Title: On the mathematics of Ã©tale gerbes inspired by physics

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Abstract: <p>For a finite group G, a G-gerbe over a space B can be thought of as a fiber bundle over B with fibers the classifying orbifold BG. Hellerman-Henriques-Pantev-Sharpe studied conformal field theories on G-gerbes. Given a G-gerbe Y-> B, they constructed a disconnected space \widehat{Y} endowed with a locally constant U(1) 2-cocycle c. They conjectured that a CFT on Y is equivalent to a CFT on \widehat{Y} twisted by the "B-field" c. In this talk, I plan to explain the constructions in this conjecture and the mathematical side of the story, in particular the viewpoints from noncommutative geometry and Gromov-Witten theory. This is based on joint work with Xiang Tang.</p>

Gerbes

Dixon - Harvey - Vafa - Witten:

String theory on orbifold ← space locally of the form $\mathbb{C}^n / G$ ← finite group.

$\Downarrow$

String theory on "crepant resolution" of orbifold

(the topic of another talk)

locally
he form

finite group.

Later on

E. Sharpe & friends:

string theory on "stacks"

essentially orbifolds.

In particular, they've studied strings on gerbes.

étale

Étal

G =

friends:

y on "stacks"

essentially orbifolds.

they've studied strings

Étale gerbes

G = finite group.

B = space (say, orbifold).

A G-gerbe over B is a stack $\mathcal{Y} \to B$ which is roughly a BG-bundle.

• BG

stack

nly a

- $BG = [\text{point}/G]$ the stack of G-bundles.

- $\mathcal{Y} \to B$ can be built as follows:

$\{U_i\}$ = open cover of B.

we glue trivial bdles $U_i \times BG \to U_i$

using ① $\varphi_{ij} \in Aut(G)$ on $U_{ij} := U_i \cap U_j$.

② $g_{ijk} \in G$ on $U_{ijk} := U_i \cap U_j \cap U_k$

w/ conditions

① on U_{ijk}, [illegible] $=$ φ_{ik}

the stack of G-bundles.

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& B.

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ut(G) on $U_{ij} := U_i \cap U_j$.

on $U_{ijk} := U_i \cap U_j \cap U_k$

w/ conditions

① on U_{ijk}, $\boxed{\varphi_{jk} \circ \varphi_{ij} = \mathrm{Ad}_{g_{ijk}} \circ \varphi_{ik}}$

② on $U_i \cap U_j \cap U_k \cap U_\ell$,

$$g_{jk\ell}\, g_{ij\ell} = \varphi_{k\ell}(g_{ijk})\, g_{ik\ell}$$

Hellerman-Henriques-Pantev-Sharpe-Ando.

Conjecture

String theory on a Gerbe $\mathcal{Y}$
CFT

is equivalent to

String theory on $\hat{\mathcal{Y}}$ twisted by
CFT a B-field c.

$$\boxed{\varphi_{jk} \circ \varphi_{ij} = \mathrm{Ad}_{g_{ijk}} \circ \varphi_{ik}}$$

$\cap U_k \cap U_\ell,$

$= \varphi_{k\ell}(g_{ijk})\, g_{ik\ell}$

Hellerman - Henriques - Pantev - Sharpe - Ando.

Conjecture

String theory on a Gerbe $\mathcal{G}$

CFT

is equivalent to

String theory on $\hat{\mathcal{G}}$ twisted by

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Construction of $\hat{\mathcal{G}}, c$

$\mathrm{Aut}(G) \longrightarrow \mathrm{Aut}(G)/\mathrm{Inn}(G) := \mathrm{Out}(G)$

$\varphi_{ij} \longmapsto \phi_{ij}$

Construction of $\hat{\mathcal{Y}}$, c

$$\mathrm{Aut}(G) \longrightarrow \mathrm{Aut}(G)/\mathrm{Inn}(G) := \mathrm{Out}(G).$$

$$\varphi_{ij} \longmapsto \phi_{ij}.$$

we get $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$. on U_{ijk}.

$\Rightarrow$ a principal $\mathrm{Out}(G)$-bundle

$\bar{\mathcal{Y}} \longrightarrow B$ "the band" of $\mathcal{Y} \to B$.

$\hat{G}$ = the set of isom. classes of irred. $\mathbb{C}$- representations of G.

