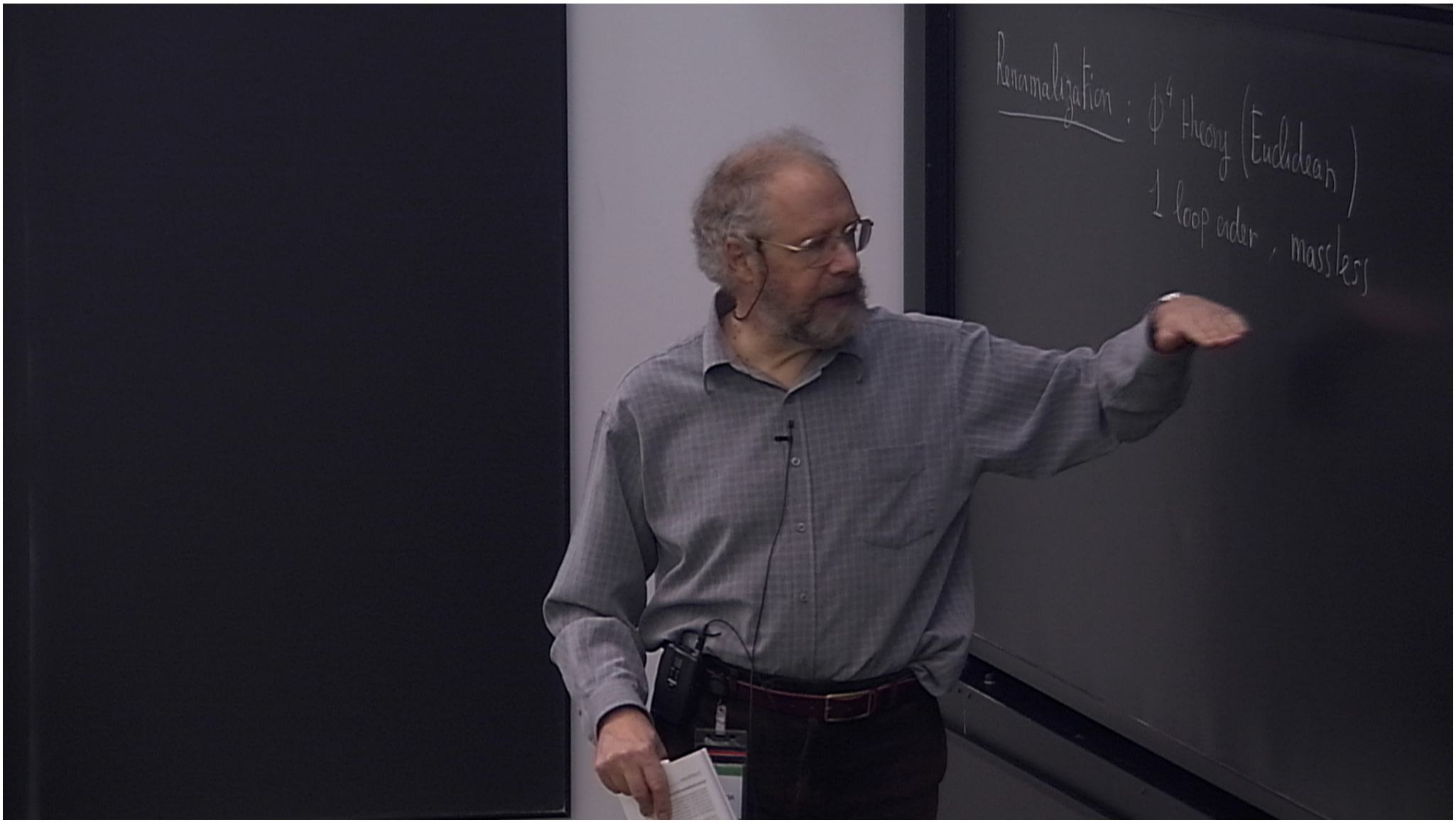
Title: PSI 2016/2017 Quantum Field Theory II - Lecture 8

