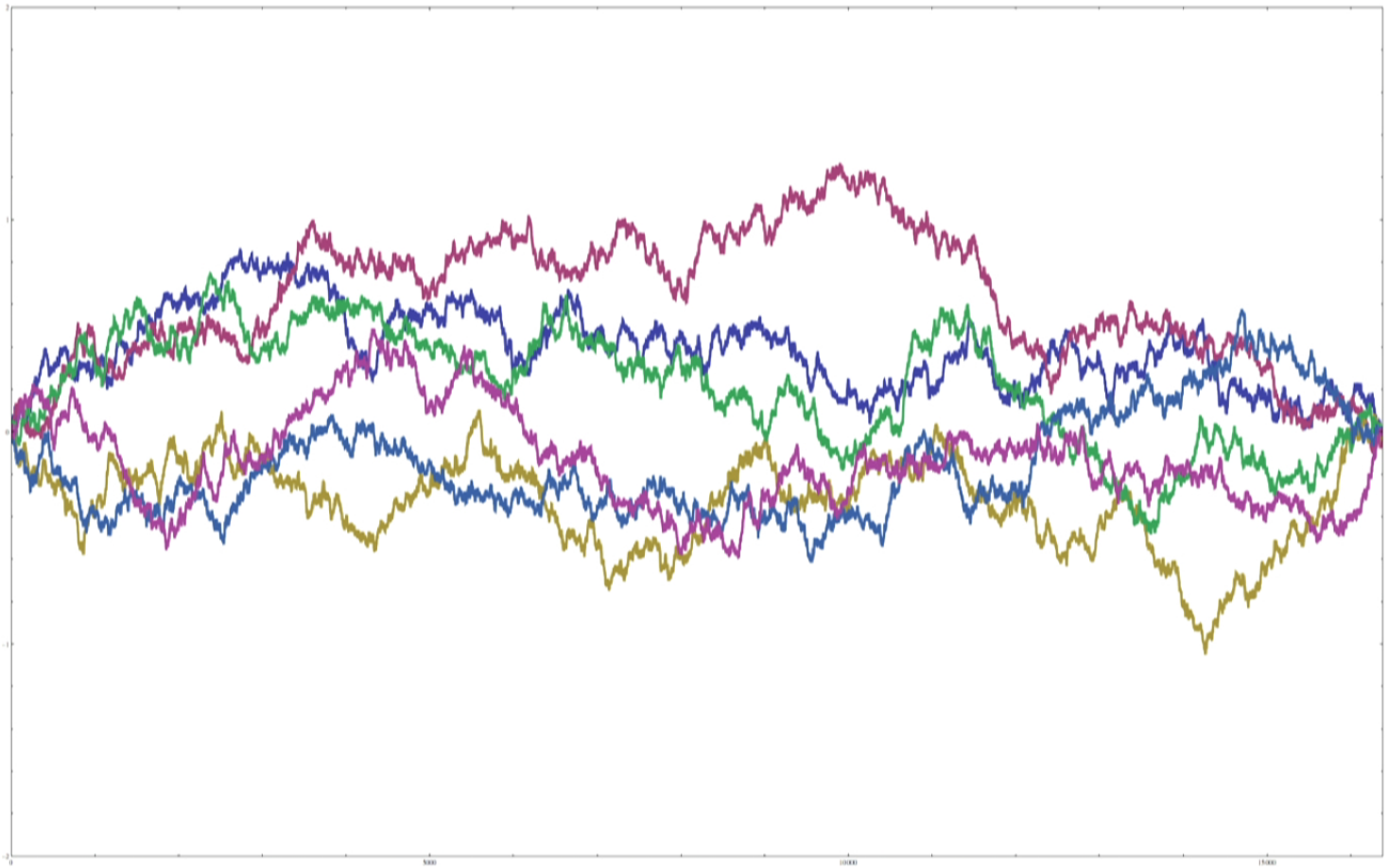
Title: PSI 2016/2017 Quantum Field Theory II - Lecture 5

Date: Nov 11, 2016 09:00 AM

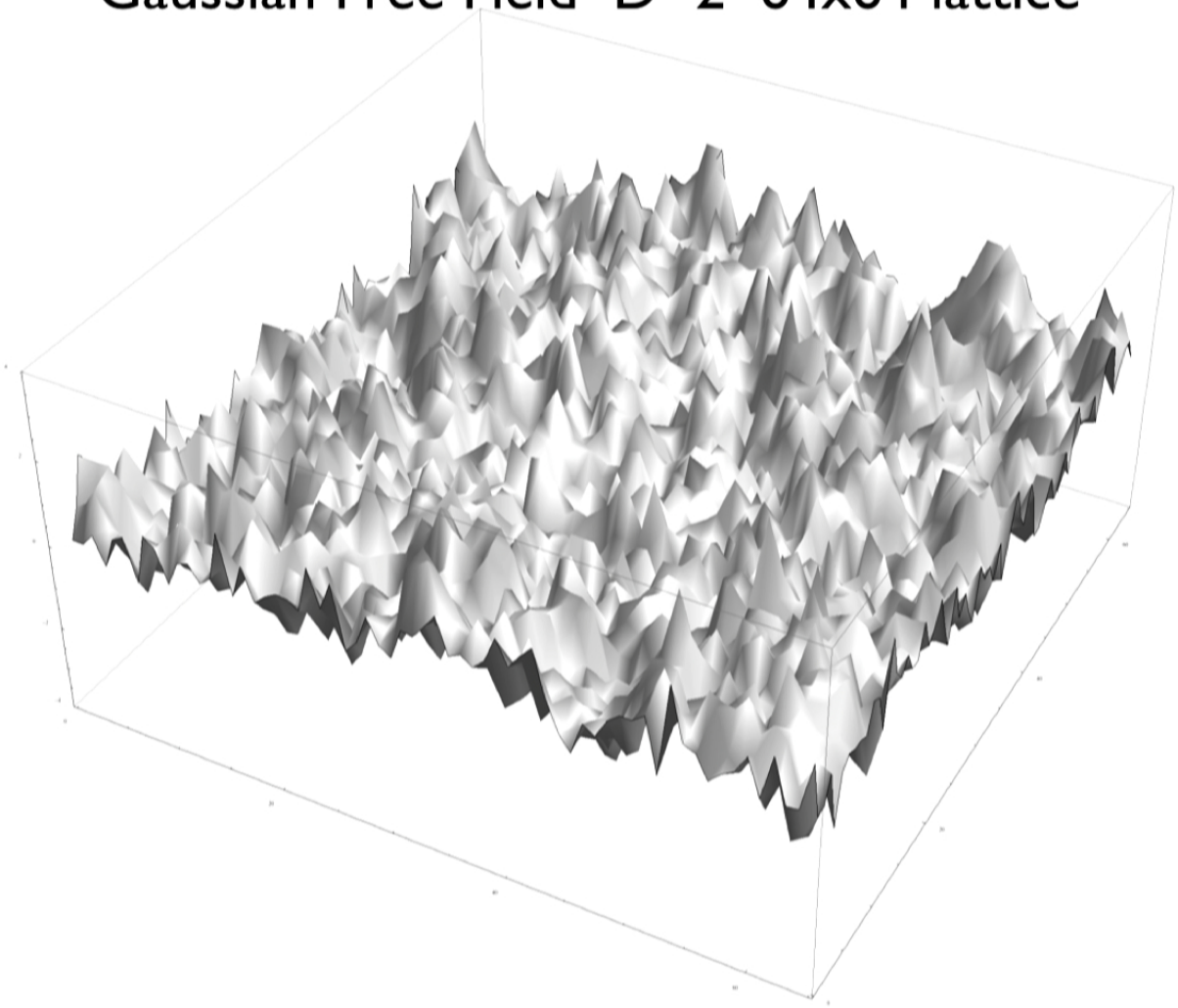
URL: http://pirsa.org/16110005

Abstract:

