Title: PSI 2016/2017 Quantum Field Theory II - Lecture 6

Date: Nov 14, 2016 09:00 AM

URL: http://pirsa.org/16110006

Abstract:

Perturbation Theory : Structure ϕ^4 theory (Euclidean)

$$S[\phi] = \int d^D x \; \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4$$

Expansion as a series (formal) in $\sum_k g^k \cdots$

$$\langle \phi(z_1)\, \phi(z_2) \int d^D x\, \phi^4(x) \cdots \rangle_0$$

external   internal   0 ← Free theory

Wick Th → Sum of amplitudes ↔ Feynman Diagrams
(Graphs)

$$Z = \left\langle \exp\left(-\frac{g}{\hbar\, 4!}\int d^D x\, \phi^4(x)\right)\right\rangle_0 = 1 + -\hbar\, g\, \frac{1}{8}\cdot$$

diagram

$\int d^D x\; G(0)^2$

Propagator

$\times \rightarrow g/\hbar$ vertex

$\text{———} \rightarrow \dfrac{\hbar}{p^2+m^2}$

2 pt function. (position space)

$\langle \phi(x_1)\, \phi(x_2) \rangle$

$$\ldots(z_1)\,\phi(z_2)\rangle_0 - \frac{g}{\hbar}\,\frac{1}{4!}\int d^D x_1 \left\langle \phi(z_1)\,\phi(z_2)\;\phi^4(x_1)\right\rangle_0 + O(g^2)$$

$$-g\hbar\,\frac{1}{8}\;8 + O(g^2)$$

$\downarrow$ Wick

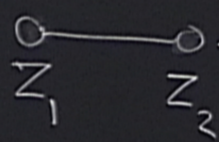
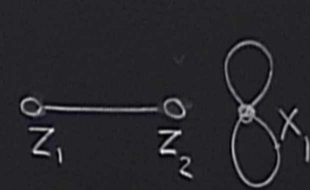
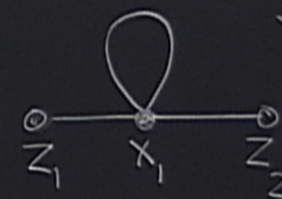
$$3\quad \langle \phi(z_1)\,\phi(z_2)\rangle_0 \left(\langle \phi(x_1)^2\rangle_0\right)^2$$

$$4\times 3\quad \langle \phi(z_1)\,\phi(x_1)\rangle_0 \langle \phi(z_2)\,\phi(x_1)\rangle_0 \langle \phi(x_1)^2\rangle_0$$

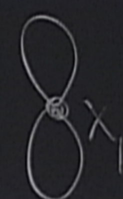
$$\langle \phi(z_1)\,\phi(z_2)\rangle = \frac{\left\langle \phi(z_1)\,\phi(z_2)\cdot \exp\left(-\frac{g}{\hbar\cdot 4!}\int d^Dx\ \phi^4(x)\right)\right\rangle_0}{\left\langle \exp\left(-\frac{g}{4!}\frac{1}{\hbar}\int d^Dx\ \phi^4(x)\right)\right\rangle_0} =$$

