Title: Wigner-Eckart theorem and Jordan-Schwinger representation for infinite-dimensional representations of the Lorentz group

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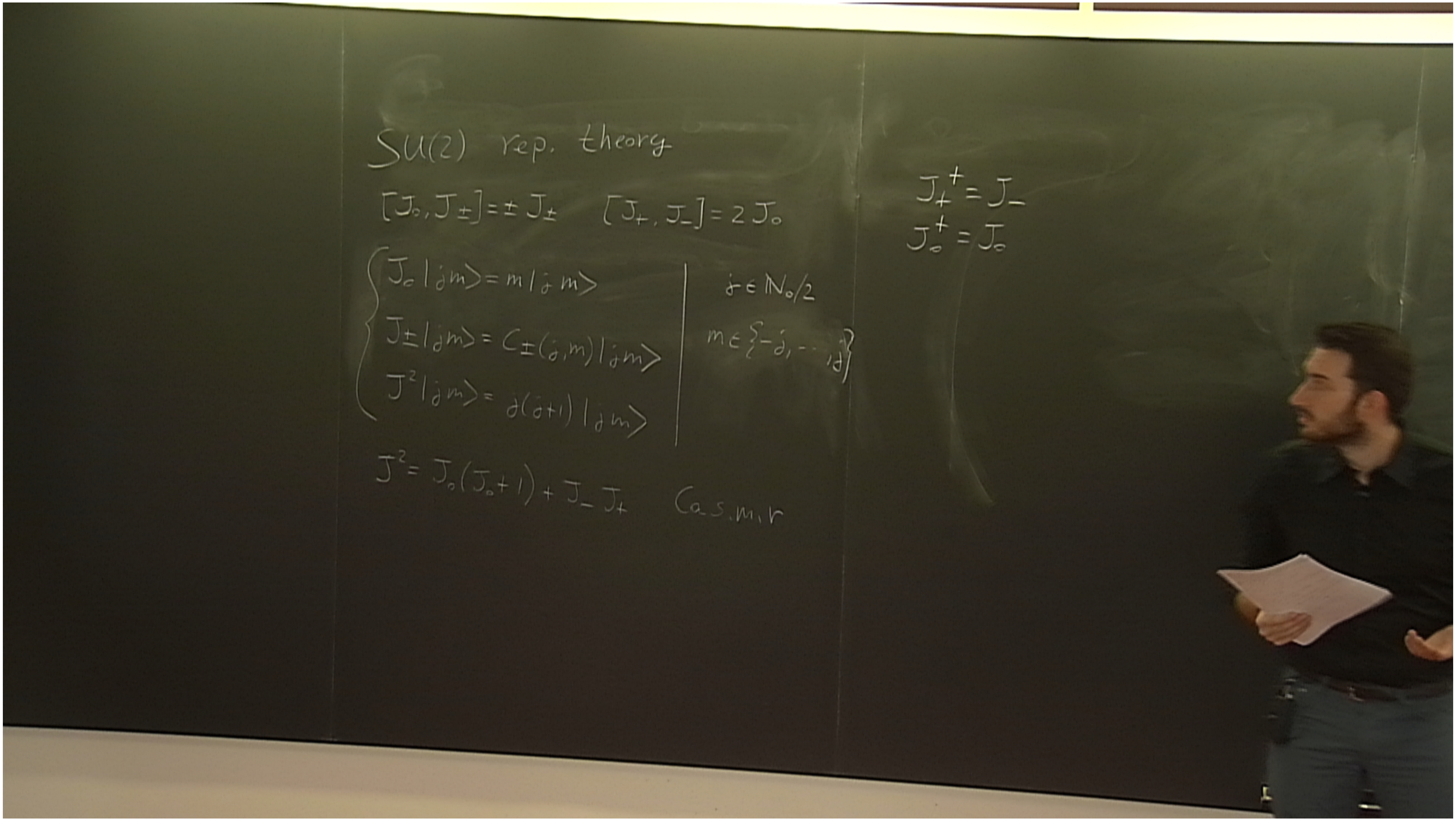
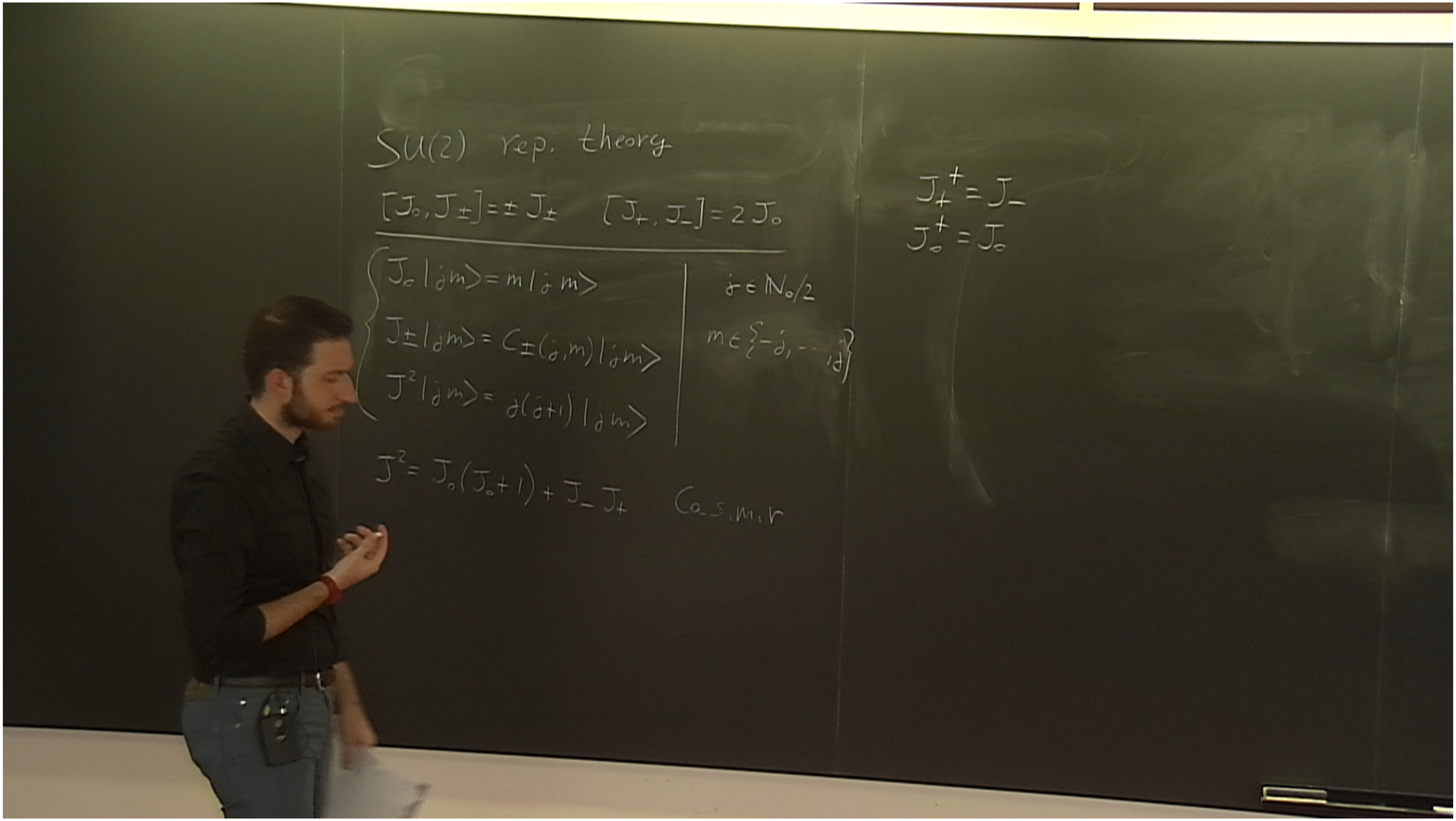
Abstract: <p>The Wigner-Eckart theorem is a well known result for tensor operators of SU(2) and, more generally, any compact Lie group. I will show how it can be generalised to arbitrary Lie groups, possibly non-compact. The result relies on the knowledge of recoupling theory between finite-dimensional and arbitrary admissible representations, which may be infinite-dimensional; the particular case of the Lorentz group will be studied in detail. As an application, the Wigner-Eckart theorem will be used to construct an analogue of the Jordan-Schwinger representation, previously known only for finite-dimensional representations of the Lorentz group, valid for infinite-dimensional ones.</p>


SU(2) rep. theory
$[J_0, J_\pm] = \pm J_\pm$
$[J_+, J_-] = 2 J_0$
$J_0 |j m\rangle = m |j m\rangle$
$J_\pm |j m\rangle = C_\pm(j,m) |j m\rangle$
$J^2 |j m\rangle = j(j+1) |j m\rangle$
$j \in \mathbb{N}_0/2$
$m \in \{-j, \dots, j\}$
$J^2 = J_0(J_0+1) + J_- J_+$
Casimir
$J_+^\dagger = J_-$
$J_0^\dagger = J_0$


SU(2) rep. theory
[J₀, J±] = ± J±   [J+, J−] = 2 J₀
J₀ |j m> = m |j m>
J± |j m> = C±(j,m) |j m>
J² |j m> = j(j+1) |j m>
j ∈ N₀/2
m ∈ {−j, … , j}
J² = J₀(J₀+1) + J− J+   Casimir
J+† = J−
J₀† = J₀

SU(2) rep. theory

$[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = 2 J_0$

$$\begin{cases} J_0 |j\, m\rangle = m |j\, m\rangle \\ J_\pm |j\, m\rangle = c_\pm(j,m) |j\, m\pm 1\rangle \\ J^2 |j\, m\rangle = j(j+1) |j\, m\rangle \end{cases}$$

$j \in \mathbb{N}_0/2$

$m \in \{-j, \dots, j\}$

$J^2 = J_0(J_0+1) + J_- J_+$ Casimir

$J_+^{\dagger} = J_-$

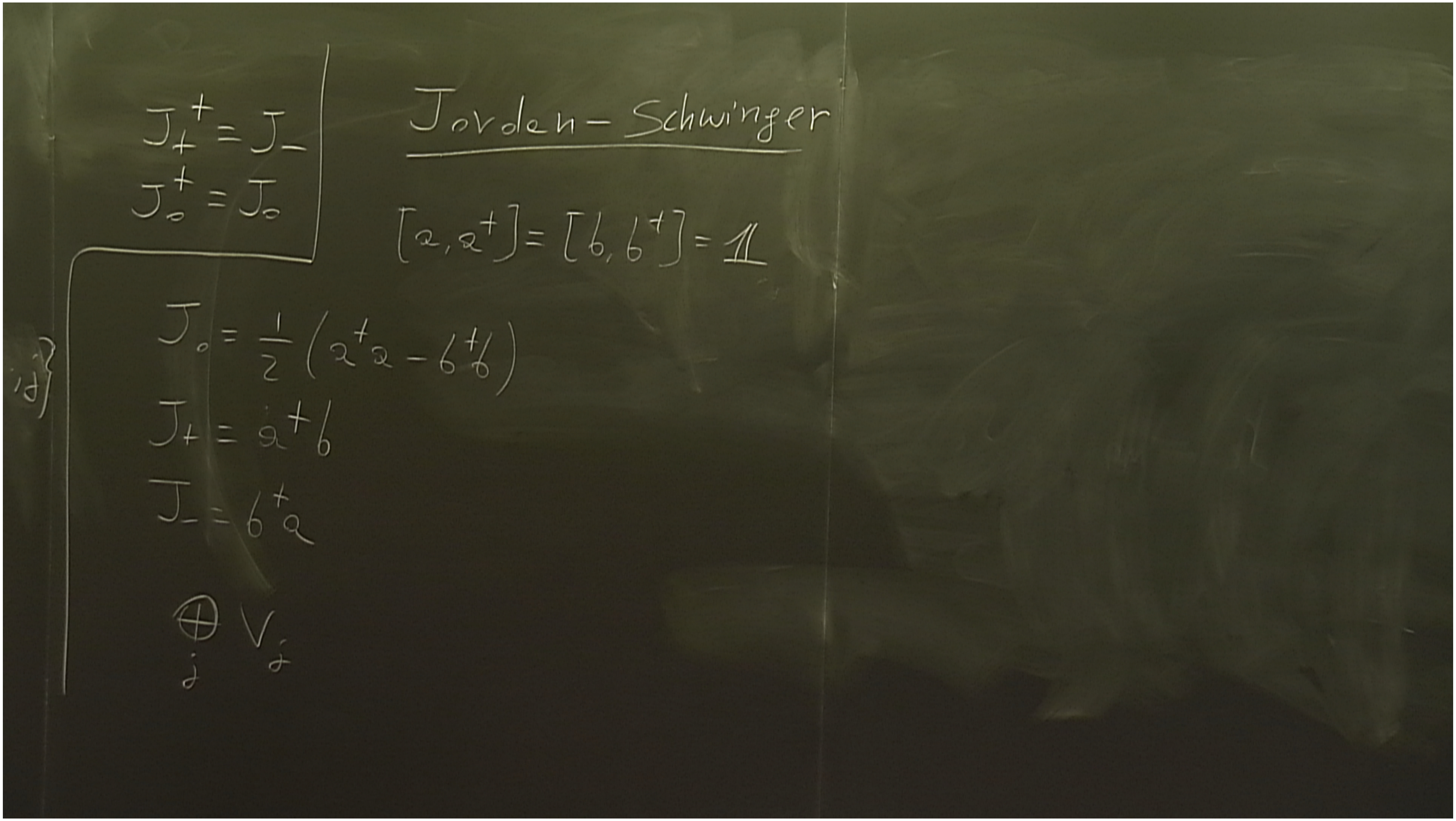
$J_0^{\dagger} = J_0$

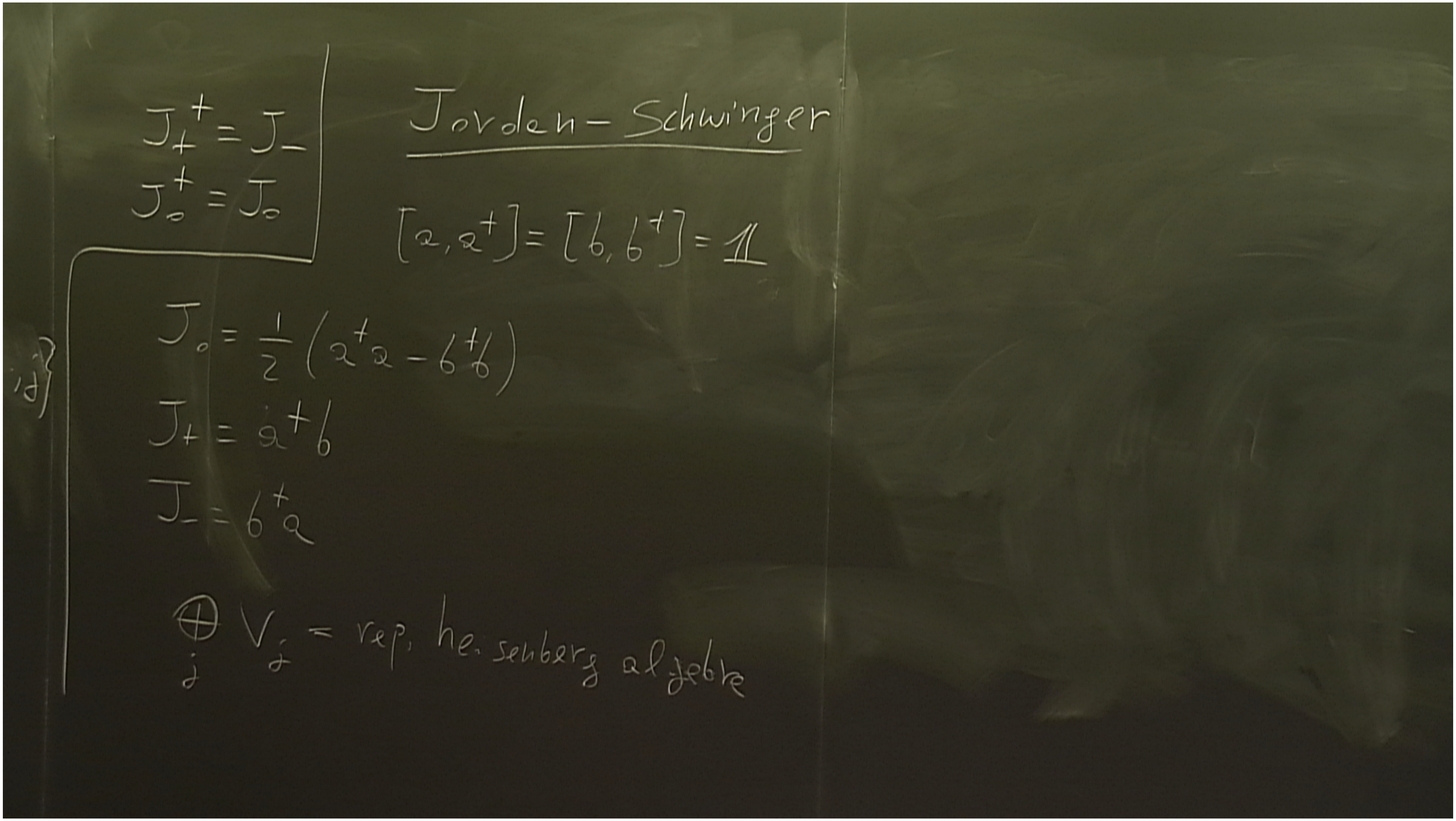
$$J_+^\dagger = J_-$$
$$J_0^\dagger = J_0$$

Jorden – Schwinger

$$[a, a^\dagger] = [b, b^\dagger] = \mathbb{1}$$

$$J_0 = \frac{1}{2}\left(a^\dagger a - b^\dagger b\right)$$
$$J_+ = a^\dagger b$$
$$J_- = b^\dagger a$$


Jorden–Schwinger
$J_+^{+} = J_-$
$J_0^{+} = J_0$
$[a, a^{+}] = [b, b^{+}] = \mathbb{1}$
$J_0 = \frac{1}{2}(a^{+}a - b^{+}b)$
$J_+ = a^{+}b$
$J_- = b^{+}a$
$\bigoplus_j V_j$


$J_+^\dagger = J_-$
$J_0^\dagger = J_0$
Jorden-Schwinger
$[a, a^\dagger] = [b, b^\dagger] = \mathbb{1}$
$J_0 = \frac{1}{2}(a^\dagger a - b^\dagger b)$
$J_+ = a^\dagger b$
$J_- = b^\dagger a$
$\bigoplus_j V_j$ = rep. heisenberg algebra

Jorden-Schwinger

$$[a, a^+] = [b, b^+] = \mathbb{1}$$

$$\ldots{}^+a - b^+b)$$

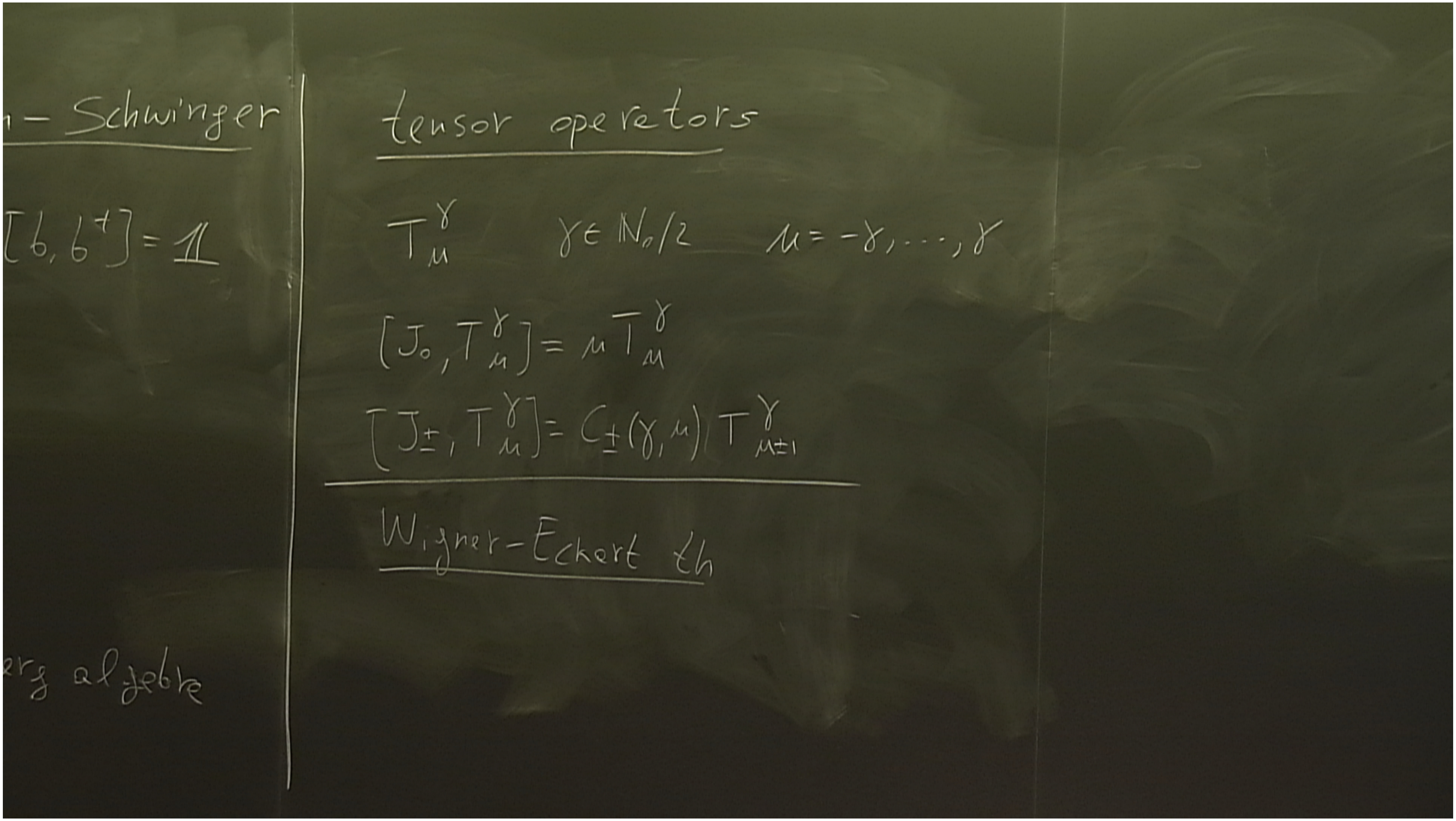
rep. heisenberg algebre

tensor operators

$$T^{\gamma}_{\mu} \qquad \gamma \in \mathbb{N}_0/2 \qquad \mu = -\gamma, \ldots, \gamma$$

$$[J_0, T^{\gamma}_{\mu}] = \mu T^{\gamma}_{\mu}$$

$$[J_\pm, T^{\gamma}_{\mu}] = C_\pm(\gamma, \mu)\, T^{\gamma}_{\mu\pm1}$$


– Schwinger
$[b, b^+] = \mathbb{1}$
erg algebra
tensor operators
T^{γ}_{μ} $\gamma \in \mathbb{N}_0/2$ $\mu = -\gamma, \dots, \gamma$
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Wigner-Eckert th

n – Schwinger

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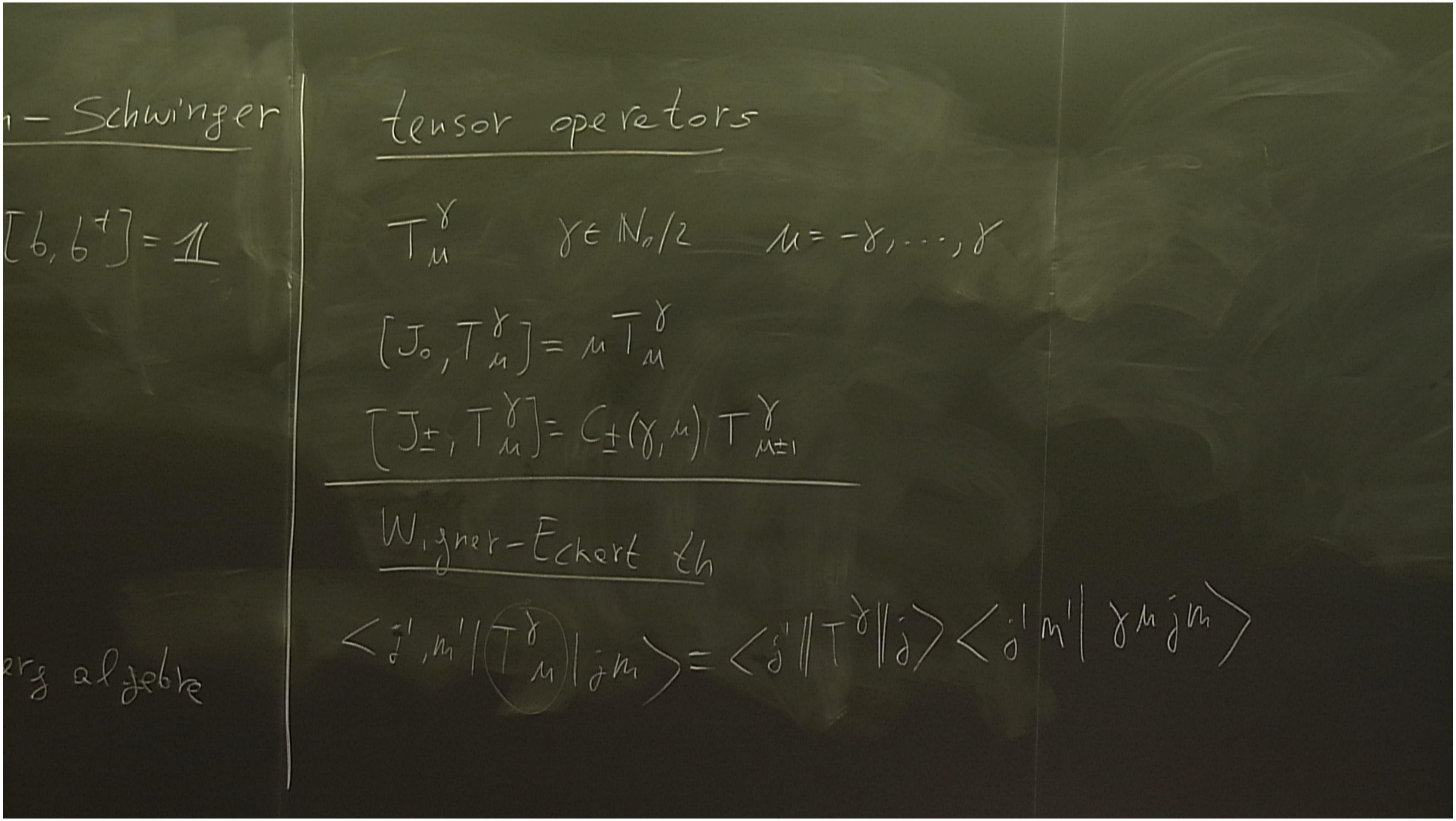
$$[J_\pm, T^\gamma_\mu] = C_\pm(\gamma, \mu)\, T^\gamma_{\mu\pm1}$$