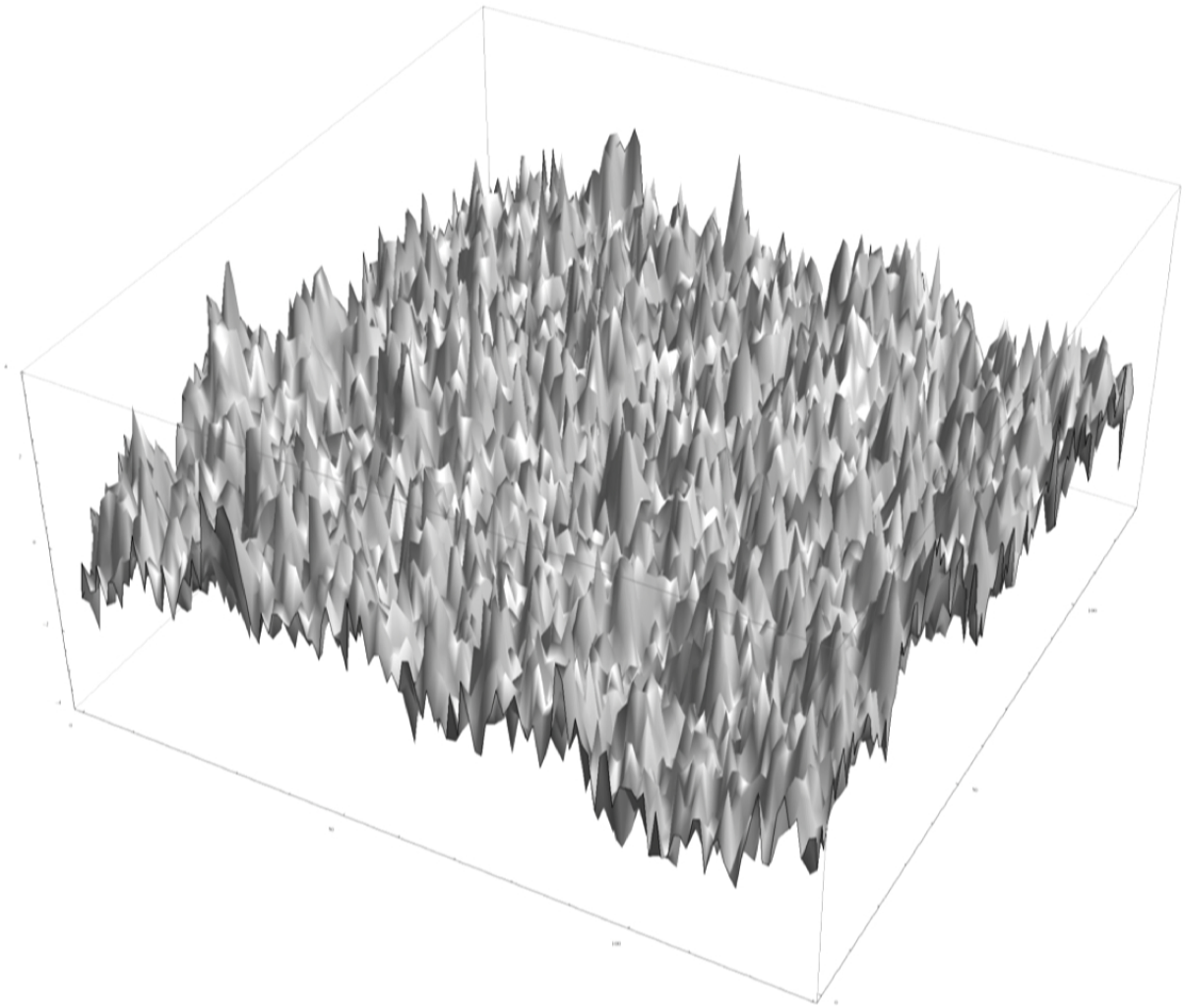
D=1 : Gaussian Free Fields = Random Walk
(i.e. Brownian or Wiener Process)