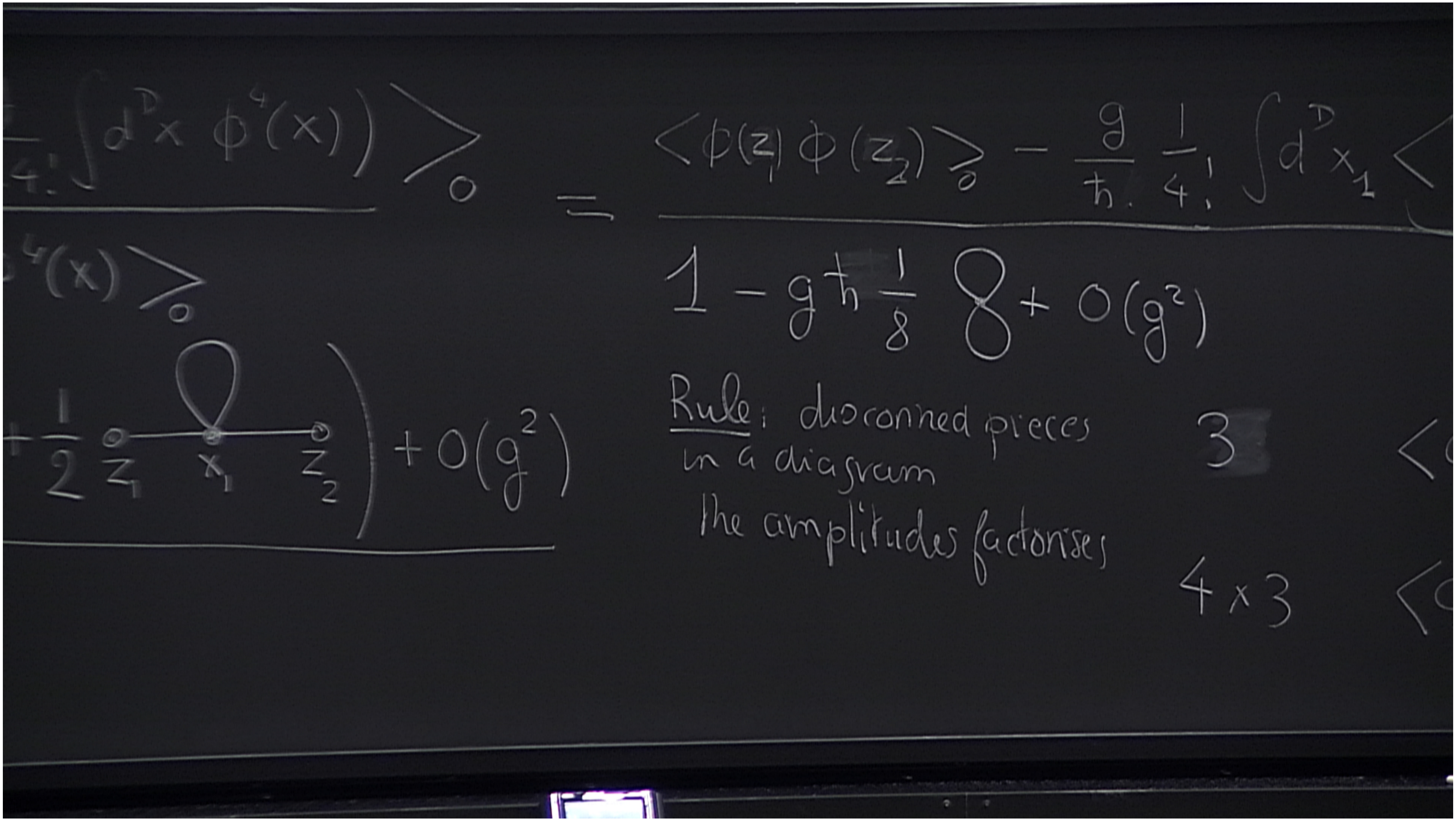
↑ interacting theory

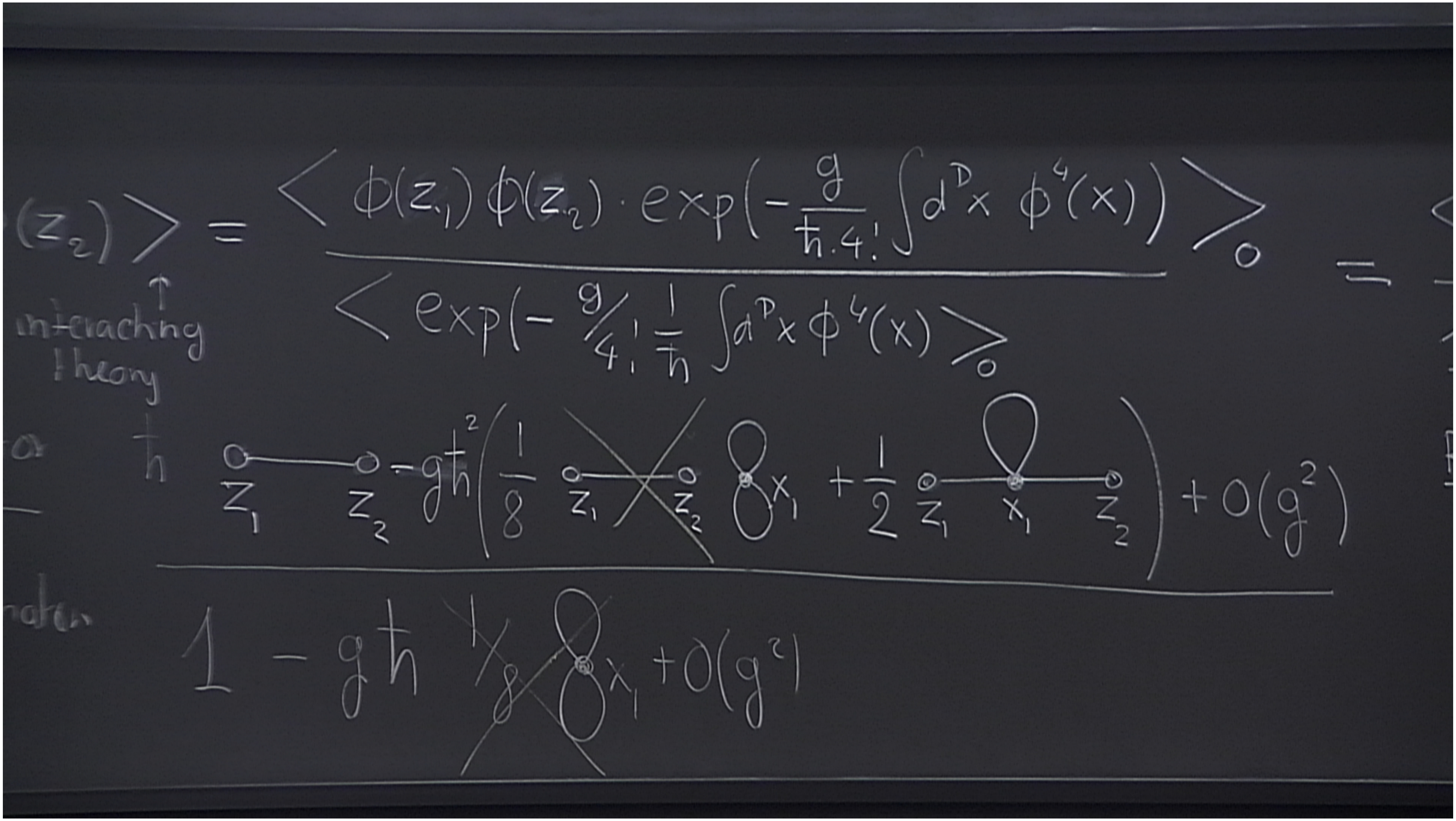
numerator

$\hbar$  $- g\hbar^2 \Big(\frac{1}{8}$  $+ \frac{1}{2}$  $\Big) + O(g^2)$

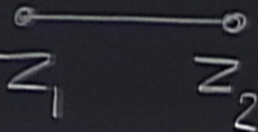
denominator

$1 - g\hbar\ 1/8$ 

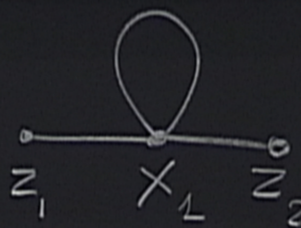

$\frac{1}{4!}\int d^Dx\, \phi^4(x)\big) \big\rangle_0 = \frac{\langle \phi(z_1)\phi(z_2)\rangle_0 - \frac{g}{\hbar}\frac{1}{4!}\int d^Dx_1 \langle}{1 - g\hbar\frac{1}{8} + O(g^2)}$
$\phi^4(x)\rangle_0$
$+\frac{1}{2}\; z_1 \quad x_1 \quad z_2 \big) + O(g^2)$
Rule: disconnected pieces in a diagram the amplitudes factorises
3
4 x 3


$(z_2)\rangle = \dfrac{\left\langle \phi(z_1)\phi(z_2)\cdot \exp\left(-\frac{g}{\hbar \cdot 4!}\int d^Dx\, \phi^4(x)\right)\right\rangle_0}{\left\langle \exp\left(-\frac{g}{4!}\frac{1}{\hbar}\int d^Dx\, \phi^4(x)\right)\right\rangle_0} =$
interacting
theory
$\hbar$
$z_1 \quad z_2 \; - g\hbar^2\left(\frac{1}{8}\; z_1\; z_2\; x_1 + \frac{1}{2}\; z_1\; x_1\; z_2\right) + O(g^2)$
$1 - g\hbar\, \frac{1}{8}\; x_1 + O(g^2)$

$$\ldots) \rangle = $$



$$\underbrace{}_{G_0(z_1 - z_2)} \quad - g \, \frac{1}{2}$$



$$\underbrace{}_{\int d^p x_1 \; G_0(z_1 - x_1)\, G_0(x_1 - z_2)\, G_0(0)}$$

$$G_{(2)}(z_1, z_2) = \langle \phi(z_1)\, \phi(z_2) \rangle = \underbrace{\overset{\circ\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\circ}{z_1 \qquad z_2}}_{G_0(z_1 - z_2)} \quad - \; g$$

"Green function"

"vaccum" diagram 8 has disappeared

This is a genral feature (all orders)

Interacting theory

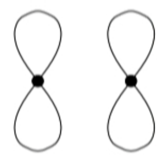
vacuum diagrams

$N = 0,\ K = 1$

$$\frac{1}{4!}\int_{x_1} \langle \Phi^4(x_1) \rangle_0 \;=\; \frac{1}{8}$$

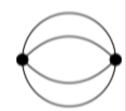
$N = 0,\ K = 2$

$$\frac{1}{2!\,(4!)^2}\int_{x_1}\int_{x_2} \langle \Phi^4(x_1)\Phi^4(x_2) \rangle_0 \;=\; \frac{1}{128}$$

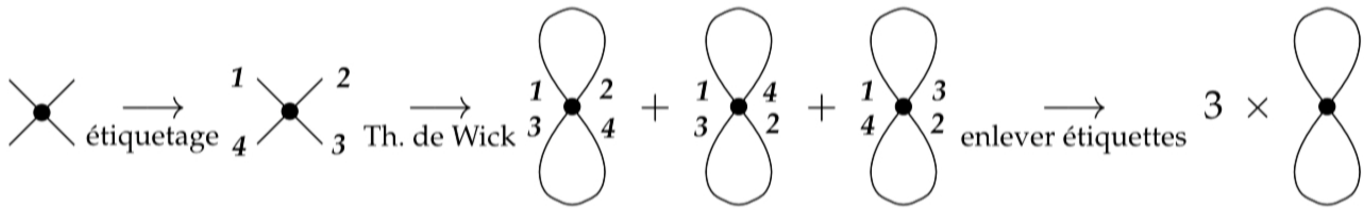


$+\frac{1}{16}$

$+\frac{1}{48}$



Origin of the combinatorial factor



$$\frac{3}{4!} = \frac{1}{8}$$

$$\frac{\text{Number of inequivalent labellings of } G}{\text{total number of labellings}} = \frac{1}{\text{number of equivalent labellings}}$$

$$= \frac{1}{\text{order of the symmetry group of } G}$$

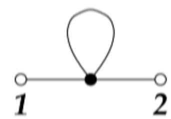
2 points diagrams

$N = 2,\ K = 0$

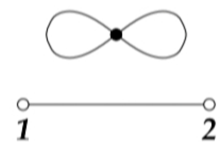
$$\langle \Phi(z_1)\Phi(z_2)\rangle_0 \ = $$

$N = 2,\ K = 1$

$$\frac{1}{4!}\int_{x_1} \langle \Phi(z_1)\Phi(z_2)\Phi^4(x_1)\rangle_0 \ = \ \tfrac{1}{2}$$



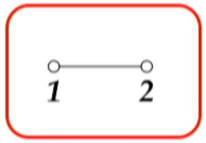
$+\ \tfrac{1}{8}$



2 points diagrams

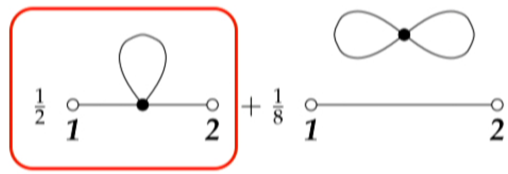
$N = 2,\ K = 0$

$$\langle \Phi(z_1)\Phi(z_2)\rangle_0 \ =$$



$N = 2,\ K = 1$

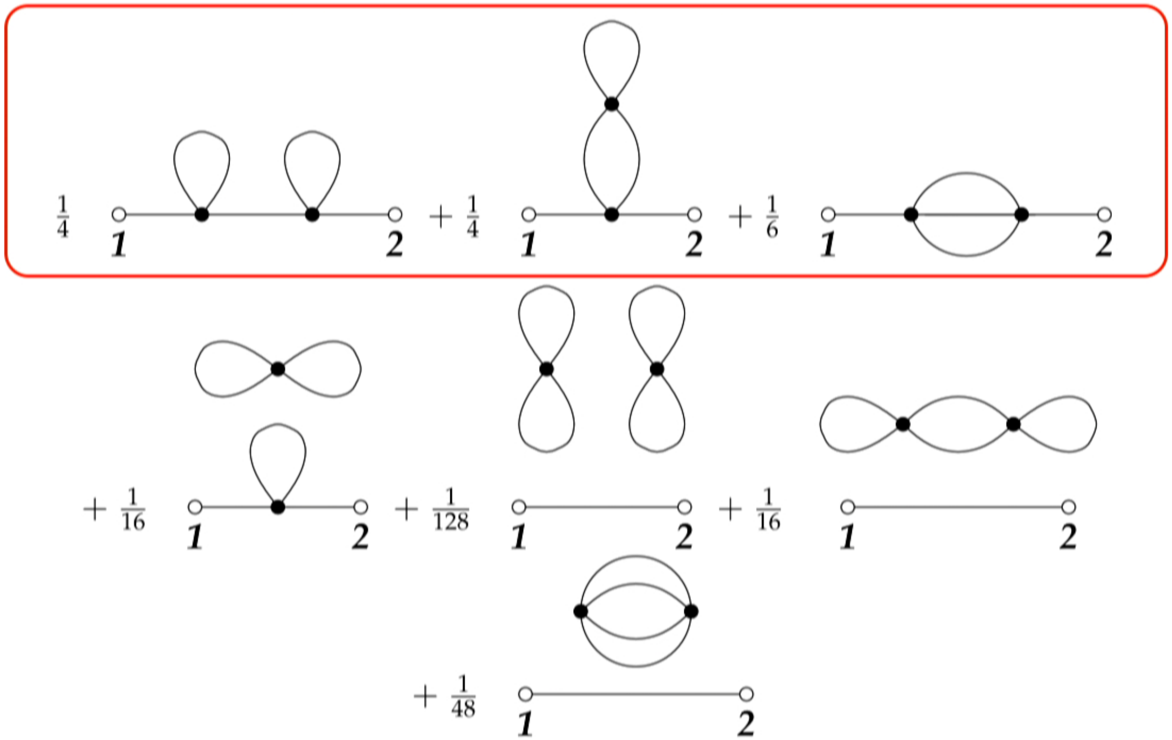
$$\frac{1}{4!}\int_{x_1} \langle \Phi(z_1)\Phi(z_2)\Phi^4(x_1)\rangle_0 \ =$$