Out(G) acts on $\hat{G}$:

$\phi \in \mathrm{Aut}(G)$, $\rho: G \to \mathrm{End}(V_\rho)$ irred.

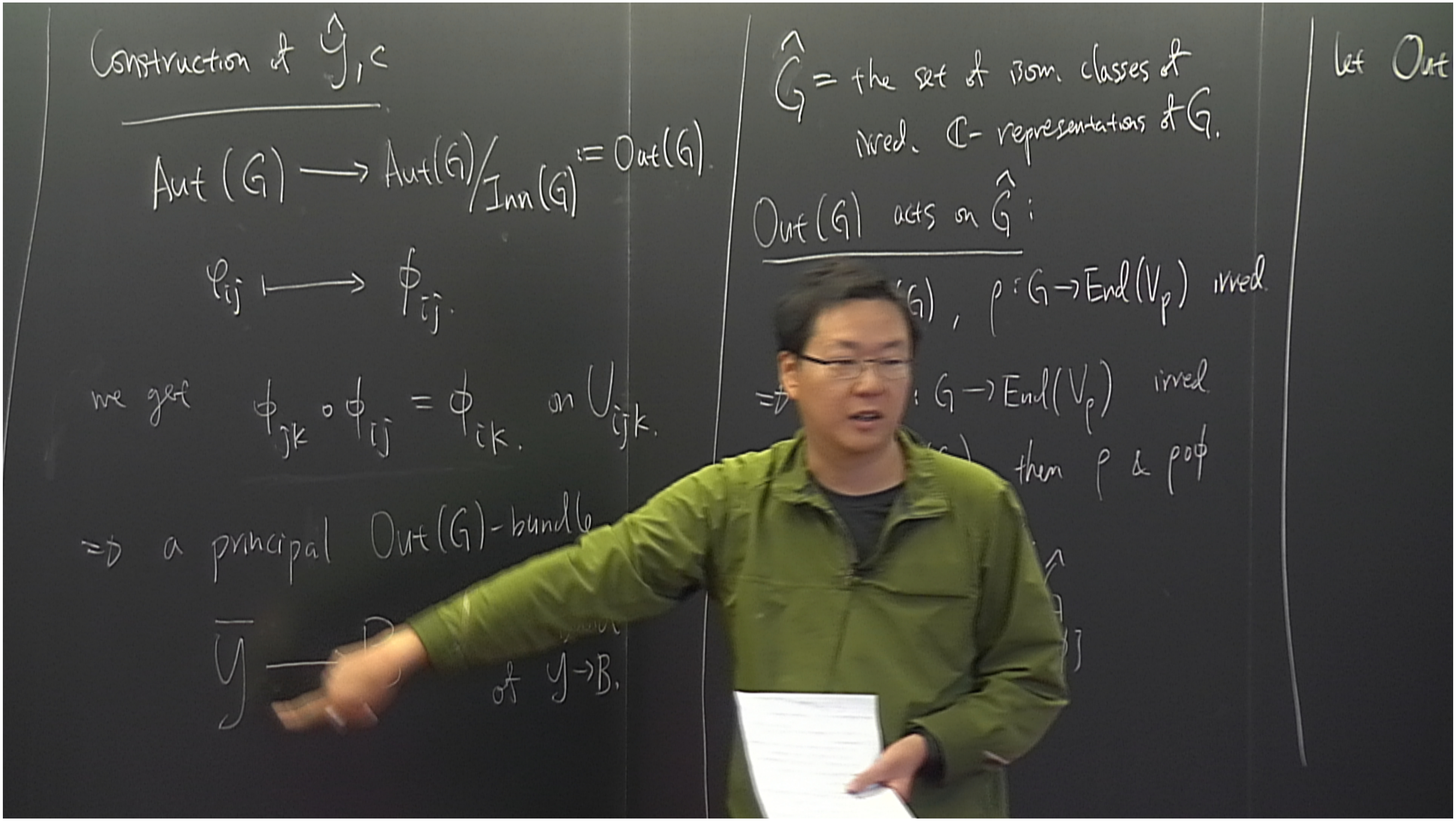
$\Rightarrow$ $\rho \circ \phi : G \to \mathrm{End}(V_\rho)$ irred.