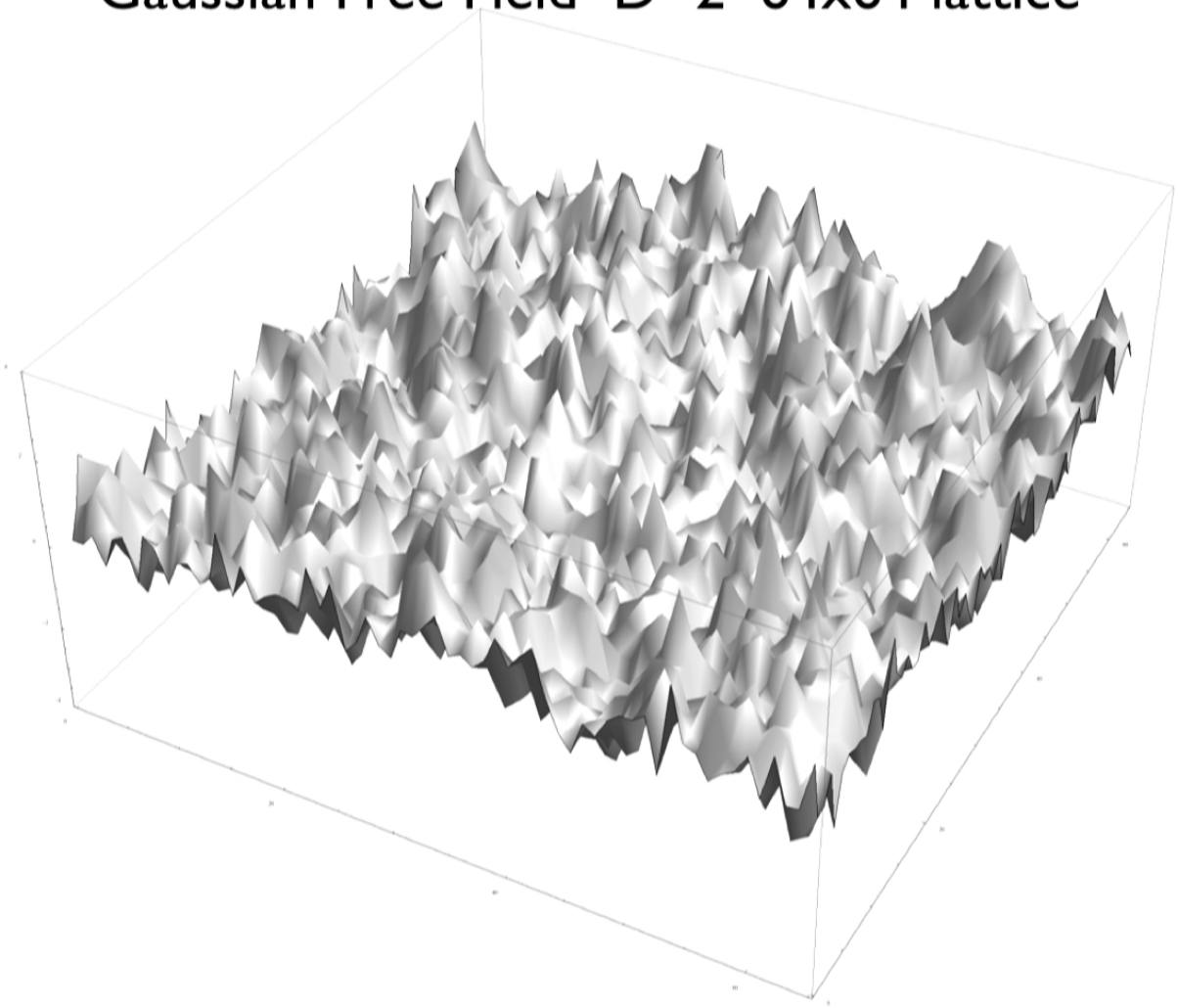
Gaussian Free Field D=2 64x64 lattice



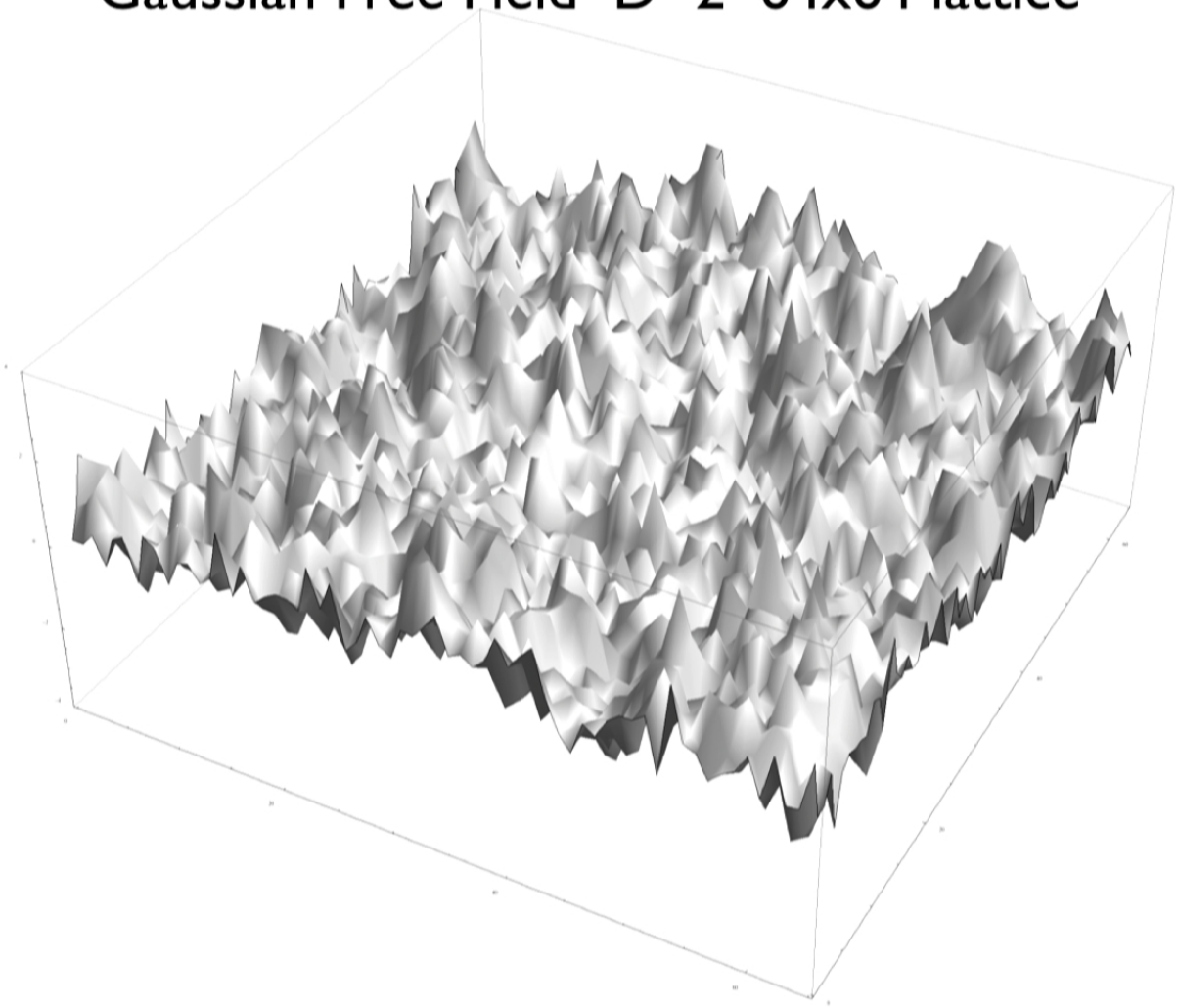
Gaussian Free Field D=2 128x128 lattice



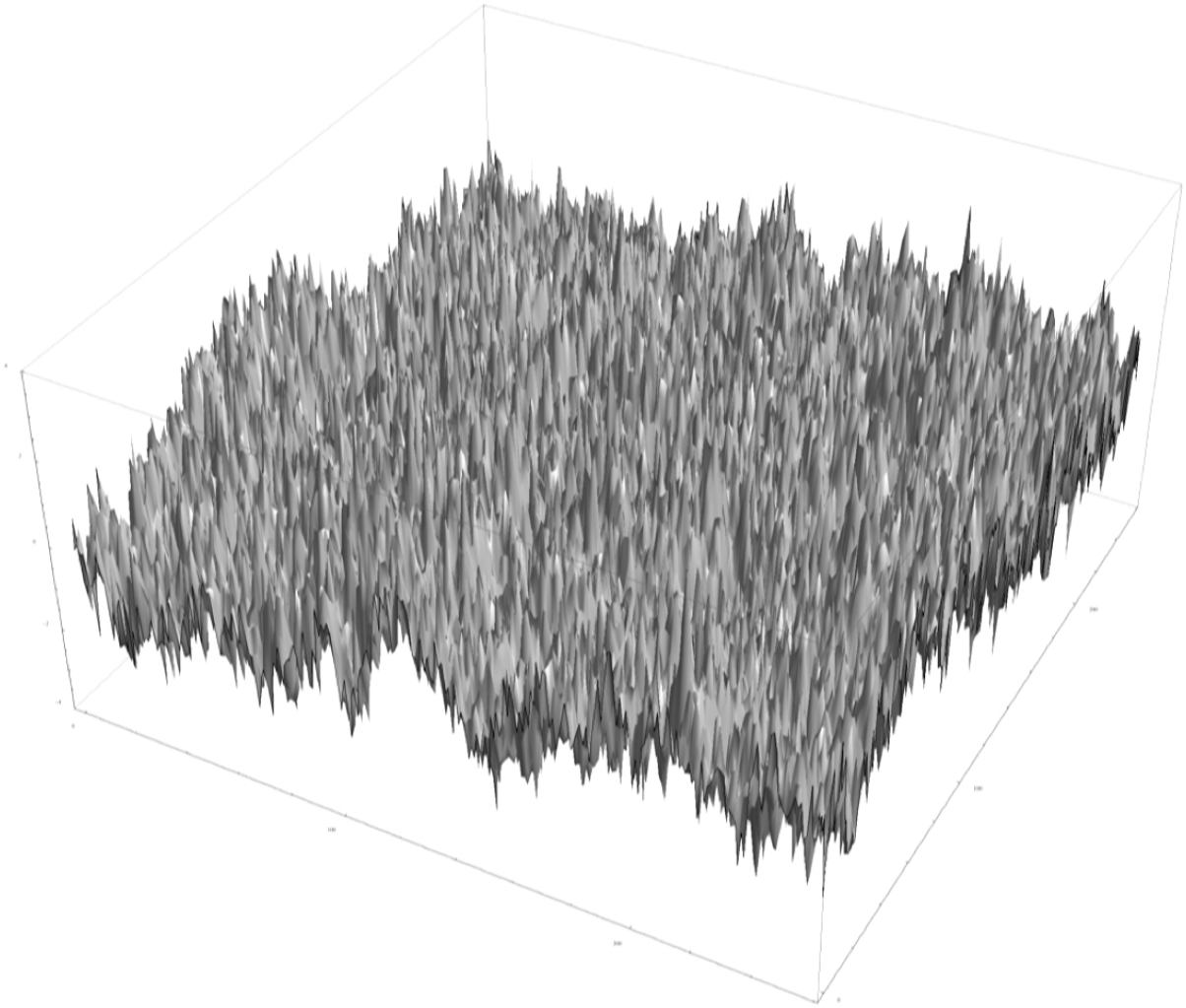