2 points diagrams (continued)

$N = 2,\ K = 2$

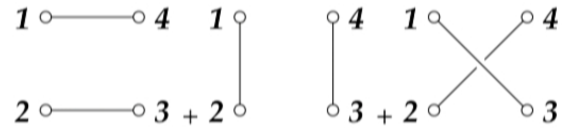
$$\frac{1}{2!\,(4!)^2} \int_{x_1} \int_{x_2} \langle \Phi(z_1)\Phi(z_2)\Phi^4(x_1)\Phi^4(x_2) \rangle_0 \ =$$



4 points diagrams

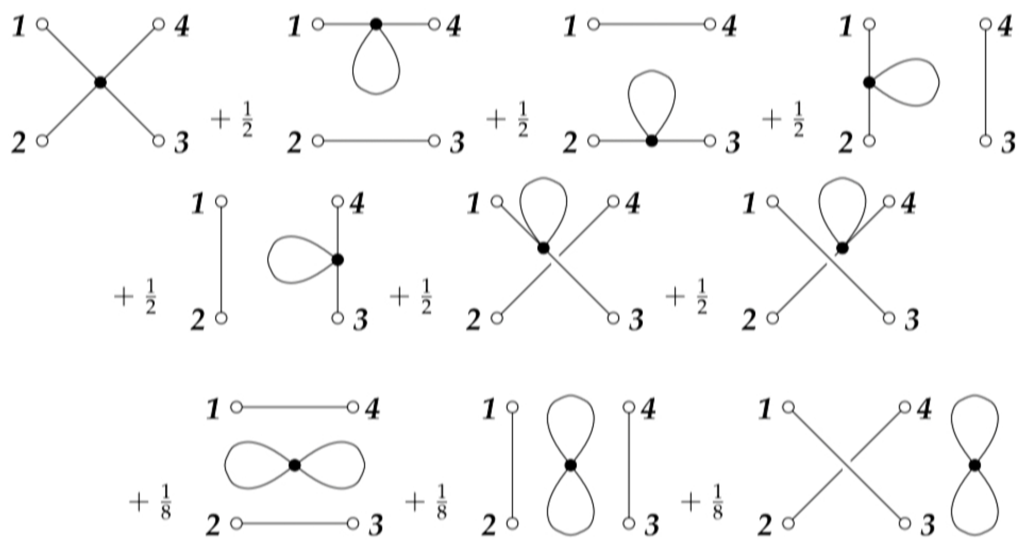
$N = 4,\ K = 0$

$$\langle \Phi(z_1)\Phi(z_2)\Phi(z_3)\Phi(z_4)\rangle_0 \ =$$



$N = 4,\ K = 1$

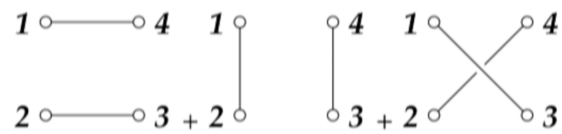
$$\frac{1}{4!}\int_{x_1} \langle \Phi(z_1)\Phi(z_2)\Phi(z_3)\Phi(z_4)\Phi^4(x_1)\rangle_0 \ =$$



4 points diagrams

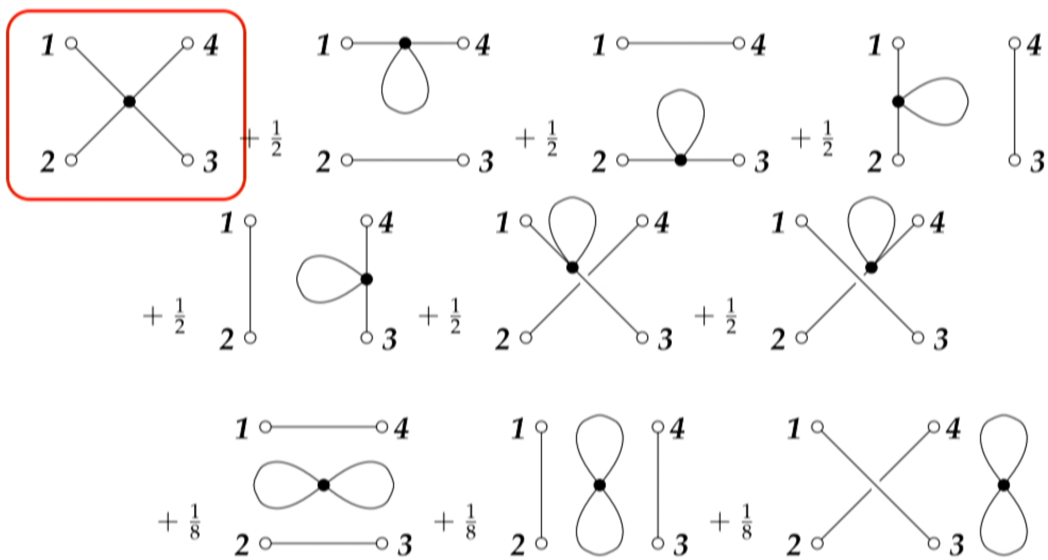
$$N = 4,\ K = 0$$

$$\langle \Phi(z_1)\Phi(z_2)\Phi(z_3)\Phi(z_4)\rangle_0 \ =$$

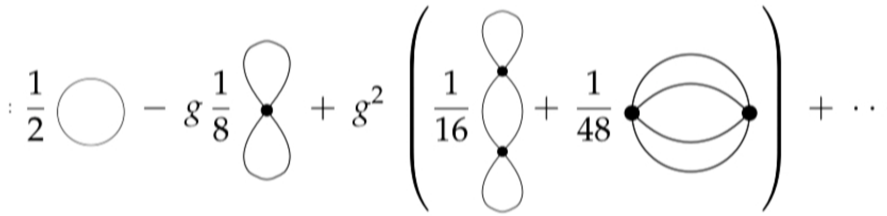


$$N = 4,\ K = 1$$

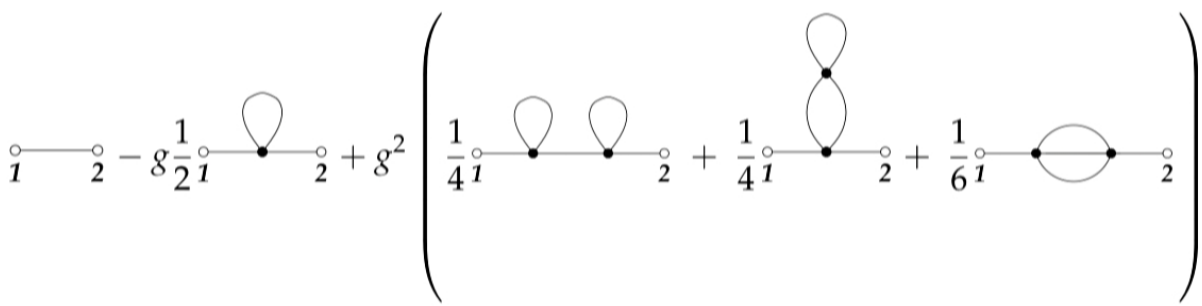
$$\frac{1}{4!}\int_{x_1} \langle \Phi(z_1)\Phi(z_2)\Phi(z_3)\Phi(z_4)\Phi^4(x_1)\rangle_0 \ =$$



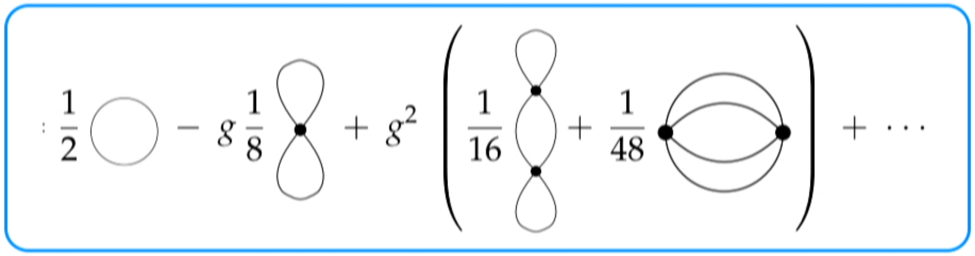
Connected vacuum diagrams



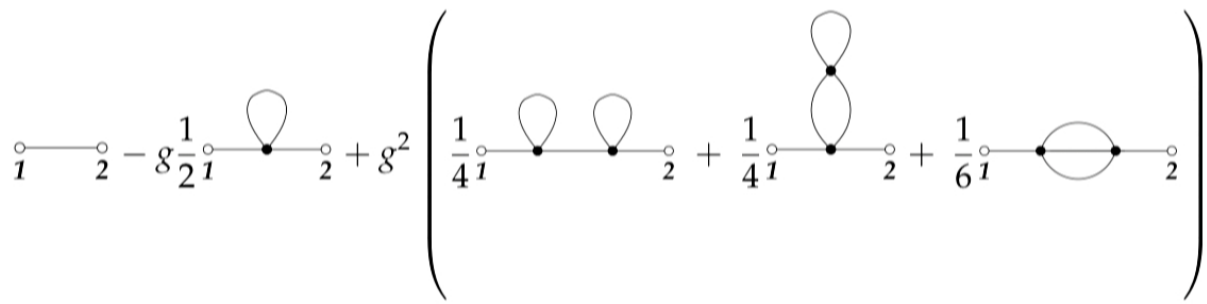
2 points function = connected 2 points function