If $\phi \in \mathrm{Inn}(G)$, then ρ & $\rho\circ\phi$ are isom. G-reps.

$\Rightarrow$ $\mathrm{Out}(G) \times \hat{G} \to \hat{G}$

$[\phi], [\rho] \mapsto [\rho\circ\phi]$

Let Out


Construction of $\hat{\mathcal{Y}}$, c
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$:= \mathrm{Out}(G)$.

U_{ijk}.

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$\Rightarrow \mathrm{Out}(G) \times \hat{G} \to \hat{G}$

$[\phi], [\rho] \longmapsto [\rho \circ \phi]$

Let Out(G) act diagonally on $\bar{\mathcal{Y}} \times \hat{G}$

Def'n $\hat{\mathcal{Y}} := \left[\bar{\mathcal{Y}} \times \hat{G} / \mathrm{Out}(G) \right]$.

(G).

$\hat{G}$ = the set of isom. classes of irred. $\mathbb{C}$-representations of G.

<u>$Out(G)$ acts on $\hat{G}$:</u>

$\phi \in Aut(G)$, $\rho : G \to End(V_\rho)$ irred

$\Rightarrow$ $\rho \circ \phi : G \to End(V_\rho)$ irred

If $\phi \in Inn(G)$, then ρ & $\rho \circ \phi$ are isom. G-reps.

$\Rightarrow$ $Out(G) \times \hat{G} \to \hat{G}$

$[\phi], [\rho] \mapsto [\rho \circ \phi]$

Let $Out(G)$ act diagonally on $\bar{\mathcal{Y}} \times \hat{G}$

<u>Def'n</u> $\hat{\mathcal{Y}} := \left[\bar{\mathcal{Y}} \times \hat{G} / Out(G) \right]$.

<u>About</u> c: $[\rho] \in \hat{G}$, fix a representative

$\rho : G \to End(V_\rho)$

$\Rightarrow$ a family of vector spaces on $\hat{\mathcal{Y}}$

$\bar{\mathcal{Y}} \times \hat{G} \ni (x, [\rho]) \mapsto V_\rho$

The obstruction for this to be a vector bdle on $\hat{\mathcal{Y}}$ is a $U(1)$-valued 2-cocycle c^{-1}.

- c is locally constant.

The most commonly seen example of G-gerbe:

$\mathcal{Y} \to B$ s.t. $\bar{\mathcal{Y}} \to B$ is trivial.

$$\bar{\mathcal{Y}} \simeq B \times \mathrm{Out}(G)$$

$$\Rightarrow \quad \hat{\mathcal{Y}} = B \times \hat{G}$$

[Giraud]: $\mathcal{Y} \to B$ w/ trivial band

is classified by classes in

$$H^2(B, Z(G)).$$

$c = U(1)$ 2-cocycle on $\hat{\mathcal{Y}} = B \times \widehat{G}$

on $B \times [p]$, c is obtained via.

$$H^2(B, Z(G)) \longrightarrow H^2(B, U(1))$$

$$[\mathcal{Y}/B] \longmapsto c$$

where

comes f

rivial band

in

$\times \widehat{G}$

$U(1)$

where $Z(G) \longrightarrow U(1)$

comes from $G \xrightarrow{\rho} \mathrm{End}(V_\rho)$

$\cup$

$Z(G) \longrightarrow \mathrm{End}(V_\rho)$

• B(

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using

What does the conj. mean?

[HHPSA] suggested the viewpoint of noncommutative geometry (A. Connes).

J. w/ Xiang Tang

$\mathcal{Y}$ and $(\hat{\mathcal{G}}, c)$ are isomorphic NC spac nes).

Spaces.

$\mathcal{Y}$ is represented by a groupoid

$$\mathcal{G} \overset{t}{\underset{s}{\rightrightarrows}} \mathcal{G}_0 .$$

$\Rightarrow$ $C^\infty(\mathcal{G})$ w/ the product

$$(f_1 * f_2)(a) := \sum_{a = a_1 a_2} f_1(a_1) f_2(a_2) .$$

l by a groupoid

$\mathcal{G}_0$

w/ the product

$\sum f_1(a_1) f_2(a_2)$

$\hat{\mathcal{G}}$ is represented by a groupoid

$$\hat{\mathcal{G}} \rightrightarrows \hat{\mathcal{G}}_0$$

c is represented by a 2 cocycle on $\hat{\mathcal{G}}$

$\Rightarrow C^\infty(\hat{\mathcal{G}})$ w/ the product

$$(f_1 *_c f_2)(a) = \sum_{a=a_1 a_2} c(a_1,a_2)\, f_1(a_1) f_2(a_2)$$

Thm

$\hat{G}$

Thm (Tang–T.)