Date: Nov 16, 2016 09:00 AM

URL: http://pirsa.org/16110008

Abstract:


Renormalization : $\phi^4$ theory (Euclidean)
1 loop order, massless

## Renormalization : $\phi^4$ theory (Euclidean)

1 loop order, massless

- perturbation theory
- UV divergences
- mass renormalization
- coupling constant renormalization
- renormalization scale and Beta-functions

$$S[\phi] = \int d^D x \; \frac{A}{2}(\partial_\mu \phi)^2 + \frac{B}{2}\phi^2 + \frac{C}{4!}\phi^4$$

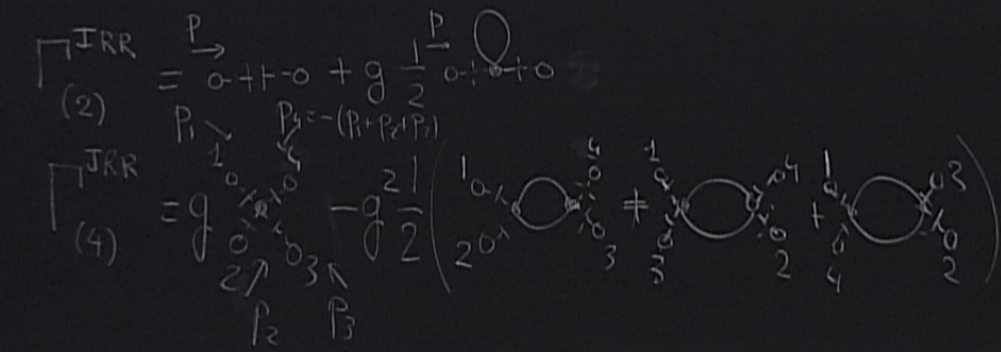
Classical

$A = 1$

$B = m^2$

$C = g$

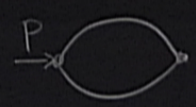
$$\int D[\phi] \exp(-S[\phi]) \longrightarrow \langle \phi \cdots \phi \rangle$$




Fourier Transform ("momentum space"

$$p^2 + m^2 + g\frac{1}{2}T(m) \qquad T(m) = \int \frac{d^D k}{(2\pi)^2} \frac{1}{k^2 + m^2}$$

$$g - g^2 \frac{1}{2}\Big[ B(p_1+p_2, m) + B(p_1+p_3, m) + B(p_1+p_4, m) \Big]$$

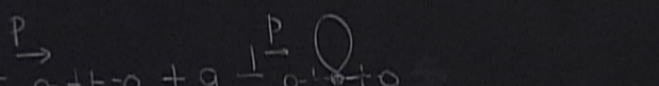


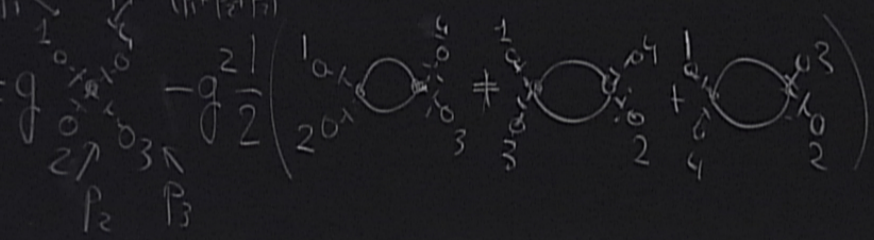

$B(p, m) =$

$$\int D[\phi]\, \exp(-S[\phi]) \longrightarrow \langle \phi \cdots \phi \rangle$$

$$B = m^2$$

$$C = g$$

$\Gamma^{IRR}_{(2)}$ = 

$\Gamma^{IRR}_{(4)}$ = 

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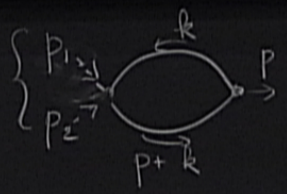
$$g - g^2\frac{1}{2}\left[B(p_1+p_2, m) + B(p_1+p_3, m) + B(p_1+p_4, m)\right]$$