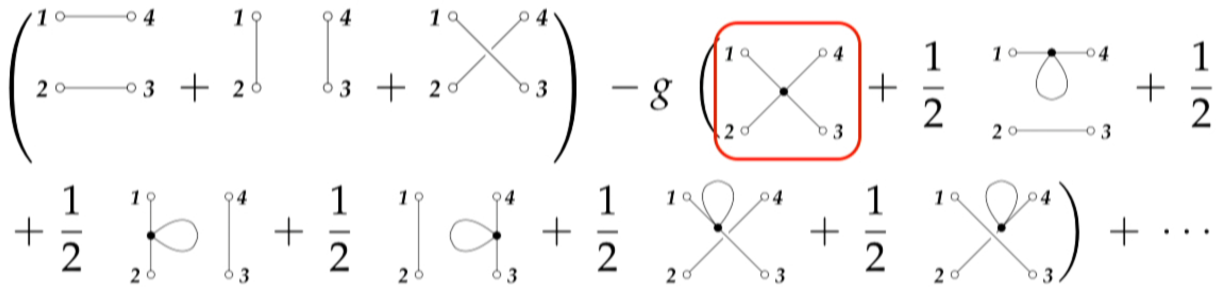
Connected vacuum diagrams



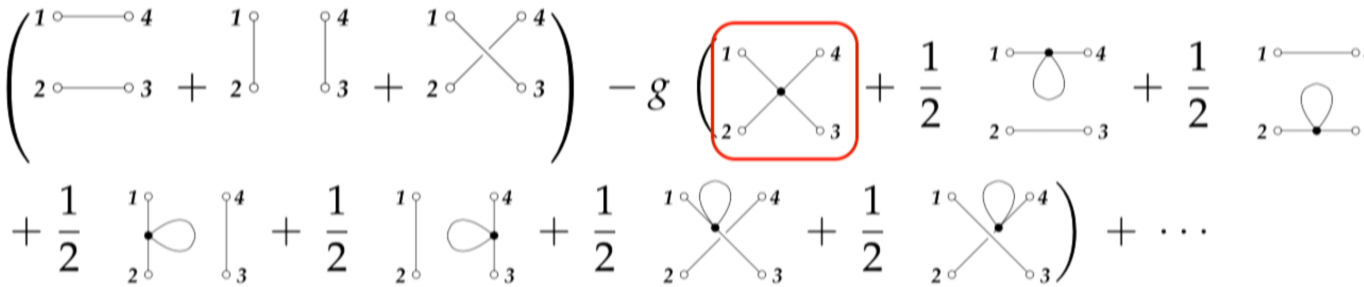
2 points function = connected 2 points function



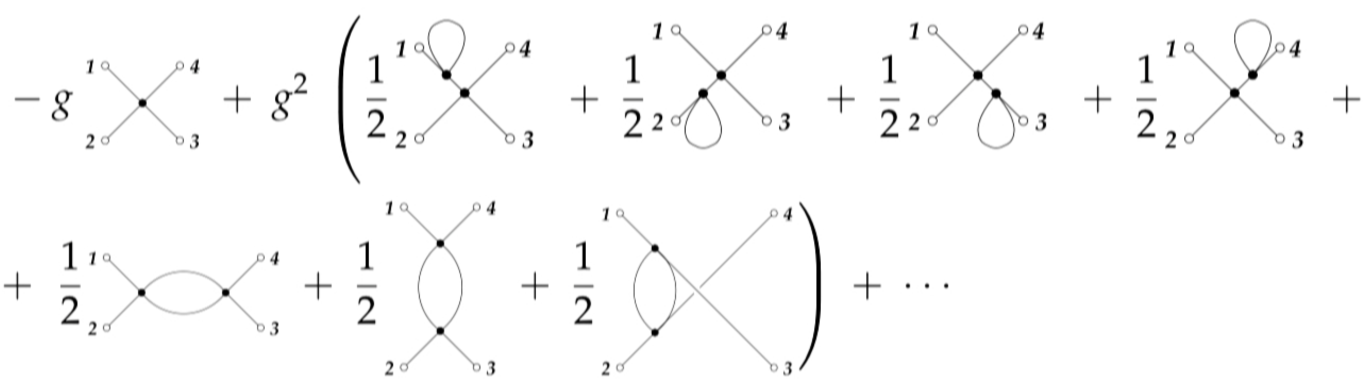
4 points function (up to order 1)



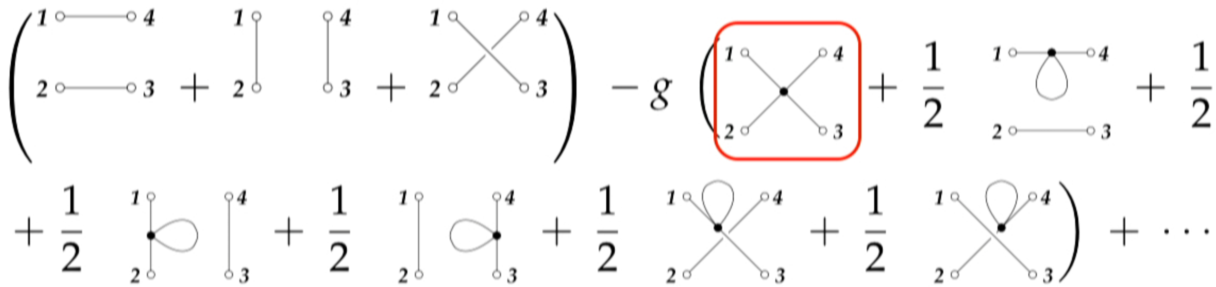
4 points function (up to order 1)



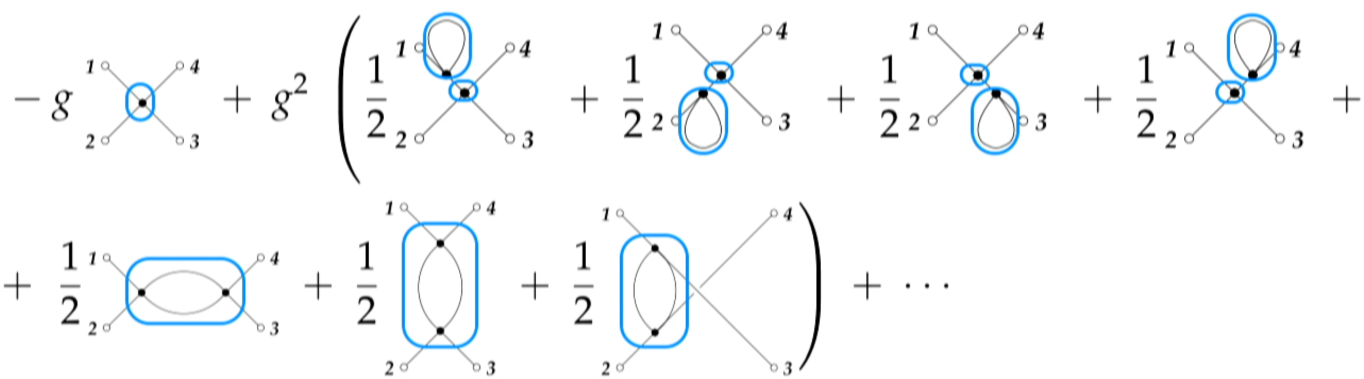
Connected 4 points function (up to order 2)



4 points function (up to order 1)

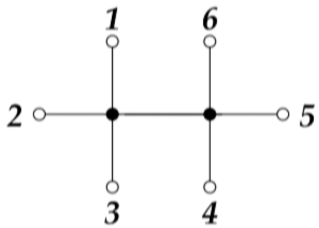


Connected 4 points function (up to order 2)