Wigner-Eckart th

$$\langle j', m' | T^\gamma_\mu | j m \rangle = \langle j' \| T^\gamma \| j \rangle \langle j' m' | \gamma \mu j m \rangle$$


–Schwinger
$[b, b^+] = \mathbb{1}$
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tensor operators
T^{γ}_{μ} $\gamma \in \mathbb{N}_0/2$ $\mu = -\gamma, \dots, \gamma$
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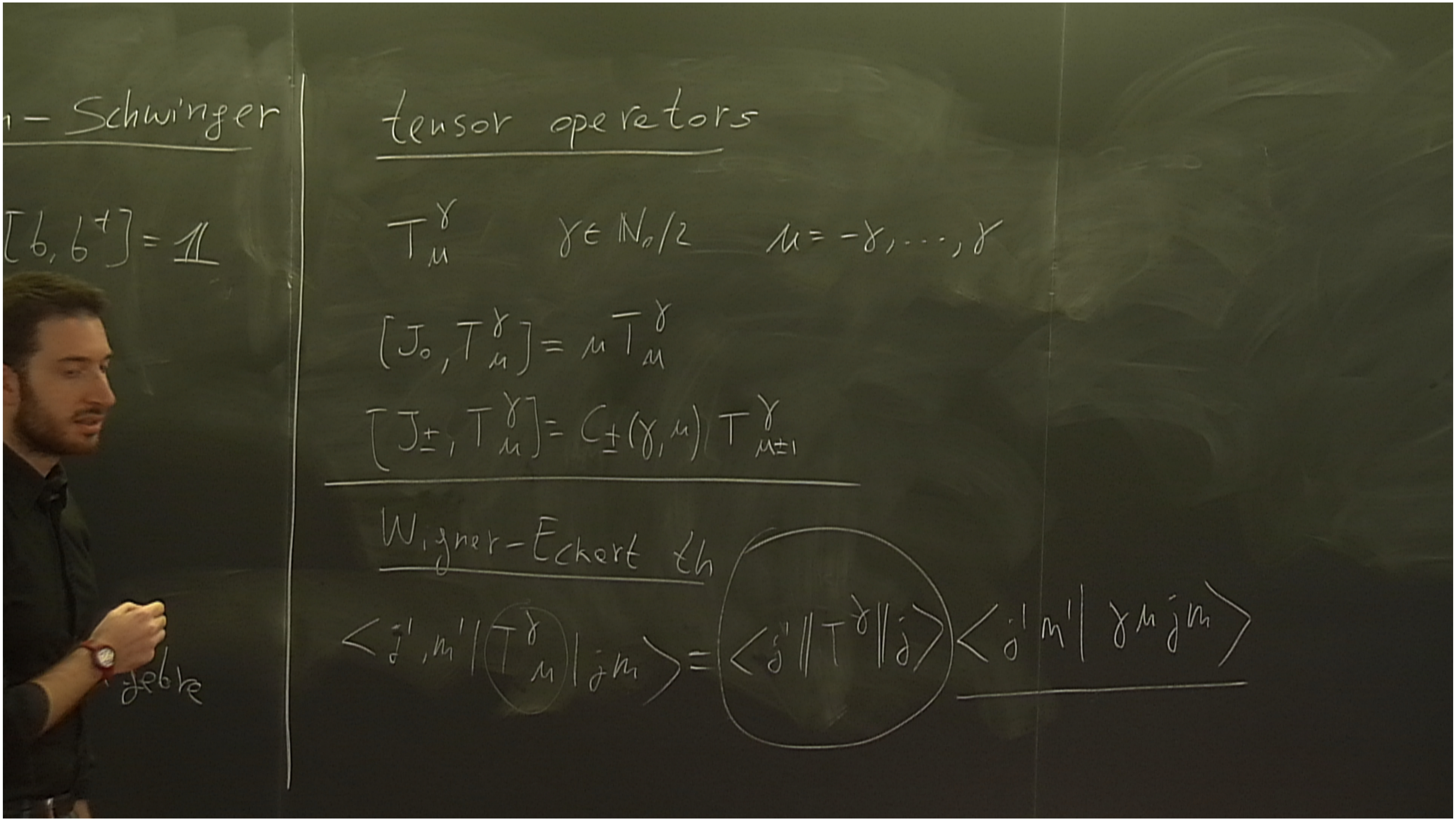
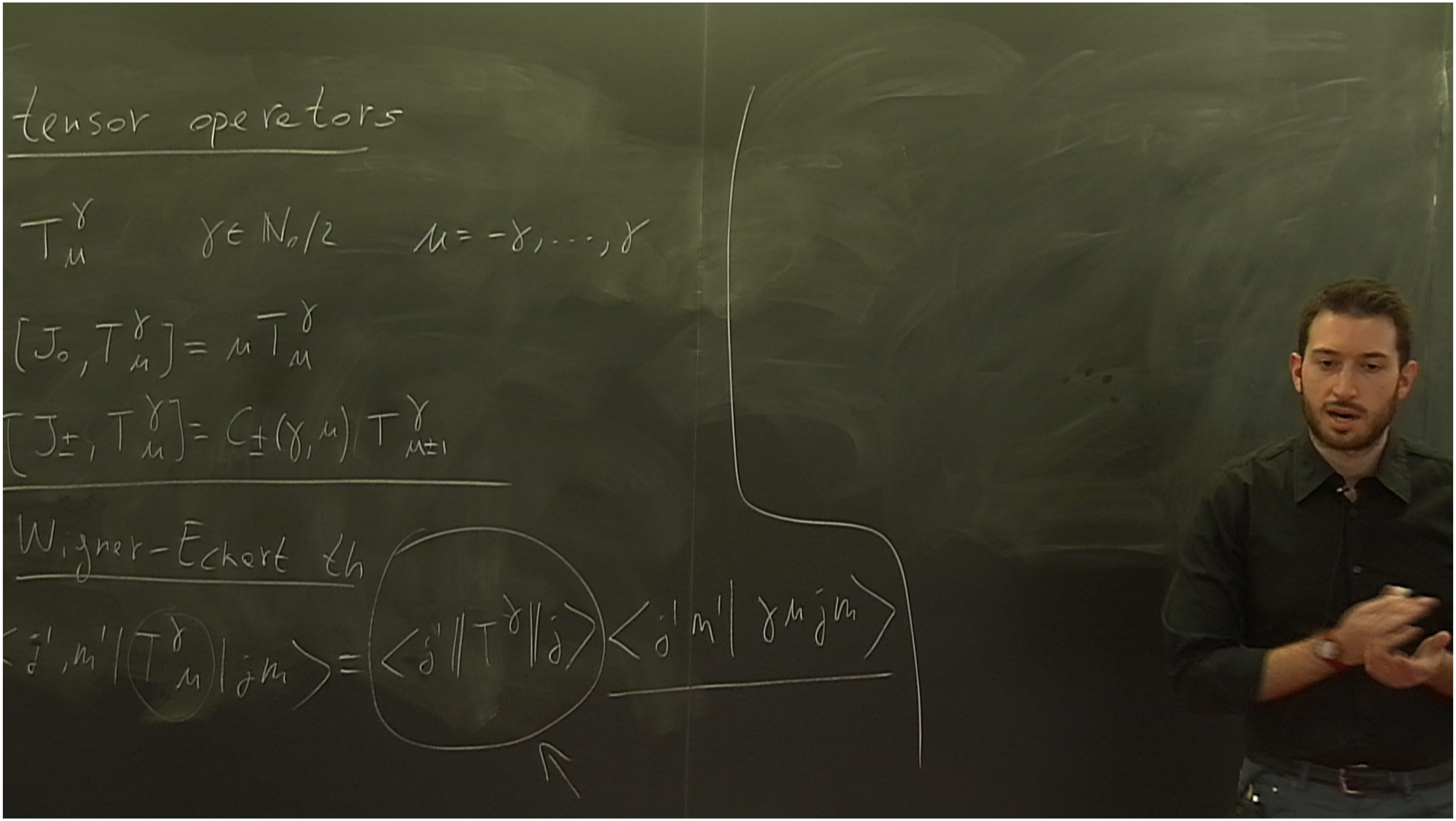
Wigner–Eckart th
$\langle j', m' | T^{\gamma}_{\mu} | j m \rangle = \langle j' \| T^{\gamma} \| j \rangle \langle j' m' | \gamma \mu j m \rangle$


-Schwinger
$[b, b^+] = \mathbb{1}$
tensor operators
T^{γ}_{μ} $\gamma \in \mathbb{N}_0/2$ $\mu = -\gamma, \dots, \gamma$
$[J_0, T^{\gamma}_{\mu}] = \mu T^{\gamma}_{\mu}$
$[J_{\pm}, T^{\gamma}_{\mu}] = C_{\pm}(\gamma, \mu) T^{\gamma}_{\mu \pm 1}$
Wigner-Eckert th
$\langle j', m' | T^{\gamma}_{\mu} | j m \rangle = \langle j' \| T^{\gamma} \| j \rangle \langle j' m' | \gamma \mu j m \rangle$
gebra



tensor operators
T^{γ}_{μ} $\gamma \in \mathbb{N}_0/2$ $\mu = -\gamma, \ldots, \gamma$
$[J_0, T^{\gamma}_{\mu}] = \mu T^{\gamma}_{\mu}$
$[J_\pm, T^{\gamma}_{\mu}] = C_\pm(\gamma, \mu) T^{\gamma}_{\mu\pm1}$
Wigner-Eckert th
$\langle j', m' | T^{\gamma}_{\mu} | jm \rangle = \langle j' \| T^{\gamma} \| j \rangle \langle j' m' | \gamma \mu j m \rangle$

nger

tensor operators

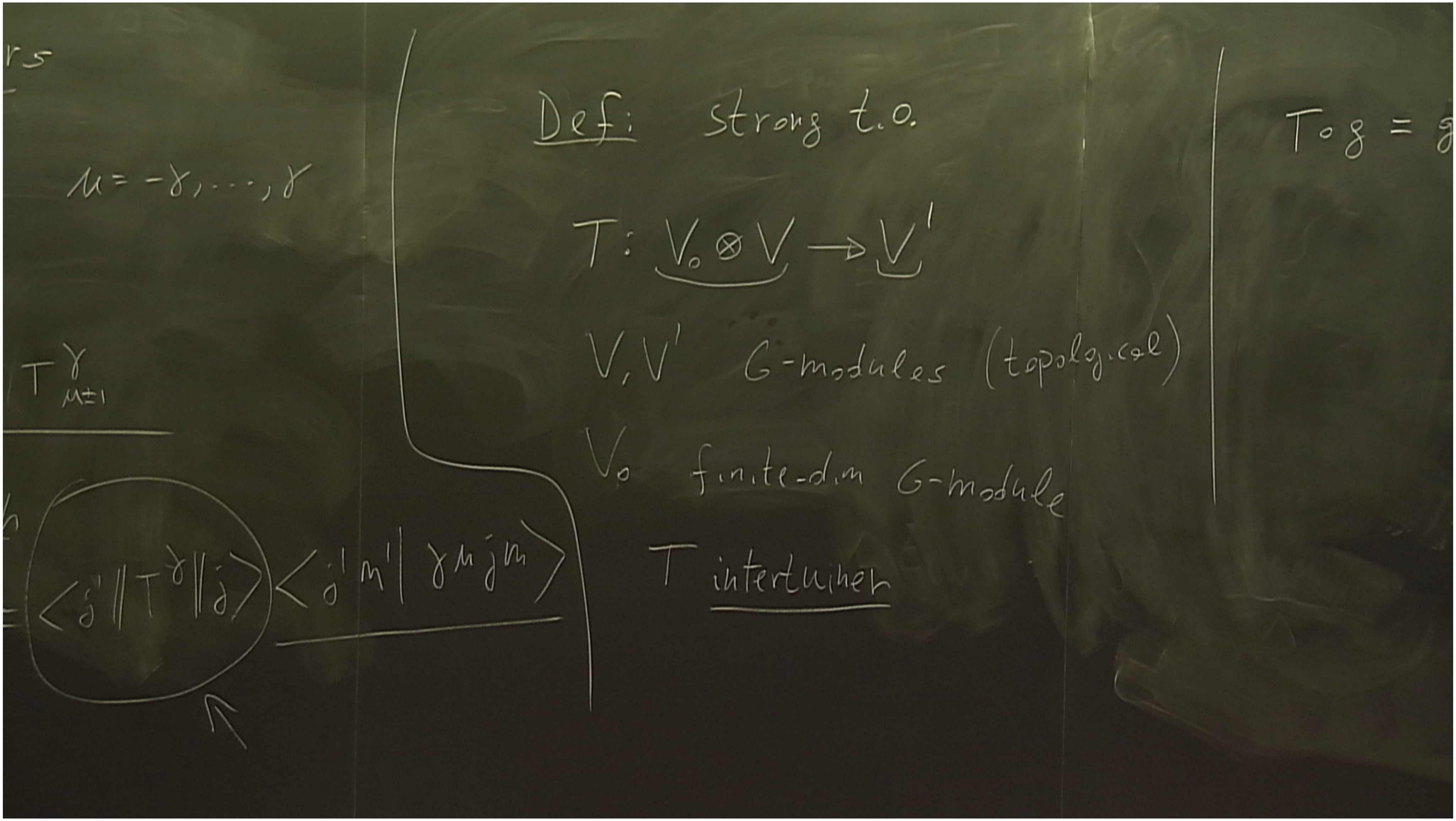
T^{γ}_{μ} $\quad \gamma \in \mathbb{N}_0/2 \quad \mu = -\gamma, \dots, \gamma$

$$[J_0, T^{\gamma}_{\mu}] = \mu T^{\gamma}_{\mu}$$

$$[J_\pm, T^{\gamma}_{\mu}] = C_\pm(\gamma, \mu)\, T^{\gamma}_{\mu\pm 1}$$

Wigner-Eckert th

$$\langle j', m' | T^{\gamma}_{\mu} | j m \rangle = \langle j' \| T^{\gamma} \| j \rangle \langle j' m' | \gamma \mu j m \rangle$$


rs
$\mu = -\gamma, \dots, \gamma$
$T^{\gamma}_{\mu\pm1}$
$\langle j' \| T^{\gamma} \| j \rangle \langle j' m' | \gamma \mu j m \rangle$
Def: strong t.o.
$T: V_0 \otimes V \to V'$
V, V' G-modules (topological)
V_0 finite-dim G-module
T intertwiner
$T \circ g = g$

efi: strong t.o.

$T : \underbrace{V_0 \otimes V} \to \underbrace{V'}$

V, V' G-modules (topological)

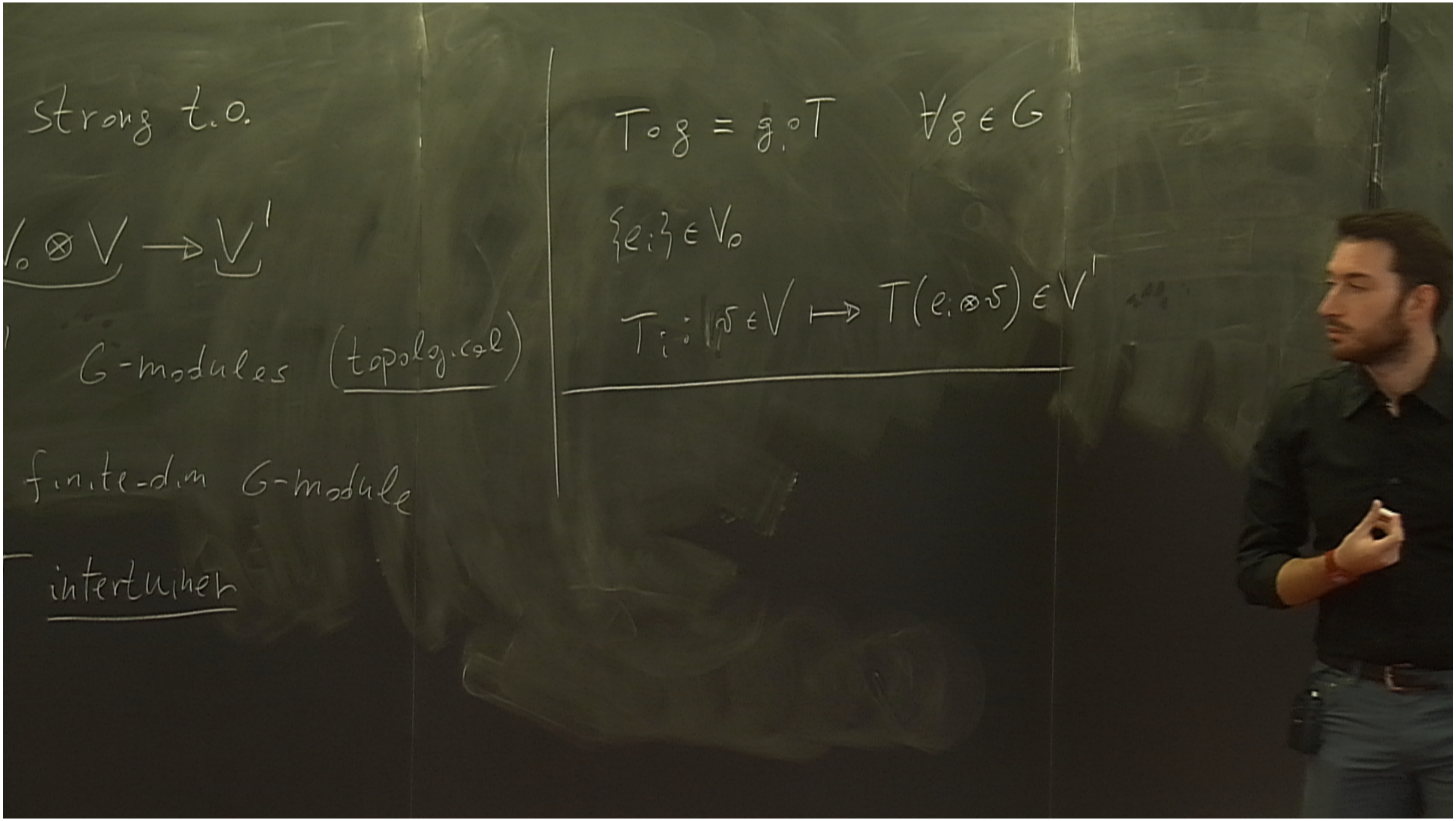
V_0 finite-dim G-module

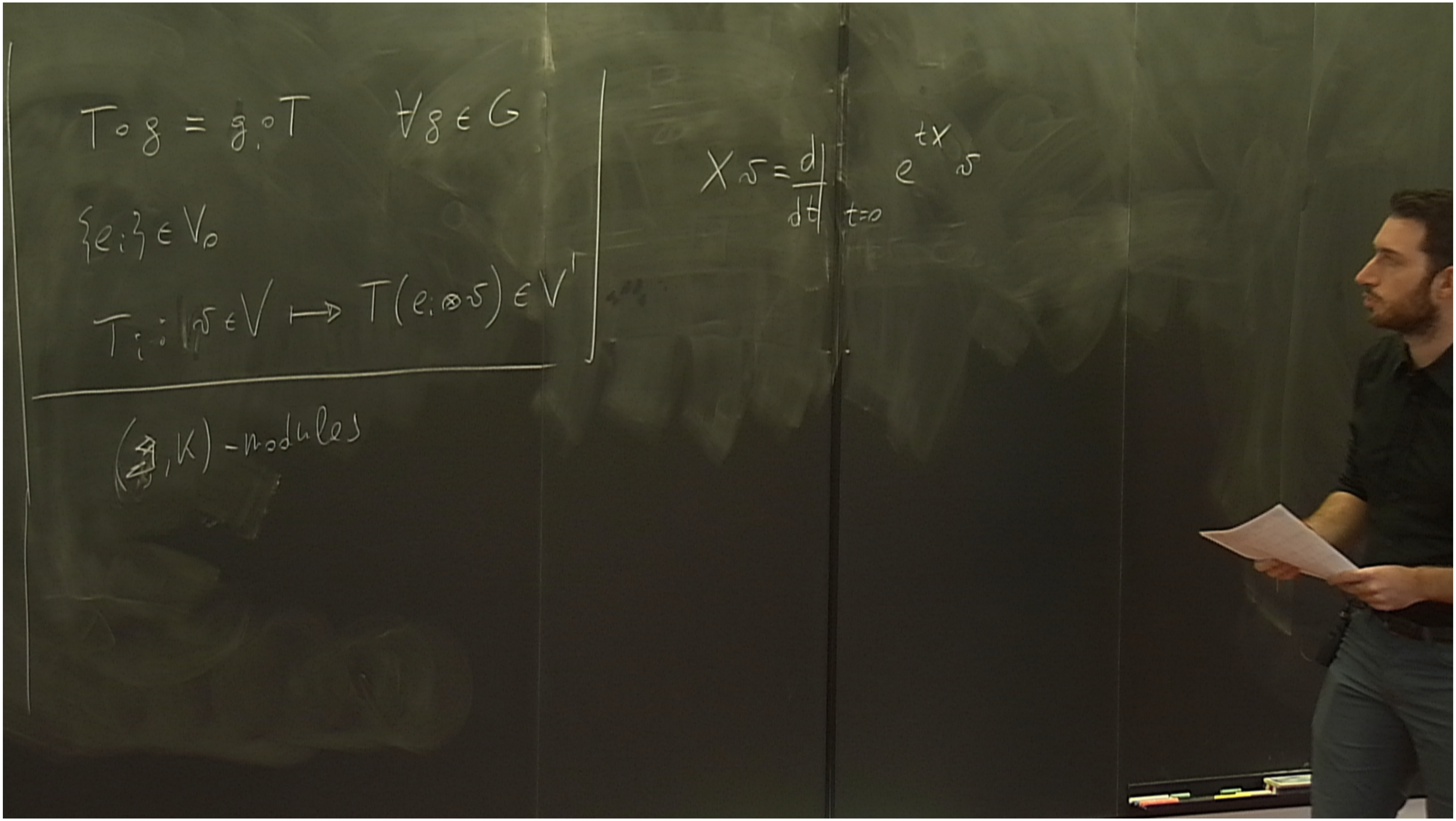
T intertwiner

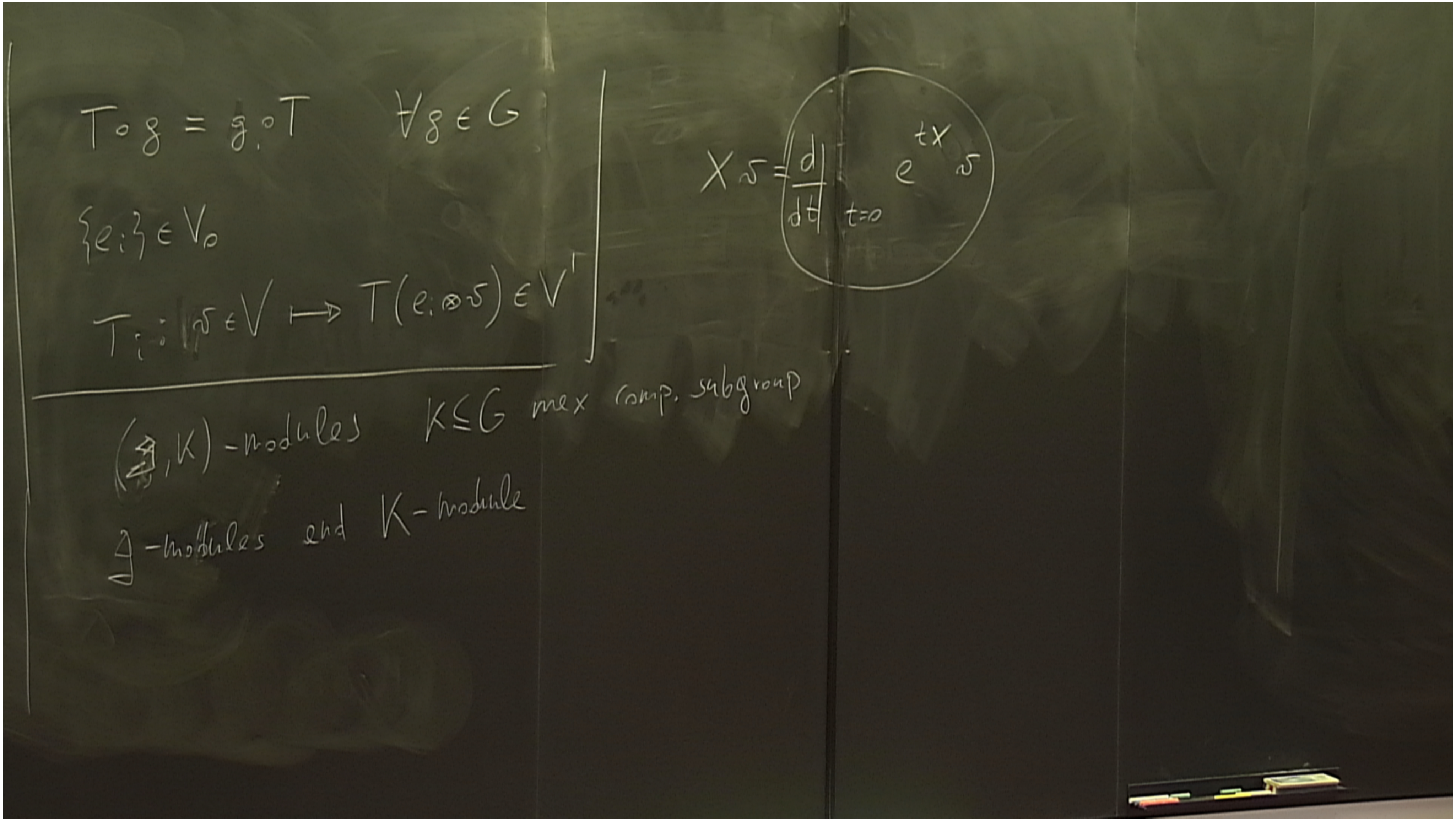
$T \circ g = g \circ T \quad \forall g \in G$