$C^\infty(G)$ and $C^\infty(\hat{G}, c)$

are Morita equivalent

(i.e. they have equivalent categories of modules).

$\hat{G}$ = the set

irred.

Out(G) acts

$\phi \in \mathrm{Aut}(G)$

$\Rightarrow \rho \circ \phi : G$

If $\phi \in \mathrm{Inn}(G$

are isom.

$\Rightarrow \mathrm{Out}(G) \times$

$C^\infty(\hat{\mathcal{G}}, c)$

valent

ve equivalent categories

es).

<u>So?</u> We can get more traditional stuff out of this.

A-model topological strings.

Let's assume B, $\mathcal{Y}$, $\mathcal{G}$ are symplectic.

$$\Rightarrow \quad C^\infty(\mathcal{G}) \underset{\text{equiv.}}{\overset{\text{Morita}}{\sim}} C^\infty(\hat{\mathcal{G}}, c)$$

$$\begin{array}{ccc} C^\infty(\mathcal{G}) & & C^\infty(\hat{\mathcal{G}}, c) \\ \text{deformation quantization [Tang]} \big\downarrow & & \big\downarrow \text{def. q.} \\ C^\infty_\hbar(\mathcal{G}) & \underset{\text{Morita.}}{\sim} & C^\infty_\hbar(\hat{\mathcal{G}}, c) \end{array}$$

Let $Out(G)$ act diagonally ...

<u>Def'n</u> $\hat{\mathcal{G}} := \left[\bar{\mathcal{Y}} \times \hat{G} \,/\, Out(G) \right.$

<u>About c</u> $[\rho] \in \hat{G}$, fix

$\rho : G \to End(V_\rho)$

$\Rightarrow$ a family of vector spaces ...

$\bar{\mathcal{Y}} \times \hat{G} \ni (x, [\rho]) \mapsto V_\rho$

get more traditional

this.

...logical strings.

..., Y, $\hat{\mathcal{G}}$ are symplectic.

$) \underset{\text{equiv.}}{\overset{\text{Morita}}{\sim}} C^\infty(\hat{\mathcal{G}}, c)$

def. q.

$\overset{\sim}{\text{Morita.}} \; C^\infty_\hbar(\hat{\mathcal{G}}, c)$

Hochschild cohomology:

$$HH^*(C^\infty_\hbar(\mathcal{G})) \cong HH^*(C^\infty_\hbar(\hat{\mathcal{G}}, c))$$

