Title: GKZ Hypergeometric Series for the Hesse Pencil, Chain Integrals and Orbifold Singularities

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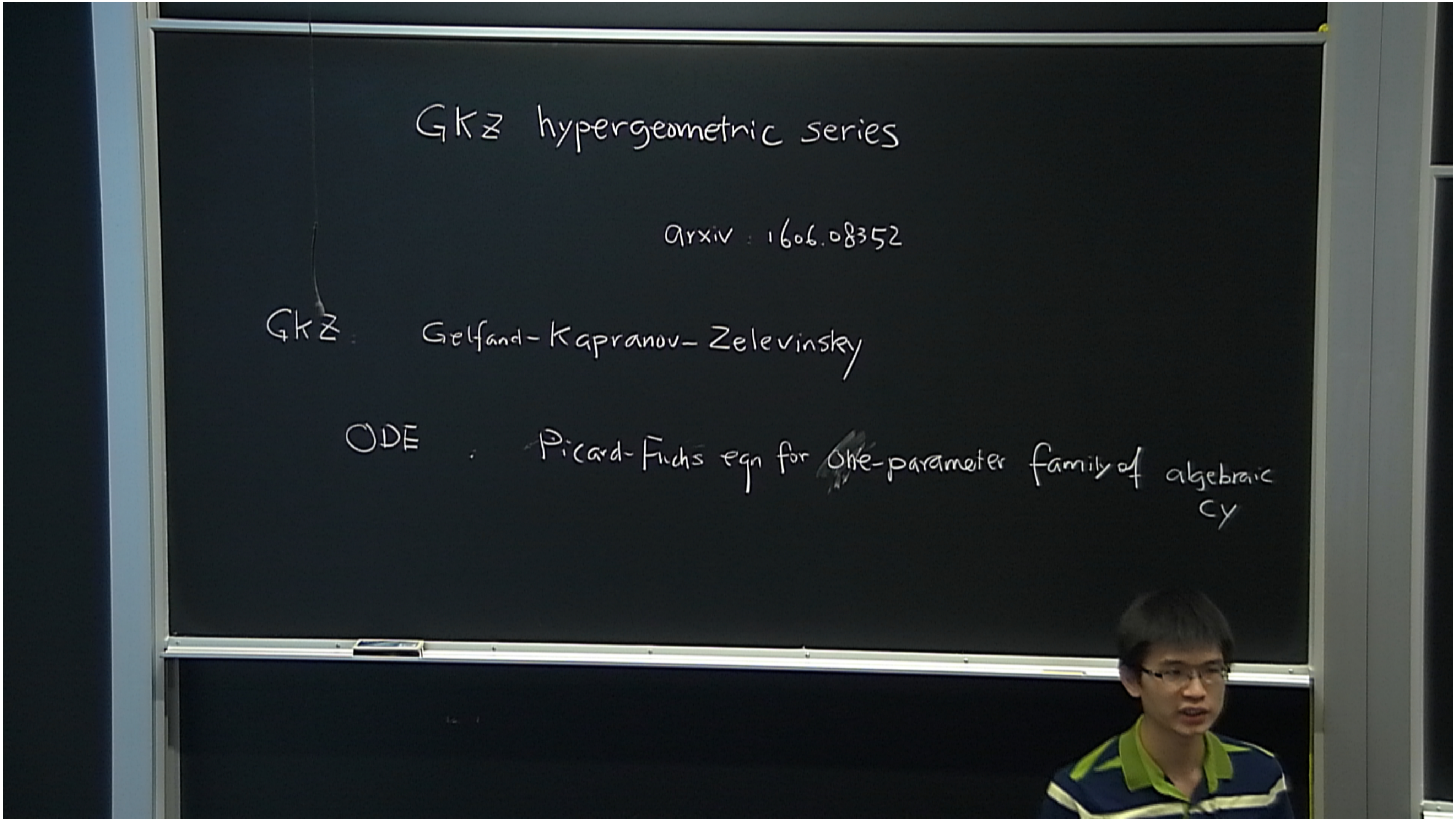
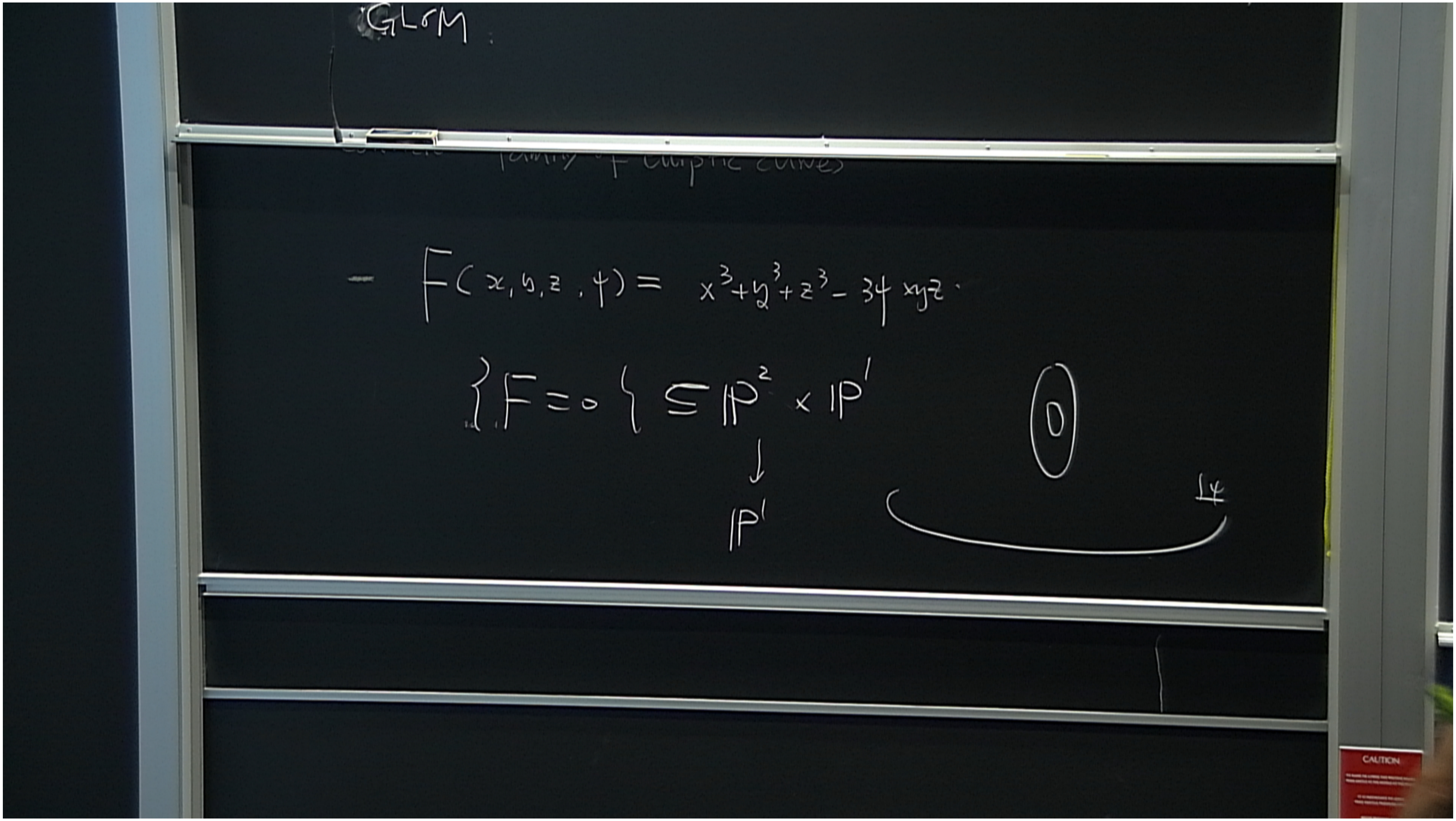
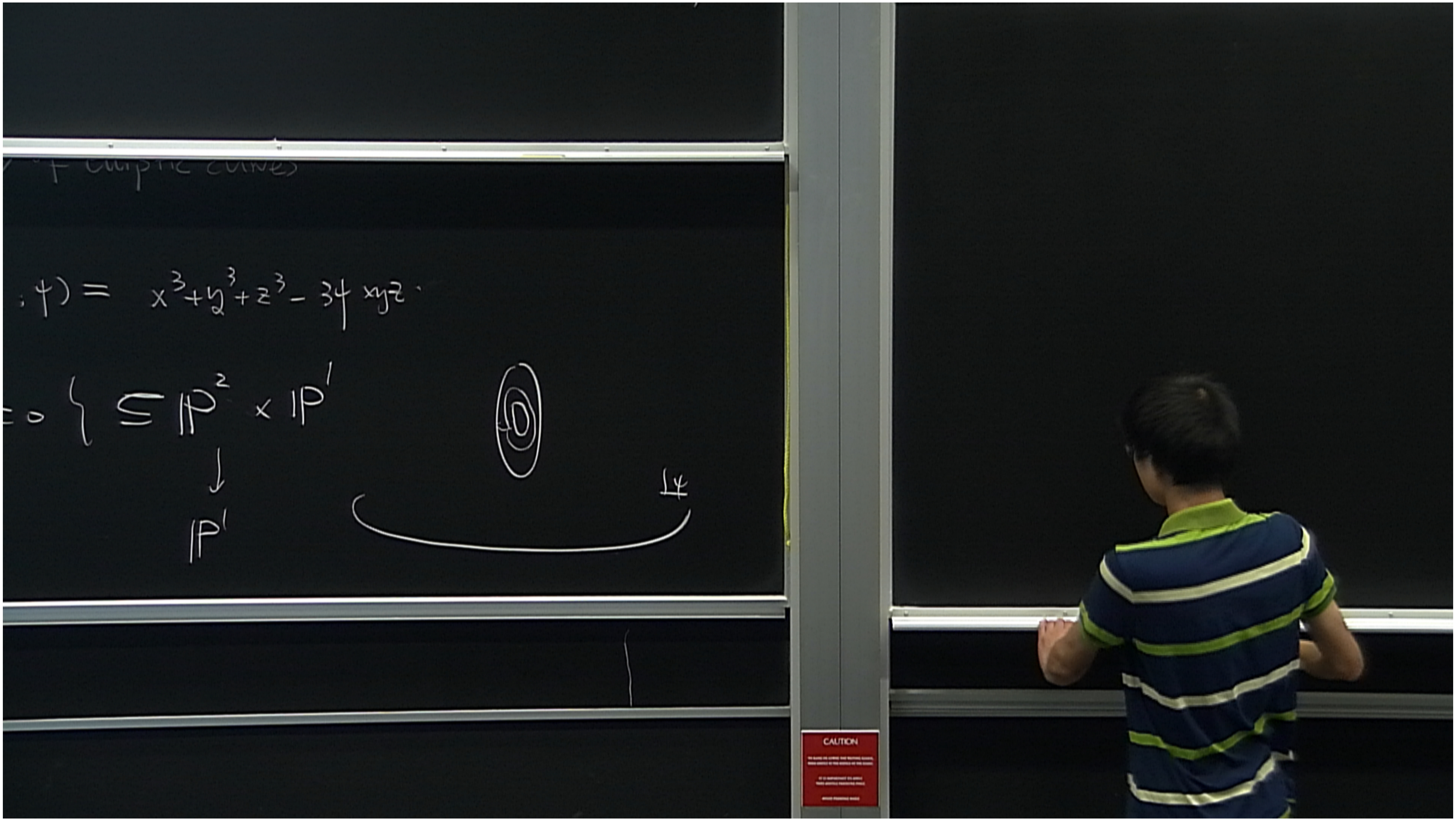
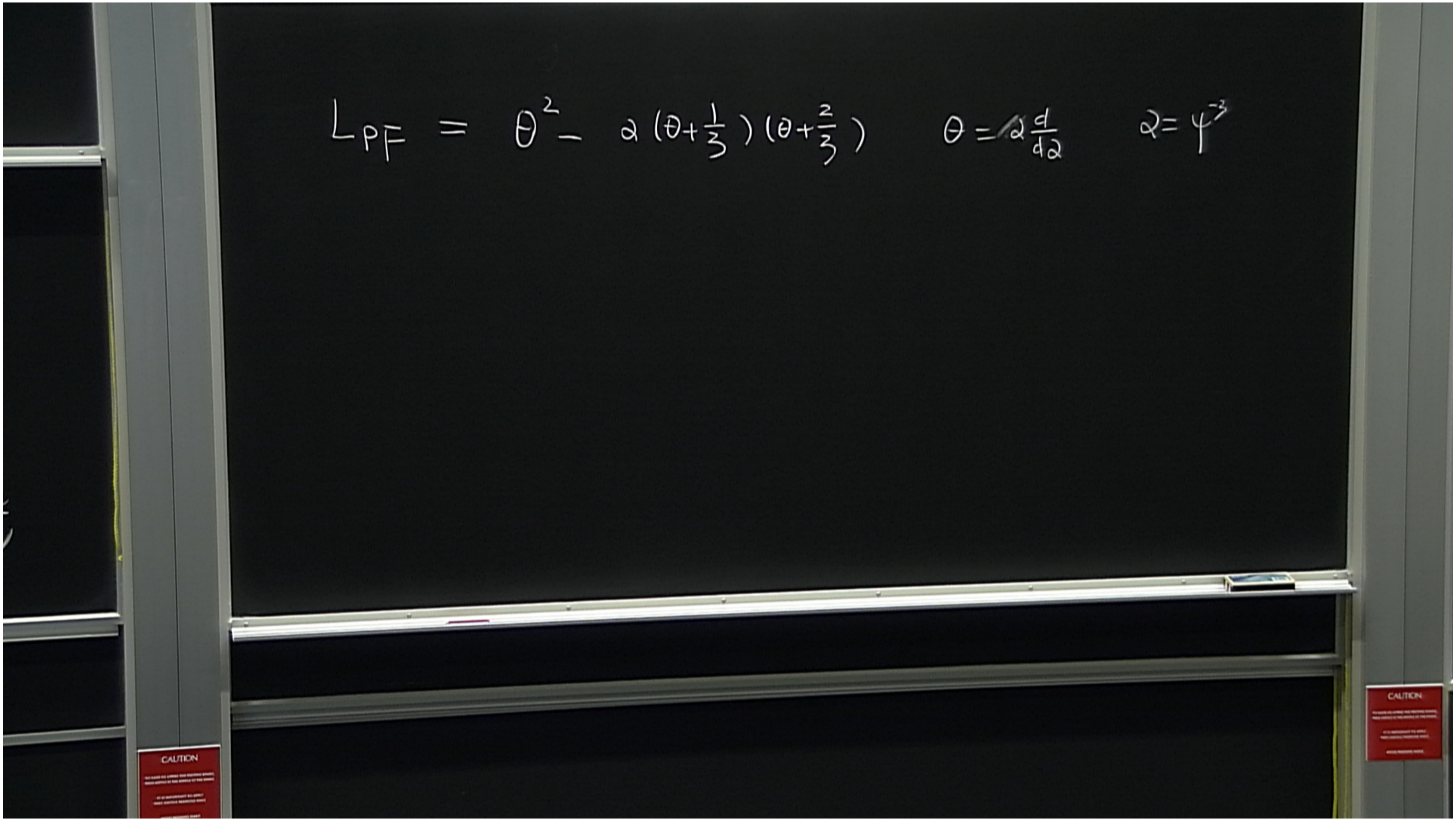
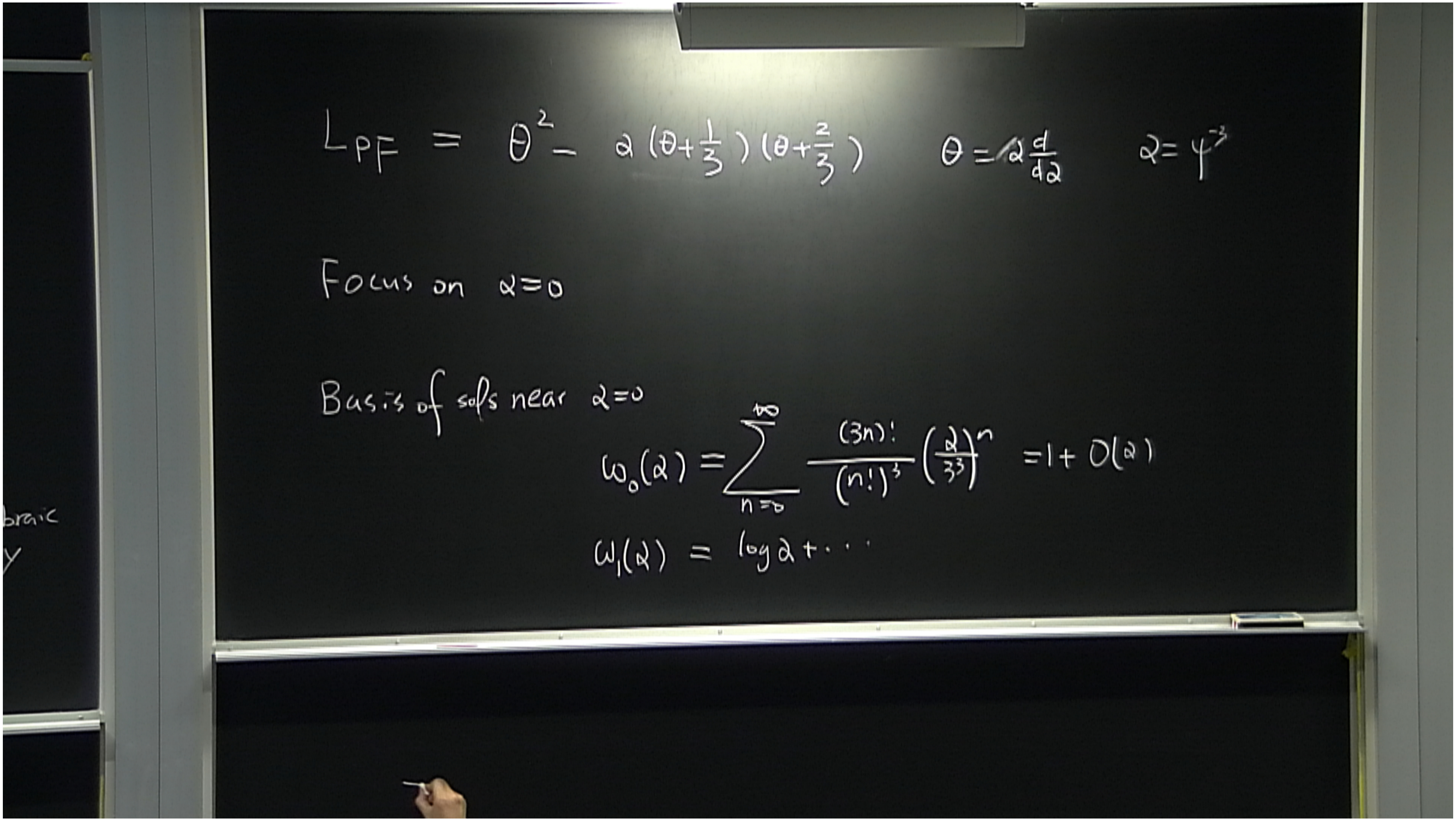
Abstract: <p>I will talk about some connections among the GKZ (introduced by Gelfand-Kapranov-Zelevinsky) hypergeometric series, orbifold singularities of the system, and chain integrals in some geometry. The GKZ hypergeometric series appeared in some very interesting contexts including arithmetic geometry, enumerative geometry and mathematical physics in the last few decades. I will report some new geometric realizations and interpretations of them.</p>