Gaussian Free Field D=2 64x64 lattice



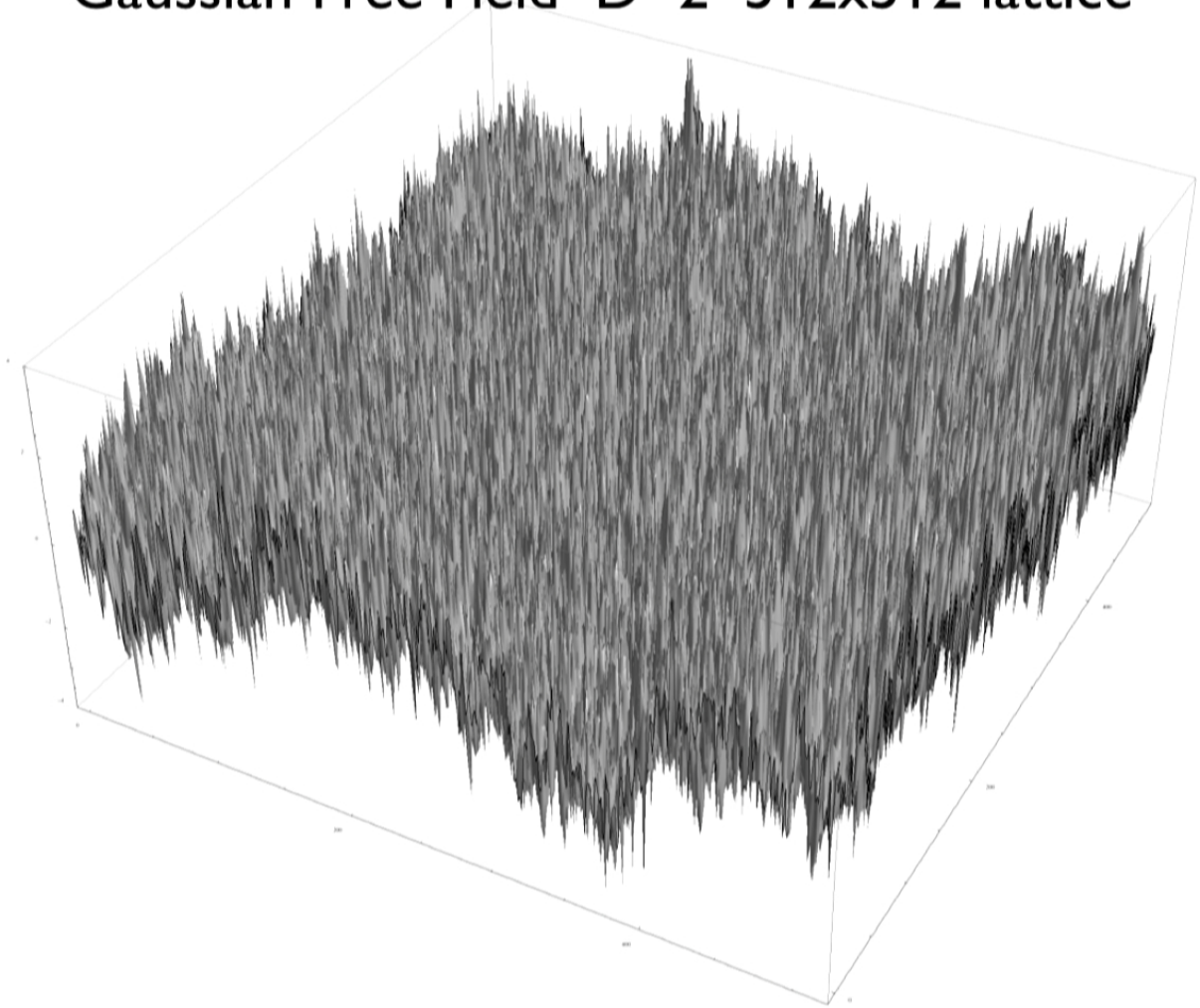
Gaussian Free Field D=2 64x64 lattice



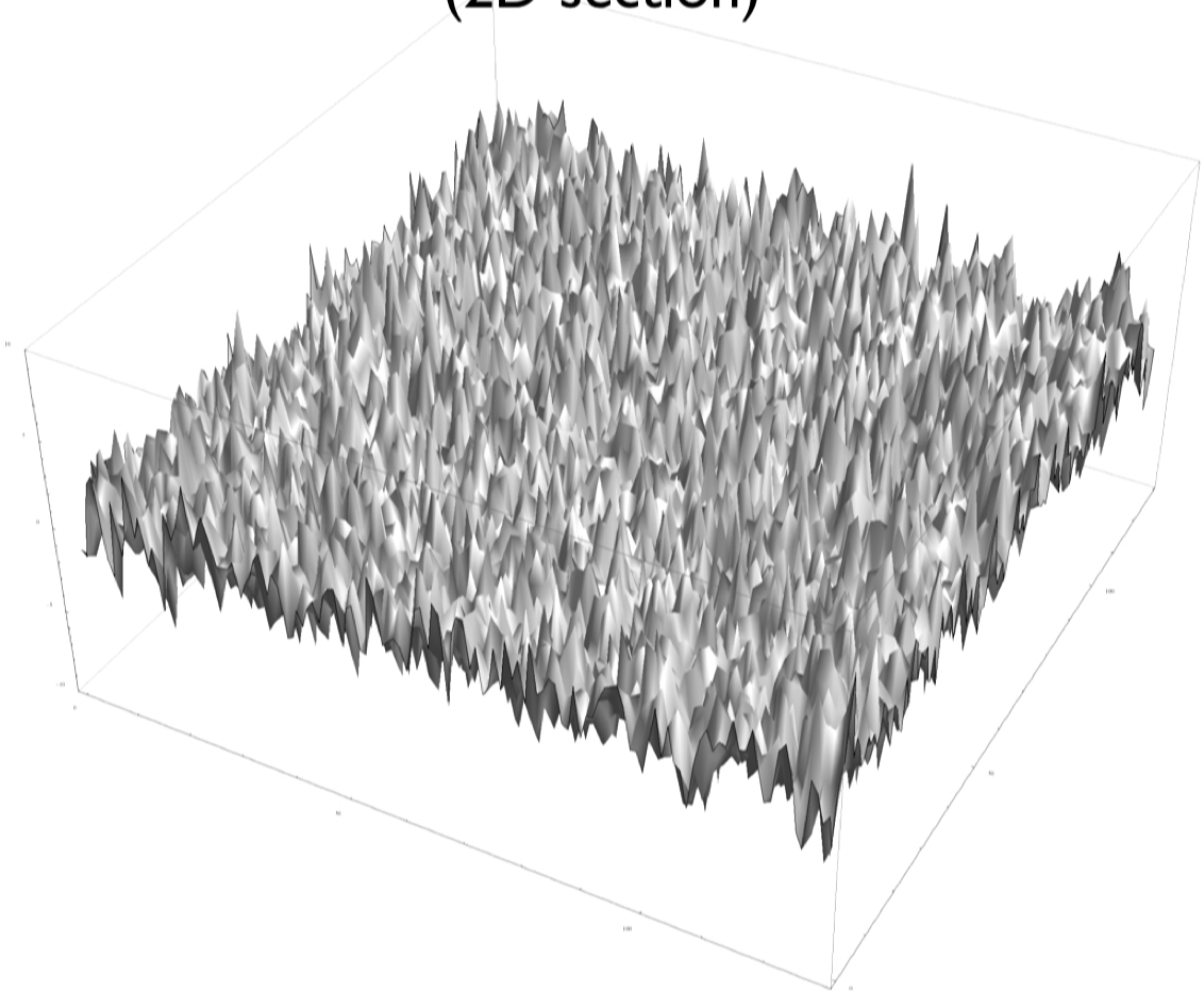
Gaussian Free Field D=2 256x256 lattice