$\wr$ Neumaier, Pflaum, Posthuma, Tang

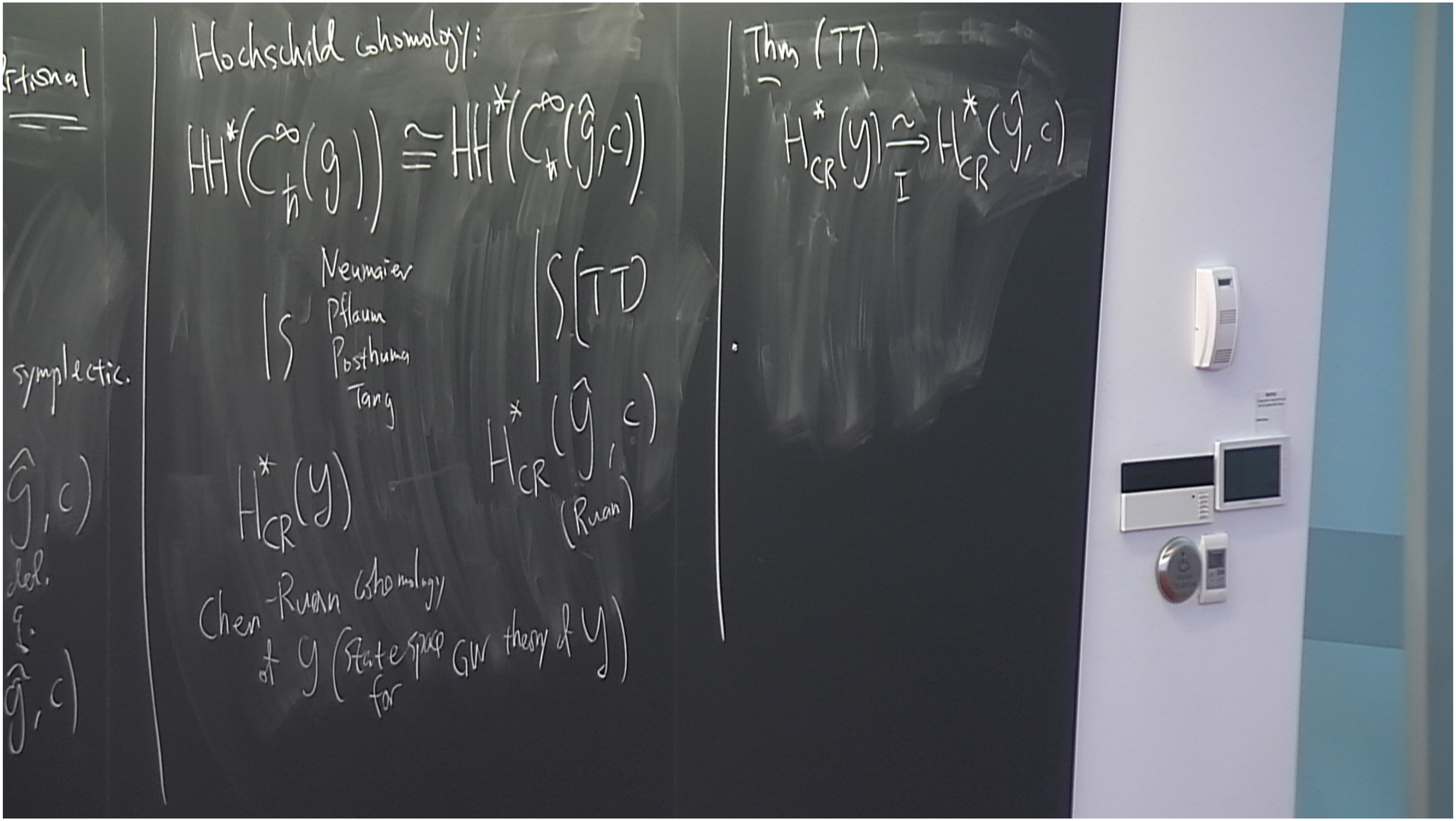
$\wr$ [TT]

$$H^*_{CR}(Y) \qquad H^*_{CR}(\hat{\mathcal{G}}, c) \text{ (Ruan)}$$

Chen-Ruan cohomology of Y (state space for GW theory of Y)

The obstruction for this to be a vector bdle on $\hat{\mathcal{G}}$ is a $U(1)$-valued 2-cocycle c^{-1}.

- c is locally constant.