GKZ hypergeometric series
arxiv : 1606.08352
GKZ : Gelfand-Kapranov-Zelevinsky
ODE : Picard-Fuchs eqn for one-parameter family of algebraic CY


GLSM :
F(x, y, z, ψ) = x³+y³+z³ − 3ψ xyz.
{F=0} ⊆ P² × P¹
P¹
ψ


; ψ) = x³+y³+z³ − 3ψ xyz.
= 0 } ⊆ ℙ² × ℙ¹
ℙ¹
ψ


$L_{PF} = \theta^2 - \alpha(\theta+\frac{1}{3})(\theta+\frac{2}{3})$
$\theta = \alpha\frac{d}{d\alpha}$
$\alpha = \psi^{-3}$


$L_{PF} = \theta^2 - \alpha(\theta + \frac{1}{3})(\theta + \frac{2}{3})$
$\theta = \alpha \frac{d}{d\alpha}$
$\alpha = \psi^{-3}$
Focus on $\alpha = 0$
Basis of sol's near $\alpha = 0$
$\omega_0(\alpha) = \sum_{n=0}^{\infty} \frac{(3n)!}{(n!)^3} \left(\frac{\alpha}{3^3}\right)^n = 1 + O(\alpha)$
$\omega_1(\alpha) = \log \alpha + \cdots$

Frobenonius

$$\omega_0(\alpha,\varepsilon) = \sum_{n=0}^{\infty} \frac{\Gamma(3n+3\varepsilon+1)}{\Gamma(n+\varepsilon+1)^3} \cdot Q(\varepsilon)\cdot\left(\frac{\alpha}{3^3}\right)^{n+\varepsilon}$$

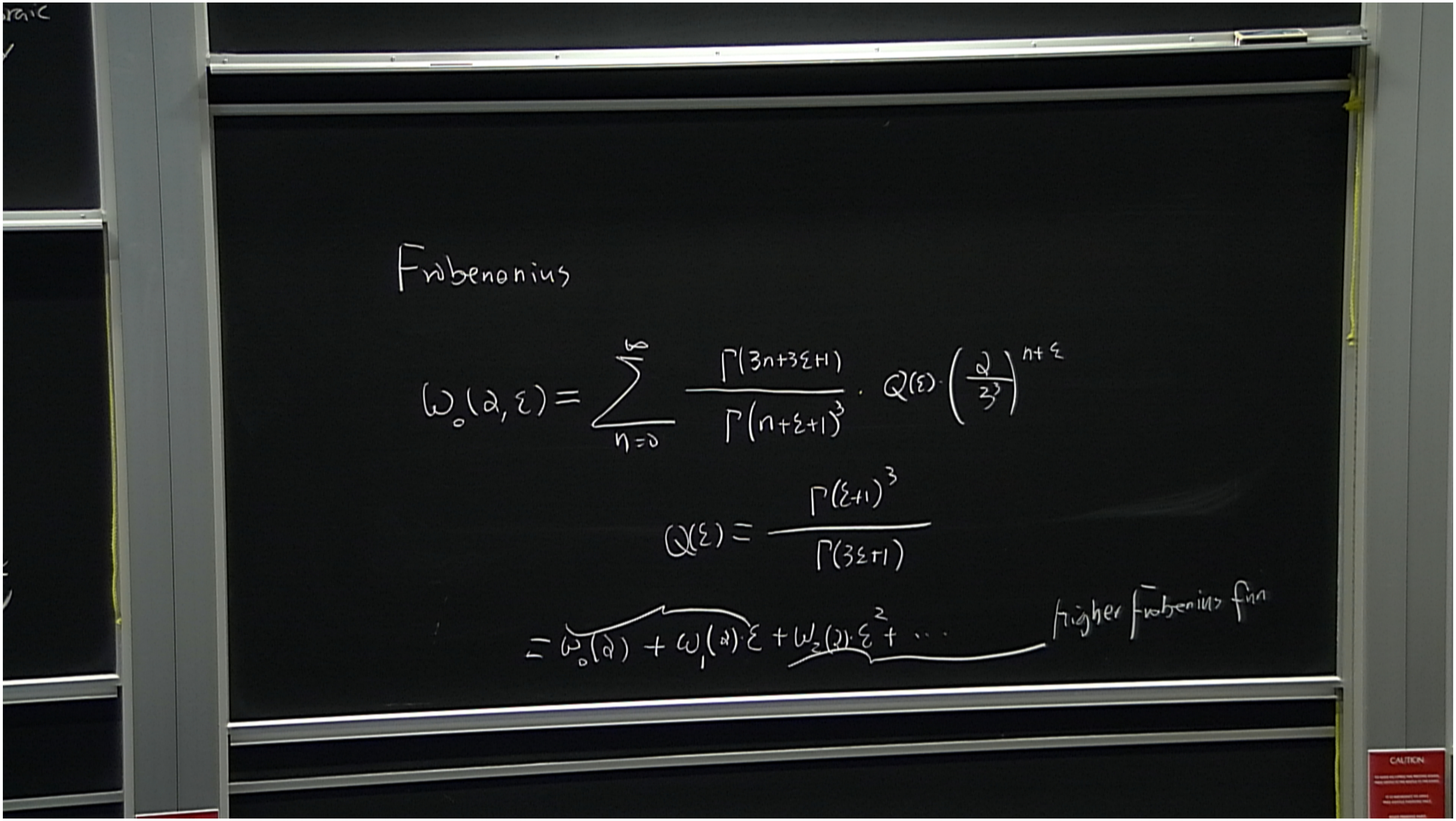
$$Q(\varepsilon) = \frac{\Gamma(\varepsilon+1)^3}{\Gamma(3\varepsilon+1)}$$