$p = p_1 + p_2$ 

$$B(p, m) = \int \frac{d^D k}{(2\pi)^D}\left(\frac{1}{k^2+m^2} \cdot \frac{1}{(p+k)^2+m^2}\right)$$

$\underline{D=4}$ • $T(m,\Lambda) = \int_{|k|<\Lambda} \frac{d^4k}{(2\pi)^2} \frac{1}{k^2+m^2} = \frac{1}{(4\pi)^2}\Lambda^2 - \frac{m^2}{(4\pi)^2}\log\Lambda^2 + O(1)$

sharp UV cutoff

$|k|<\Lambda$ regulator in momentum space

leading

subleading divergence

$\frac{1}{k^2} - \frac{m^2}{(k^2)^2} + \cdots$ expansion when $\Lambda \to \infty$

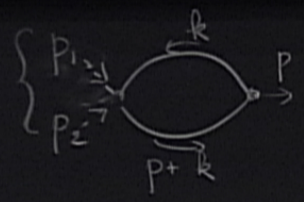
$\Lambda \gg m$ and $|p|$ external momenta

•  $= B(p,m;\Lambda) = \frac{1}{(4\pi)^2}\log(\Lambda^2) + O(1)$

leading divergence $\Lambda \nearrow \infty$ indep of $p$ and $m$

$$+ B(p_3+p_4, m)\Big]$$

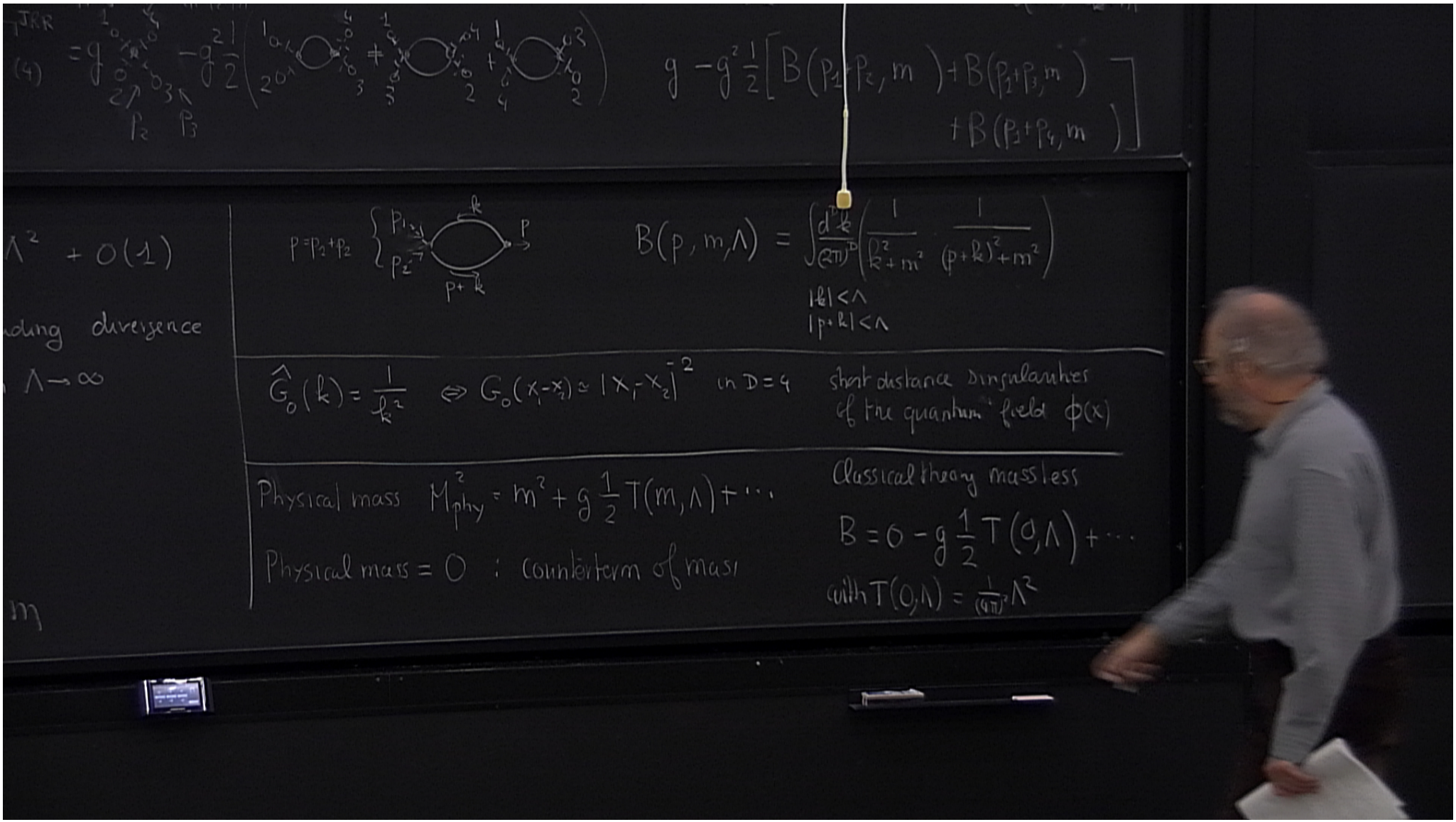
$p = p_1 + p_2$




$$B(p, m, \Lambda) = \int_{\substack{|k|<\Lambda \\ |p+k|<\Lambda}} \frac{d^D k}{(2\pi)^D} \left( \frac{1}{k^2+m^2} \cdot \frac{1}{(p+k)^2+m^2} \right)$$

$$\hat{G}_0(k) = \frac{1}{k^2} \Leftrightarrow G_0(x_1 - x_2) \simeq |x_1 - x_2|^{-2} \quad \text{in } D=4$$

short distance singularities of the quantum field $\phi(x)$


$g - g^2 \frac{1}{2}\left[ B(p_1+p_2, m) + B(p_1+p_3, m) + B(p_1+p_4, m) \right]$