Hochschild cohomology:
$HH^*(C^\infty_\hbar(g)) \cong HH^*(C^\infty_\hbar(\hat{g}, c))$
Neumaier
Pflaum
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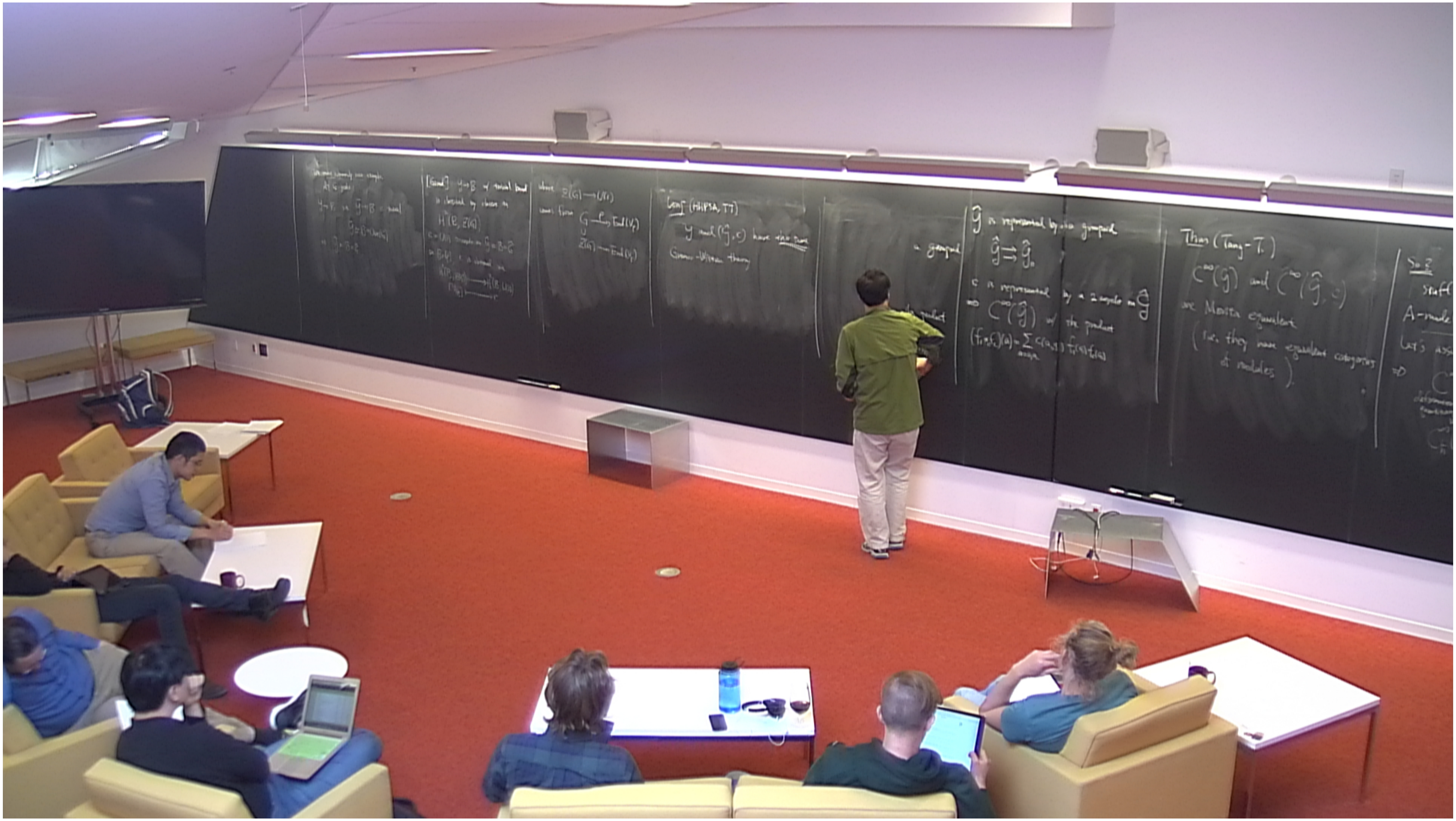
Tang
[TT]
$H^*_{CR}(y)$
$H^*_{CR}(\hat{g}, c)$
(Ruan)
Chen-Ruan cohomology of y (state space for GW theory of y)
Thm (TT).
$H^*_{CR}(y) \xrightarrow[I]{\simeq} H^*_{CR}(\hat{g}, c)$
symplectic.
$(\hat{g}, c)$


Gromov-Witten theory
Thm (Tang - T.)
are Morita equivalent
(i.e. they have equivalent categories of modules)

Conj (HHPSA, TT).

$\mathcal{Y}$ and $(\hat{\mathcal{Y}}, c)$ have the same Gromov-Witten theory

Main evidence:

Thm (TT). If $\mathcal{Y} \to B$ has trivial band, then the conjecture is true. More precisely we have an equality