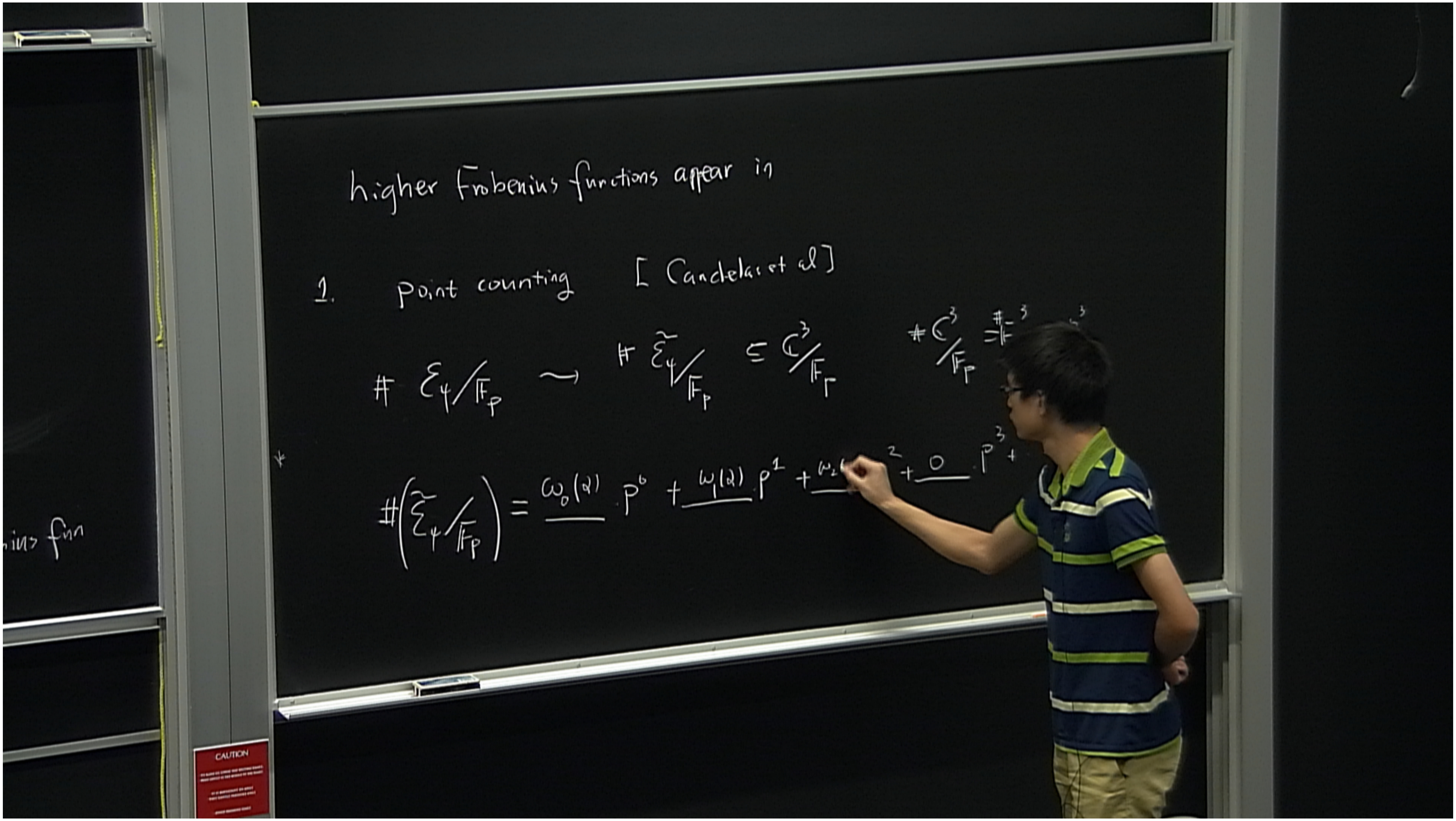
Frobenonius

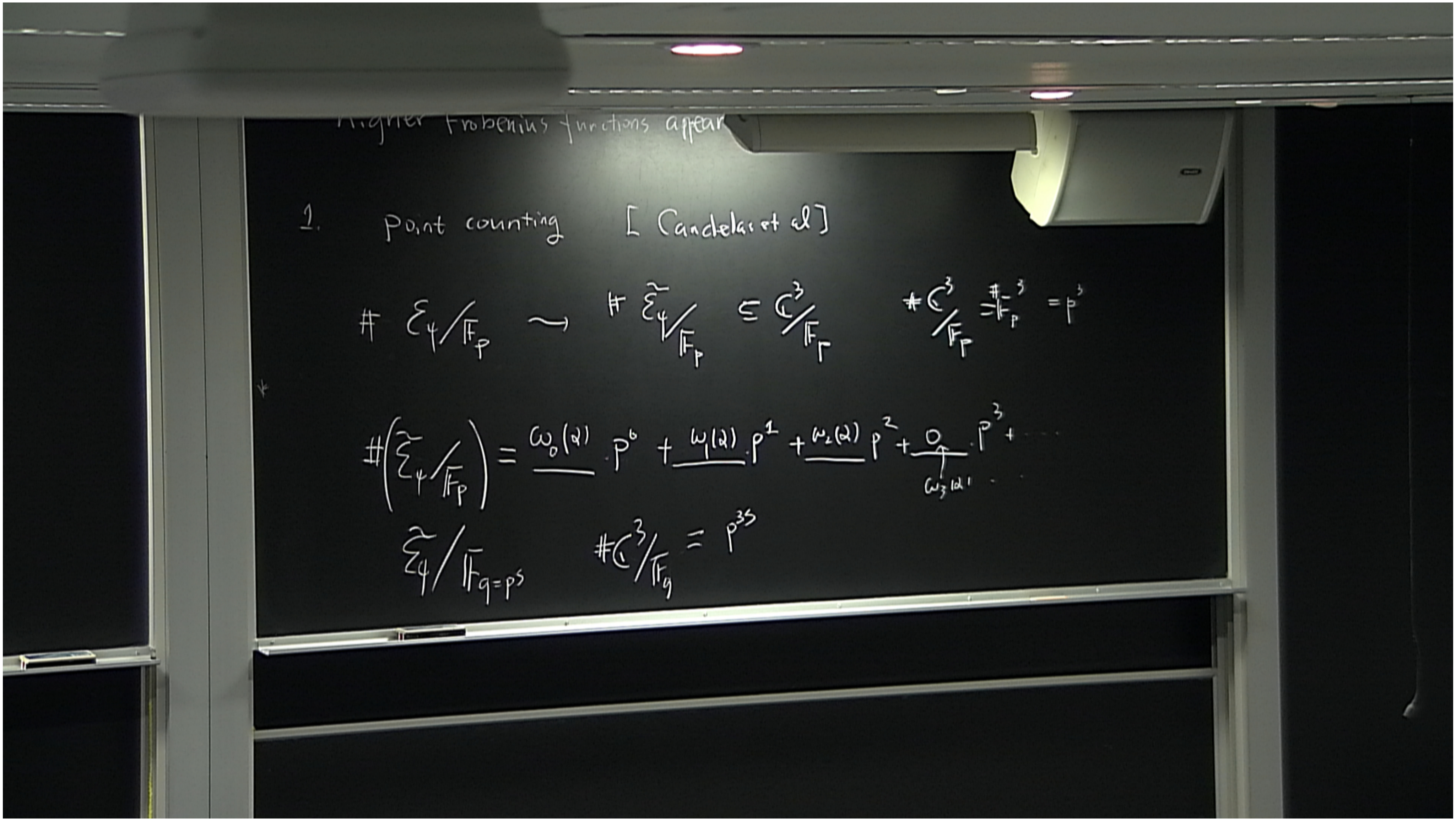
$$\omega_0(a,\varepsilon) = \sum_{n=0}^{\infty} \frac{\Gamma(3n+3\varepsilon+1)}{\Gamma(n+\varepsilon+1)^3} \cdot Q(\varepsilon) \cdot \left(\frac{a}{3^3}\right)^{n+\varepsilon}$$