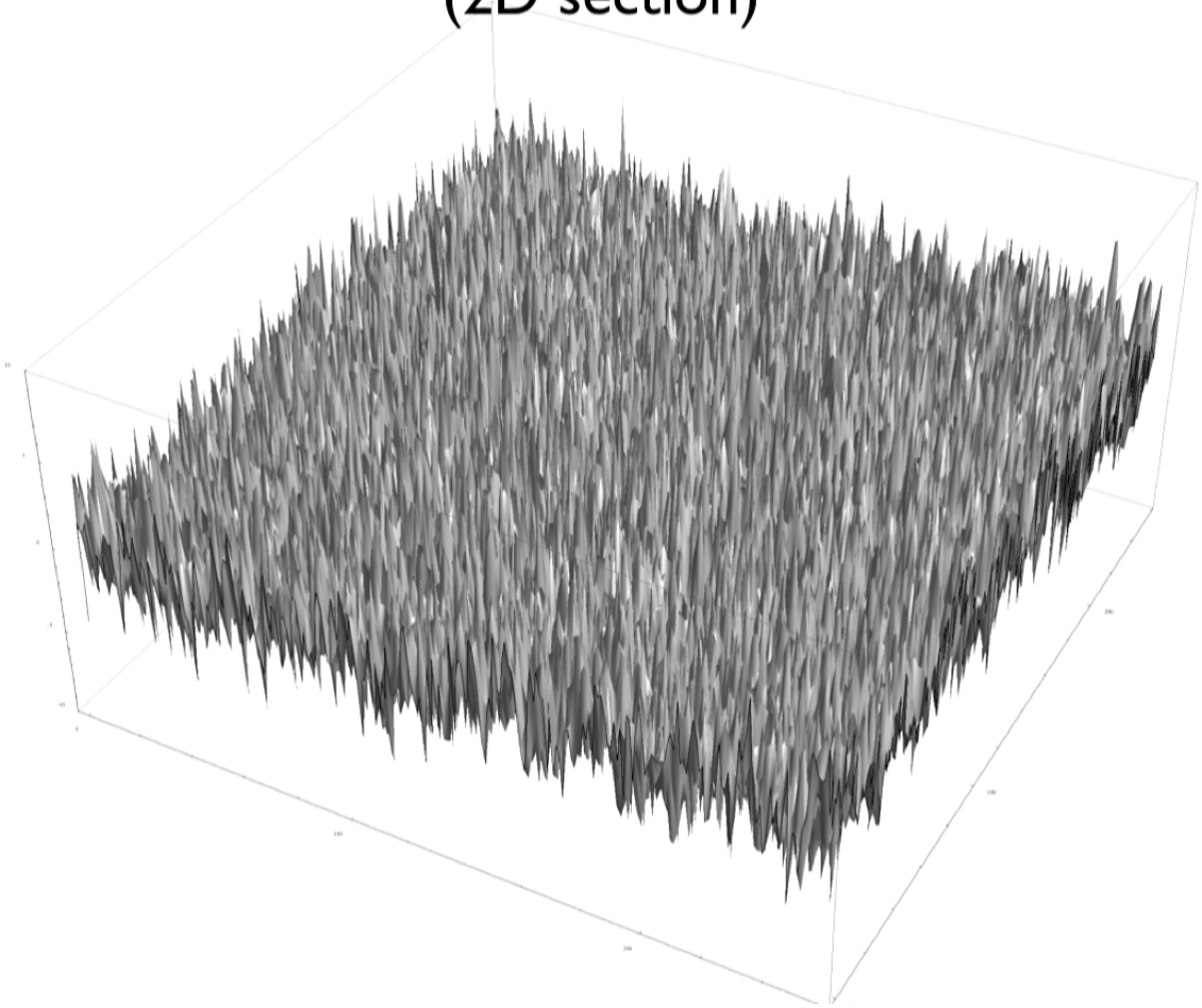
Gaussian Free Field D=2 512x512 lattice



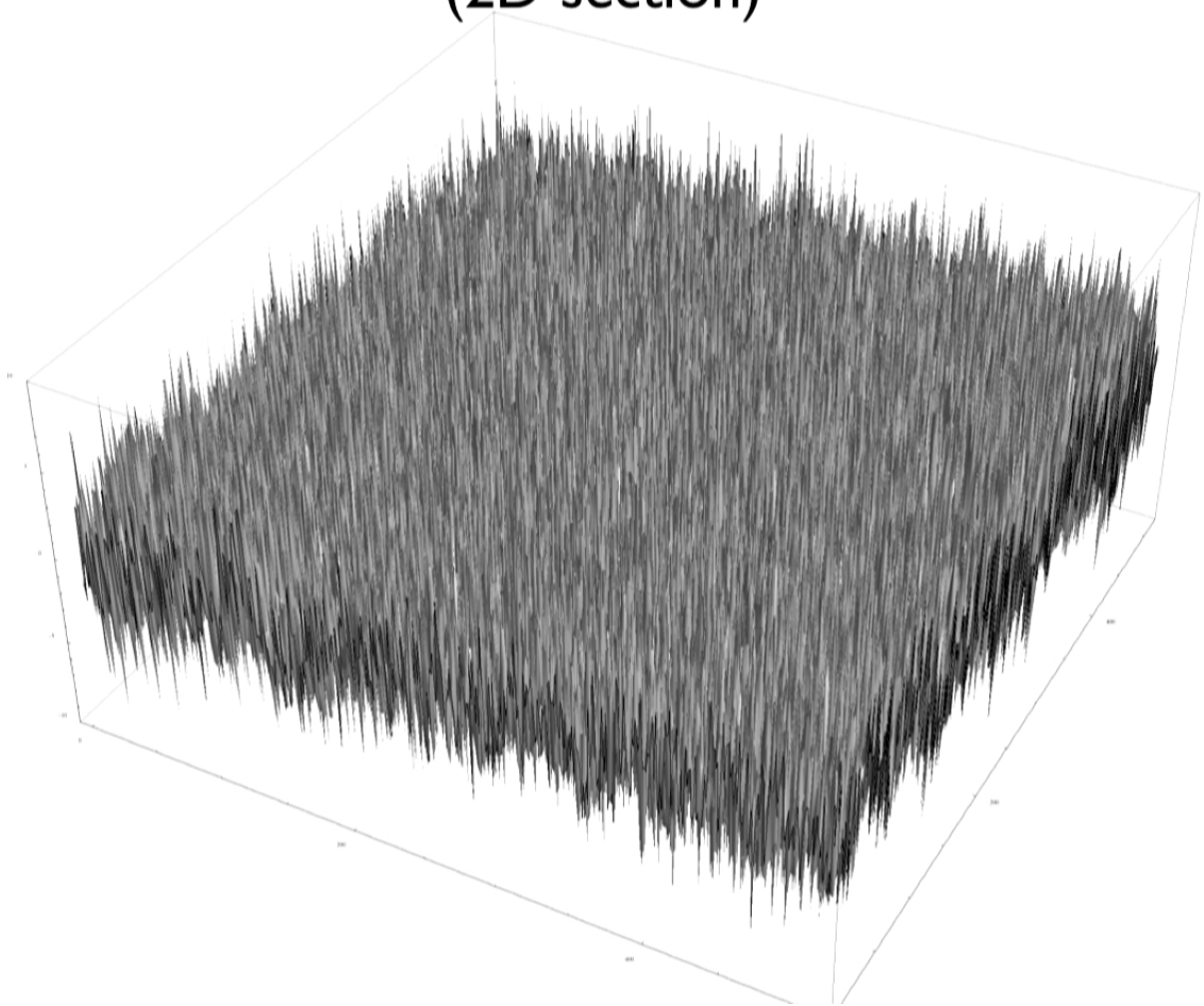
Gaussian Free Field D=3 128x128x128 lattice (2D section)