$\{e_i\} \in V_0$

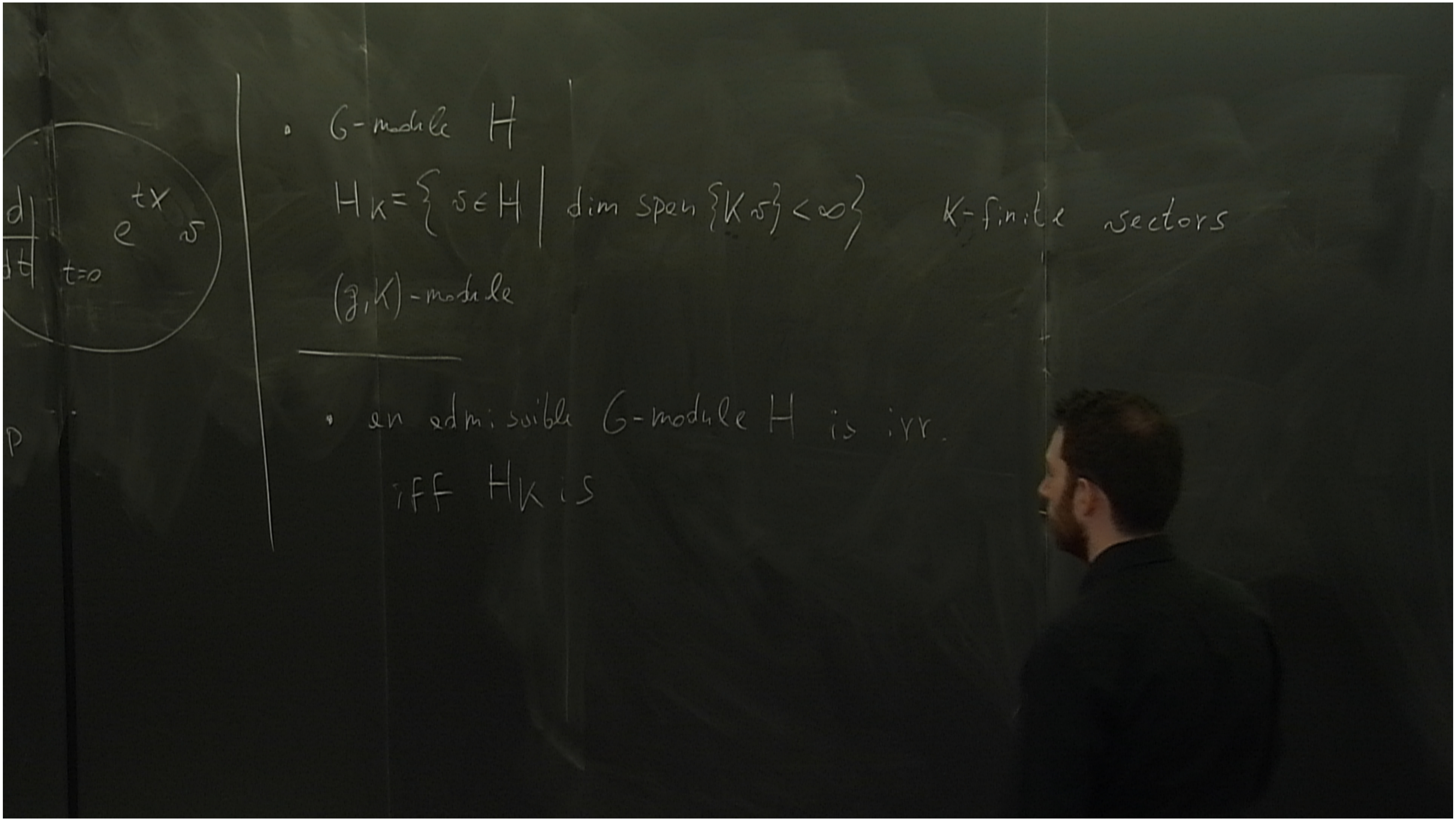
$T_i : v \in V \mapsto T(e_i \otimes v) \in V'$

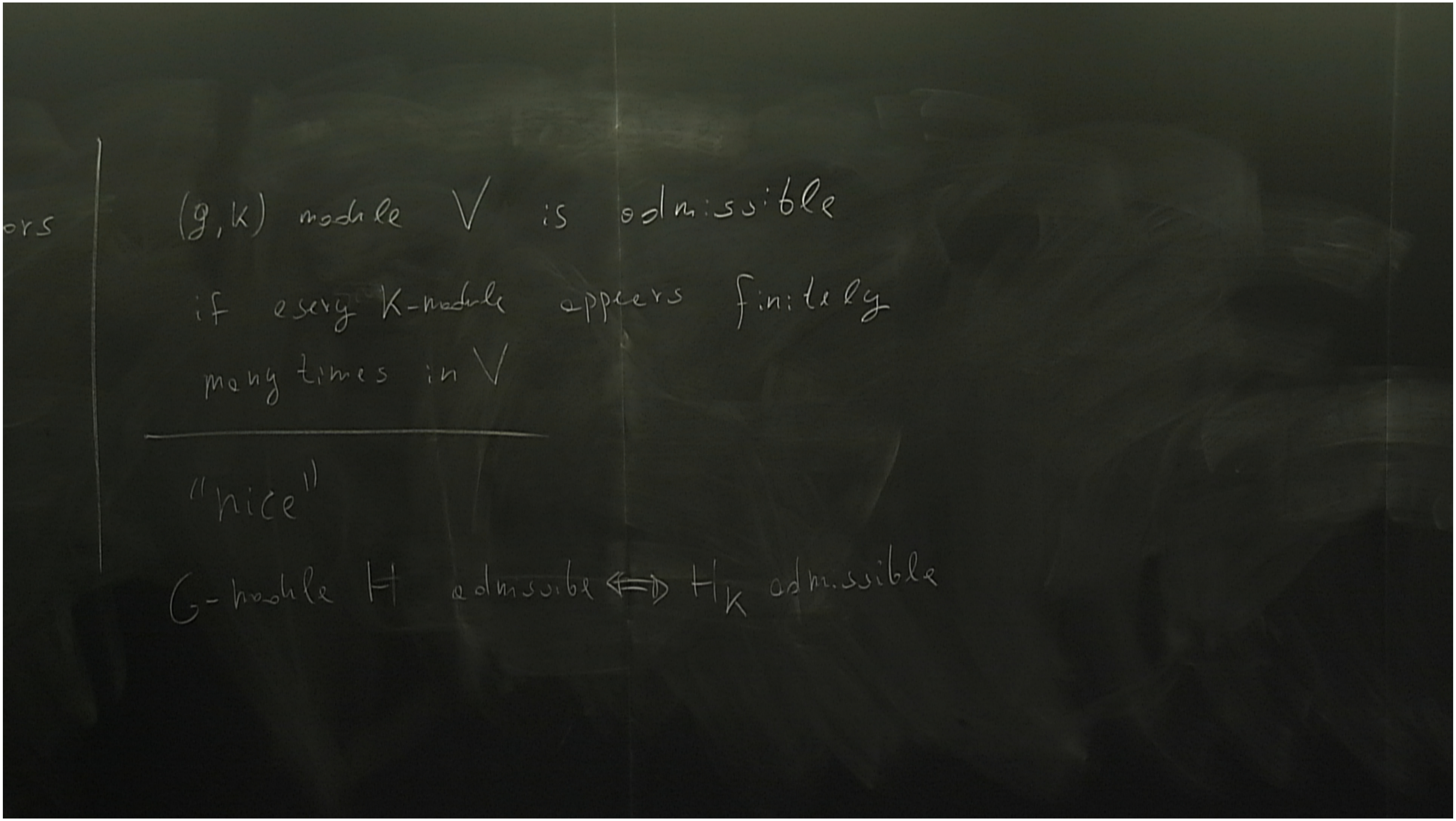

strong t.o.
V₀ ⊗ V → V'
G-modules (topological)
finite-dim G-module
intertwiner
T∘g = g∘T ∀g ∈ G
{eᵢ} ∈ V₀
Tᵢ : v ∈ V ↦ T(eᵢ ⊗ v) ∈ V'

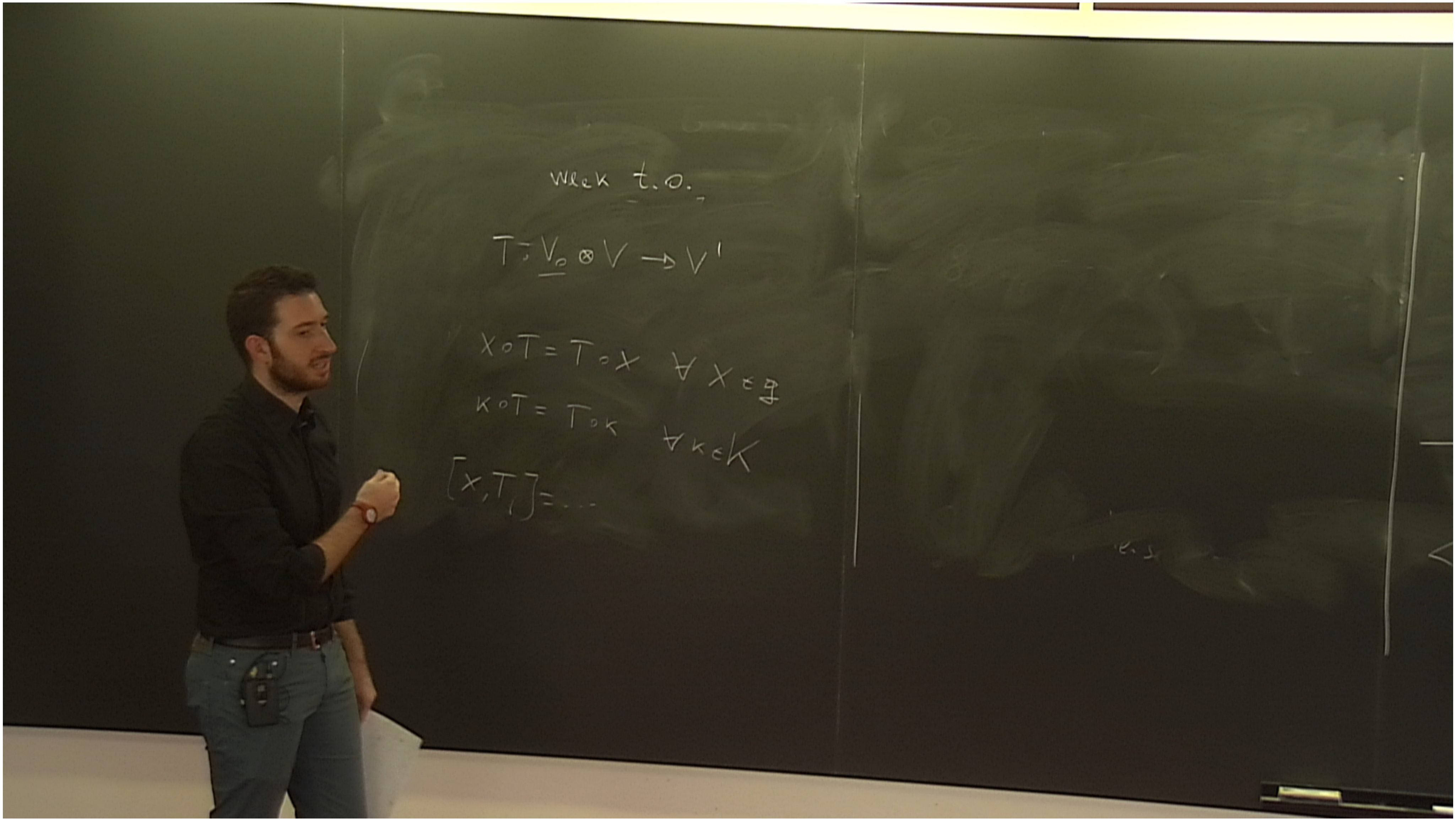

T ∘ g = g_i ∘ T   ∀ g ∈ G
{e_i} ∈ V_0
T_i : v ∈ V ⟼ T(e_i ⊗ v) ∈ V'
(g, K) - modules
X v = d/dt |_{t=0} e^{tX} v

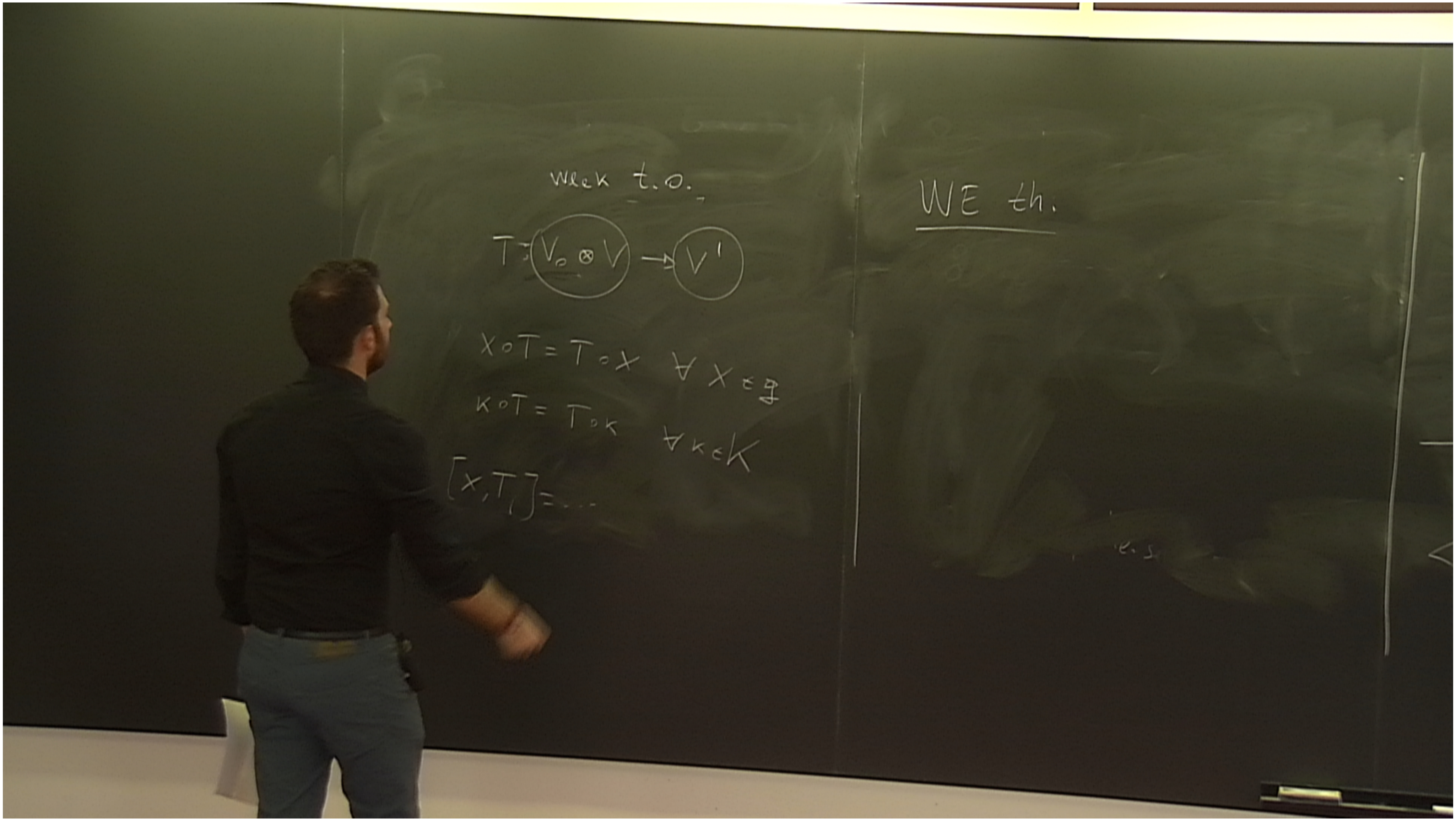


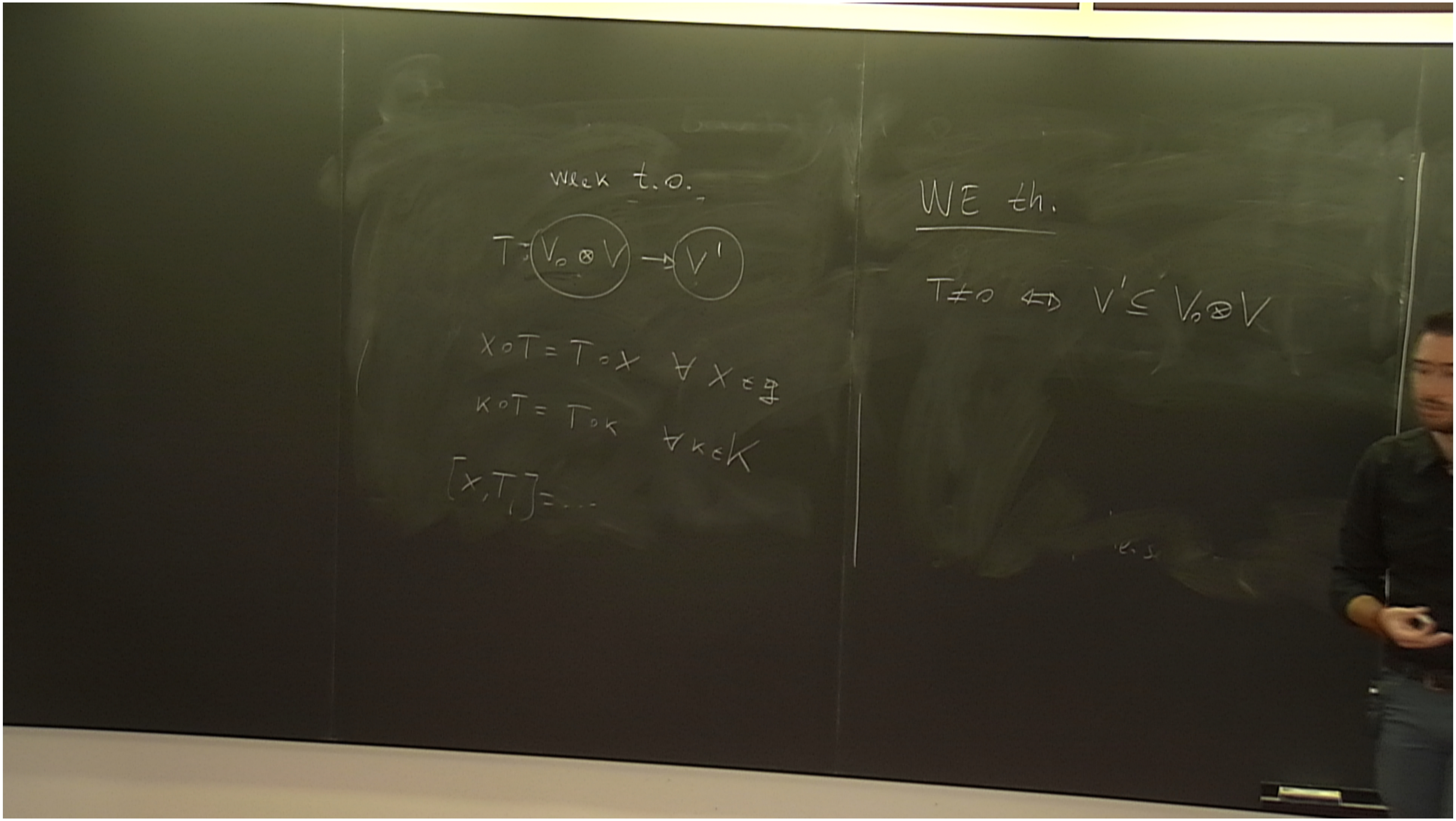
T ∘ g = g ∘ T ∀ g ∈ G
{e_i} ∈ V_0
T_i : v ∈ V ↦ T(e_i ⊗ v) ∈ V'
X v = d/dt|_{t=0} e^{tX} v
(g, K) - modules K ≤ G max comp. subgroup
g - modules and K - module