$$Q(\varepsilon) = \frac{\Gamma(\varepsilon+1)^3}{\Gamma(3\varepsilon+1)}$$

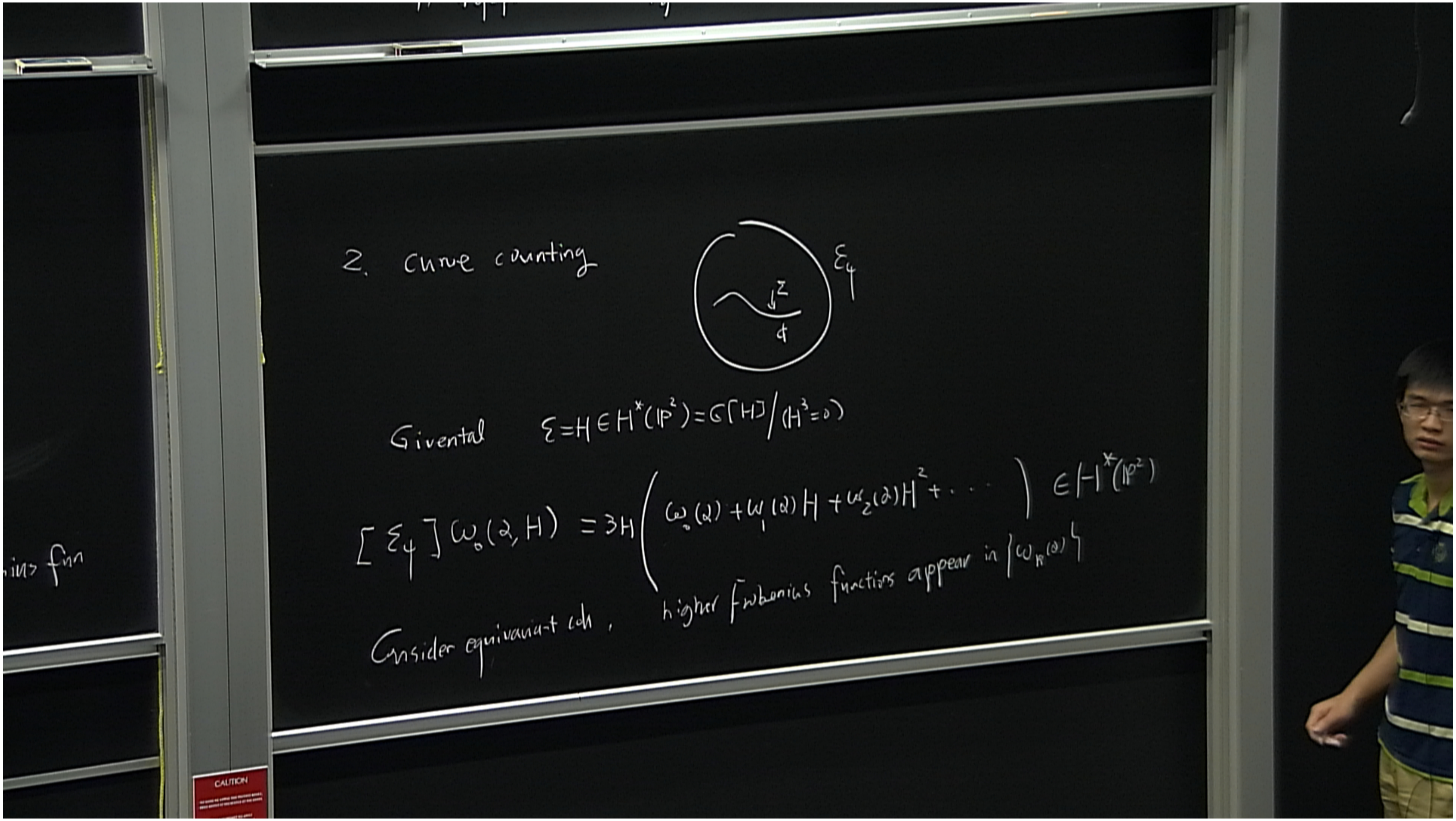
$$= \omega_0(a) + \omega_1(a)\cdot\varepsilon + \omega_2(a)\cdot\varepsilon^2 + \cdots$$