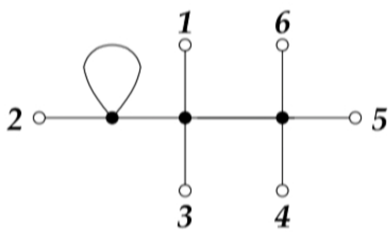
Connected 6 points function (up to order 3)

g^2



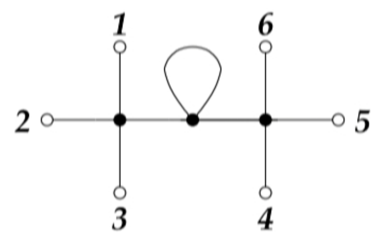
$+\cdots$

$-g^3\frac{1}{2}$



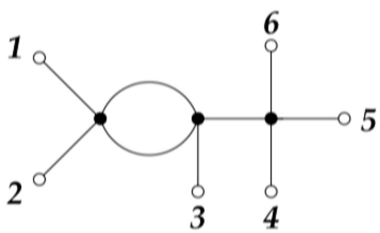
$+\cdots$

$-g^3\frac{1}{2}$



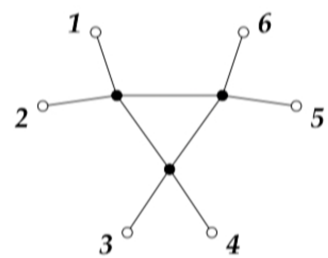
$+\cdots$

$-g^3\frac{1}{2}$



$+\cdots$

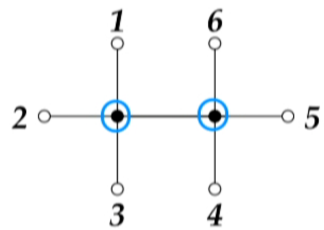
$-g^3$



$+\cdots$

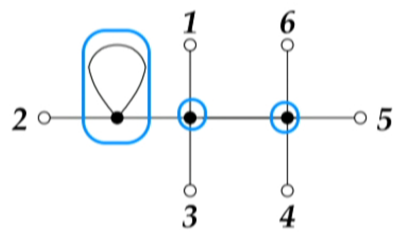
Connected 6 points function (up to order 3)

g^2



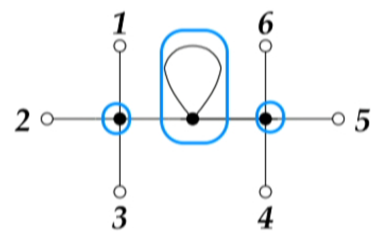
$+\cdots$

$-g^3\frac{1}{2}$



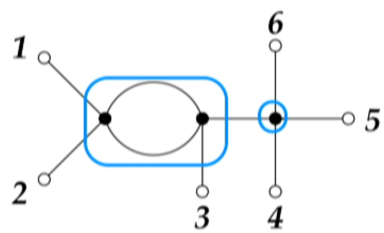
$+\cdots$

$-g^3\frac{1}{2}$



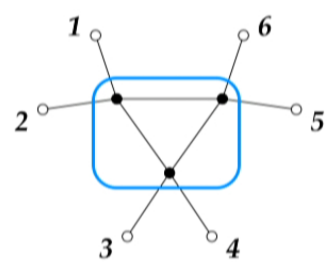
$+\cdots$

$-g^3\frac{1}{2}$



$+\cdots$

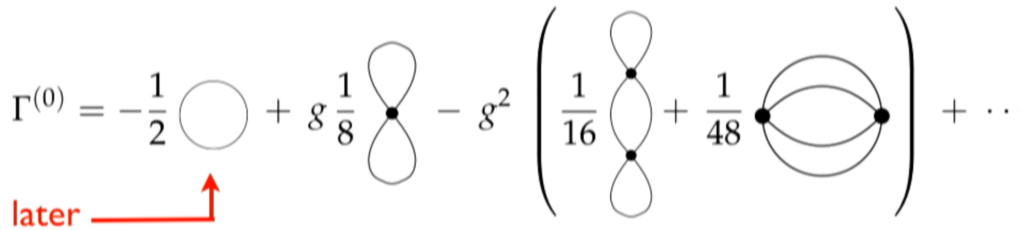
$-g^3$



$+\cdots$

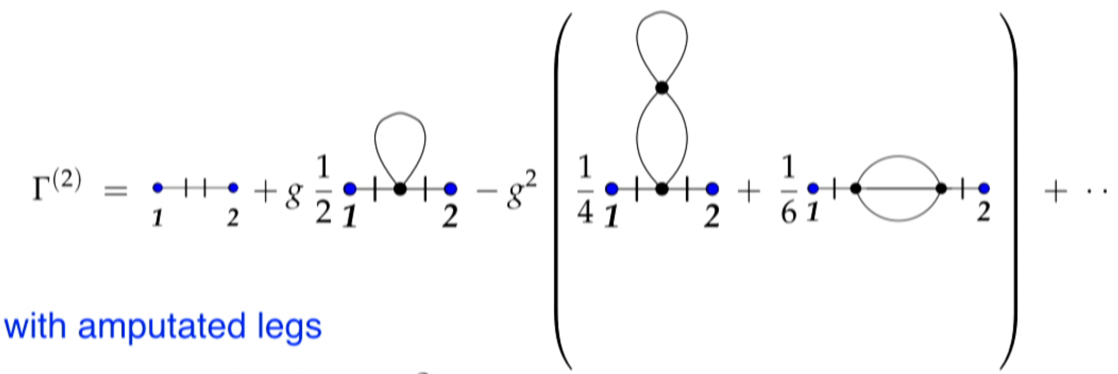
One Particle Irreducible (1PI) functions

Irreducible vacuum diagrams = vacuum self energy
(= connected vacuum diagrams for Φ^4)

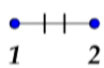


to be explained later

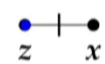
Irreducible 2 points function = inverse of the 2 points function



with amputated legs



$$= (-\Delta + m^2)_{z_1 z_2}$$
$$= p^2 + m^2$$



$$= \delta(z - x)$$
$$= 1$$

Correlation Functions / Connected / Irreducible

Feynman Diagram $G \longrightarrow$ Amplitude/Integral

N external vertices

K internal vertices

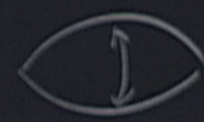
$$I_G(z_1 \dots z_N) = \int \prod_{i=1}^{K} d^D x_i \prod_{\text{lines}} \underset{\text{propagators}}{G_0}(\cdot\,,\cdot)$$

may be a z or a x

symmetry factor

$$C(G) = \frac{\#\text{ ways to construct } G \text{ out of Wick Theorem}}{K!\,(4!)^K} = \frac{1}{\#\text{ of elements of the sym. group of } G}$$

/ Irreducible

 $\to 2$

mplitude/Integral

N external vertices

K internal vertices

$x_i \prod_{\text{lines}} G_0(\cdot\,,\cdot)$

propagators

may be a Z or a X

G

$$\text{em} = \frac{1}{\#\text{ of elements of the sym. group of } G}$$

- Correlation Function N points

$$G_{(N)}(z_1 \dots z_N) := \sum_{K=0}^{\infty} (-g)^K \sum_{\substack{\text{diagrams } G \\ \text{with } N \text{ external vertices} \\ K \text{ internal vertices}}} C(G)\, I_G(z_1, \dots, z_N)$$

- Connected correlation function

$$G^{CONN}_{(N)}(z_1 \dots z_N) := \sum_{K} (-g)^K \sum_{G \text{ CONNECTED}} C(G)\, I_G(z_1 \dots z_N)$$

Correlations functions $\longleftarrow$ Connected Functions.

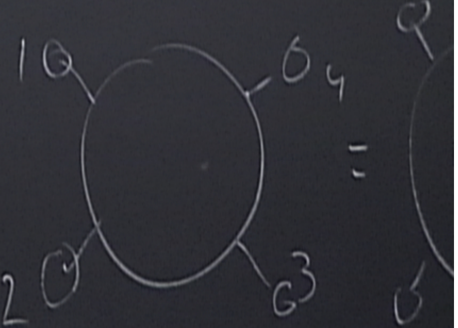
$$G_{(4)}(z_1, \dots z_4) = G^{CONN}_{(4)}(z_1..z_4) + G^{CONN}_{(2)}(z_1,z_2) \cdot G^{CONN}_{(2)}(z_3,z_4)$$

$$+ \;'' \;+\; ''$$

$$G_{(2)}(z_1,z_2) = G^{CONN}_{(2)}$$

$$G^{CONN}_{(4)}(z_1..z_4) = \sum_{\text{connected diagrams}} g^{\#\text{vertices}}$$

Integral$(z_1 .. z_4)$



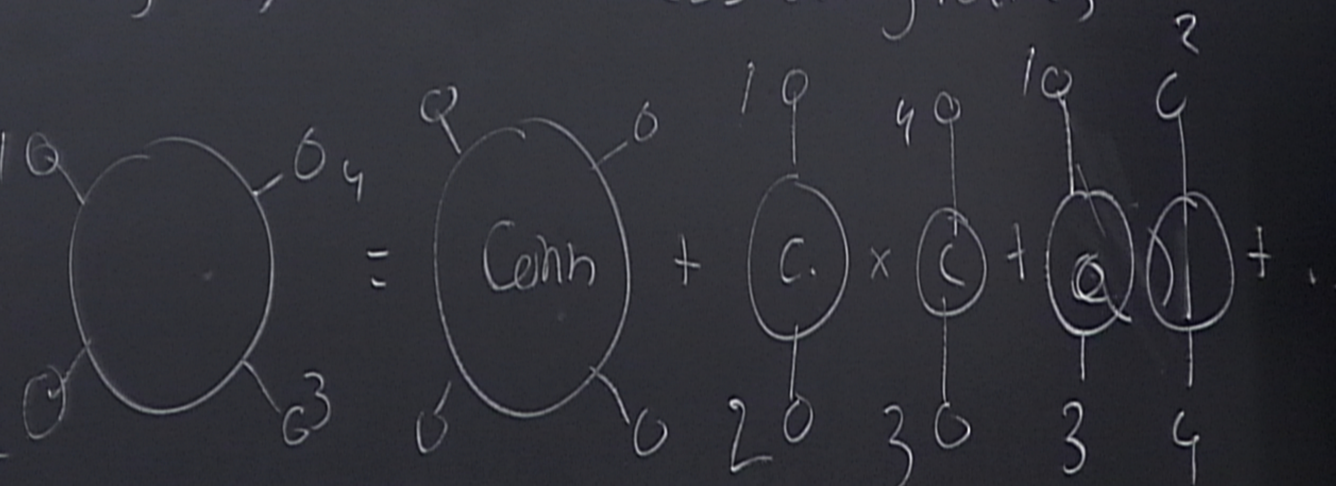
— Connected Functions ← Irreducible "Functions"