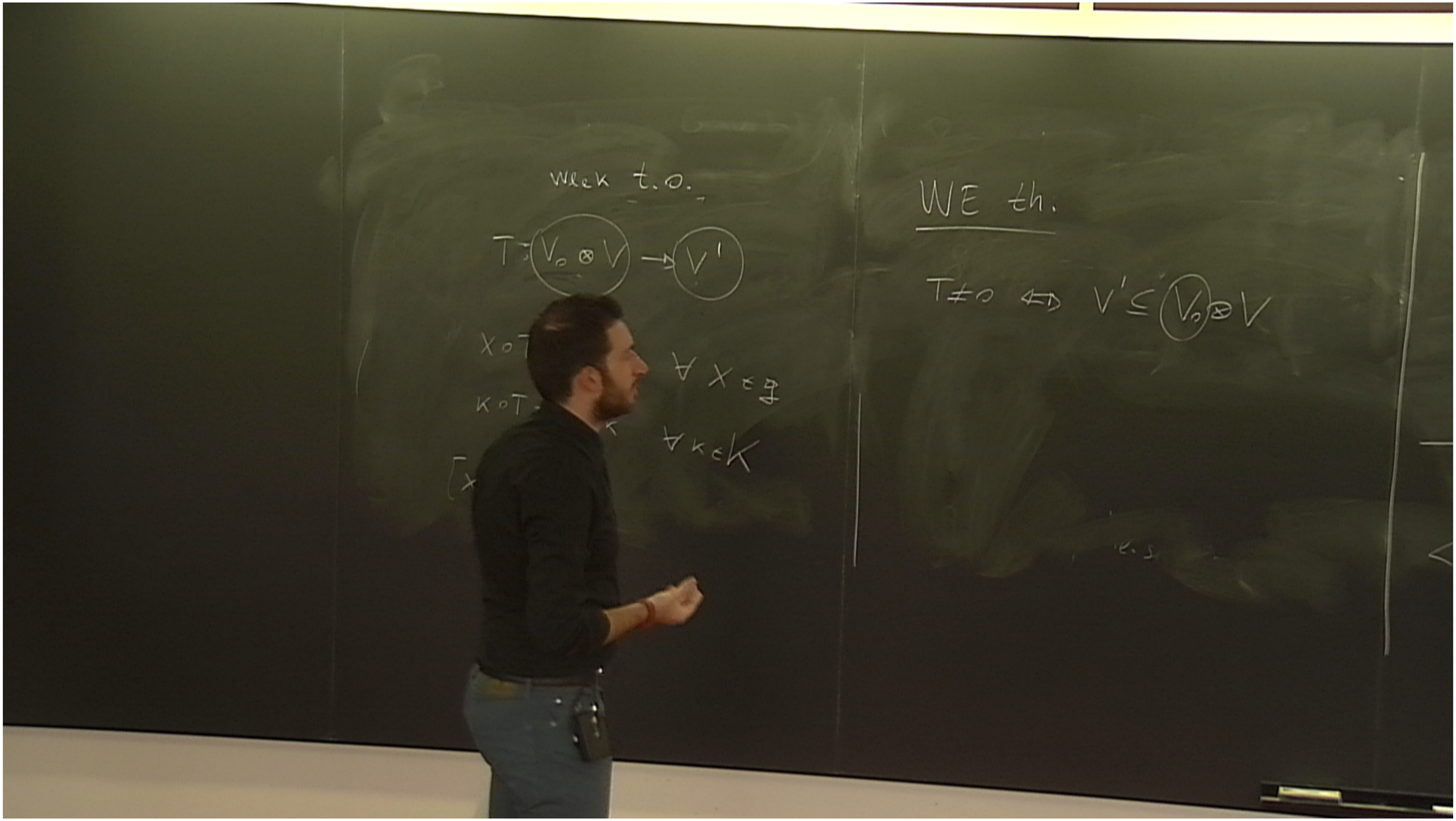

• G-module H
$H_K = \{ v \in H \mid \dim \text{span}\{K v\} < \infty \}$
K-finite vectors
$(\mathfrak{g}, K)$-module
• an admissible G-module H is irr.
iff H_K is
$\frac{d}{dt}\Big|_{t=0} e^{tX} v$

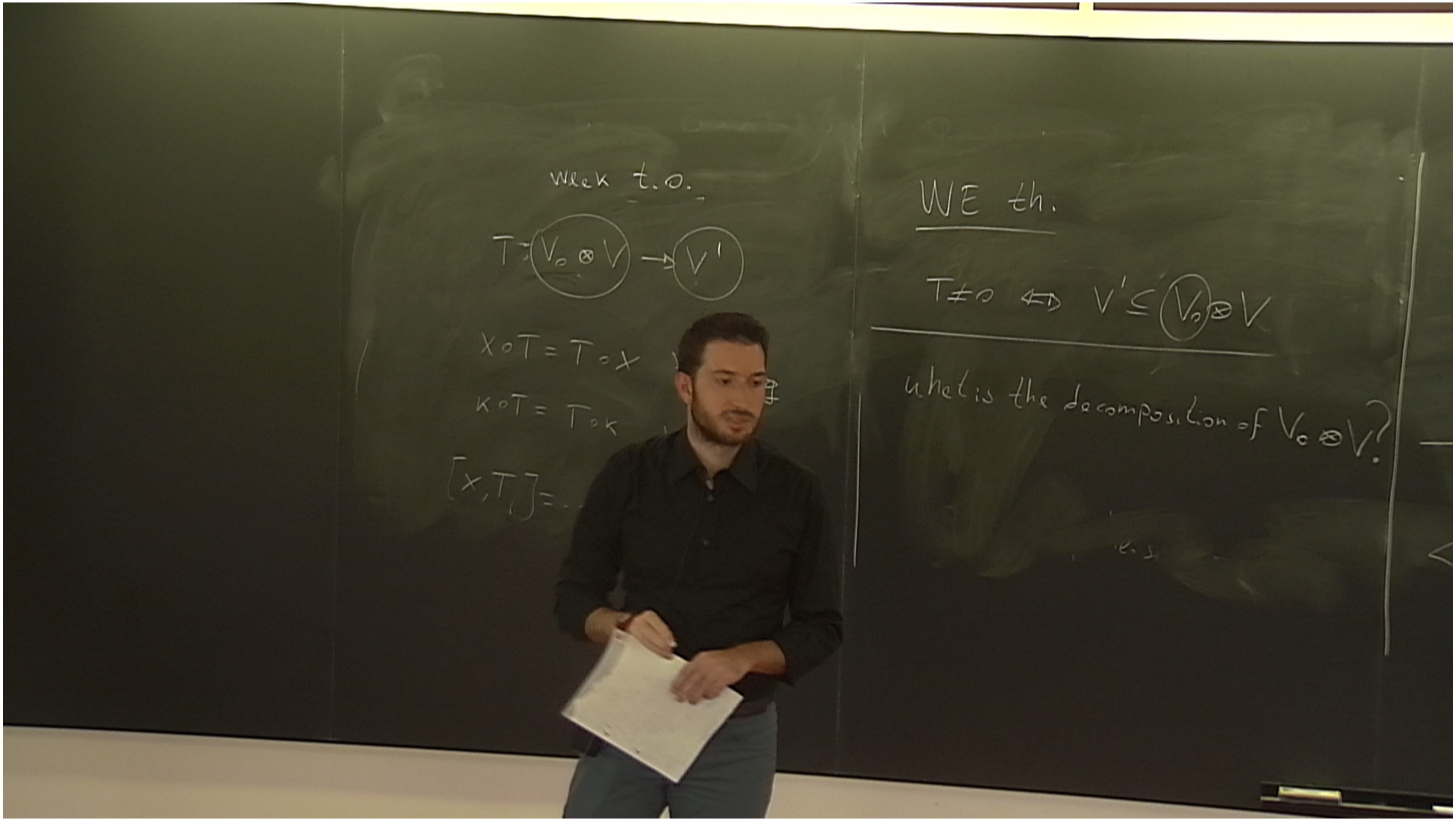

ors
(g, k) module V is admissible
if every K-module appears finitely
many times in V
"nice"
G-module H admissible ⟺ H_K admissible

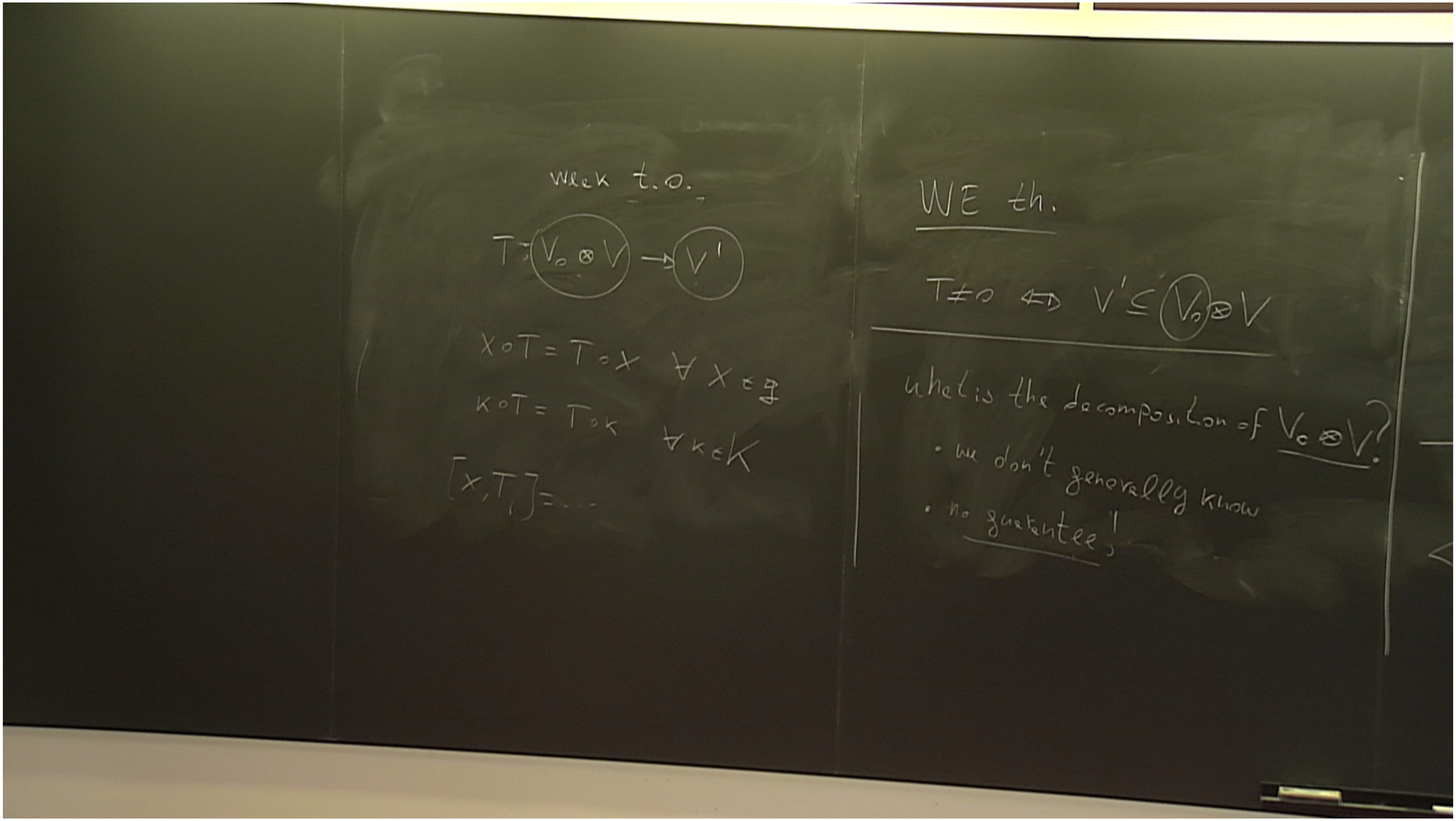

weak t.o.
T: V_0 ⊗ V → V'
X∘T = T∘X ∀X∈g
k∘T = T∘k ∀k∈K
[X,T_i] = ...

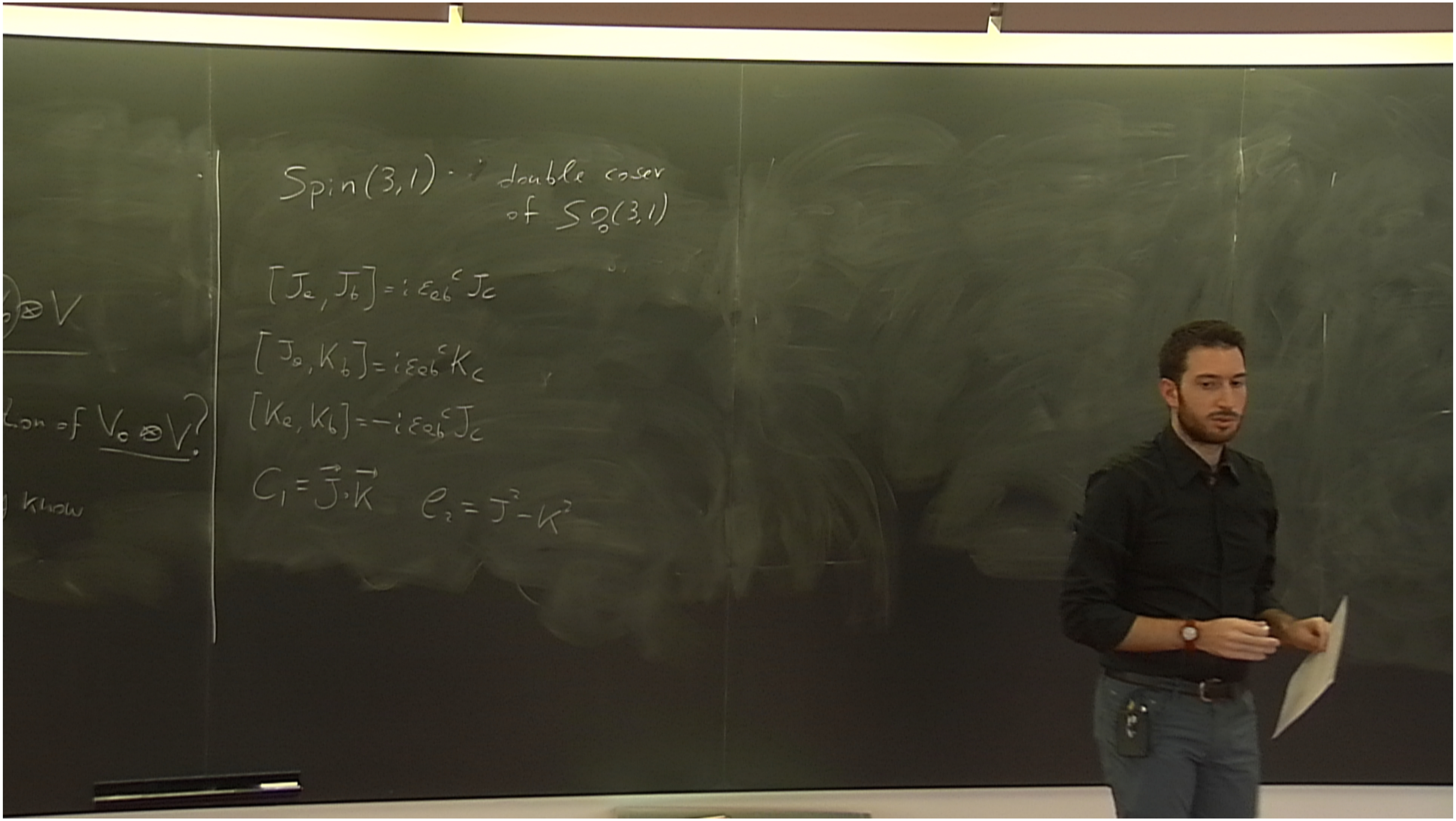

weak t.o.
WE th.
T: V0 ⊗ V → V'
X∘T = T∘X ∀ X ∈ g
k∘T = T∘k ∀ k ∈ K
[X, T] = ...

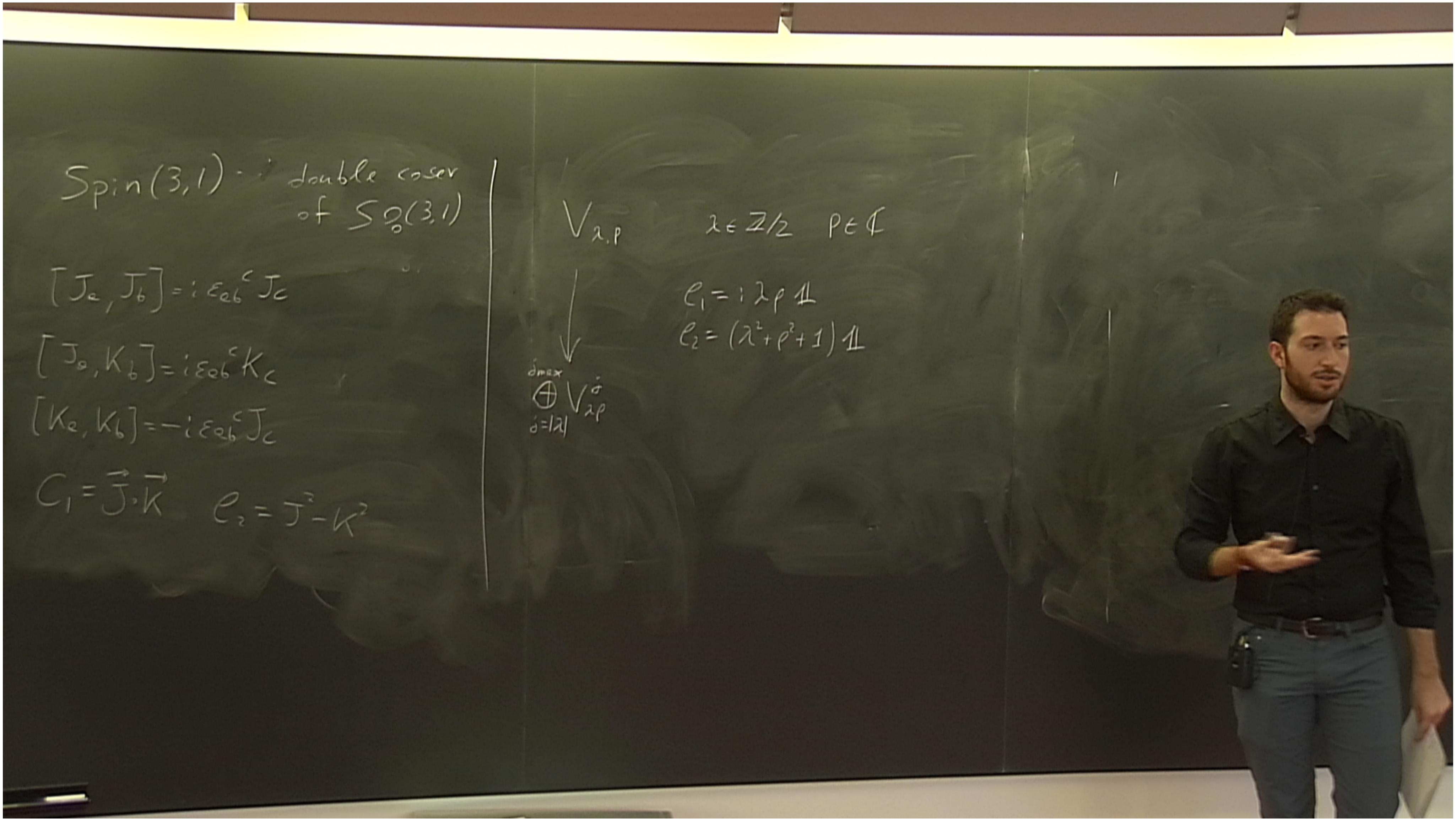

week t.o.
T: V₀ ⊗ V → V'
X∘T = T∘X ∀ X ∈ g
K∘T = T∘K ∀ K ∈ K
[X,T] = ...
WE th.
T≠0 ⇔ V' ⊆ V₀ ⊗ V


week t.o.
WE th.
T≠0 ⟺ V' ≤ V₀⊗V
∀x∈g
∀k∈K

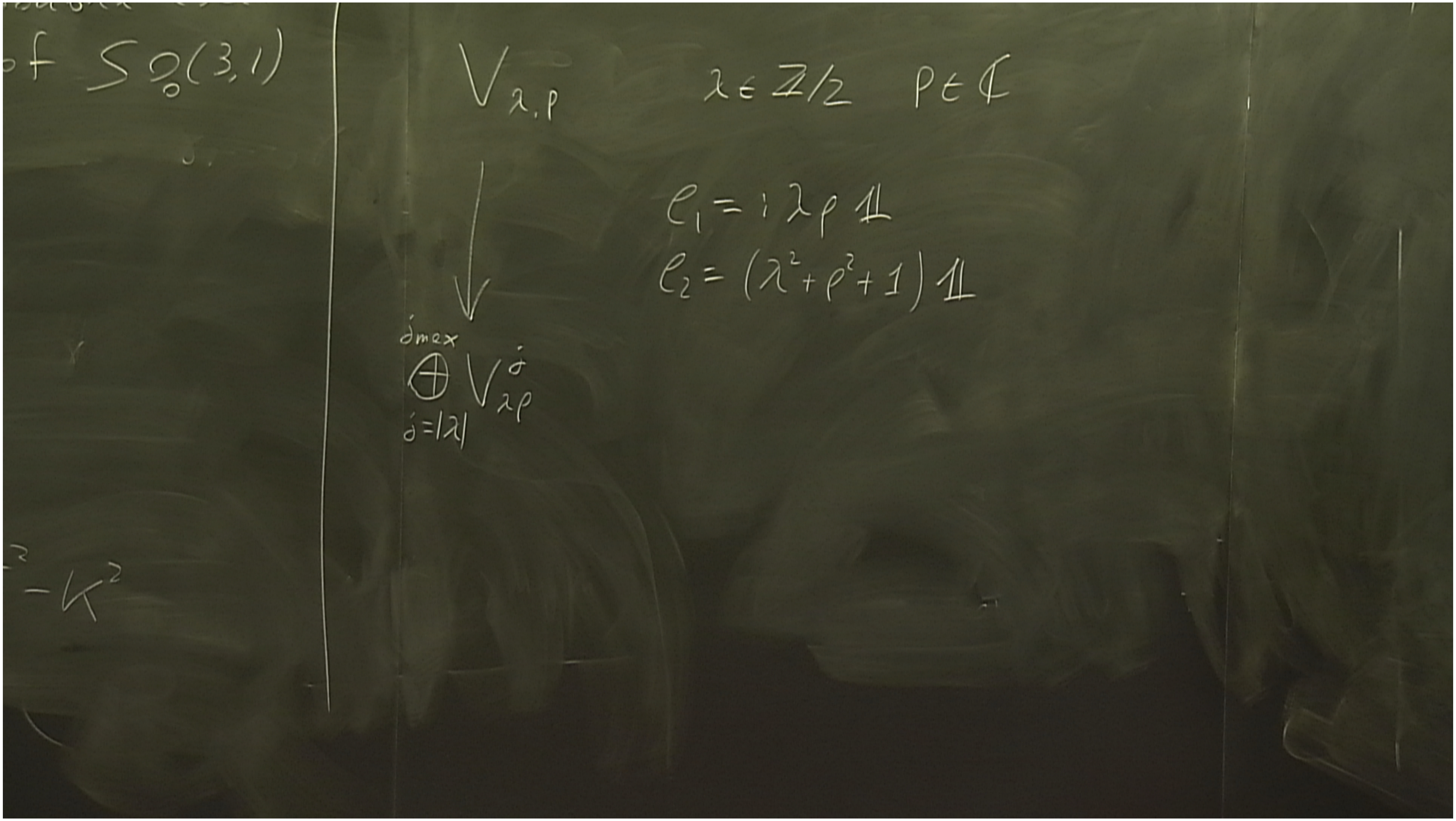

week t.o.
T: V₀ ⊗ V → V'
x∘T = T∘x
k∘T = T∘k
[x,T]=
WE th.
T≠0 ⇔ V' ≤ V₀ ⊗ V
what is the decomposition of V₀ ⊗ V?

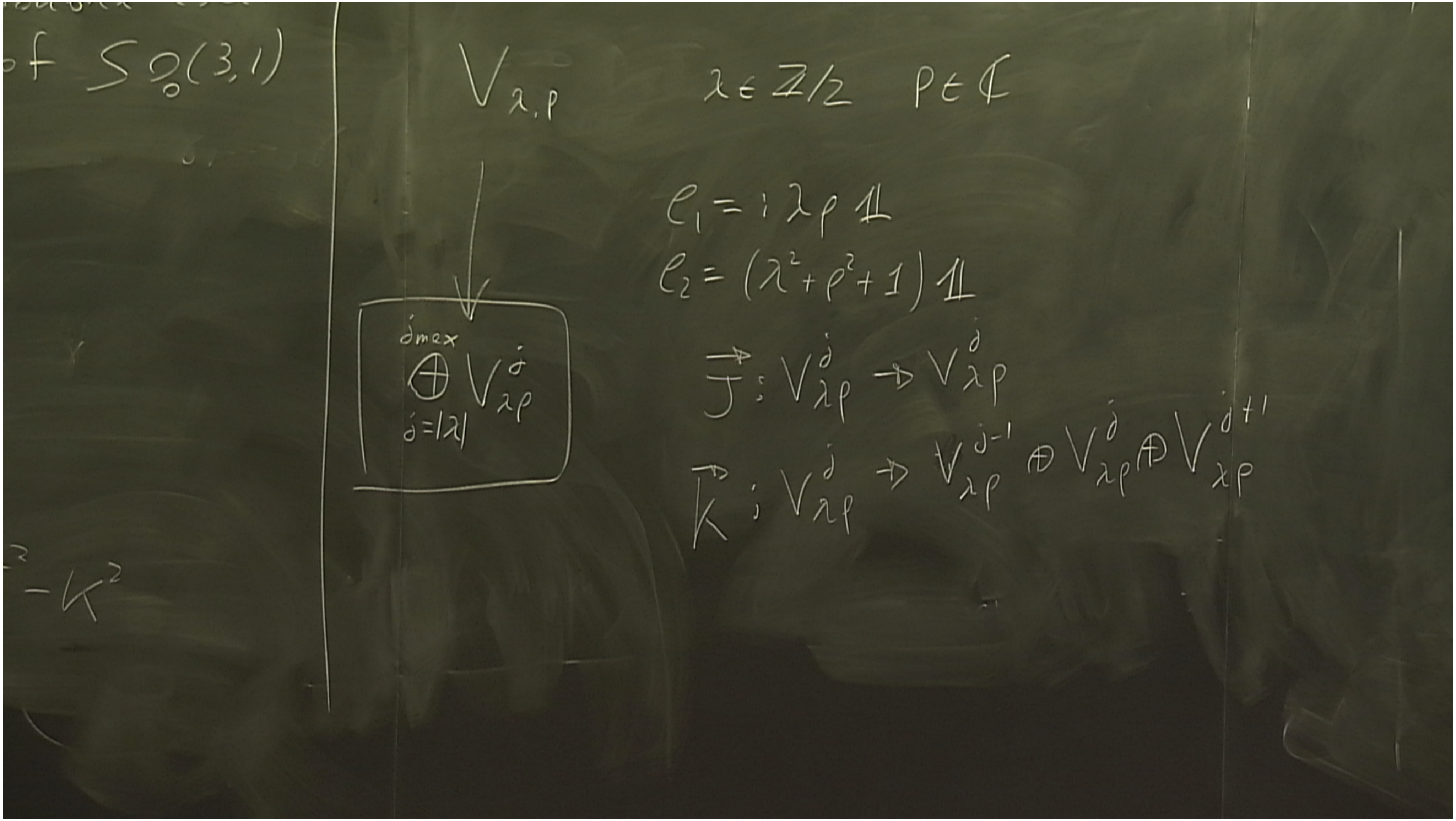

week t.o.
T: V₀ ⊗ V → V'
X∘T = T∘X ∀ X ∈ g
K∘T = T∘K ∀ K ∈ K
[X,T] = ...
WE th.
T≠0 ⇔ V' ⊆ V₀ ⊗ V
what is the decomposition of V₀ ⊗ V?
• we don't generally know
• no guarantee!