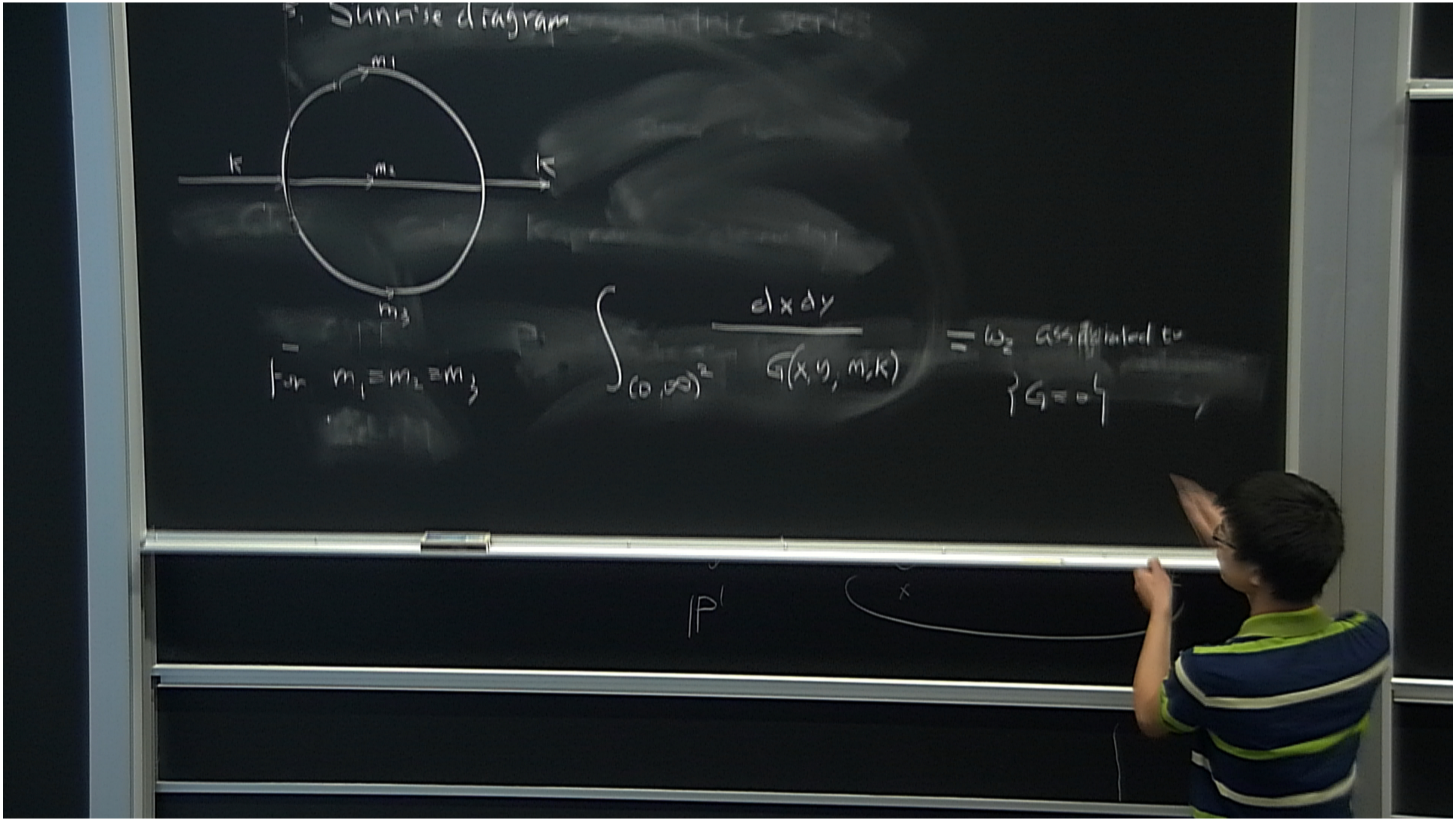

Frobenonius
$\omega_0(a,\varepsilon) = \sum_{n=0}^{\infty} \frac{\Gamma(3n+3\varepsilon+1)}{\Gamma(n+\varepsilon+1)^3} \cdot Q(\varepsilon) \cdot \left(\frac{a}{3^3}\right)^{n+\varepsilon}$
$Q(\varepsilon) = \frac{\Gamma(\varepsilon+1)^3}{\Gamma(3\varepsilon+1)}$
$= \omega_0(a) + \omega_1(a)\cdot\varepsilon + \omega_2(a)\cdot\varepsilon^2 + \dots$
higher Frobenius fcn

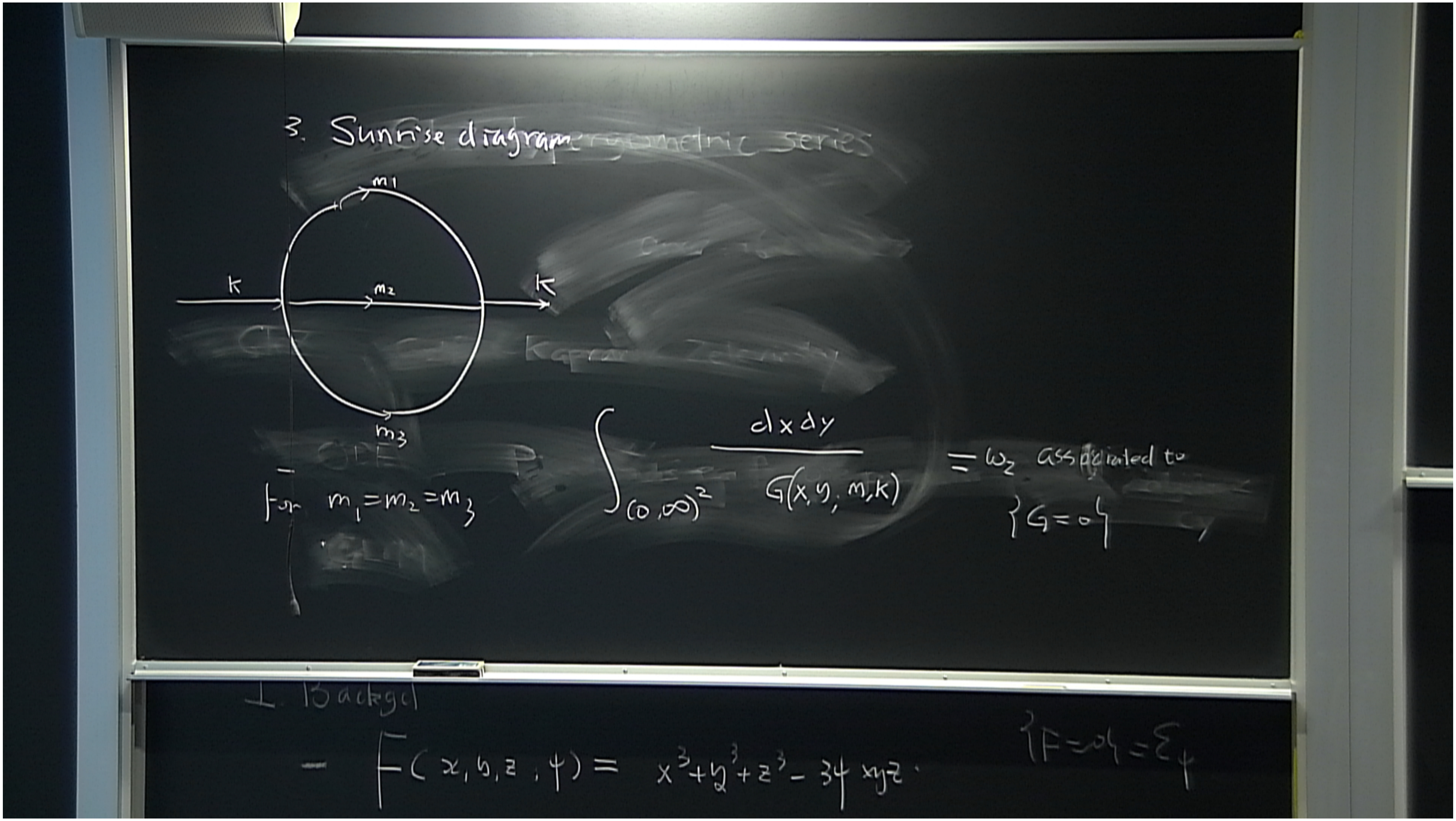

higher Frobenius functions appear in
1. point counting [Candelas et al]
$\mathcal{E}_\psi / \mathbb{F}_p$ ⟶ # $\tilde{\mathcal{E}}_\psi / \mathbb{F}_p \subseteq \mathbb{C}^3 / \mathbb{F}_p$
$\#\left(\tilde{\mathcal{E}}_\psi / \mathbb{F}_p\right) = \omega_0(\lambda)\, p^0 + \omega_1(\lambda)\, p^1 + \omega_2(\ldots)\, p^2 + 0\, p^3 +$