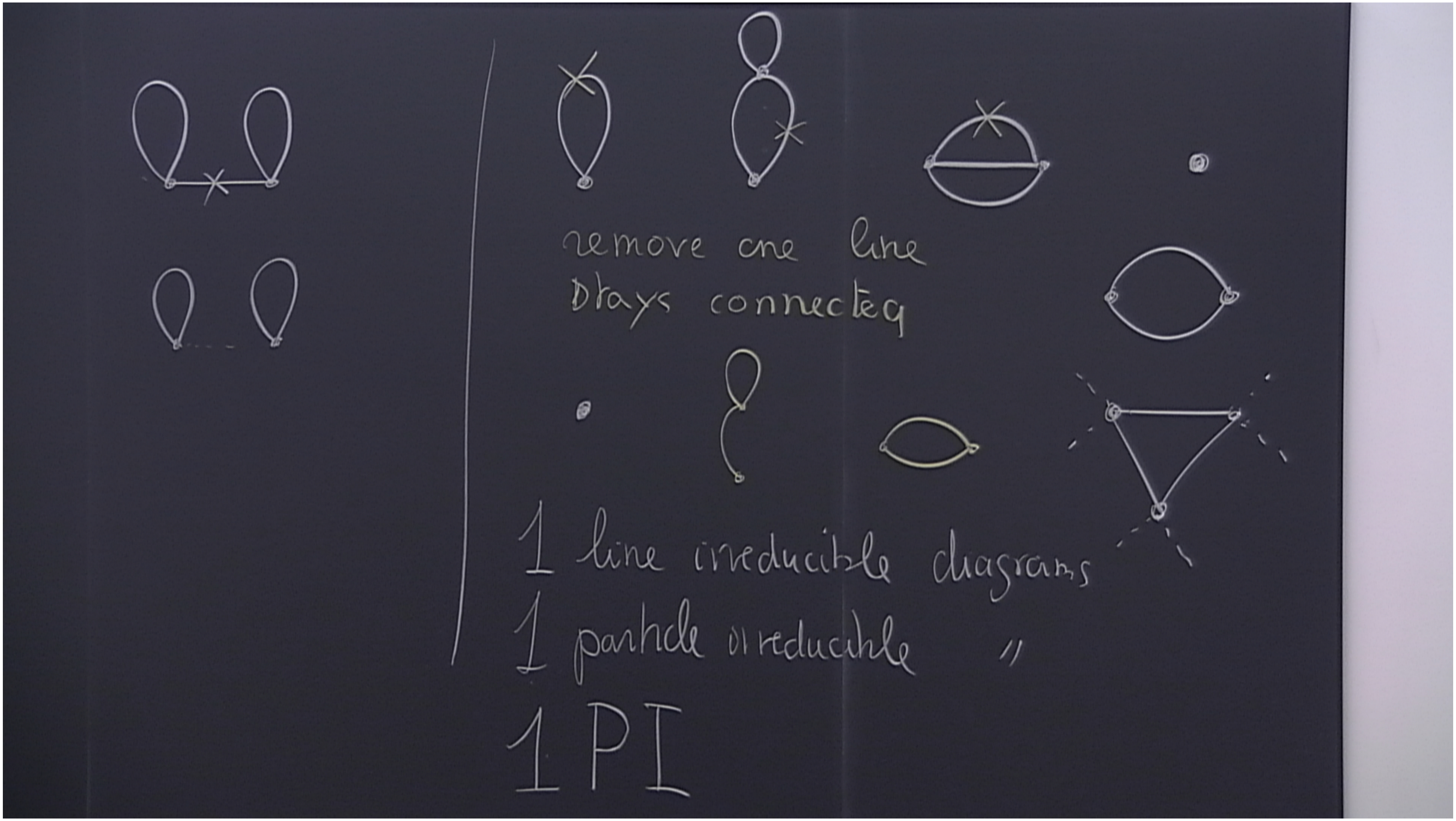
less diagrams

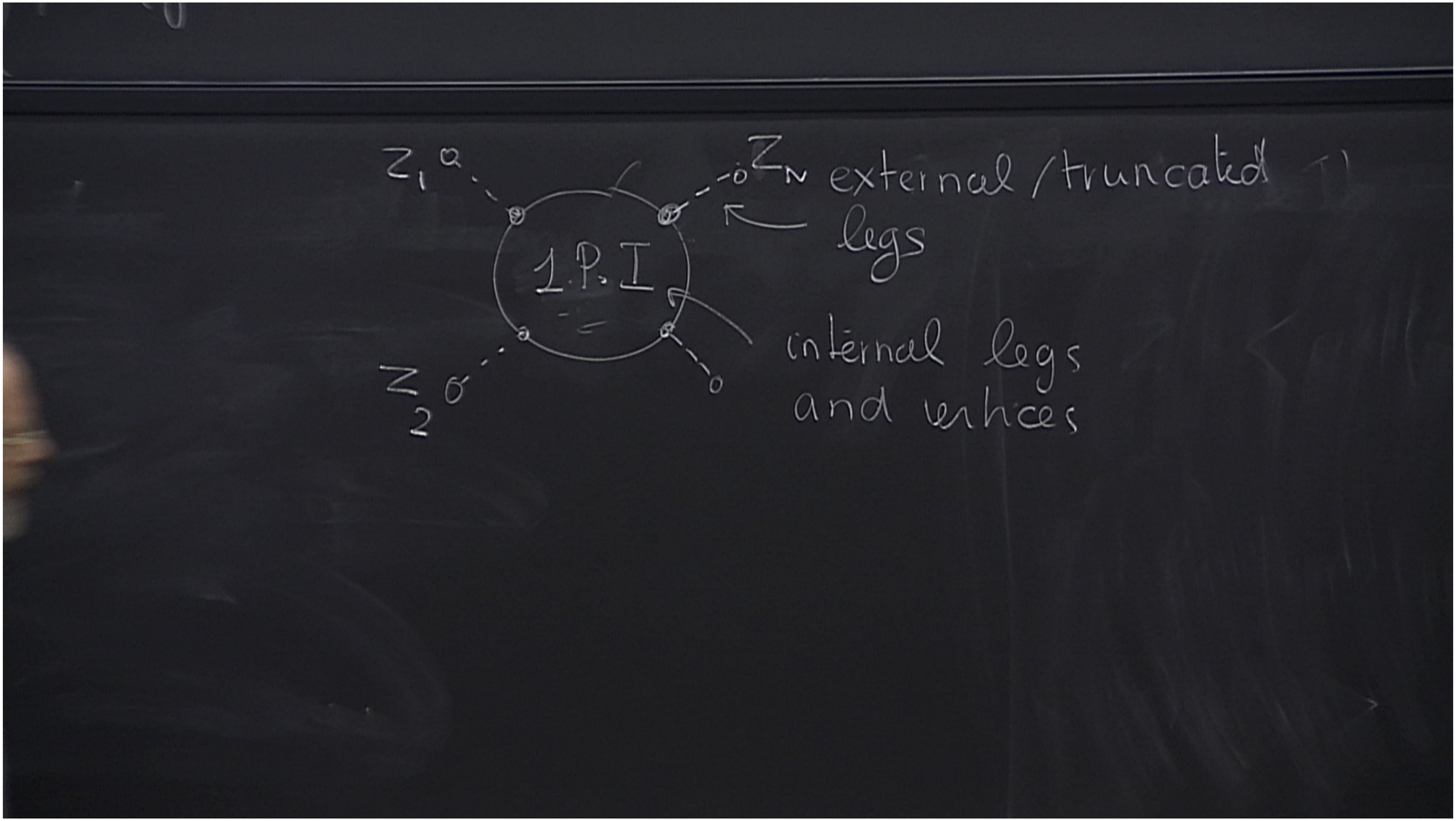
$$) + G^{\mathrm{CONN}}_{(2)}(z_1, z_2) \cdot G^{\mathrm{CONN}}_{(2)}(z_3, z_4)$$

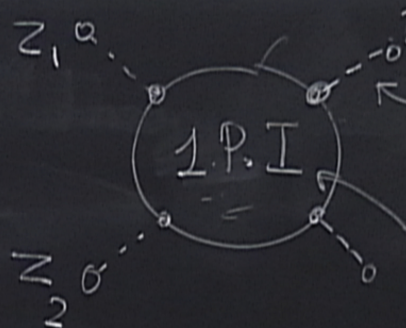
$$+ \;'' \;+\; ''$$

Integral$(z_1 \cdots z_4)$



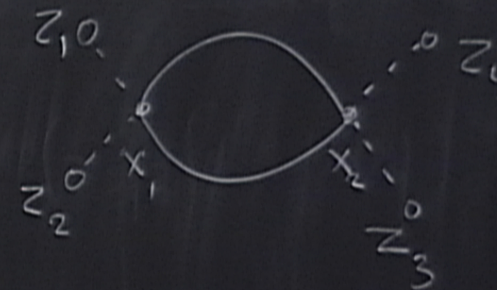

remove one line
stays connected
1 line irreducible diagrams
1 particle irreducible "
1PI