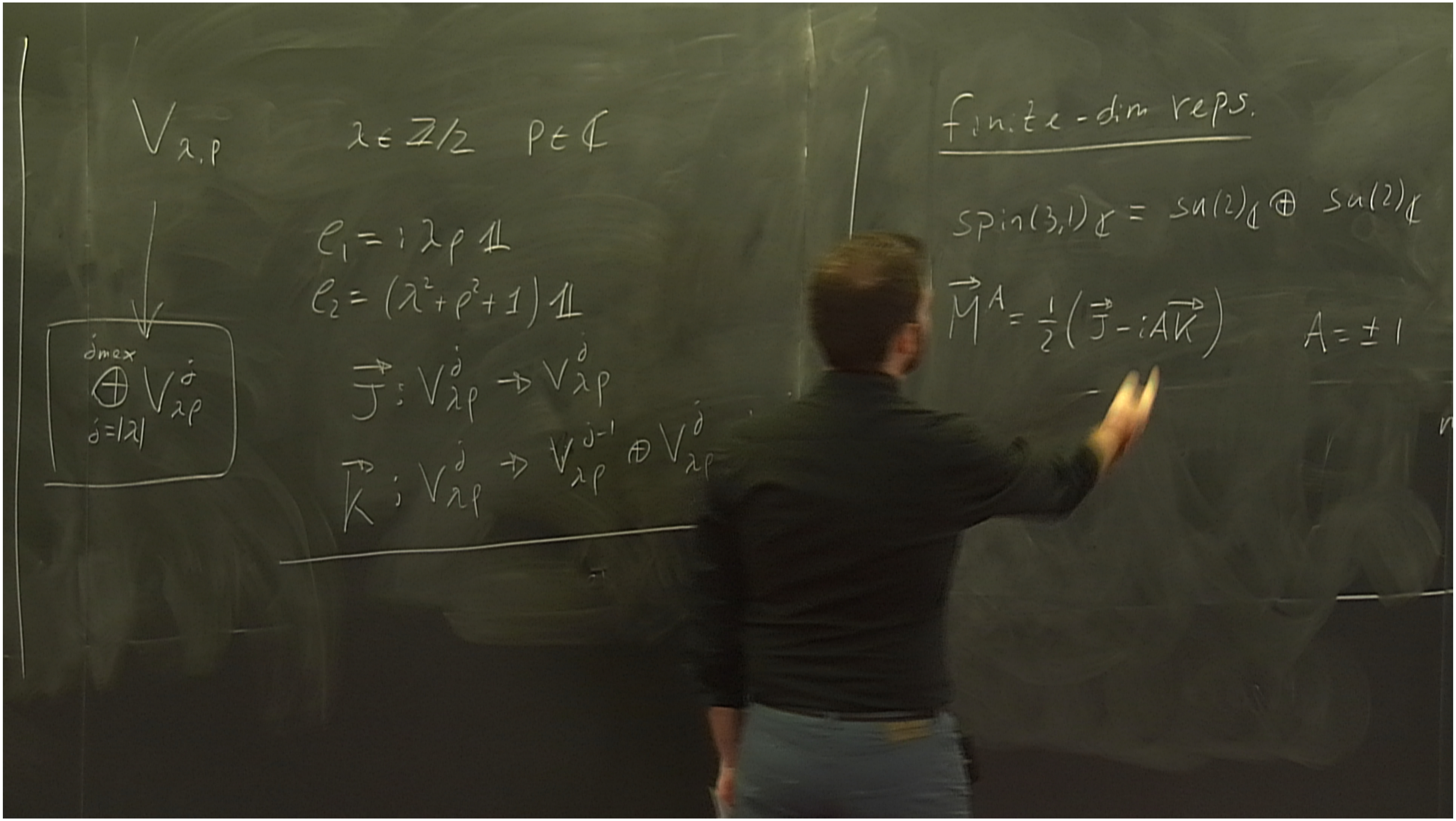

Spin(3,1) double cover
of SO(3,1)
$[J_a, J_b] = i \varepsilon_{ab}{}^{c} J_c$
$[J_a, K_b] = i \varepsilon_{ab}{}^{c} K_c$
$[K_a, K_b] = -i \varepsilon_{ab}{}^{c} J_c$
$C_1 = \vec{J} \cdot \vec{K}$
$C_2 = J^2 - K^2$
$V_0 \otimes V$?

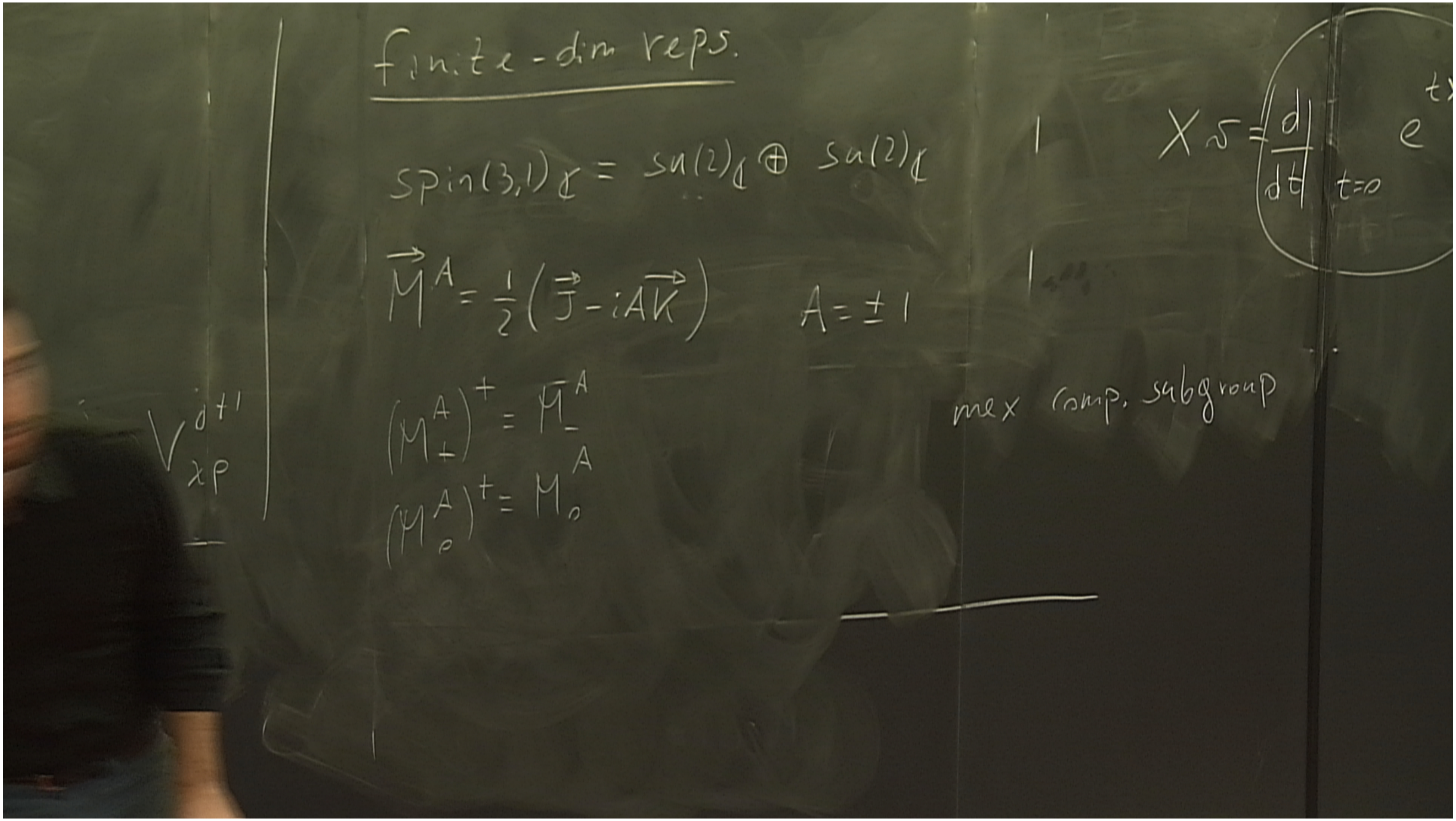


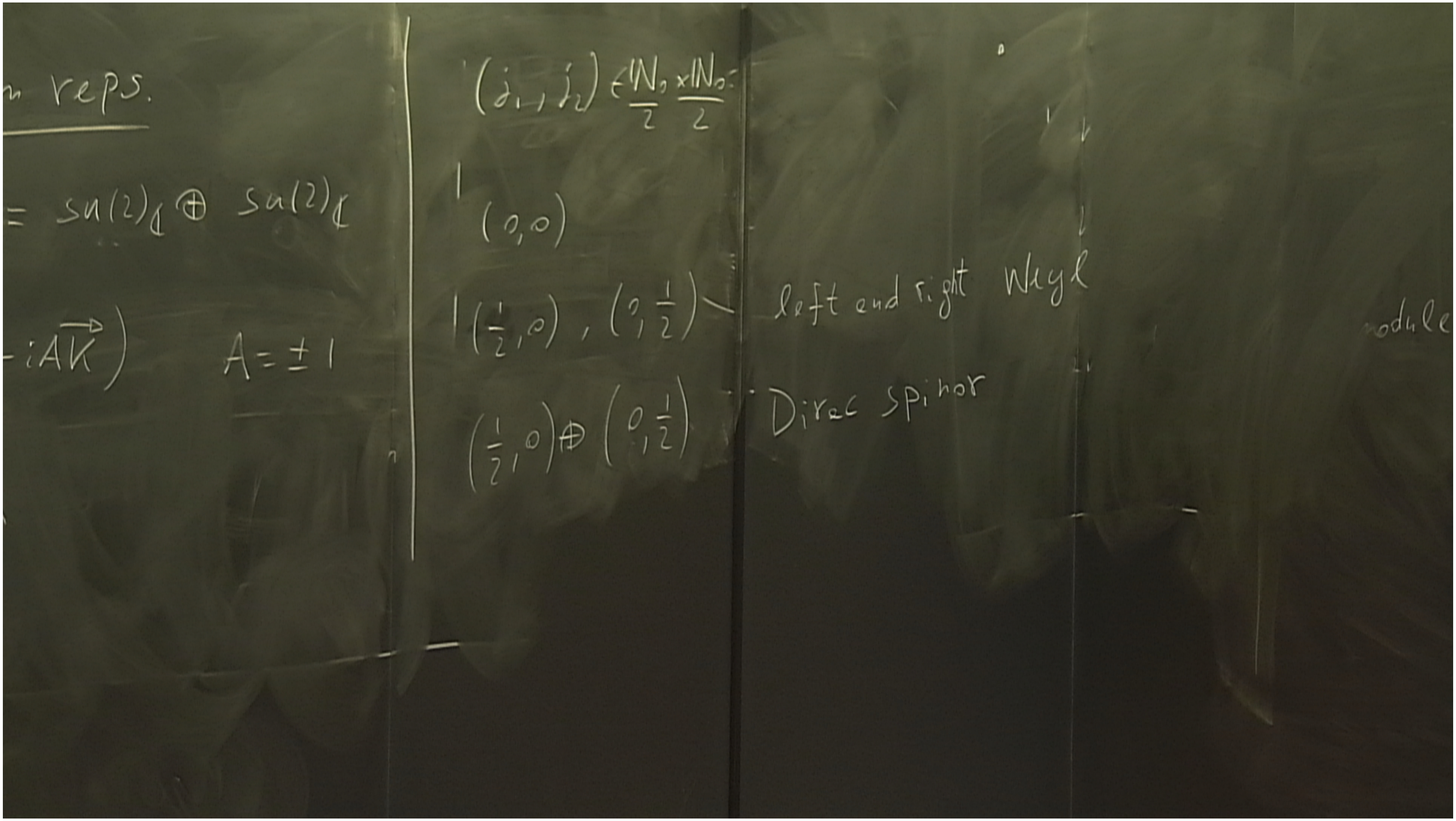
$Spin(3,1)$: double coser of $SO(3,1)$
$[J_a, J_b] = i\varepsilon_{ab}{}^c J_c$
$[J_a, K_b] = i\varepsilon_{ab}{}^c K_c$
$[K_a, K_b] = -i\varepsilon_{ab}{}^c J_c$
$C_1 = \vec{J}\cdot\vec{K}$ $C_2 = J^2 - K^2$
$V_{\lambda,\rho}$ $\lambda \in \mathbb{Z}/2$ $\rho \in \mathbb{C}$
$C_1 = i\lambda\rho\, \mathbb{1}$
$C_2 = (\lambda^2 + \rho^2 + 1)\, \mathbb{1}$
$\bigoplus_{j=|\lambda|}^{j_{max}} V^{j}_{\lambda\rho}$

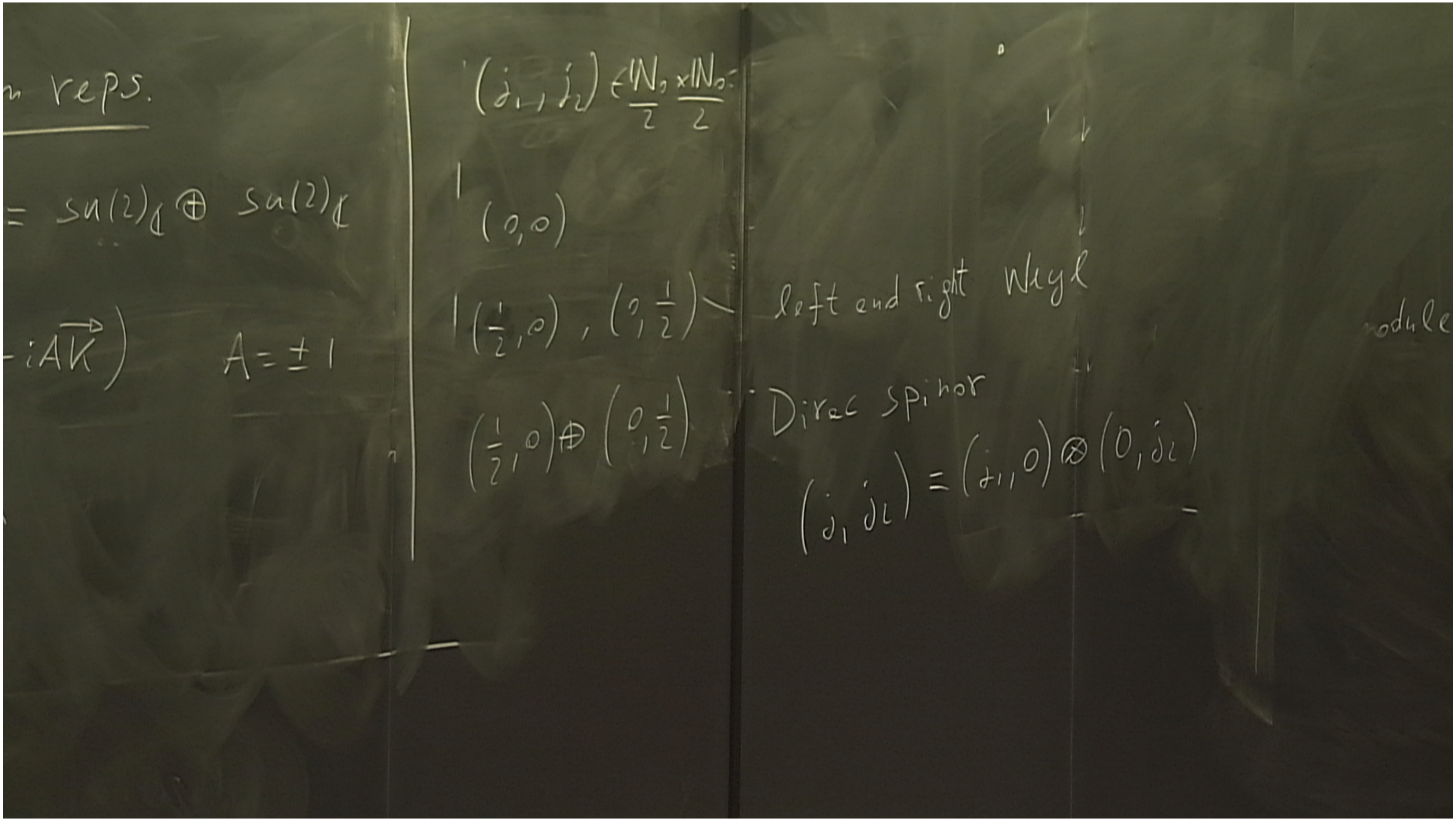

of SO(3,1)
V_λ,ρ
λ ∈ Z/2   ρ ∈ C
C_1 = iλρ 1
C_2 = (λ^2 + ρ^2 + 1) 1
⊕ V^j_λρ
j_max
j = |λ|
-K^2

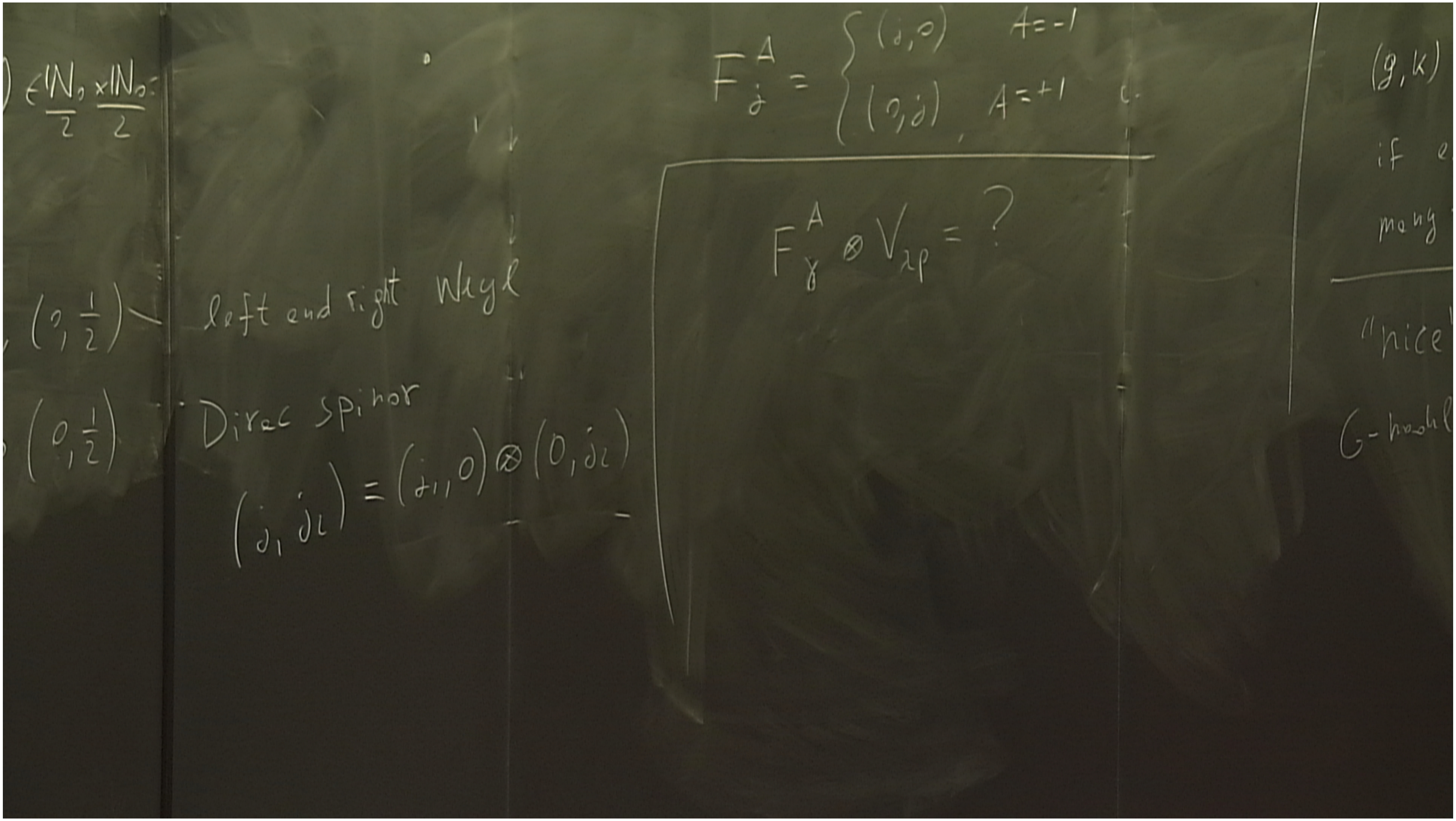

of $SO(3,1)$
$V_{\lambda,\rho}$
$\lambda \in \mathbb{Z}/2 \quad \rho \in \mathbb{C}$
$C_1 = i\lambda\rho \, \mathbb{1}$
$C_2 = (\lambda^2+\rho^2+1)\, \mathbb{1}$
$\bigoplus_{j=|\lambda|}^{j_{max}} V^j_{\lambda\rho}$
$\vec{J} : V^j_{\lambda\rho} \to V^j_{\lambda\rho}$
$\vec{K} : V^j_{\lambda\rho} \to V^{j-1}_{\lambda\rho} \oplus V^j_{\lambda\rho} \oplus V^{j+1}_{\lambda\rho}$
$-K^2$