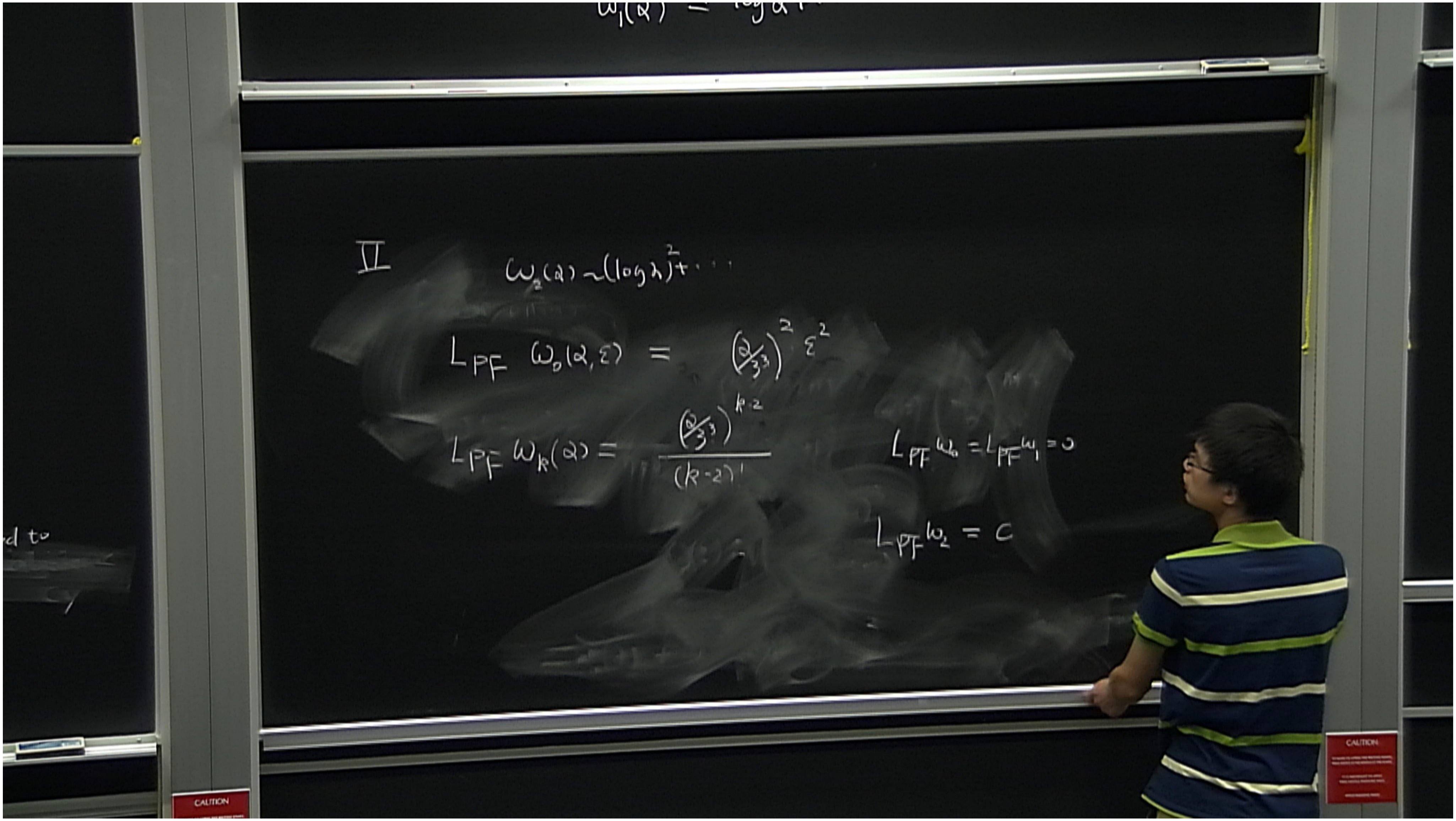


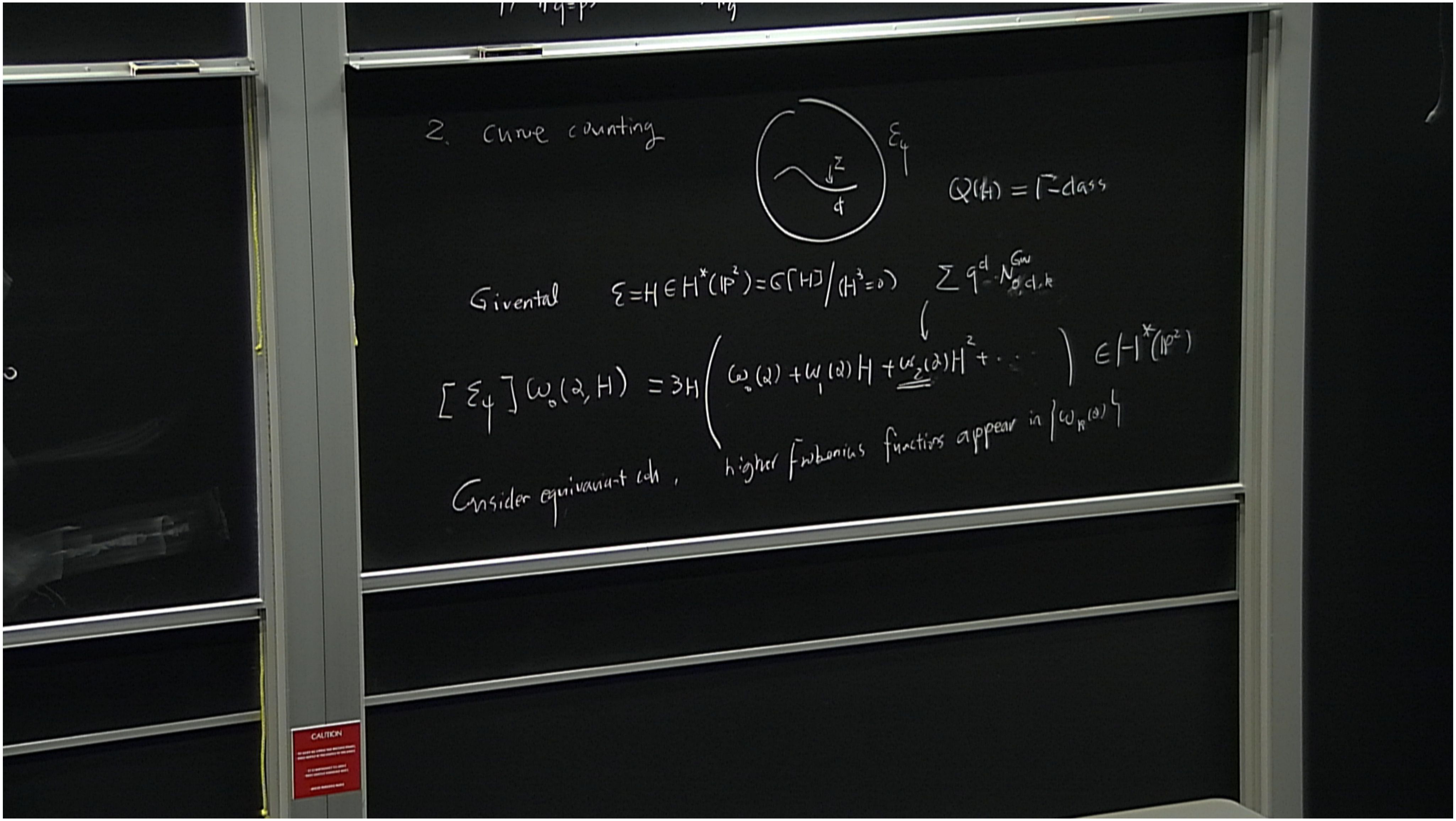
Higher Frobenius functions appear
1. Point counting [Candelas et al]
$\# \mathcal{E}_\psi/\mathbb{F}_p \sim \# \tilde{\mathcal{E}}_\psi/\mathbb{F}_p \subseteq \mathbb{C}^3/\mathbb{F}_p$
$\# \mathbb{C}^3/\mathbb{F}_p = \# \mathbb{F}_p^3 = p^3$
$\#(\tilde{\mathcal{E}}_\psi/\mathbb{F}_p) = \omega_0(\lambda) \cdot p^0 + \omega_1(\lambda) \cdot p^1 + \omega_2(\lambda) p^2 + 0 \cdot p^3 + \cdots$
$\omega_3(\lambda)$
$\tilde{\mathcal{E}}_\psi/\mathbb{F}_{q=p^s}$
$\# \mathbb{C}^3/\mathbb{F}_q = p^{3s}$