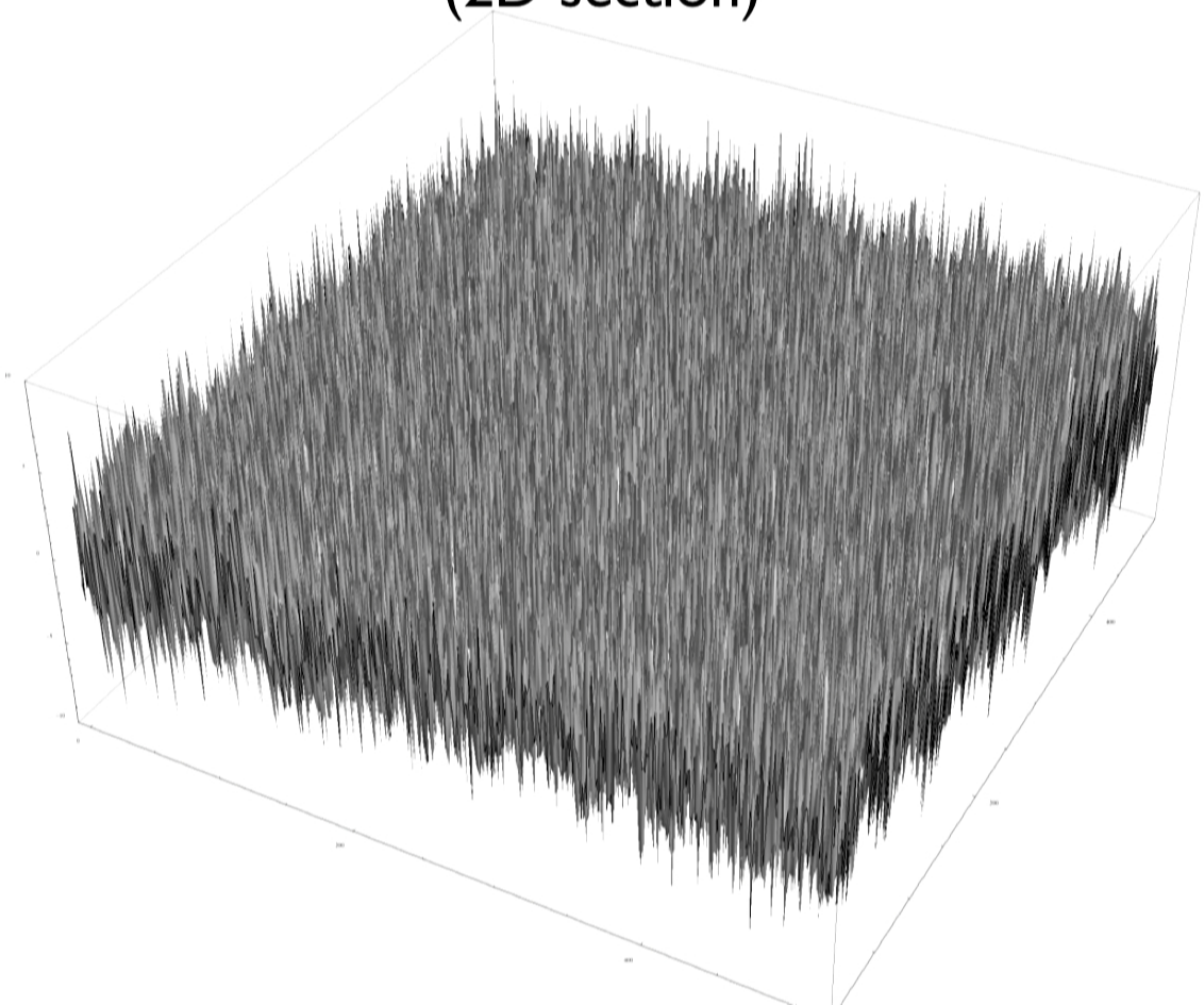
Gaussian Free Field D=3 256x256x256 lattice (2D section)



Gaussian Free Field D=3 512x512x512 lattice (2D section)



Gaussian Free Field D=3 512x512x512 lattice (2D section)



Scalar Free Field $\phi(x)$

$x_E \in \mathbb{R}^D$ = Euclid

- Euclidean Theory $\longrightarrow$ Real time QFT

$$\int \mathcal{D}_E[\phi] \exp\left(-\frac{1}{\hbar} S_E[\phi]\right) \qquad S_E[\phi] = \int d^D x_E \, \frac{1}{2}(\partial_\mu \phi)^2 +$$

- 2 points function: $\langle \phi(x_1)\phi(x_2)\rangle = G(x_1 - x_2) \longrightarrow G_{\text{Feynman}}(x$

"propagator"

- $G(x) \simeq |x|^{2-D}$ $\quad x \gg 0$ singular at short distance $\Longrightarrow$ UV divergen

$\phi(x)$

$x_E \in \mathbb{R}^D$ = Euclidean Space

$\longrightarrow$ Real time QFT

$S_E[\phi])$

$$S_E[\phi] = \int d^D x_E \, \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2$$

$$\langle \phi(x_1)\phi(x_2)\rangle = G(x_1 - x_2) \longmapsto G_{\text{Feynman}}(x_1 - x_2) \text{ in } \mathbb{M}^{1,D-1}$$

$x \gtrsim 0$ singular at short distance $\Longrightarrow$ UV divergences stronger as $D \nearrow$

N-points Functions and the Wick Theorem

$$G_N(x_1, \dots x_N) = \langle \phi(x_1) \cdots\cdots \phi(x_N) \rangle$$

$$= \frac{\int D_E[\phi]\, e^{-\frac{1}{\hbar} S_E[\phi]}\, \phi(x_1) \cdots \phi(x_N)}{\int D_E[\phi]\, e^{-1/\hbar\, S_E[\phi]}}$$

Functional Integral

N odd $\quad G_N = 0$

$N = 2M$ even

ts Functions and the Wick Theorem

Exp. (Value) of a product of Gaussian rand. variable

$$\ldots X_N) = \langle \phi(x_1) \cdots\cdots \phi(x_N) \rangle$$

$$= \frac{\int D_E[\phi]\, e^{-\frac{1}{\hbar} S_E[\phi]}\, \phi(x_1) \cdots \phi(x_N)}{\int D_E[\phi]\, e^{-1/\hbar\, S_E[\phi]}}$$

Integral

$$G_N = 0$$

even $$G_N = \sum_{\text{pairing}} \prod_{\text{pairs}} \langle \phi(x_i)\phi(x_i) \rangle$$

$\phi(x_N)\rangle$

of a product of Gaussian rand. variable

$[\phi]$

$\phi(x_1)\cdots\phi(x_N)$

$S_E[\phi]$

$G_2 = G$ propagat

$= 2$ pt Fuct.