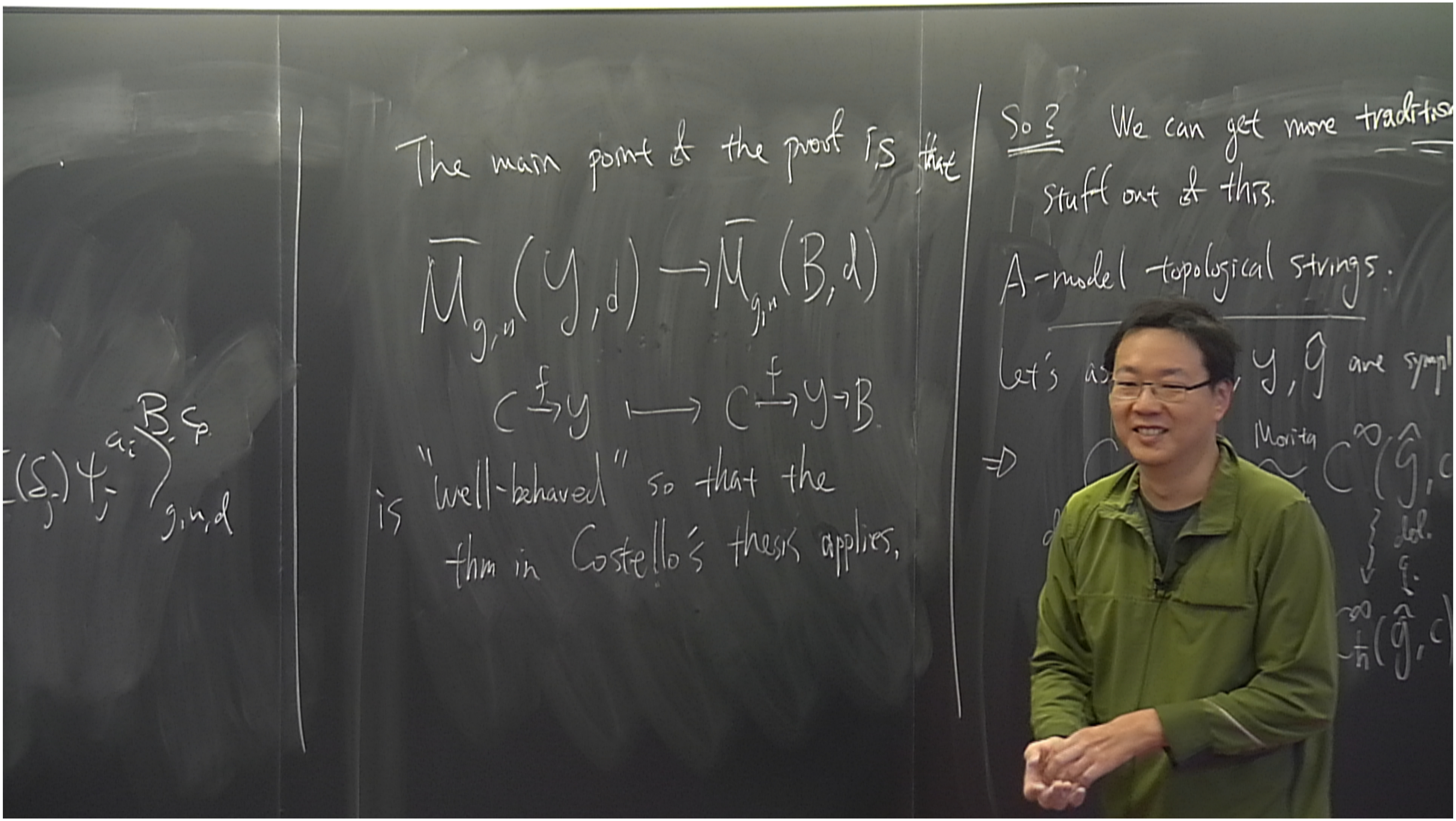
$\mathcal{Y} \to B$

then the

e. More

an equality

$$\left\langle \prod_{j=1}^{n} \delta_j \psi_j^{a_i} \right\rangle^{\mathcal{Y}}_{g,n,d} = \sum_{\rho \in \hat{G}} \left(\frac{\dim V_\rho}{|G|} \right)^{2-2g} \left\langle \prod_{j=1}^{n} I(\delta_j) \psi_j^{a_i} \right\rangle^{B, c_\rho}_{g,n,d}$$


The main point of the proof is that
$\overline{M}_{g,n}(Y,d) \to \overline{M}_{g,n}(B,d)$
$C \xrightarrow{f} Y \longmapsto C \xrightarrow{f} Y \to B$
is "well-behaved" so that the
thm in Costello's thesis applies.
So? We can get more
stuff out of this.
A-model topological strings.
Morita