Z1
ZN
external / truncated
legs
1.P.I
internal legs
and vertices
Z2

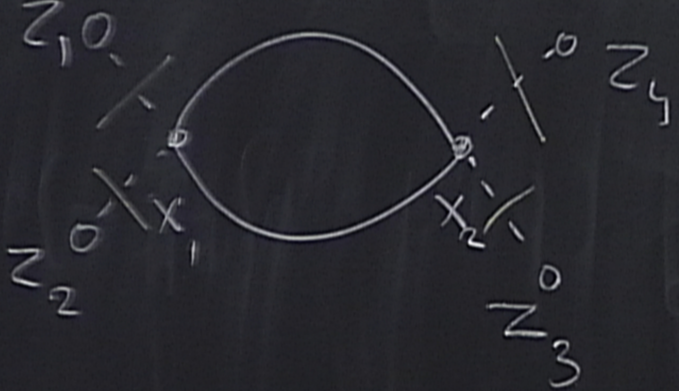


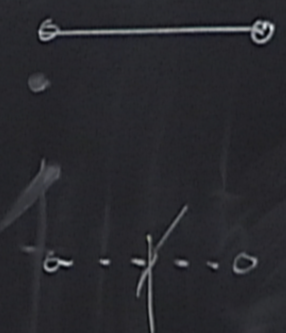
external / truncated legs

internal legs and vertices



$$G_0(x_\circ, x_\circ) \underbrace{\prod}_{\text{external/truncated lines}} \delta(z_\circ - x_\circ)$$





Fourier transform

$$\frac{1}{p^2+m^2}$$

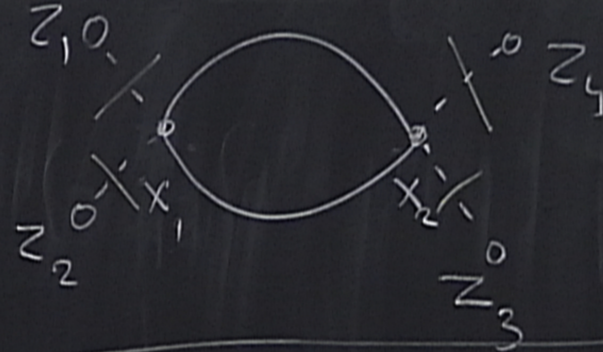
$$1$$

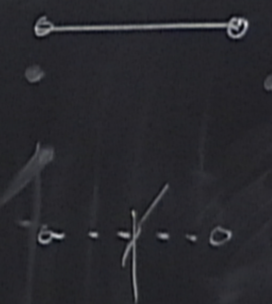
$$\int d^D x_1 \, d^D x_2 \; G_0(x_1-x_2)^2 \, \delta(z_1-x_1)\,\delta(z_2-x_1)\,\delta(z_3-x_2)\,\delta(z_4-x_2)$$

nal / truncated

l legs

rtices





Fourier tra

$$\frac{1}{p^2+m^2}$$

$$1$$

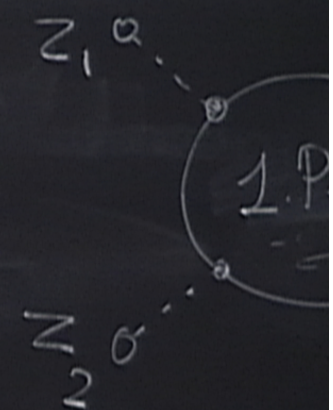
$$I^{IRR}(z_1,\dots,z_4) = \int d^D x_1\, d^D x_2\; G_0(x_1-x_2)^2\, \delta(z_1-x_1)\,\delta(z_2-x_1)\,\delta(z_3-x_2)\,\delta(z_4$$

$$= \delta(z_1-z_2)\,\delta(z_3-z_4)\,G_0(z_1-z_3)^2$$

Position Space

Irreducible Function

1 P. I graph G



$$I^{IRR}_G(z_1 \cdots z_N) = \int \prod_{\substack{\text{internal}\\ \text{vertices}\\ i=1,\cdots K}} dx_i \prod_{\substack{\text{internal}\\ \text{lines}}} G_0(x_\cdot, x_\cdot) \prod_{\substack{\text{external/truncated}\\ \text{lines}}} \delta(z_\cdot - x_\cdot)$$

N-point irreducible Function

$$\Gamma_{(N)}(z_1 \cdots z_N) = \sum_K (-1)^{K-1} g^K \sum_{\text{1PI Graphs N ext. vertices K int. vertices}} C(G)\, I^{IRR}_G(z_1 \cdots z_N)$$

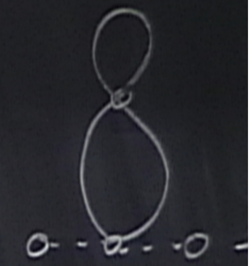
Irreducible 2 point function

$$\Gamma_{(2)}(z_1, z_2) = g\,\frac{1}{2}$$

$$\downarrow$$

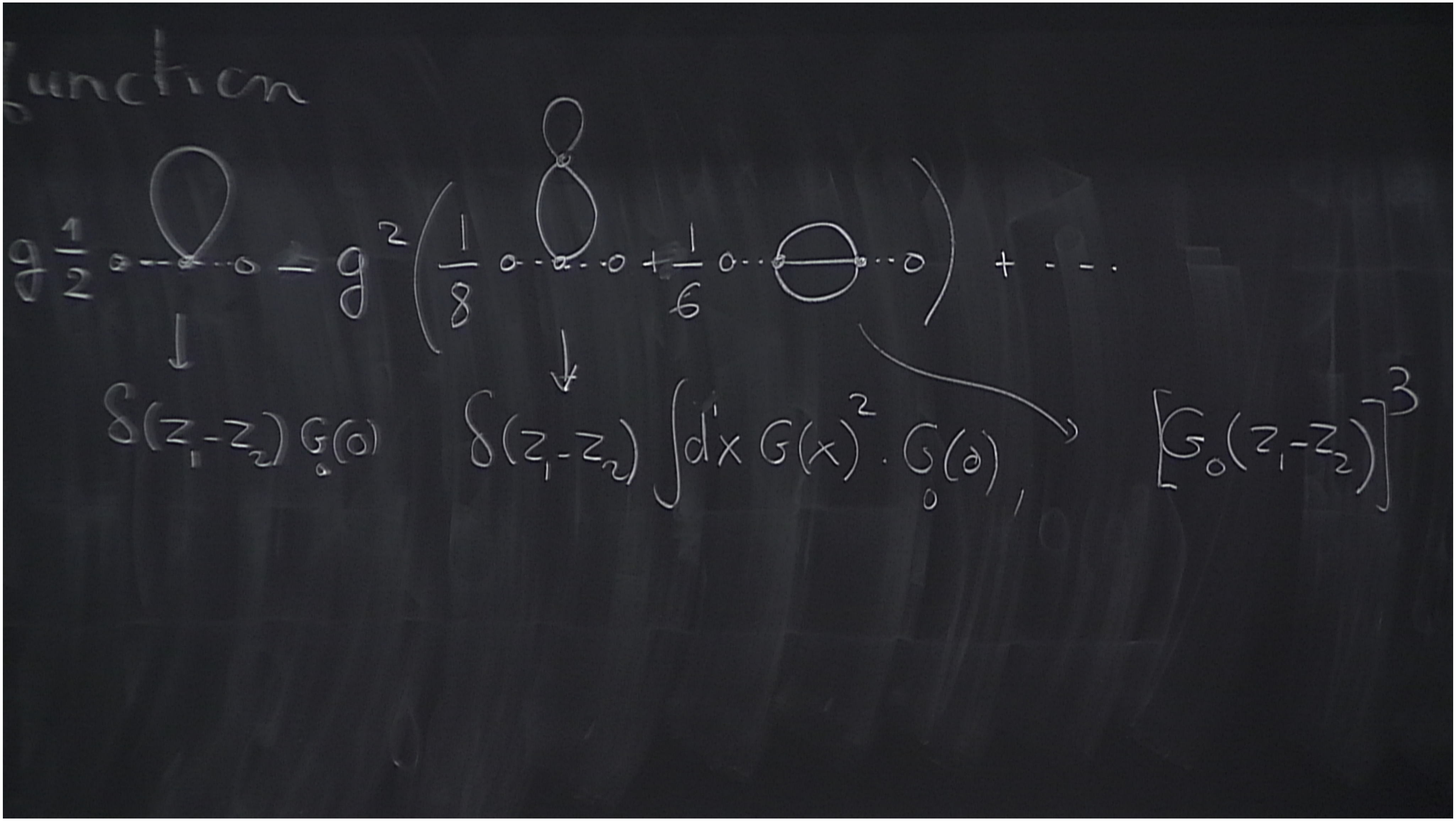
$$\delta(z_1 - z_2)\, G_0(0)$$

$$- g^2\left(\frac{1}{8}\right.$$



$$\downarrow$$

$$\delta(z_1 - z_2)$$



unction
g 1/2 — g² (1/8 + 1/6) + . . .
δ(z₁-z₂) G₀(0)
δ(z₁-z₂) ∫dᵈx G(x)² · G₀(0)
[G₀(z₁-z₂)]³