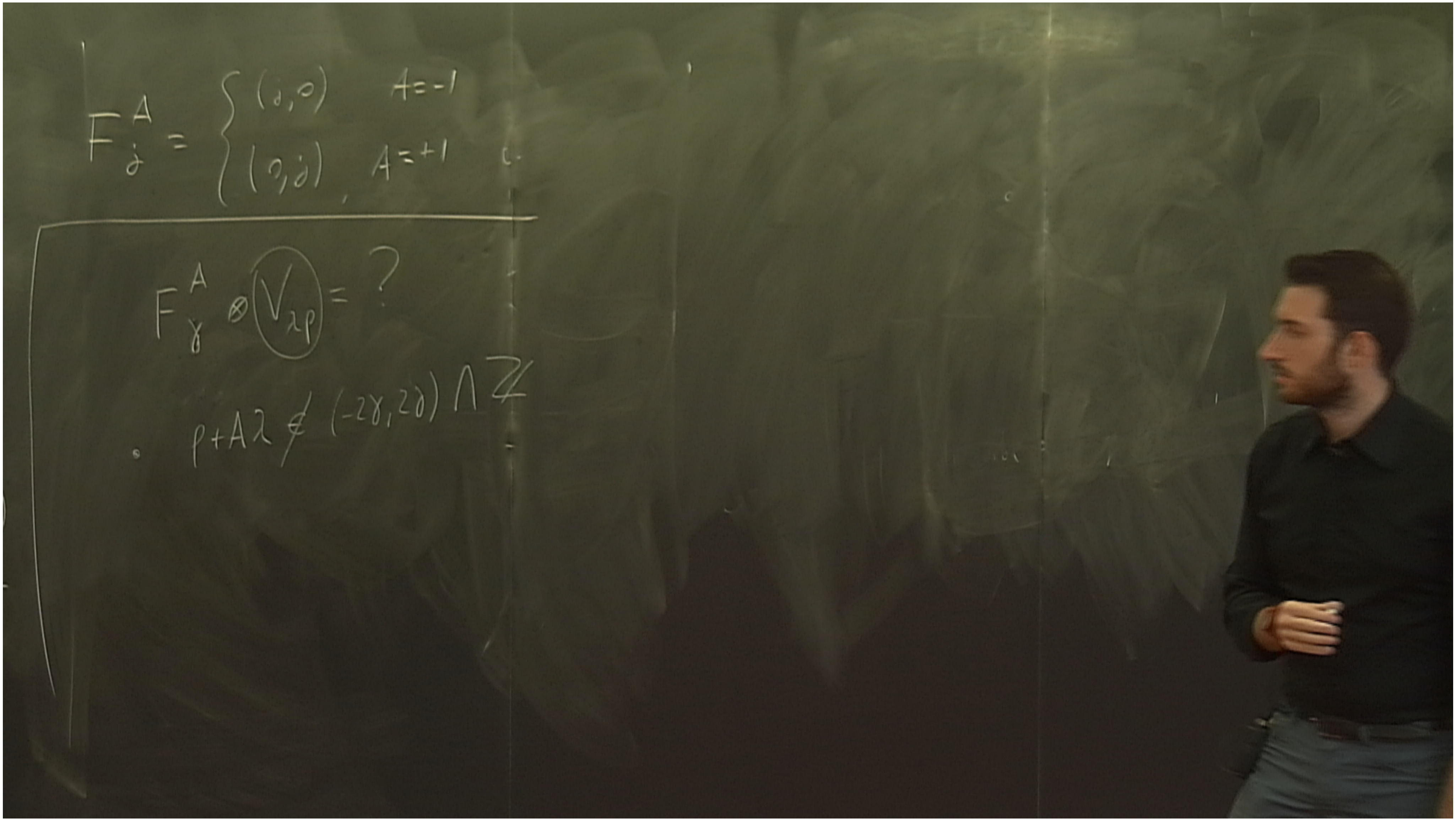

V λ,ρ
λ ∈ ℤ/2   ρ ∈ ℂ
C1 = i λ ρ 𝟙
C2 = (λ² + ρ² + 1) 𝟙
jmax
⊕ V j λρ
j=|λ|
J : V j λρ → V j λρ
K : V j λρ → V j-1 λρ ⊕ V j λρ
finite-dim reps.
spin(3,1)ℂ = su(2)ℂ ⊕ su(2)ℂ
M A = ½ (J - iA K)
A = ± 1

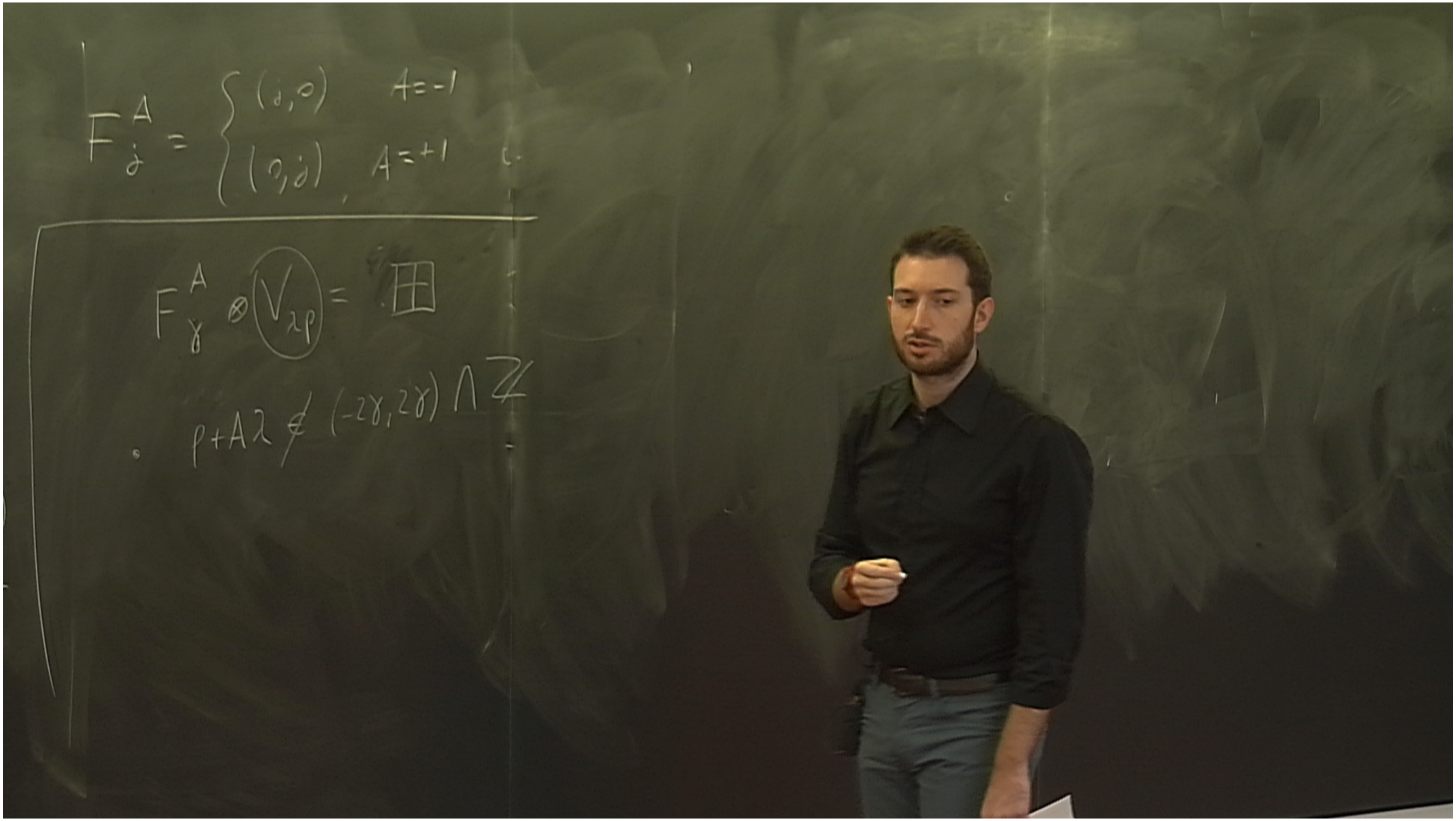

finite-dim reps.
spin(3,1)ℂ = su(2)ℂ ⊕ su(2)ℂ
M^A = 1/2 (J - iAK)
A = ±1
(M^A_+)^† = M^A_-
(M^A_0)^† = M^A_0
max comp. subgroup
Xv = d/dt |t=0 e^tX

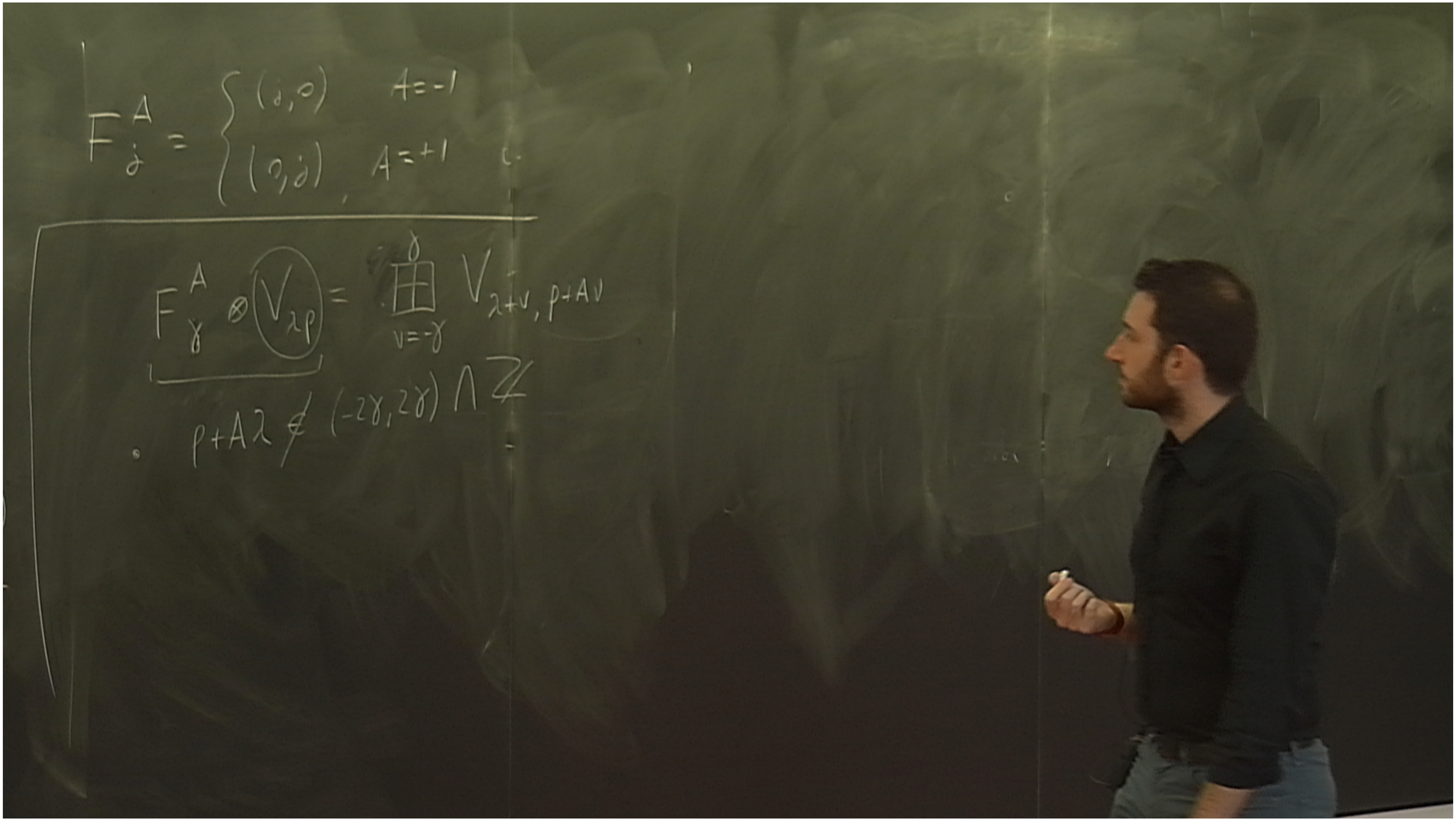

reps.
= su(2)_ℂ ⊕ su(2)_ℂ
iAK⃗)
A = ±1
(j₁, j₂) ∈ ℕ₀/2 × ℕ₀/2
(0,0)
(1/2, 0), (0, 1/2)
left and right Weyl
(1/2, 0) ⊕ (0, 1/2)
Dirac spinor
odule

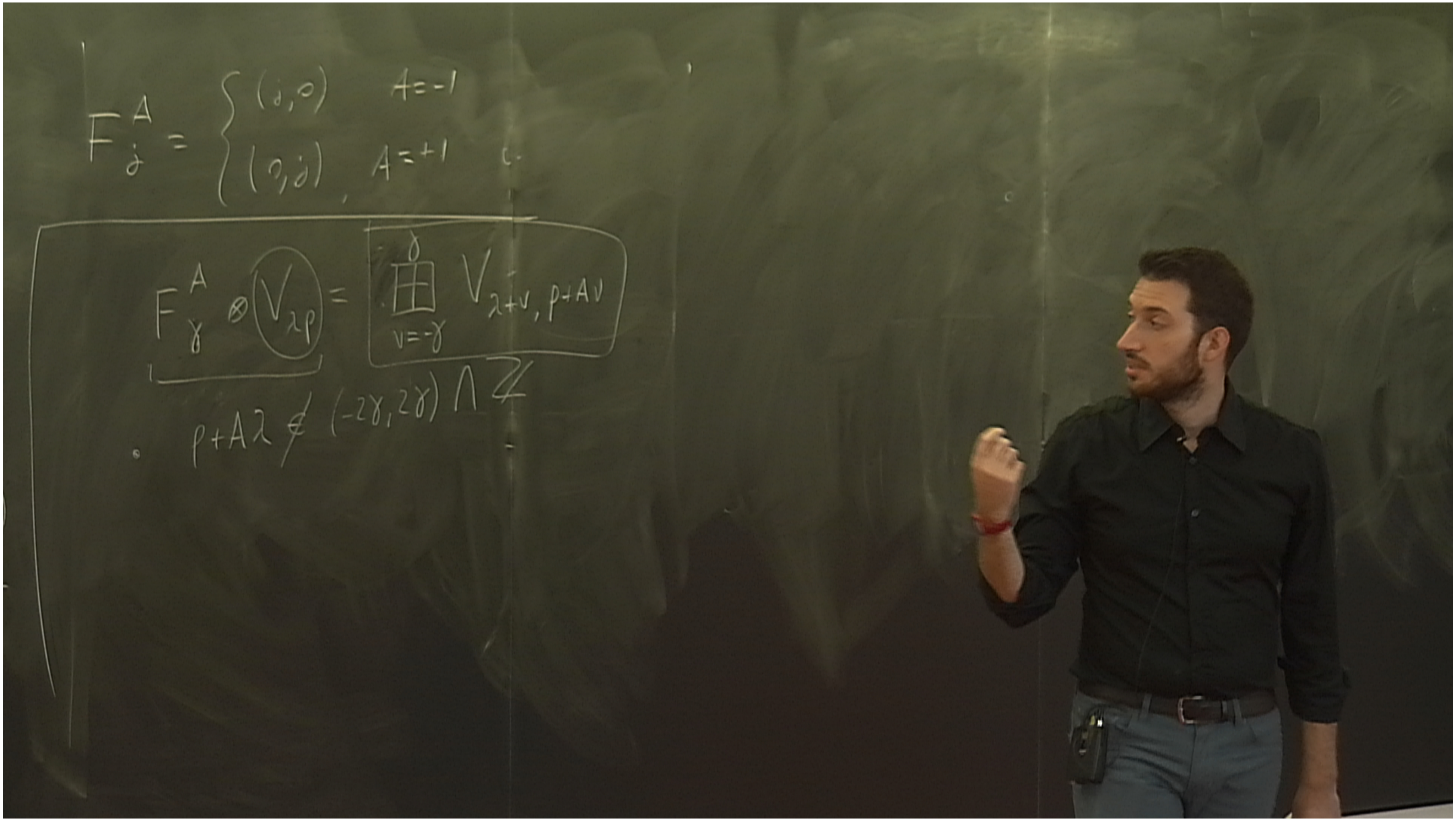

reps.
= su(2) ⊕ su(2)
iAK
A = ±1
(j1, j2) ∈ N0/2 × N0/2
(0,0)
(1/2, 0), (0, 1/2)
left and right Weyl
(1/2, 0) ⊕ (0, 1/2)
Dirac spinor
(j1, j2) = (j1, 0) ⊗ (0, j2)
module

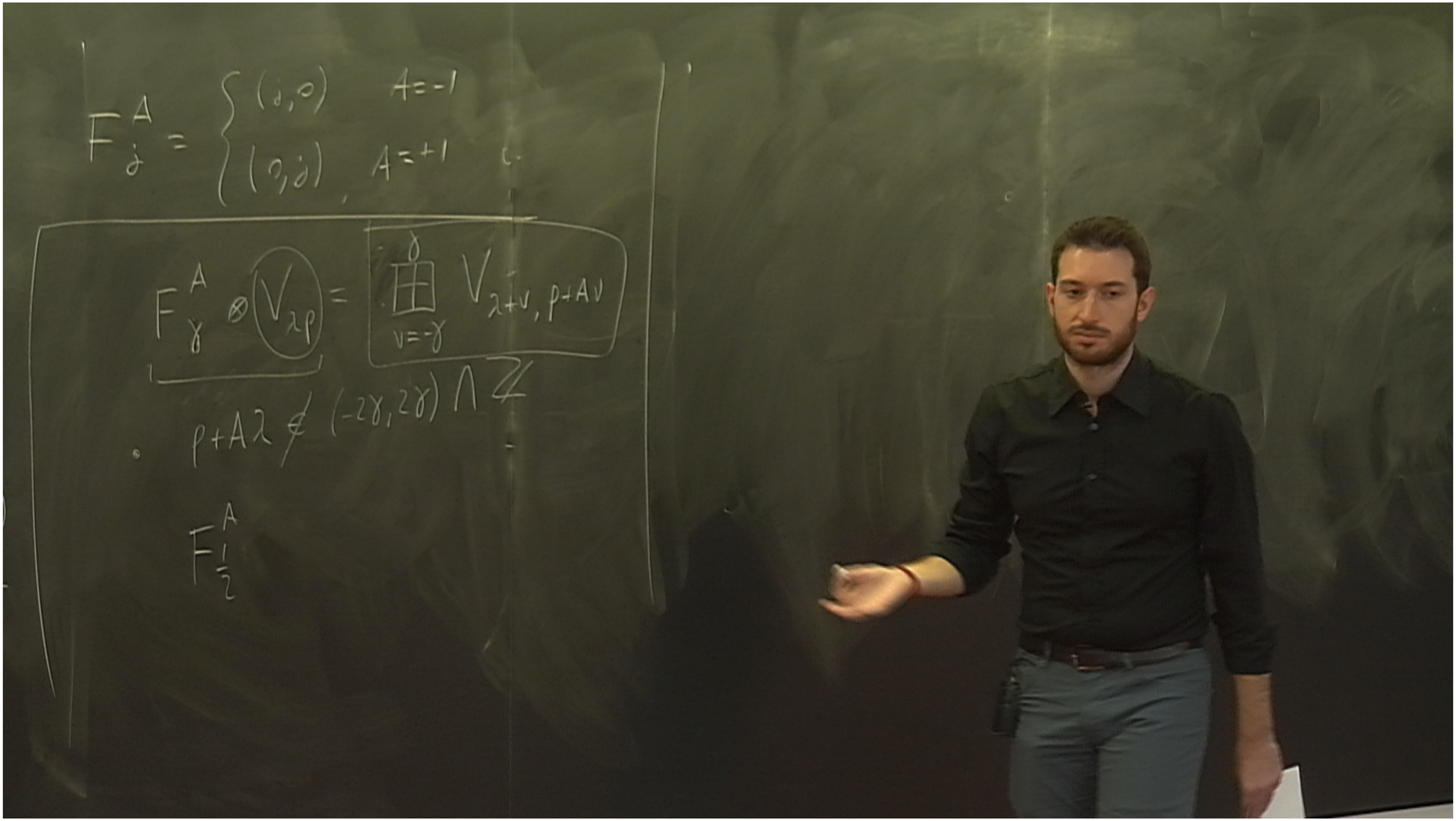

$F^A_j = \begin{cases} (j,0) & A=-1 \\ (0,j) & A=+1 \end{cases}$
$F^A_\gamma \otimes V_{\lambda p} = ?$
left and right Weyl
Dirac spinor
$(j_1, j_2) = (j_1, 0) \otimes (0, j_2)$
(g,k)
"nice"