$p = p_1 + p_2$
$B(p, m, \Lambda) = \int \frac{d^D k}{(2\pi)^D} \left( \frac{1}{k^2+m^2} \cdot \frac{1}{(p+k)^2+m^2} \right)$
$|k| < \Lambda$
$|p+k| < \Lambda$
$\hat{G}_0(k) = \frac{1}{k^2} \Leftrightarrow G_0(x_1 - x_2) \simeq |x_1 - x_2|^{-2}$ in $D=4$
short distance singularities of the quantum field $\phi(x)$
Physical mass $M^2_{phy} = m^2 + g \frac{1}{2} T(m, \Lambda) + \cdots$
Physical mass = 0 : counterterm of mass
Classical theory massless
$B = 0 - g \frac{1}{2} T(0, \Lambda) + \cdots$
with $T(0,\Lambda) = \frac{1}{(4\pi)^2} \Lambda^2$
$\Lambda^2 + O(1)$
$\Lambda \to \infty$

$\boxed{\text{Physical mass} = 0}$ : counterterm of mass

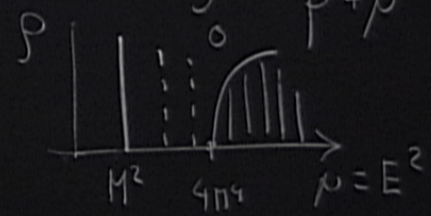
with $T(0,\Lambda) = \frac{1}{(4\pi)^2}\Lambda^2$

with this B $\quad \hat{\Gamma}^{IRR}_{(2)}(p) = p^2 + O(g^2)$ $\quad$ zero at $p^2 = -M^2_{phys} = 0$

$G_{(2)}(p) = \frac{1}{p^2}$ pole at $p^2 = 0$

Källen–Lehman Theorem

$$G_{(2)}(p) = \int_0^\infty d\mu \, \frac{\rho(\mu)}{p^2+\mu}$$




$G_{(2)}(p^2)$ has a pole in $p^2$ space at $p^2 = -M^2$

in momentum space

$$\frac{1}{k^2} - \frac{m^2}{(k^2)^2} + \cdots \quad \text{expansion when } \Lambda \to \infty$$