$\phi(x_i)\phi(x_i)\rangle$

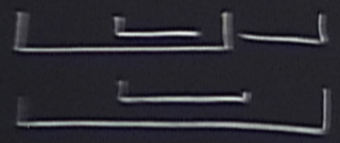
x_1 x_2

3 pa

$N = 4$

$$G_4(x_1, \cdots x_4) = G(x_1 - x_2)G(x_3 - x_4) + G(x_1 - x_3)G(x_2 - x_4) + G(x_1 - x_4)G(x_2 - x_3)$$

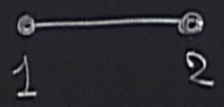
$x_1 \; x_2 \; x_3 \; x_4$

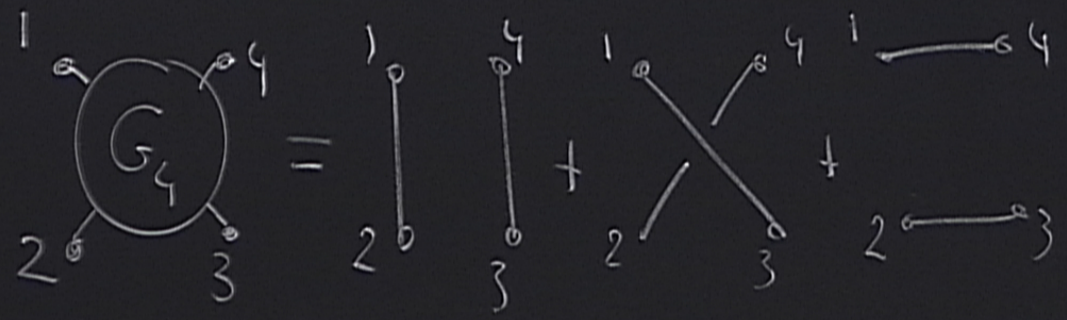


3 pairings

Feynman diagrams (representation)

$G(x_1, x_2) \rightarrow$





- We have defined the local $\phi(x)$ operator & its correlations.
- Composite operators $\phi^2(x)$

$$\langle 0 | \phi \cdots \cdots \phi | 0 \rangle$$

$$\langle \text{OUT} | \text{IN} \rangle$$

$$\langle 0 | \phi \cdots \cdots \phi \cdots | 0 \rangle$$

$$\langle \text{OUT} | \mathcal{O} | \text{IN} \rangle$$

Correlation Functs $\Longleftrightarrow$ Hilbert Space + Alg. of operators

Algebraic Q.F.T

$$= \lim_{y \to x} \left[\phi(x)\phi(y) - \langle \phi(x)\phi(y) \rangle \mathbb{1} \right]$$

operator

function

Identity Operator

exists and is finite

local $\phi(x)$ operator & its correlations

rs $\phi^2(x)$ ⚠ ∞

take care of UV sing.

$$\boxed{:\phi^2:}(x) = \lim_{y \to x} \left[\phi(x)\phi(y) - \langle \phi(x)\phi(y) \rangle \mathbb{1} \right]$$

operator

function

Identity Operator

$\phi(x_n)\rangle$ exists and is finite

- We have defined the local $\phi(x)$ operator & its correlations.
- Composite operators $\phi^2(x)$ ⚠ ∞

take care of UV sing.

Normal product prescriptions $\boxed{:\phi^2:}(x) = \lim_{y \to x} \left[\phi(x)\phi(y) - \langle \phi(x)\phi(y) \rangle \mathbb{1} \right]$

operator function Identity Operator

$\langle \phi(x_1) \cdots :\phi^2:(z) \cdots \phi(x_n) \rangle$ exists and is finite

is finite

Interacting ϕ^4 theory $\longleftrightarrow$ Landau-Ginzburg theory

$$S_0[\phi] = \int d^D x \, \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2$$

bosons + repulsive
local 2 body interac

Euclidean theory

$$S[\phi] = S_0[\phi] + \frac{g}{4!}\int d^D x \, \phi^4(x)$$

g coupling constant

$\frac{1}{4!} = \frac{1}{24}$ normalization (convenient)

interacting theory

Free action

to be defined properly (Renormalization)

Construct a functional integral (Euclidean → Mi

$$\int D[\phi] \exp\left(-\frac{1}{\hbar} S[\phi]\right)$$ to be defined

ung theory

repulsive

2 body interaction

$$G_N(x_1 \cdots x_N) = \langle \phi(x_1) \cdots \phi(x_N) \rangle$$

Functional integral (Euclidean → Minkowski)

$$\exp\left(-\frac{1}{\hbar} S[\phi]\right) \quad \text{to be defined}$$

vacuum of the interacting theory

$$\langle X_N \rangle = \frac{\int D[\phi]\, e^{-\frac{1}{\hbar} S}\, \phi \cdots \phi}{\int D[\phi]\, e^{-\frac{1}{\hbar} S}} \Rightarrow \langle \Omega | T(\phi, \cdots \phi) | \Omega \rangle$$

$$|\Omega\rangle \neq |0\rangle$$

Free theory

Landan-Ginzburg theory

bosons + repulsive
local 2 body interaction

Construct a functional inte

$$\int D[\phi] \exp\left(-\frac{1}{\hbar} S[\phi]\right)$$

$$G_N(x_1 \cdots x_N) = \langle \phi(x_1) \cdots \phi(x_N) \rangle = \frac{\int D[\phi}{\int D[\phi}$$

pling constant

normalization

(convenient)

tion)

Formalism; Generating Functionals

$$\int \mathcal{D}_0[\phi] \exp\left(-\frac{1}{\hbar}\left(S[\phi] - J\cdot\phi\right)\right)$$

Free Field Measure

$$J\cdot\phi = \int d^D x\, j(x)\, \phi(x)$$

random variable

$$Z[j] = \int D_0[\phi] \exp\left(-\frac{1}{\hbar}\left(S[\phi] - J\cdot\phi\right)\right)$$

functional of the function j

Free Field Measure

$$J\cdot\phi = \int d^D x\, j(x)\, \phi(x)$$

classical function source term

random variable

$$\hbar \frac{\delta}{\delta j(x)} \longrightarrow \phi(x)$$

Functional derivatives

$$\bar{Z}(x_1 \cdots x_N) = \frac{1}{Z_0} \hbar^N \frac{\delta}{\delta j(x_1)} \cdots \frac{\delta}{\delta j(x_N)} Z[j] \Big|_{j=0}$$

$$= \frac{1}{Z_0} \int D_0[\phi] \exp\left(-\frac{1}{\hbar} S[\phi]\right) \phi(x_1) \cdots \phi(x_N)$$

$$G(x_1 \cdots x_N) = \frac{\bar{Z}(x_1 \cdots x_N)}{\bar{Z}(\cdot)}$$

$\leftarrow$ depends on g

$\leftarrow$ depend on g

$$\bar{Z}(\cdot) = \int D_0[\phi] \exp\left(-\frac{1}{\hbar} S[\phi]\right) / Z_0$$

$$= \frac{1}{\hbar^N} \frac{\delta}{\delta j(x_1)} \cdots \frac{\delta}{\delta j(x_N)} Z[j] \Big|_{j=0}$$

$$Z_0 = \int D_0[\phi] \, e^{x}$$

free theory

$$= \frac{1}{Z_0} \int D_0[\phi] \exp\left(-\frac{1}{\hbar} S[\phi]\right) \phi(x_1) \cdots \phi(x_N)$$

$\phi(x_N)$ ← depends on g

← depend on g

$$\left(-\frac{1}{\hbar} S[\phi]\right) / Z_0$$

$$D_0[\phi] = \prod_x d\phi(x) \left(\frac{2\pi\hbar}{\epsilon^{D-2}}\right)^{-1/2}$$

$(x_1)\ \delta J(x_N)$ $\Big|_{J=0}$ free theory

$\exp\left(-\frac{1}{\hbar}S[\phi]\right)\phi(x_1)\cdots\phi(x_N)$

ends on g

end on g

$/Z_0$

$$D_0[\phi] = \prod_x d\phi(x)\left(\frac{2\pi\hbar}{\epsilon^{D-2}}\right)^{-1/2}$$

ϵ = mesh of the lattice

Perturbation theory : g is small, series expansion in g

$$Z(z_1, \dots z_N) = \Big\langle \underbrace{\exp\left(-\frac{1}{\hbar}\frac{g}{4!}\int d^D x\, \phi^4(x)\right)}_{\text{non-local}} \cdot \underbrace{\phi(z_1) \cdots \phi(z_N)}_{\text{local}} \Big\rangle$$

change of notations

z_i coordinates of "external points"