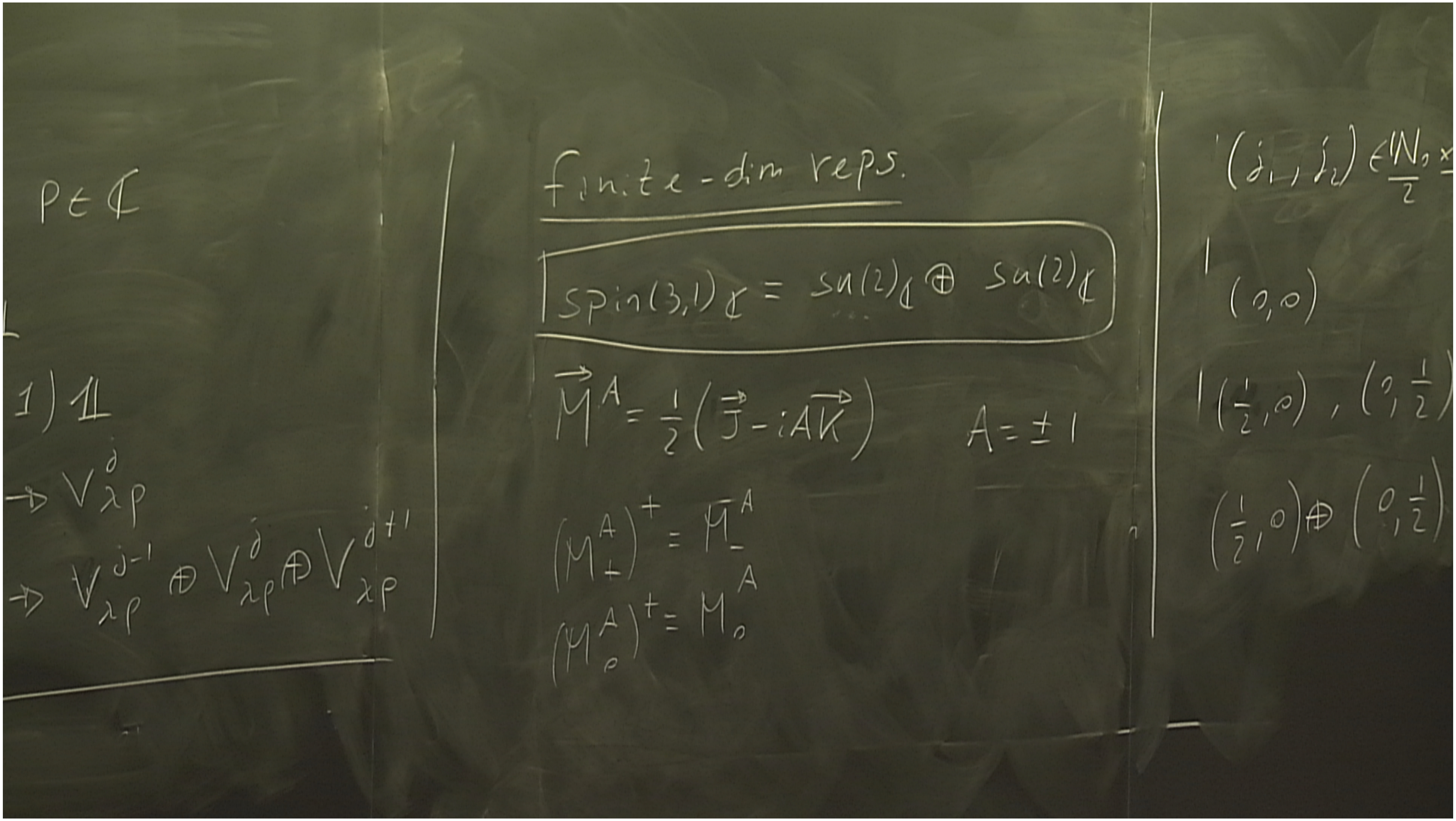

F_j^A = { (j,0) A=-1 ; (0,j) A=+1
F_γ^A ⊗ V_λρ = ?
ρ+Aλ ∉ (-2γ,2γ) ∩ Z

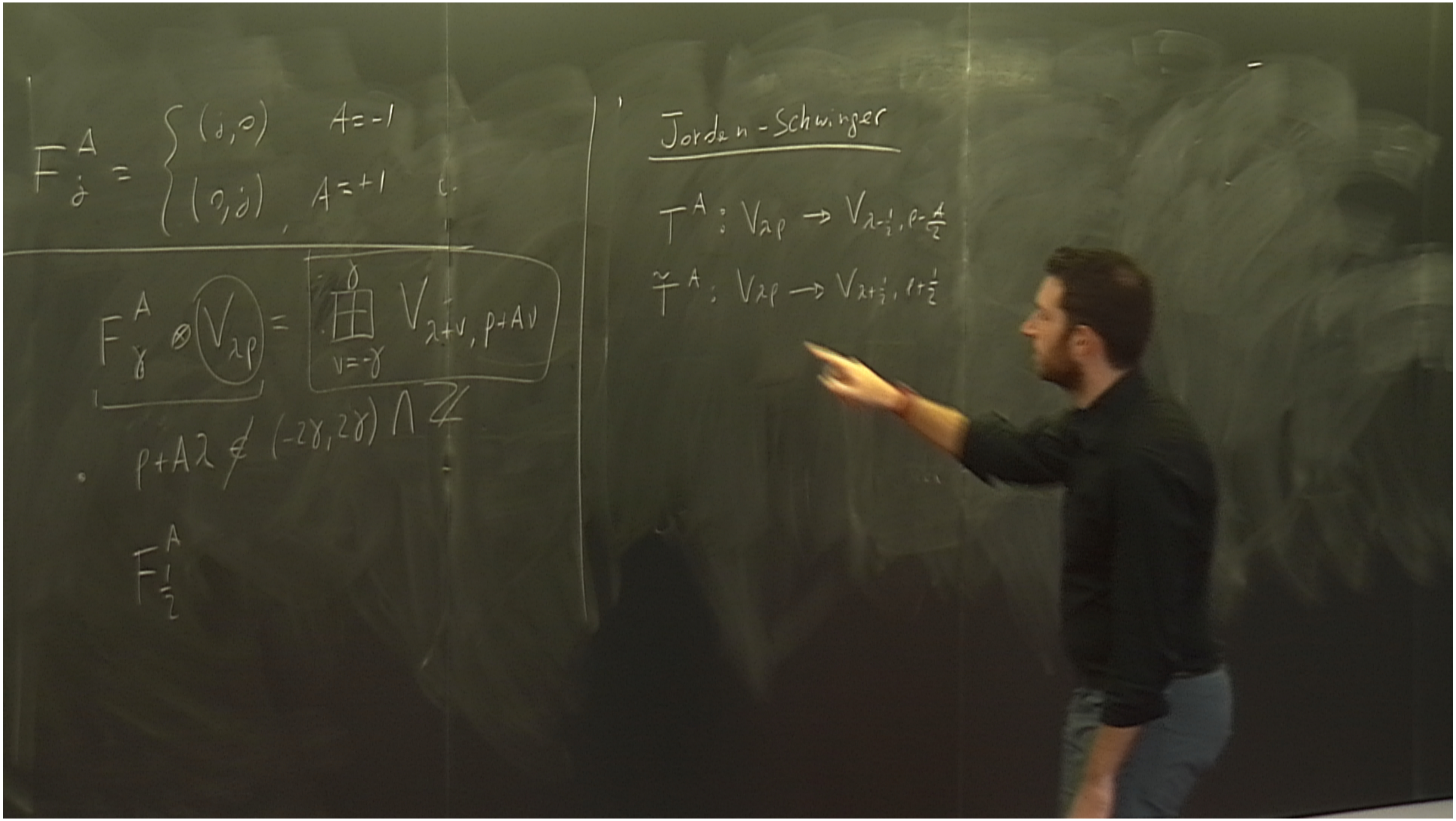

F_j^A = { (j,0) A=-1 ; (0,j) A=+1
F_γ^A ⊗ V_λρ = ⊞
ρ+Aλ ∉ (-2γ,2γ) ∩ Z

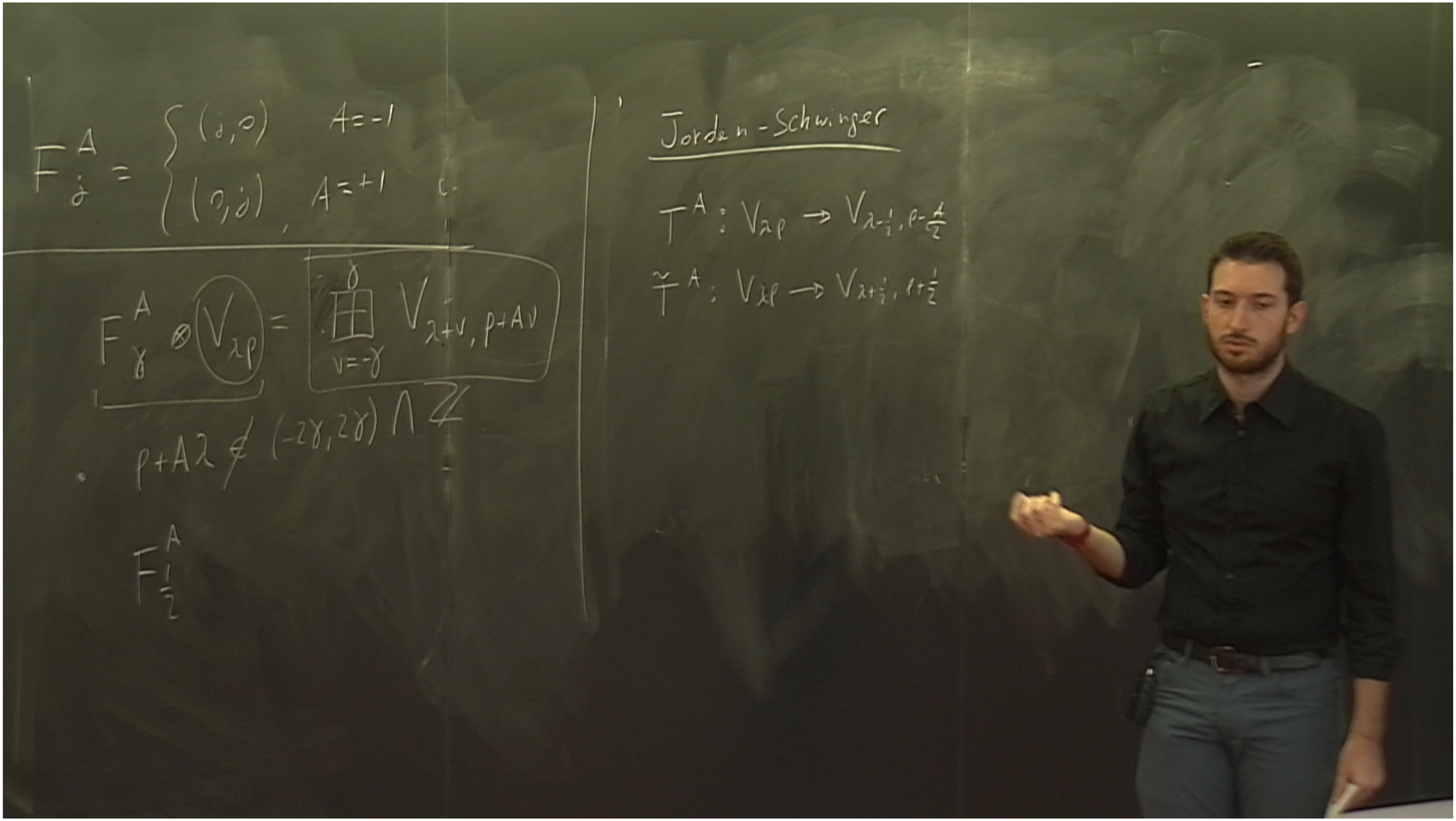

$F^A_j = \begin{cases} (j,0) & A=-1 \\ (0,j) & A=+1 \end{cases}$
$F^A_\gamma \otimes V_{\lambda\rho} = \bigoplus_{\nu=-\gamma}^{\gamma} V_{\lambda+\nu,\, \rho+A\nu}$
$\rho + A\lambda \notin (-2\gamma, 2\gamma) \cap \mathbb{Z}$

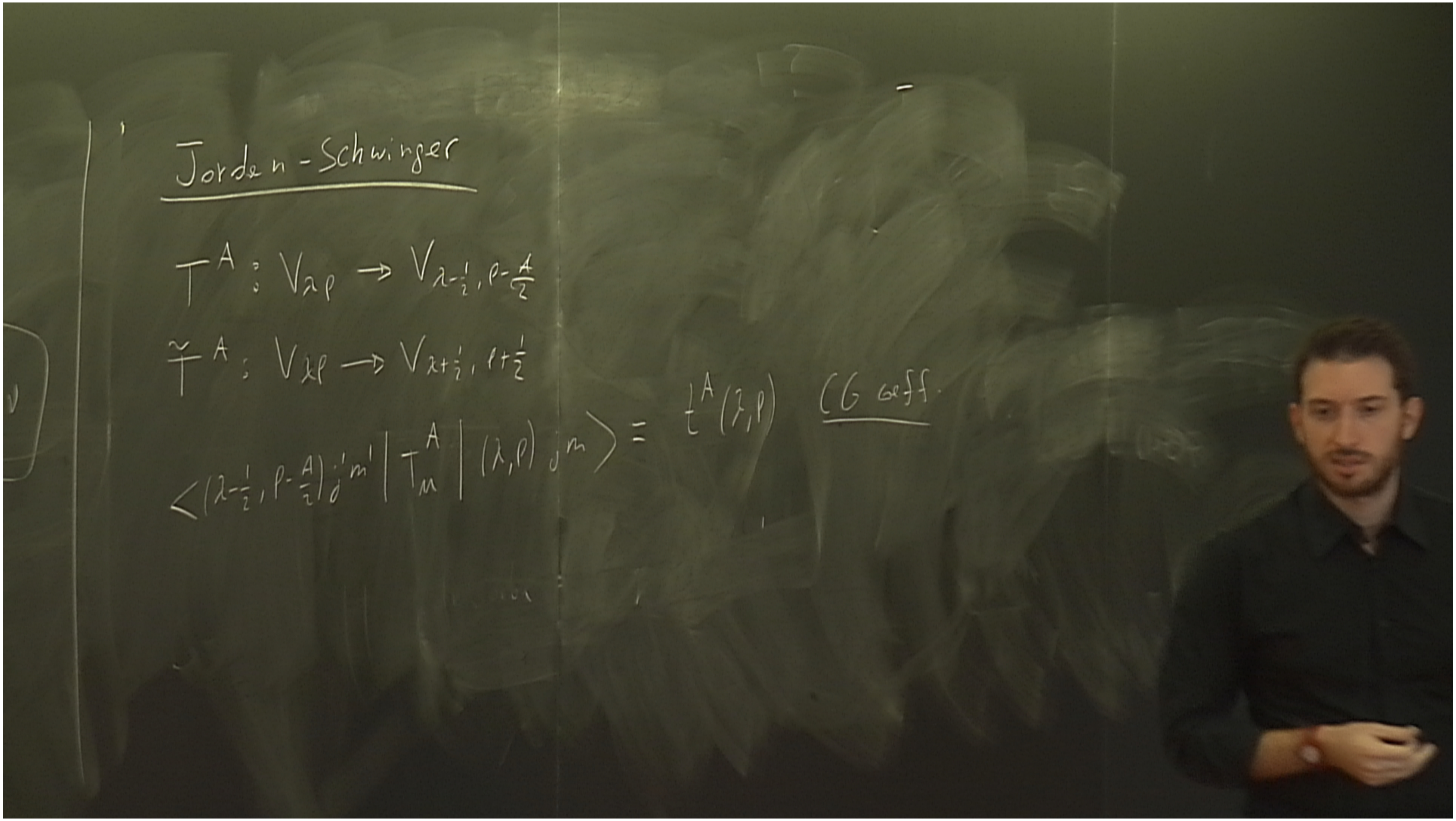

$F_j^A = \begin{cases} (j,0) & A=-1 \\ (0,j) & A=+1 \end{cases}$
$F_\gamma^A \otimes V_{\lambda\rho} = \bigoplus_{\nu=-\gamma}^{\gamma} V_{\lambda+\nu,\, \rho+A\nu}$
$\rho + A\lambda \notin (-2\gamma, 2\gamma) \cap \mathbb{Z}$

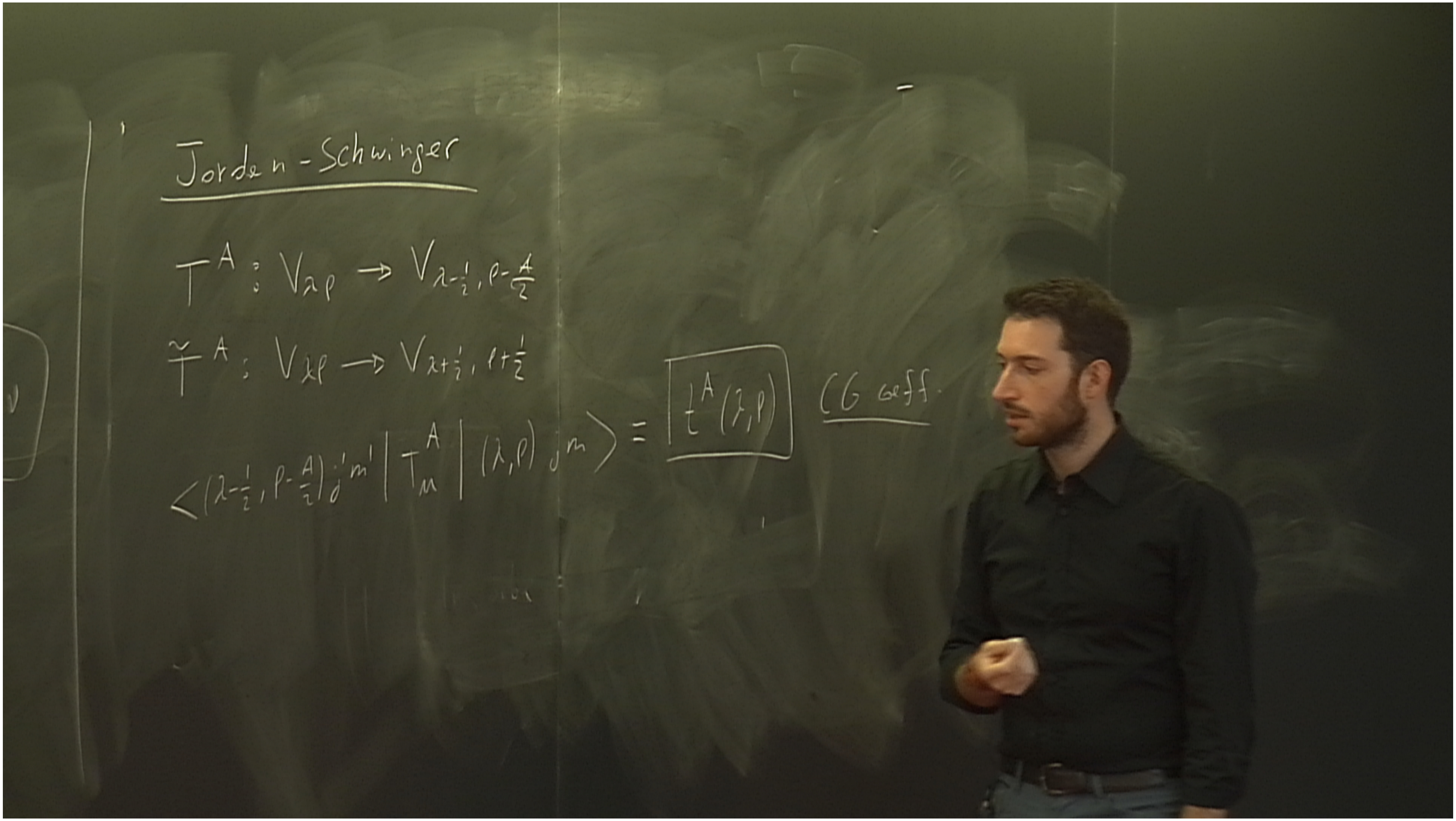

F_j^A = (j,0) A=-1
(0,j) A=+1
F_γ^A ⊗ V_λρ = ⊕_{ν=-γ}^{γ} V_{λ+ν, ρ+Aν}
ρ+Aλ ∉ (-2γ,2γ) ∩ Z
F_{1/2}^A

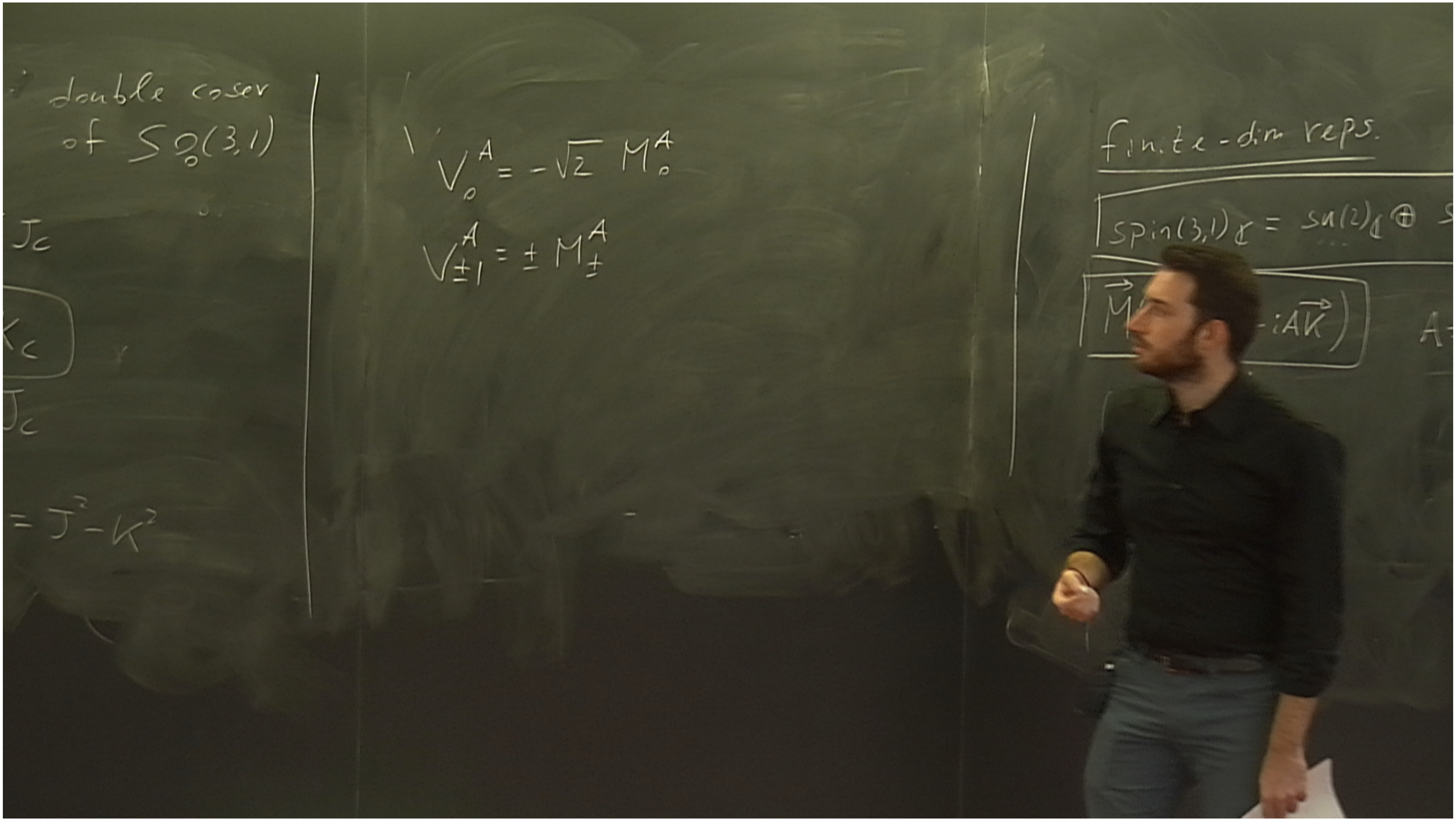

finite-dim reps.
spin(3,1)ℂ = su(2)ℂ ⊕ su(2)ℂ
A = ±1
(0,0)
(1/2, 0) ⊕ (0, 1/2)