$\Lambda \gg m$ and $|p|$ external momenta

$$\hat{G}_0(k) = \frac{1}{k^2} \Leftrightarrow G_0(x_1 - x_2) = |x_1 - x_2|^{-2} \quad \text{in } D=4$$

$$= B(p, m; \Lambda) = \frac{1}{(4\pi)^2} \log(\Lambda^2) + O(1)$$

leading divergence $\Lambda \nearrow \infty$ indep. of $p$ and $m$

Physical mass $M^2_{phy} = m^2 + g\frac{1}{2}T(m,\Lambda) + \cdots$

$\boxed{\text{Physical mass} = 0}$ : counterterm of mass

③. Renormalization of coupling constant : $B$ is fixed, $C$ not fixed yet

$$\Gamma^{IRR}_{(4)} = C - C^2 \frac{1}{2}\left[\mathbb{B}(p_1+p_2, B, \Lambda) + \mathbb{B}(p_1+p_3, B, \Lambda) + \mathbb{B}(p_1+p_4, B, \Lambda)\right]$$

$B$ depends on $C$ and $\Lambda$

with this $B$

$$\hat{\Gamma}^{IRR}_{(2)}(p) = p^2 + O(g^2)$$

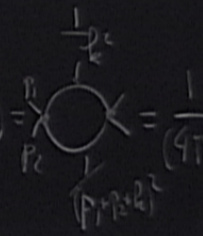
$$G_{(2)}(p) = \frac{1}{p^2} \quad \text{pole at } p^2 = 0$$

$$\Gamma^{IRR}_{(2)} = p^2 + \cdots$$

with $B = -\frac{1}{2}\frac{1}{(4\pi)^2}\Lambda^2 \cdot C \Rightarrow \Gamma^{IRR}_{(4)} = C - C^2\frac{1}{2}\left(\mathbb{B}(p_1+p_2, 0, \Lambda) + \text{"}+\text{"}\right)$

of order $O(0)$

$\mathbb{B}(p_1+p_2, 0, \Lambda) =$




$$= \frac{1}{(4\pi)^2}\log\left(\frac{\Lambda^2}{|p_1+p_2|^2}\right)$$

amplitude

② $\underline{D=4}$ • $T(m,\Lambda) = \int_{|k|<\Lambda} \frac{d^4k}{(2\pi)^2} \frac{1}{k^2+m^2} = \frac{1}{(4\pi)^2}\Lambda^2 - \frac{m^2}{(4\pi)^2}\log\Lambda^2 + O(1)$

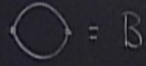
leading — subleading divergence

sharp UV cutoff

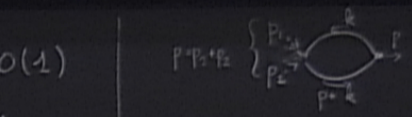
$|k|<\Lambda$ regulator in momentum space

$\Lambda \gg m$ and $|p|$ external momenta

$\frac{1}{k^2} - \frac{m^2}{(k^2)^2} + \cdots$ expansion when $\Lambda \to \infty$

•  $= B(p,m,\Lambda) = \frac{1}{(4\pi)^2}\log(\Lambda^2) + O(1)$

leading divergence $\Lambda \nearrow \infty$ indep. of $p$ and $m$




$B(p,m,\Lambda) = \int_{\substack{|k|<\Lambda \\ |p-k|<\Lambda}} \frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^2+m^2}\right.$

$\hat{G}_0(k) = \frac{1}{k^2} \Leftrightarrow G_0(x_1-x_2) = |x_1-x_2|^{-2}$ in $D=4$

Physical mass $M^2_{phy} = m^2 + g\frac{1}{2}T(m,\Lambda) + \cdots$

$\boxed{\text{Physical mass} = 0}$ : counterterm of mass

③. Renormalization of coupling constant : $B$ is fixed, $C$ not fixed yet with this $B$

