{1}{(4\pi)^2} g_R^2(\mu) \Rightarrow g_R(\mu) = \frac{g_R^0}{1 - \frac{3}{(4\pi)^2} g_R^0 \log(\mu/\mu_0)}$$

~ subclass of leading diagrams

at this scale the theory breaks down!
very large, non-perturbative phenomenon

Landau

Landau "Ghosts"

Problem for ϕ^4, QED, Not a problem for QCD

⑦ Renormalizable versus non-renormalizable theories

1 loop ϕ^4 $D=4$, N-loops

Theorem BPHZ Bogoliubov, Parasiuk, Hepp, Zimmermann

Renormalization is possible at all orders

Power Couplings:

$\log \Lambda$ $\quad d^4k \quad \frac{1}{k^2}$

loop $\quad$ lines

$D \cdot \#\text{loops} - 2\#\text{lines} > 0$ divergence

$= 0$ log div

ω_G d° of divergence < 0 convergent

if $D > 4$

$$\omega = \omega_{D=4} + (D-4)\#\text{loops}$$

$\omega_4 = -2 \Rightarrow$ in $D=6$ divergent

in $D=6$

ϕ^6 counterterm

∞ # of counterterms

non-renormalizable

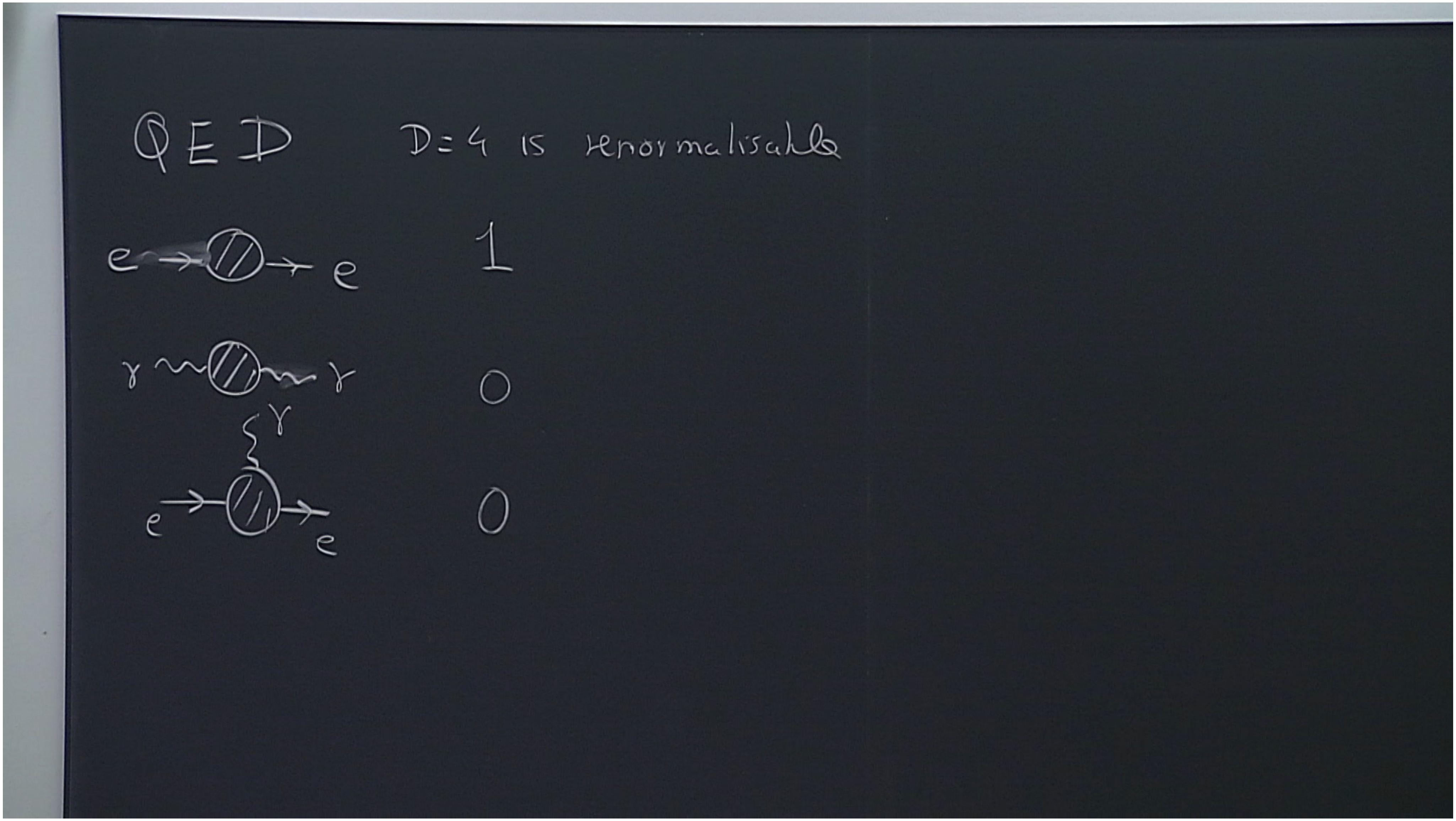
Exercise

$D=4$ counterterm

$\omega = 0 \quad \log \Lambda \quad \phi^4$

$\omega = 2 \quad \Lambda^2, p^2 \log \Lambda \quad \phi^2, (\partial \phi)^2$

$\omega < 0$ renormalizable


QED
D = 4 is renormalisable
e
e
1
γ
γ
0
γ
e
e
0