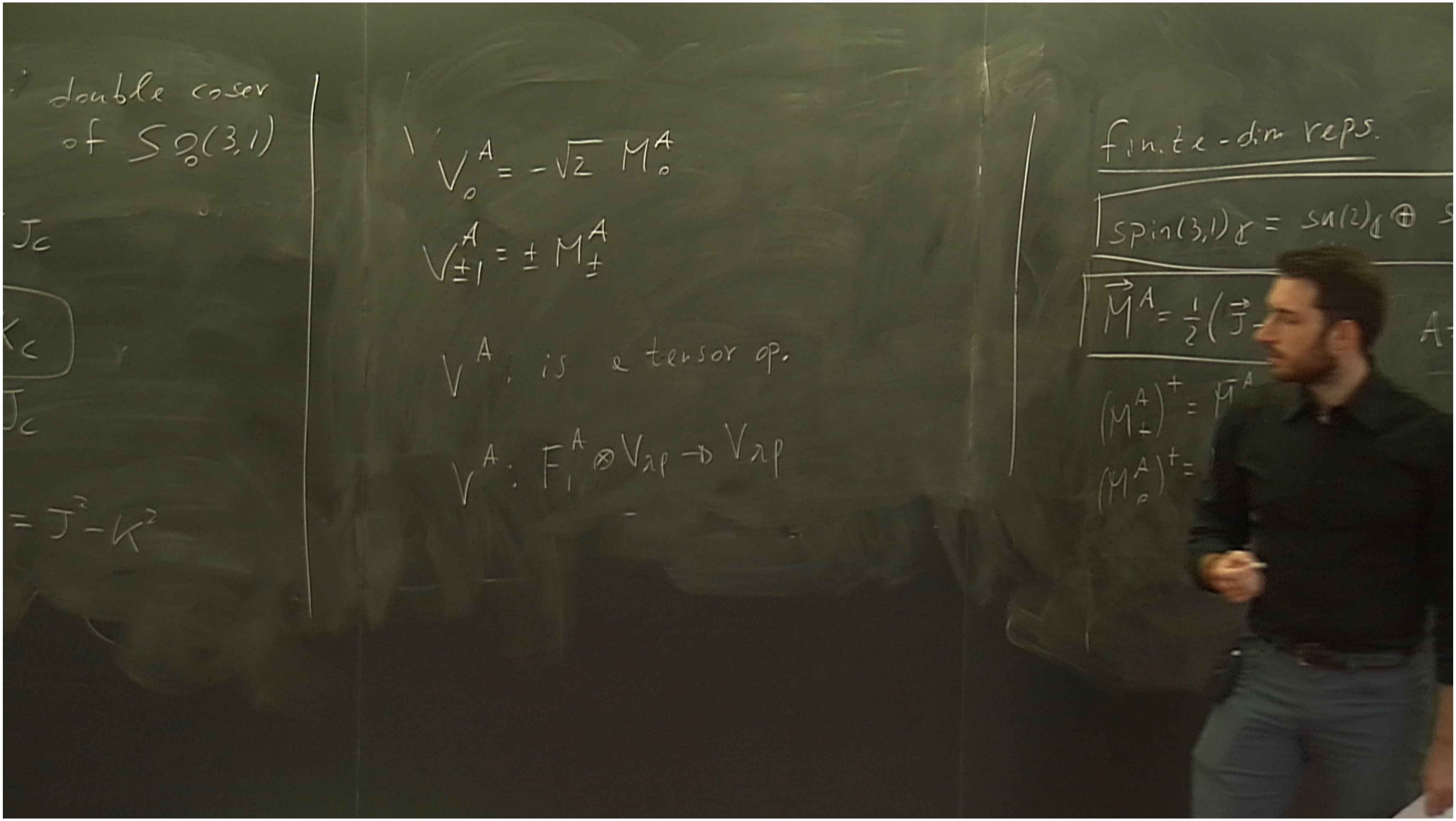

Jordan-Schwinger

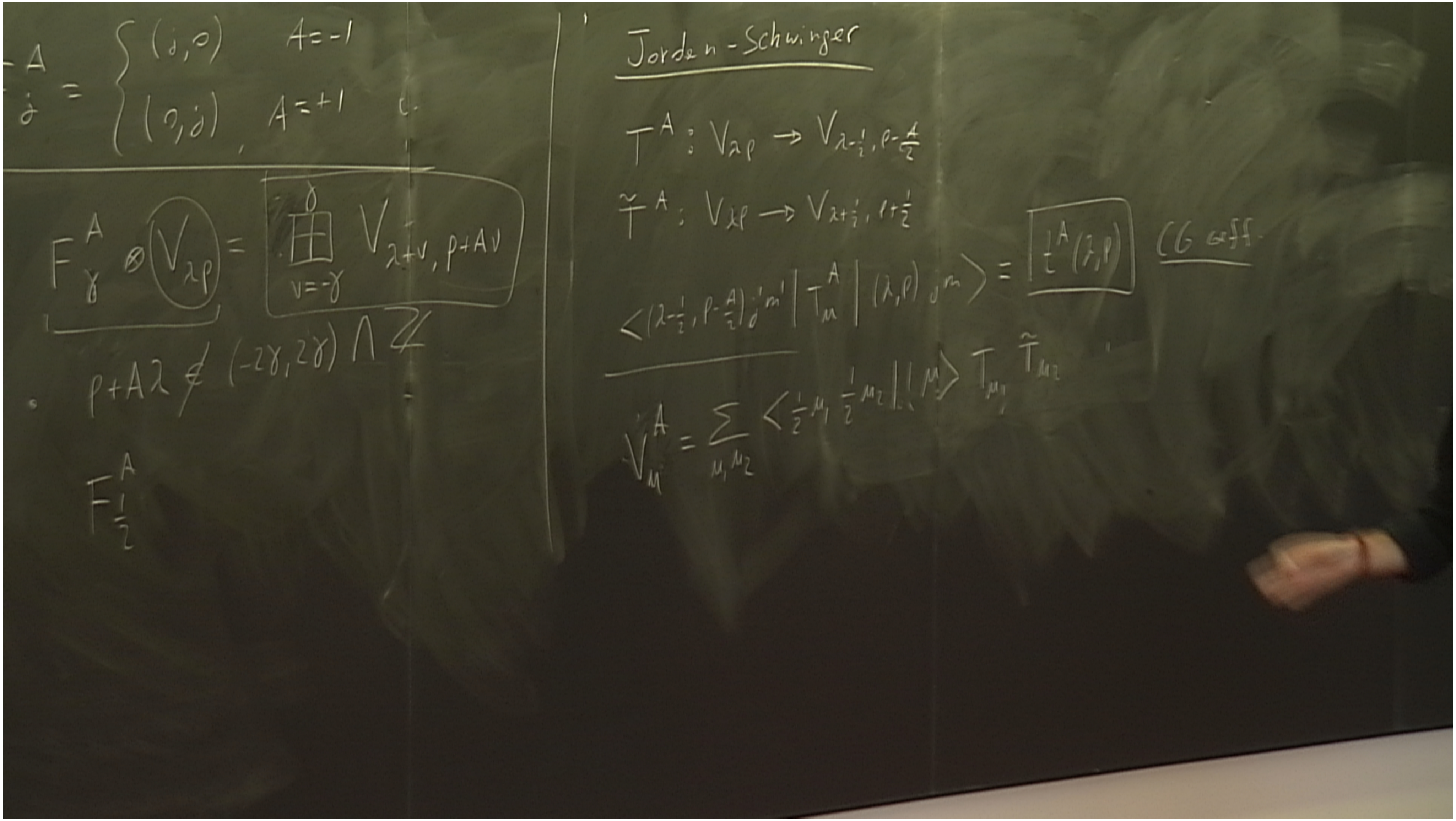

Jordan-Schwinger

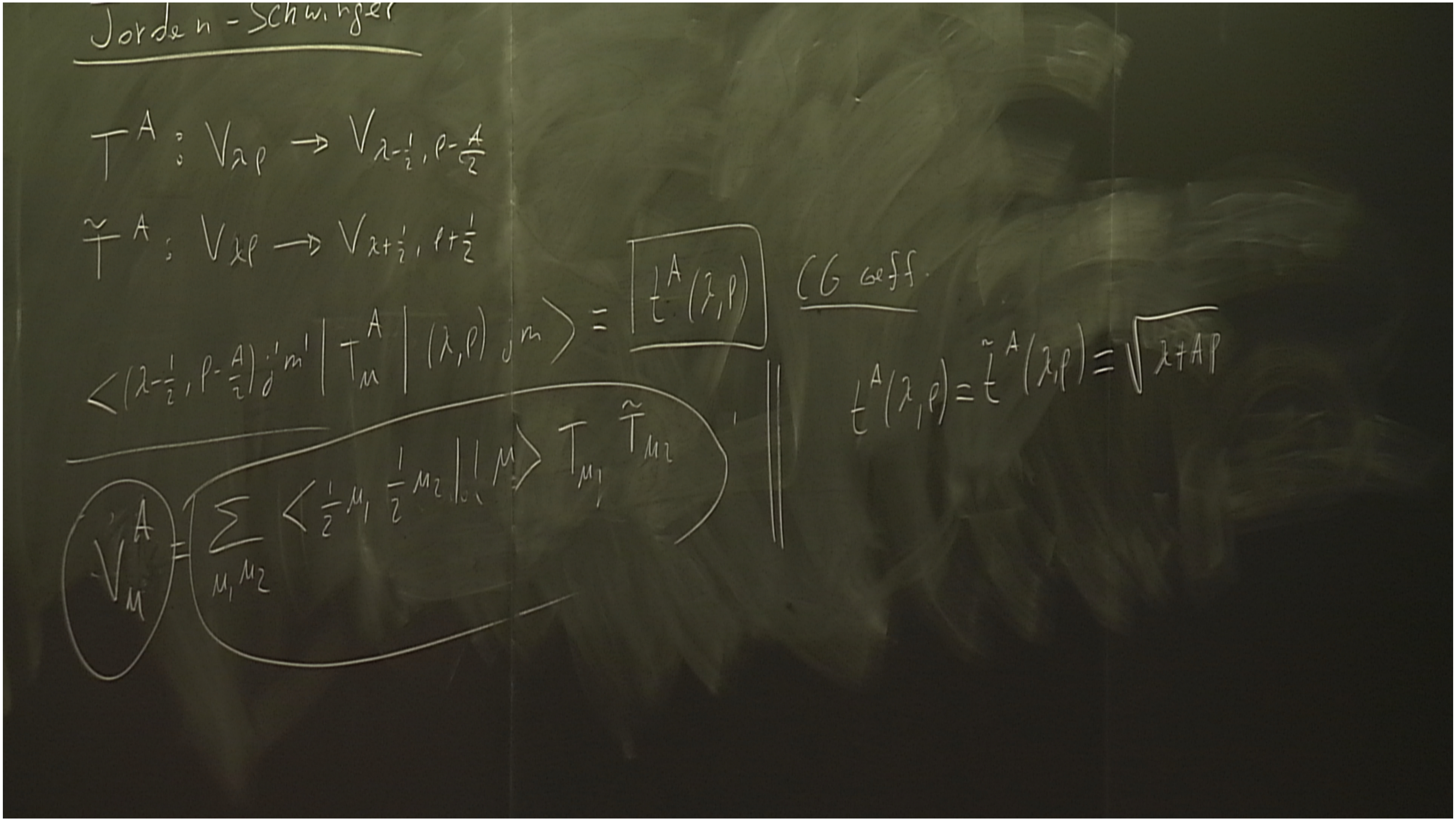

Jordan - Schwinger
$T^A : V_{\lambda\rho} \to V_{\lambda-\frac{1}{2},\,\rho-\frac{A}{2}}$
$\tilde{T}^A : V_{\lambda\rho} \to V_{\lambda+\frac{1}{2},\,\rho+\frac{1}{2}}$
$\langle (\lambda-\frac{1}{2}, \rho-\frac{A}{2}), j' m' | T^A_m | (\lambda,\rho), j m \rangle = t^A(\lambda,\rho)$
CG coeff.

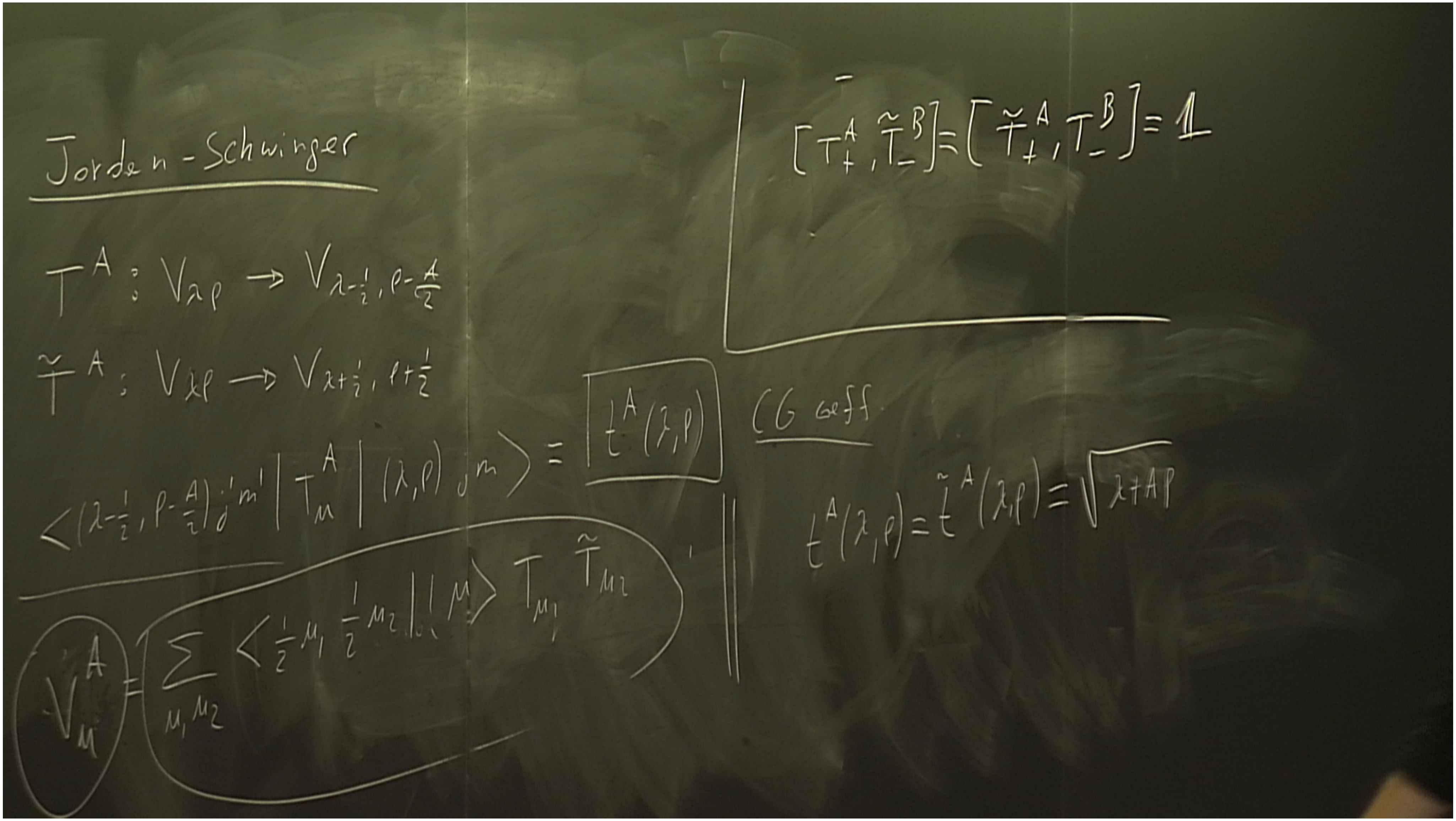

Jorden-Schwinger
$T^A : V_{\lambda\rho} \to V_{\lambda-\frac{1}{2},\, \rho-\frac{A}{2}}$
$\tilde{T}^A : V_{\lambda\rho} \to V_{\lambda+\frac{1}{2},\, \rho+\frac{1}{2}}$
$\langle (\lambda-\frac{1}{2}, \rho-\frac{A}{2}) j' m' | T^A_m | (\lambda,\rho) j m \rangle = t^A(\lambda,\rho)$
CG coeff.


double cover
of SO(3,1)
$V_0^A = -\sqrt{2}\, M_0^A$
$V_{\pm 1}^A = \pm M_\pm^A$
finite-dim reps.
$= J^2 - K^2$


double cover
of SO(3,1)
$V_0^A = -\sqrt{2}\, M_0^A$
$V_{\pm 1}^A = \pm M_\pm^A$
V^A is a tensor op.
$V^A : F_1^A \otimes V_{\lambda\rho} \to V_{\lambda\rho}$
$= J^2 - K^2$
finite-dim reps.


Jorden - Schwinger
CG coeff.


Jordan-Schwinger
CG coeff.


Jorden-Schwinger
$T^A : V_{\lambda\rho} \to V_{\lambda-\frac{1}{2}, \rho-\frac{A}{2}}$
$\tilde{T}^A : V_{\lambda\rho} \to V_{\lambda+\frac{1}{2}, \rho+\frac{1}{2}}$
$\langle (\lambda-\frac{1}{2}, \rho-\frac{A}{2}) j'm' | T^A_\mu | (\lambda,\rho) jm \rangle = t^A(\lambda,\rho)$
$V^A_\mu = \sum_{\mu_1 \mu_2} \langle \frac{1}{2}\mu_1 \frac{1}{2}\mu_2 || \mu \rangle T_{\mu_1} \tilde{T}_{\mu_2}$
$[T^A_+, \tilde{T}^B_-] = [\tilde{T}^A_+, T^B_-] = 1$
CG coeff.
$t^A(\lambda,\rho) = \tilde{t}^A(\lambda,\rho) = \sqrt{\lambda + A\rho}$

Jorden-Schwinger

$$T^A : V_{\lambda\rho} \to V_{\lambda-\frac{1}{2}, \rho-\frac{A}{2}}$$

$$\tilde{T}^A : V_{\lambda\rho} \to V_{\lambda+\frac{1}{2}, \rho+\frac{1}{2}}$$

$$\langle (\lambda-\tfrac{1}{2}, \rho-\tfrac{A}{2}) j' m' | T^A_\mu | (\lambda,\rho) j m \rangle = t^A(\lambda,\rho)$$

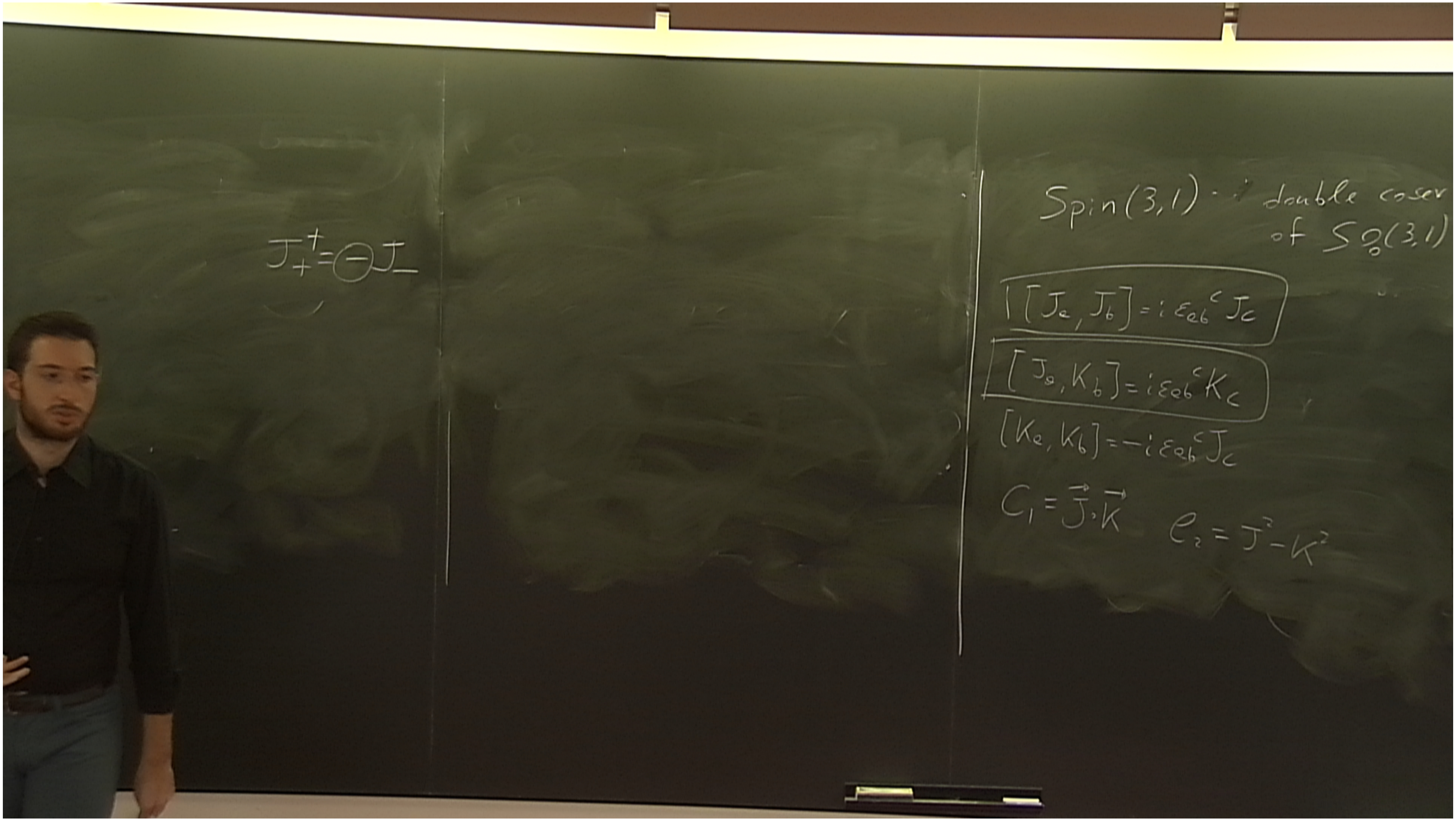
$$V^A_\mu = \sum_{\mu_1 \mu_2} \langle \tfrac{1}{2}\mu_1 \tfrac{1}{2}\mu_2 | 1 \mu \rangle T_{\mu_1} \tilde{T}_{\mu_2}$$

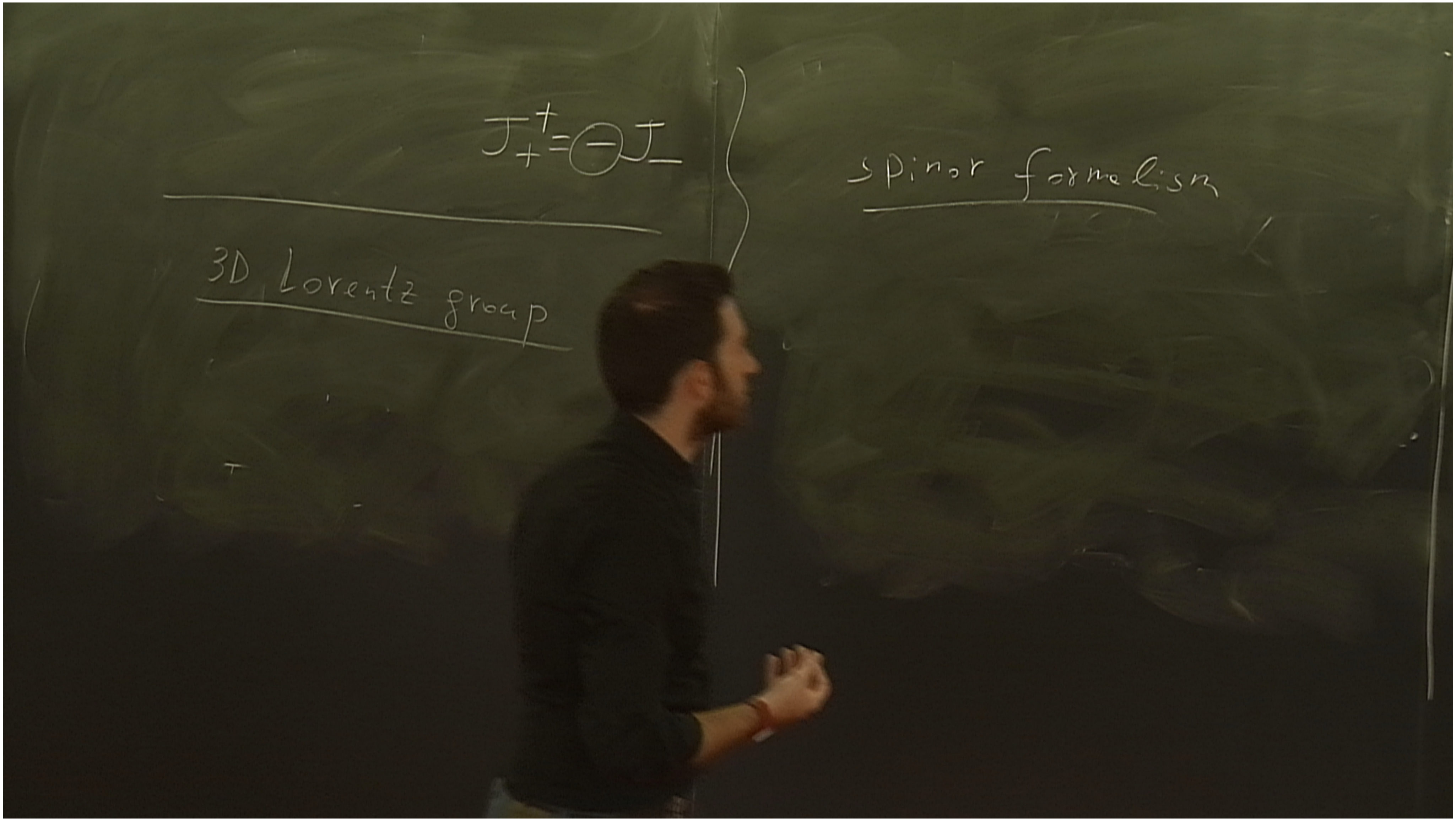
$$[T^A_+, \tilde{T}^B_-] = [\tilde{T}^A_+, T^B_-] = \mathbb{1}$$

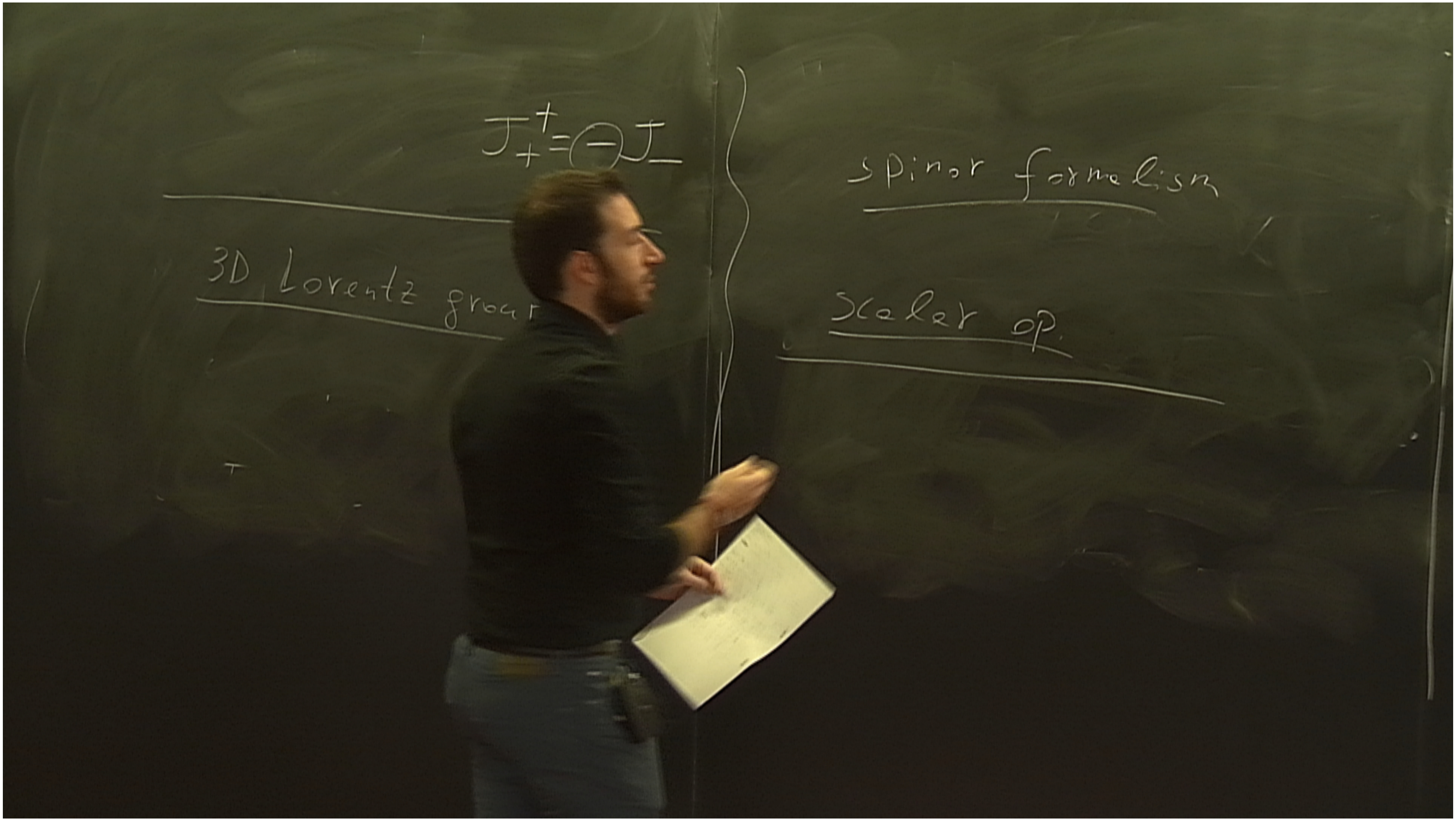
$$T^A_\pm \neq \sigma (\tilde{T}^A_\mp)^\dagger$$

CG coeff.

$$t^A(\lambda,\rho) = \tilde{t}^A(\lambda,\rho) = \sqrt{\lambda + A\rho}$$


Spin(3,1) ... double cover of SO(3,1)
$J_+^\dagger = -J_-$
$[J_a, J_b] = i\varepsilon_{ab}{}^c J_c$
$[J_a, K_b] = i\varepsilon_{ab}{}^c K_c$
$[K_a, K_b] = -i\varepsilon_{ab}{}^c J_c$
$C_1 = \vec{J}\cdot\vec{K}$
$C_2 = J^2 - K^2$


$J_+^\dagger = (-) J_-$
3D Lorentz group
spinor formalism


$J_+^\dagger = (-) J_-$
3D Lorentz group
spinor formalism
Scalar op.