$\Gamma^{IRR}_{(4)} = C - C^2\frac{1}{2}\left[B(p_1+p_2, B, \Lambda) + B(p_1+p_3, B, \Lambda) + B(p_1+p_4, B, \Lambda)\right]$

$B$: depends on $C$ and $\Lambda$

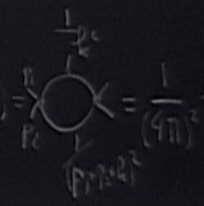
$\Gamma^{IRR}_{(2)} = p^2 + \cdots$

$\hat{\Gamma}^{IRR}_{(2)}(p) = p^2 + O(g^2)$

$G_{(2)}(p) = \frac{1}{p^2}$ pole at $p^2=0$

with $B = -\frac{1}{2}\frac{1}{(4\pi)^2}\Lambda^2 \cdot C \Rightarrow \Gamma^{IRR}_{(4)} = C - C^2\frac{1}{2}\left(B(p_1+p_2, 0, \Lambda) + \text{"}+\text{"}\right)$

of order $O(C)$

$B(p_1+p_2, 0, \Lambda) =$ 

$= \frac{1}{(4\pi)^2}\log\left(\frac{\Lambda^2}{|p_1+p_2|^2}\right)$

amplitude

$\Lambda$ fixed, Expand in $C$

then $\Lambda \to \infty$

Recipe: is this consistent or not?

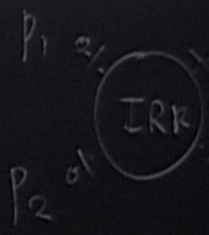
$\hbar \to 0$ Loop expansion vs $\Lambda \to \infty$ continuum limit

Q: Does those limits commute?

A: This requires a RG analysis

$$\Gamma^{IRR}_{(4)}(p_1, p_2, p_3, p_4) = C - C^2 \frac{1}{2} \frac{1}{(4\pi)^2} \left( \log\left( \frac{\Lambda^2}{(p_1+p_2)^2} \right) + \text{''} + \text{''} \right)$$