Irreducible 2 point function

$$\Gamma_{(2)}(z_1, z_2) = \quad g\,\frac{1}{2} \quad - g^2\left(\frac{1}{8} \quad + \frac{1}{6} \right.$$

twice truncated propagator

$\delta(z_1 - z_2)\, G_0(0)$

$\delta(z_1 - z_2) \int d^d x \; G($

IRREDUCIBLE

CONN

$$\underset{z_1}{\circ}\!\!-\!\!-\!\!-\!\!\underset{z_2}{\circ} \longrightarrow \underset{z_1}{\circ}\!/\cdots/\!\underset{z_2}{\circ}$$

position space

$$\underset{1}{\circ}\!-\!|\!-\!\underset{2}{\circ} = (-\Delta+m^2)_{z_1} G_0(z_1-z_2) = \delta(z_1-z_2) \longrightarrow 1$$

$$\underset{1}{\circ}\!-\!|\!|\!-\!\underset{2}{\circ} = (-\Delta+m^2)^2_{z_1} G_0(z_1-z_2) = (-\Delta+m^2)_{z_1}\delta(z_1-z_2)$$

distribution

$$\hookrightarrow p^2+m^2$$

$\circ - - - \circ$

$\to p^2$

K 1PI Graphs N ext. vertices K int vertices

Irreducible 2 point function

$$\Gamma_{(2)}(z_1, z_2) = \; + g\,\frac{1}{2} \; - g^2\left(\frac{1}{8} \; + \frac{1}{6} \;\right) + \cdots$$

twice truncated propagator

$$\delta(z_1 - z_2)\, G_0(0) \qquad \delta(z_1 - z_2) \int d^dx\, G(x)^2 \cdot G_0(0),$$

F.T.

$$\hat{\Gamma}_{(2)}(p) = p^2 + m^2 + g\,\frac{1}{2}\, G_0(0) - g^2\left[\frac{1}{8} \; + \frac{1}{6} \;\right.$$

$\overrightarrow{p}$

Functn of p non triv

...tices k int vertices

$+\frac{1}{6}$ $+ \ldots$

$\ldots z_1 - z_2) \int d^d x \, G(x)^2 \cdot G_0(0)$, $\left[G_0(z_1-z_2)\right]^3$

$+\frac{1}{6}$ p

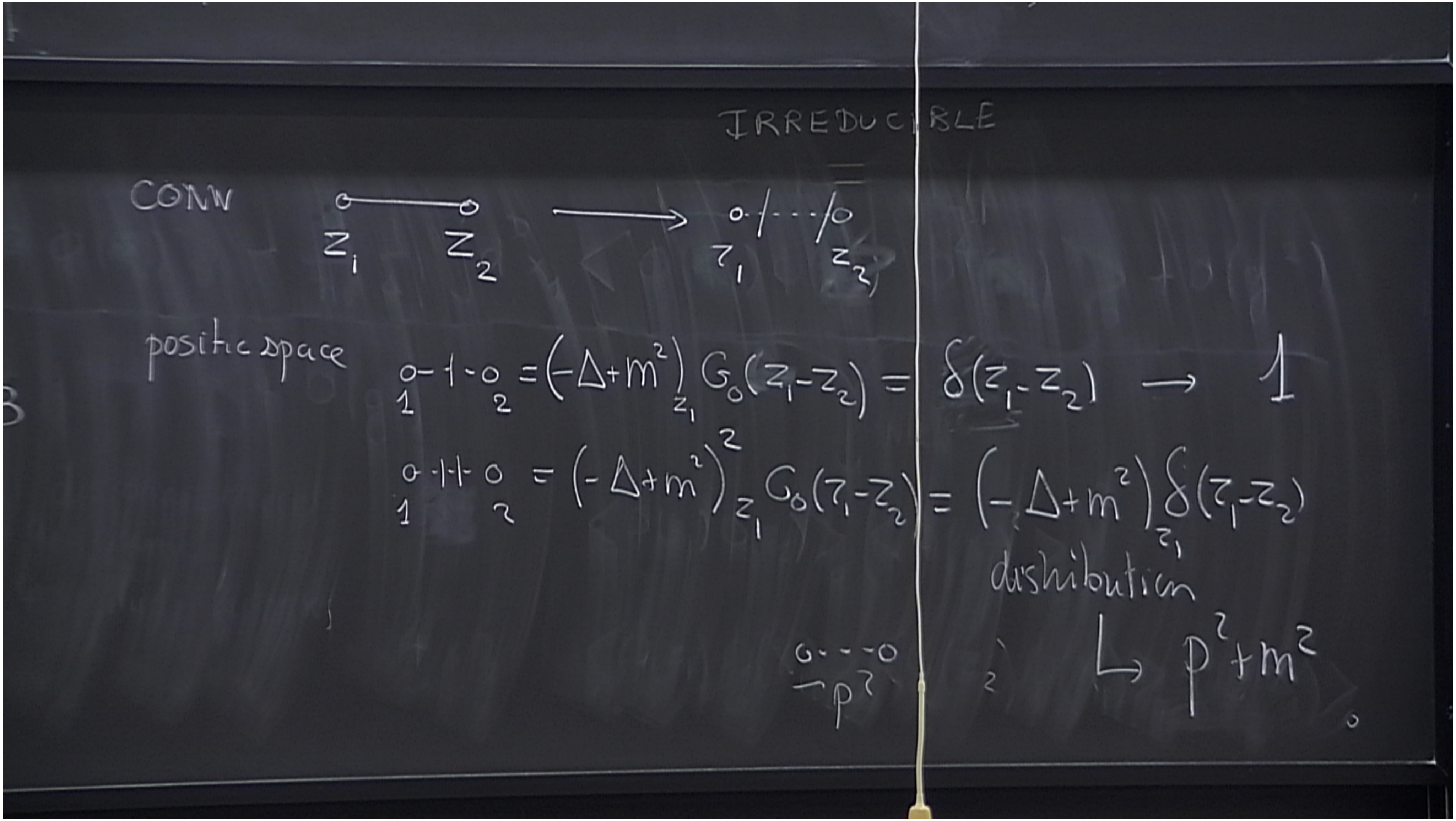
Functio of p non trivial

CONN z_1 z_2 $\longrightarrow$ z_1

positio space

$o\!-\!|\!-\!o_{\,1\;2} = (-\Delta+m^2)_{z_1} G_0($

$o\!-\!||\!-\!o_{\,1\;2} = (-\Delta+m^2)^2_{z_1}$


IRREDUCIBLE
CONN
z1 z2
z1 z2
position space
= (-Δ+m²)z1 G0(z1-z2) = δ(z1-z2) → 1
1 2
= (-Δ+m²)²z1 G0(z1-z2) = (-Δ+m²)z1 δ(z1-z2)
1 2
distribution
↳ p²+m²
→ p²

Generating Function

$\boxed{\text{Funct. Int.}}$ $Z[j] = \int D[\phi] \exp\left(- \frac{1}{\hbar} \right.$

$Z[j] \leftarrow$ Correlation Functions

Functional of source term j

expand in j

$$\boxed{\text{Diagrammatic}} \quad \frac{Z[j]}{Z[0]} = \sum_{N=0}^{\infty} \frac{\hbar^{-N}}{N!} \int \prod_{a=1}^{N} dz_a \, j(z_a) \, G_{(N)}(z_1, \ldots z_N)$$

$\ldots] - j\cdot\phi)$

$j(x)$ source term

$$j\cdot\phi = \int d^Dx\, j(x)\cdot\phi(x)$$

…enerating Function for the connected correlations

$$\ldots[j] = \sum_{N=0}^{\infty} \frac{\hbar^{1-N}}{N!}\int \prod_{a=1}^{N} dz_a\, j(z_a)\, G^{\mathrm{CONN}}_{(N)}(z_1,\ldots z_N)$$

Diagrammatic Definition

$$\frac{1}{\hbar}\left(S[\phi] - j\cdot\phi\right)$$

$j(x)$ source term

$$j\cdot\phi = \int d^D x \, j(x)\,\phi(x)$$

Generating Function for the connected correlations

$$W[j] = \sum_{N=0}^{\infty} \frac{\hbar^{1-N}}{N!} \int \prod_{a=1}^{N} dz_a \, j(z_a) \, G^{\mathrm{CONN}}_{(N)}(z_1 \ldots z_N)$$

Diagrammatic Definition

$$\boxed{W[j] = \hbar \cdot \mathrm{Log}(Z[j])}$$

Funct. Int def.

$$j \cdot \phi = \int d^D x \; j(x) \cdot \phi(x)$$

n the connected correlations

1-N

$$\frac{\quad}{!} \int \prod_{a=1}^{N} dZ_a \; j(Z_a) \; G^{\mathrm{CONN}}_{(N)}(Z_1 \ldots Z_N)$$

Diagrammatic Definition

$$\boxed{g\left(Z[j]\right)}$$

Tomorrow

Irreducible $\Rightarrow$ UV divergens

Funct. Int def.