$z_i, i = 1, N \in \mathbb{R}^D$

- Taylor Expand the exp
- $\left\langle \sum_K g^K \dots \right\rangle_0 = \sum_K g^K \langle \dots \rangle_0$

$$\frac{1}{K!}\int dx_1 \cdots dx_K \left\langle \phi(z_1)\cdots\phi(z_N)\,\phi^4(x_1)\cdots\phi^4(x_K)\right\rangle_0$$

is small, series expansion in g

$$\left(-\frac{1}{\hbar}\frac{g}{4!}\int d^D x\, \phi^4(x)\right) \cdot \underbrace{\phi(z_1)\cdots\phi(z_N)}_{\text{local}}\Big\rangle_0 = \sum_{k=0}^{\infty} g^k \left(\frac{-1}{\hbar\, 4!}\right)^k \int dx_1 \cdots d$$

exact

non-local

e.v. in the free theory

Warning ⚠ : interessa di ∞ sums.

$$\int D_0[\phi] \sum_k = \sum_k \int D_0[\phi]$$

$$\frac{g}{4!}\int d^D x\, \phi^4(x)\Big) \cdot \underbrace{\phi(z_1)\cdots\phi(z_N)}_{\text{local}}\Big\rangle_0 = \sum_{K=0}^{\infty} g^K \left(\frac{-1}{\hbar\, 4!}\right) \int d^D x$$

on-local

e.v. in the free theory

Warning ⚠ : interversion of ∞ sums.

$$\int D_0[\phi] \sum_K = \sum_K \int D_0[\phi]$$

Entire-functioning

Asymptotic series ing

$$) = \Big\langle \underbrace{\exp\Big(-\frac{1}{\hbar}\frac{g}{4!}\int d^D x\, \phi^4(x)\Big)}_{\text{non-local}} \cdot \underbrace{\phi(z_1)\cdots\phi(z_N)}_{\text{local}} \Big\rangle_0 =$$

e.v. in the free theory

two "external points"

$\in \mathbb{R}^D$

and the exp

$$\Big\langle \Big\rangle_0 = \sum_k g^k \langle \cdots \rangle_0$$

Warning ⚠ interchange of ∞ sums.

$$\int D_0[\phi] \cdot \sum_k = \sum_k \int D_0[\phi]$$

Entire function

Asymptotic series only

$$\left(\frac{-1}{\sqrt{\ }\cdot 4!}\right)^K \int dx_1 \cdots dx_K \left\langle \phi(z_1)\cdots\phi(z_N)\,\phi^4(x_1)\cdots\phi^4(x_K)\right\rangle_0$$

internal points

ick Theorem

$$= \sum_{\text{pairings}} \prod G(\cdot,\cdot)$$

X's or Z's

propagator of the free theory

$$\langle \phi(z_1) \cdots \phi(z_N)\, \phi(x_1) \cdots \phi(x_K) \rangle$$

el points

$$= \sum_{\text{pairings}} \prod G(\cdot\,,\cdot)$$

X's or Z's

propagator of the free theory

Diagram $\longrightarrow$ Feynman Integral

Feynman rules (Euclidean theory) in position space

external vertex $\circ\!-$ Z_i $i=1,..N$ "1 leg" 1 factor

internal vertex $\times$ X_j $j=1,..K$ "4 legs" $-g/\hbar$ Factor

propagator Y_1 —— Y_2 $\hbar \cdot \int \frac{d^D K}{(2\pi)^D} \frac{e^{iK(Y_1-Y_2)}}{K^2+m^2} = \hbar\, G_0(Y_1-Y_2)$

integrate over internal vertex positions $\int d^D X_j$

0 point function $N=0$

order $K=0$ $\quad 1$

order $K=1$ $\quad -g\frac{1}{4!\hbar}\int d^D x_1 \langle \phi^4(x_1)\rangle_0$

ng only

function $N=0$

$k=0$ $\quad 1$

$k=1$ $\quad -g\frac{1}{4!\hbar}\int d^D x_1 \langle \phi^4(x_1)\rangle_0 = -\frac{g}{\hbar}\frac{1}{8}\int d^D x_1 \langle \phi^2(0)\rangle^2$

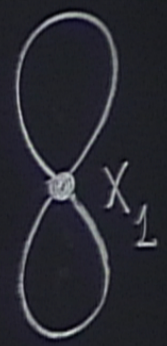
unction $N=0$

$=0 \qquad 1$

$=1 \qquad -g\frac{1}{4!\hbar}\int d^D x_1 \langle \phi^4(x_1)\rangle_0 = -\frac{g}{\hbar}\frac{1}{8}\int d^D x_1 \langle \phi^2(0)\rangle^2 = -\frac{g}{8}$

Feynman Diagram

$$\langle \phi^4(x_1) \rangle_0 = -\frac{g}{\hbar}\frac{1}{8}\int d^D x_1 \, \langle \phi^2(0) \rangle^2 = -\frac{g}{8}\hbar$$



amplitude of  $= G_0(0)^2$

it is ∞ !

$G_0(0) =$  tadpole diagram

let us close our eyes

exact

$$\Big\rangle_0 = \sum_{K=0}^{\infty} \frac{g^K}{K!}\left(\frac{-1}{\hbar \cdot 4!}\right)^K \int dx_1 \cdots dx_K \, \langle \phi(z_1) \cdots \phi(z$$

internal points

e.v. in the free theory

Apply Wick Theorem

$$\sum_{\text{pairings}} \prod$$

Diagram

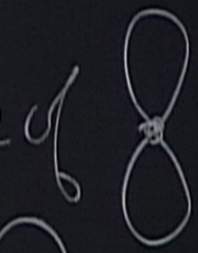
sums.

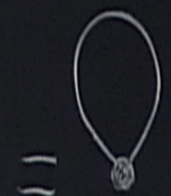
mptotic

nt function $N=0$

$K=0$ $\quad 1$

$K=1$ $\quad -g\frac{1}{4!\hbar}\int d^D x_1 \langle \phi^4(x_1)\rangle_0 = -\frac{g}{\hbar}\frac{1}{8}\int d^D x_1 \langle$

$K=2$ $\quad g^2 \frac{1}{2!(4!)^2}\frac{1}{\hbar^2}\int dx_1 dx_2 \langle \phi^4(x_1)\phi^4(x_2)\rangle$ amplitude of  =

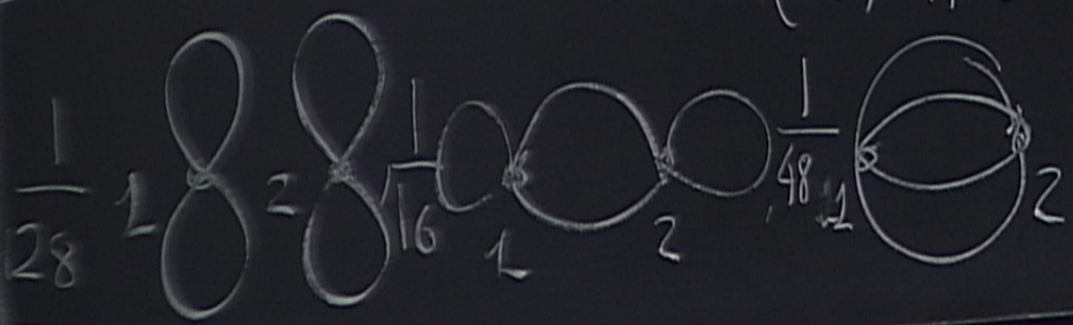
$G_0(0) =$  tadpo

0 point function $N=0$

order $K=0$ $\quad 1$

order $K=1$ $\quad -g\frac{1}{4!\hbar}\int d^D x_1 \langle \phi^4(x_1)\rangle_0 = -\frac{g}{\hbar}$

order $K=2$ $\quad g^2 \frac{1}{2!(4!)^2}\frac{1}{\hbar^2}\int dx_1 dx_2 \langle \phi^4(x_1)\phi^4(x_2)\rangle$ amplitud

$\left[\frac{1}{28}\right.$  $\left.\right] g^2\frac{1}{\hbar^2}$ $\quad G_0(0) =$