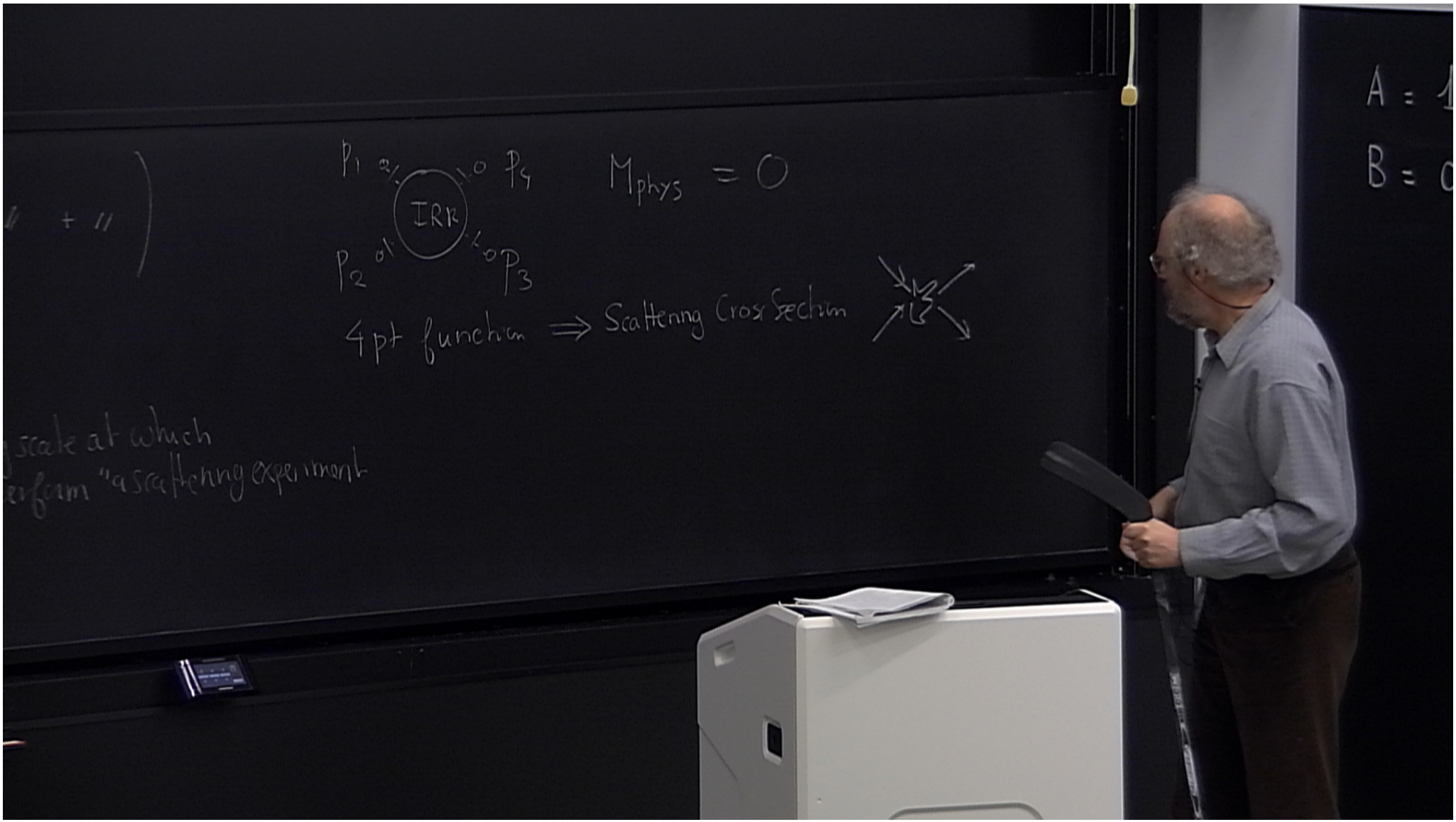
$p_1 + p_2 + p_3 + p_4 = 0$




4pt func

"strength of the interaction" $\Leftarrow$ <u>renormalized</u> coupling $g_R$

with reference to a choosen momentum scale $\mu$ = Energy scale at which "we" perform "a scattering experiment


P1
P4
IRR
P2
P3
Mphys = 0
4pt function ⟹ Scattering Cross Section
scale at which
perform "a scattering experiment
A =
B =

$$\Gamma^{IRR}_{(4)}(p_1, p_2, p_3, p_4) = C - C^2 \frac{1}{2} \frac{1}{(4\pi)^2} \left( \log\left(\frac{\Lambda^2}{(p_1+p_2)^2}\right) + // + // \right)$$

$$p_1 + p_2 + p_3 + p_4 = 0$$

"strength of the interaction" $\Leftarrow$ renormalized coupling $g_R$

with reference to a choosen momentum scale $\mu$ = Energy scale at which we "perform" a scattering experiment

$\boxed{\mu = \text{renormalization scale}}$

Here choice

$$\boxed{(p_1^{(R)} + p_2^{(R)})^2 = (p_1^{(R)} + p_3^{(R)})^2 = (p_1^{(R)} + p_4^{(R)})^2 = \mu^2}$$

Then I define $\boxed{g_R := \Gamma^{IRR}_{(4)}(p_1^{(R)}, p_2^{(R)}, p_3^{(R)}, p_4^{(R)})}$

$p_1$

$p_2$

4pt

$$\boxed{p_1+p_2+p_3+p_4=0}$$

"Strength of the interaction" $\Leftarrow$ renormalized coupling $g_R$

4pt function $\Rightarrow$ Scatt

with reference to a choosen momentum scale $\mu$ = Energy scale at which we "perform" a scattering experiment

$$\boxed{\mu = \text{renamalization scale}}$$

$$\boxed{g_R = C - C^2 \frac{3}{2}}$$

$$C = g_R + g_R^2 \frac{3}{2}$$

Here, choice

$$\boxed{(p_1^{(R)}+p_2^{(R)})^2 = (p_1^{(R)}+p_3^{(R)})^2 = (p_1^{(R)}+p_4^{(R)})^2 = \mu^2}$$

then I define $\boxed{g_R := \Gamma^{IRR}_{(4)}(p_1^{(R)}, p_2^{(R)}, p_3^{(R)}, p_4^{(R)})}$

Reexpress $\Gamma^{IRR}_{(4)}$ in terms of $g_R$ rather than $C$, it is UV finite, no $\log\Lambda$ any more

$$\Gamma^{IRR}_{(4)} = g_R - g_R^2 \frac{1}{2}\frac{1}{(4\pi)^2}\left[\log\frac{\mu^2}{(p_1+p_2)^2} + \log\frac{\mu^2}{(p_1+p_3)^2} + \log\frac{\mu^2}{(p_1+p_4)^2}\right] + O(g_R^3)$$

2 loops diagrams + counterterms that contains $\log\Lambda$ ..

this makes sense for any momenta $(p_1, p_2, p_3, p_4)$

with a renormalized action

$$S_R[\phi] = \int d^4x \left( \frac{A}{2} (\partial_\mu \phi)^2 + \frac{B}{2} \phi^2 + \frac{C}{4!} \phi^4 \right)$$

with this choice of A, B, C

2pt, 4pt functions are well defined as function of $p_1, p_2 \ldots$ and $g_R$

$\Lambda \to \infty$ exists

with reference to a choosen momentum scale $\mu$ = Energy scale at which we "perform" a scattering experiment

$\boxed{\mu = \text{renormalization scale}}$

$g_R = C - C^2 \ldots$

Here choice

$$\boxed{(p_1^{(R)} + p_2^{(R)})^2 = (p_1^{(R)} + p_3^{(R)})^2 = (p_1^{(R)} + p_4^{(R)})^2 = \mu^2}$$

then I define

$$\boxed{g_R := \Gamma^{IRR}_{(4)}(p_1^{(R)}, p_2^{(R)}, p_3^{(R)}, p_4^{(R)})}$$

$C = g_R + g_R^2 \ldots$

Reexpress $\Gamma^{IRR}_{(4)}$ in terms of $g_R$ rather than $C$, it is UV finite. no $\log \Lambda$ any more

$$\Gamma^{IRR}_{(4)} = g_R - g_R^2 \frac{1}{2} \frac{1}{(4\pi)^2} \left[ \log \frac{\mu^2}{(p_1+p_2)^2} + \log \frac{\mu^2}{(p_1+p_3)^2} + \log \frac{\mu^2}{(p_1+p_4)^2} \right] + O(g_R^3)$$

2 loops diagrams + counterterms that contains $\log \Lambda$ ..

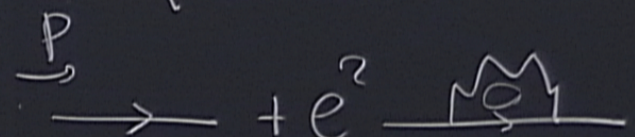
this makes sense for any momenta $(p_1, p_2, p_3, p_4)$

$$A = 1$$

$$B = 0 + g_R\left(-\frac{1}{2}\frac{1}{(4\pi)^2}\Lambda^2\right)$$

$$C = g_R + g_R^2\,\frac{3}{2}\frac{1}{(4\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)$$

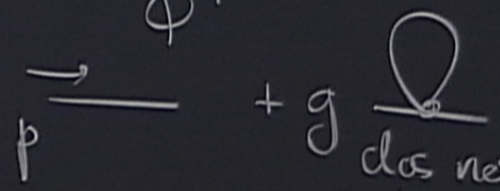
QED




$m_e = 0$

depends on p

$\phi^4$




does not depend on p

classical

$g$

1 parameter

Quantum

$\underbrace{\mu \text{ and } g_R}_{\text{2 parameter?}}$

only 1 parameter