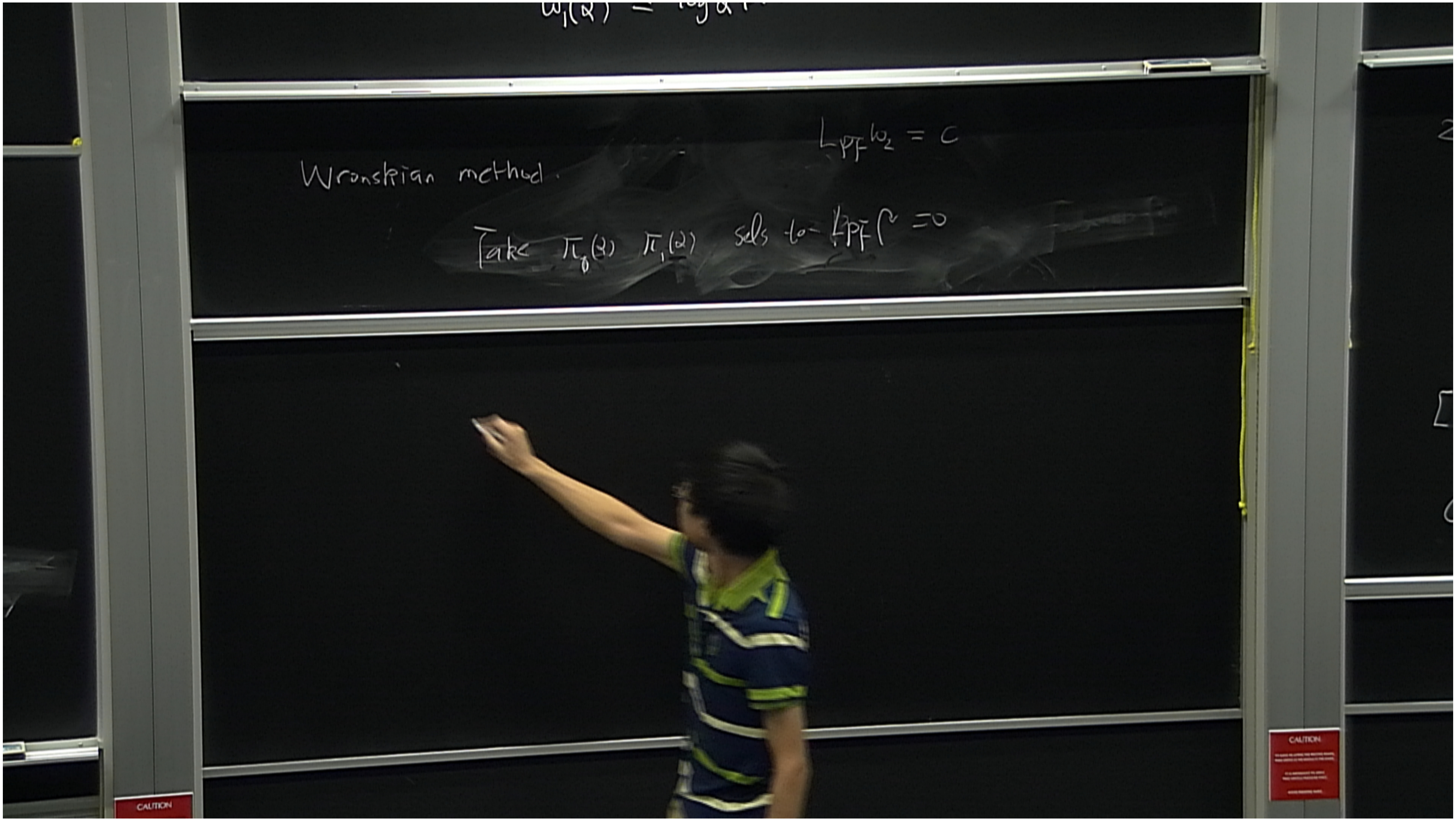

2. Curve counting
E_φ
z
φ
Givental ξ = H ∈ H*(P²) = C[H]/(H³=0)
[E_φ] ω₀(z, H) = 3H(ω₀(z) + ω₁(z) H + ω₂(z) H² + . . .) ∈ H*(P²)
higher Frobenius functions appear in {ω_k(z)}
Consider equivariant coh ,
CAUTION

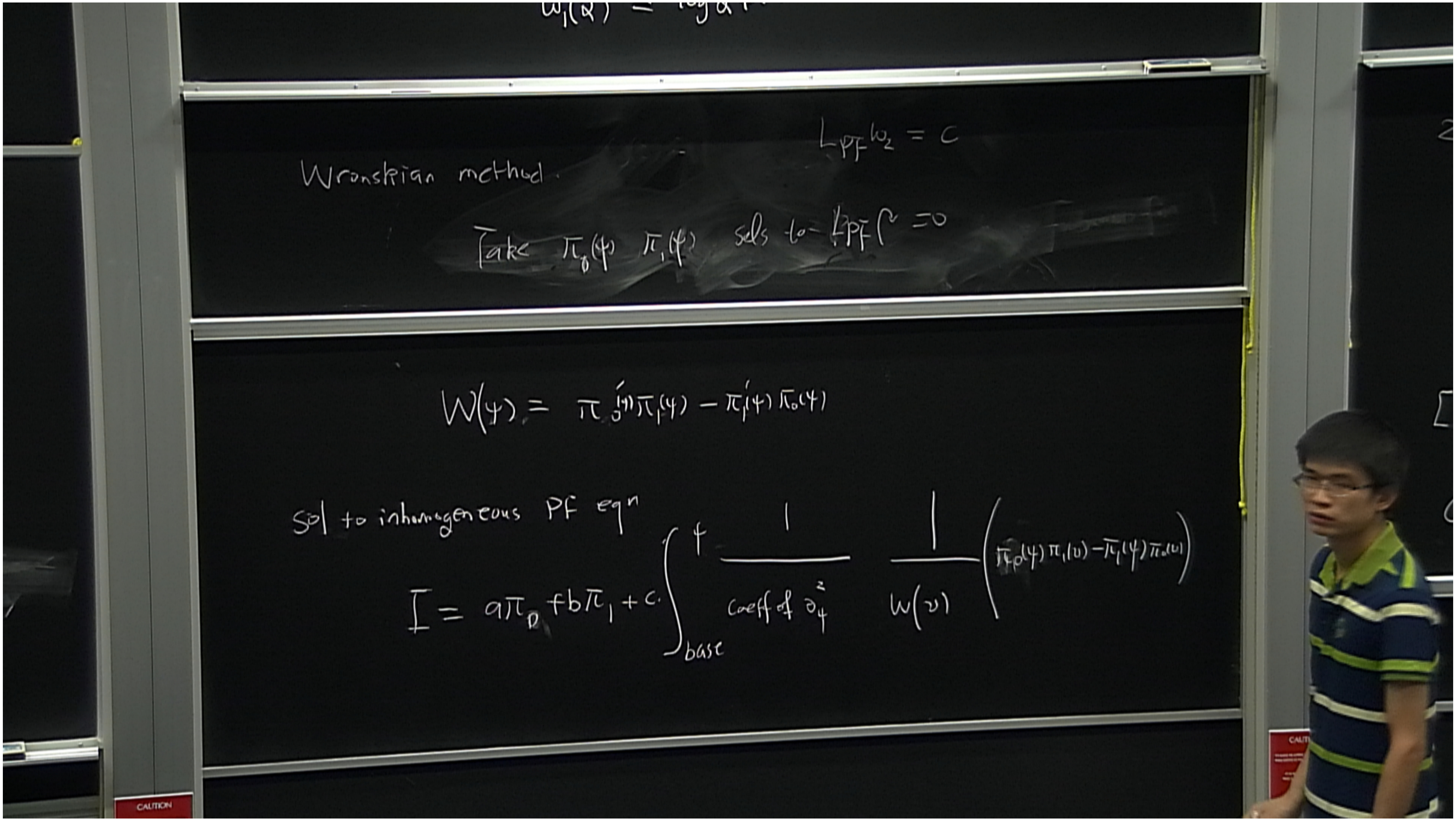

Sunrise diagram
m1
k
m2
k
m3
for m1=m2=m3
dx dy
G(x,y,m,k)
(0,∞)²
= ω2 associated to
{G=0}
P¹
x


3. Sunrise diagram
m1
k
m2
k
m3
for m1=m2=m3
∫(0,∞)^2 dx dy / G(x,y,m,k) = ω2 associated to {G=0}
1. Backgd
F(x,y,z;ψ) = x^3+y^3+z^3−3ψxyz.
{F=0}=Eψ

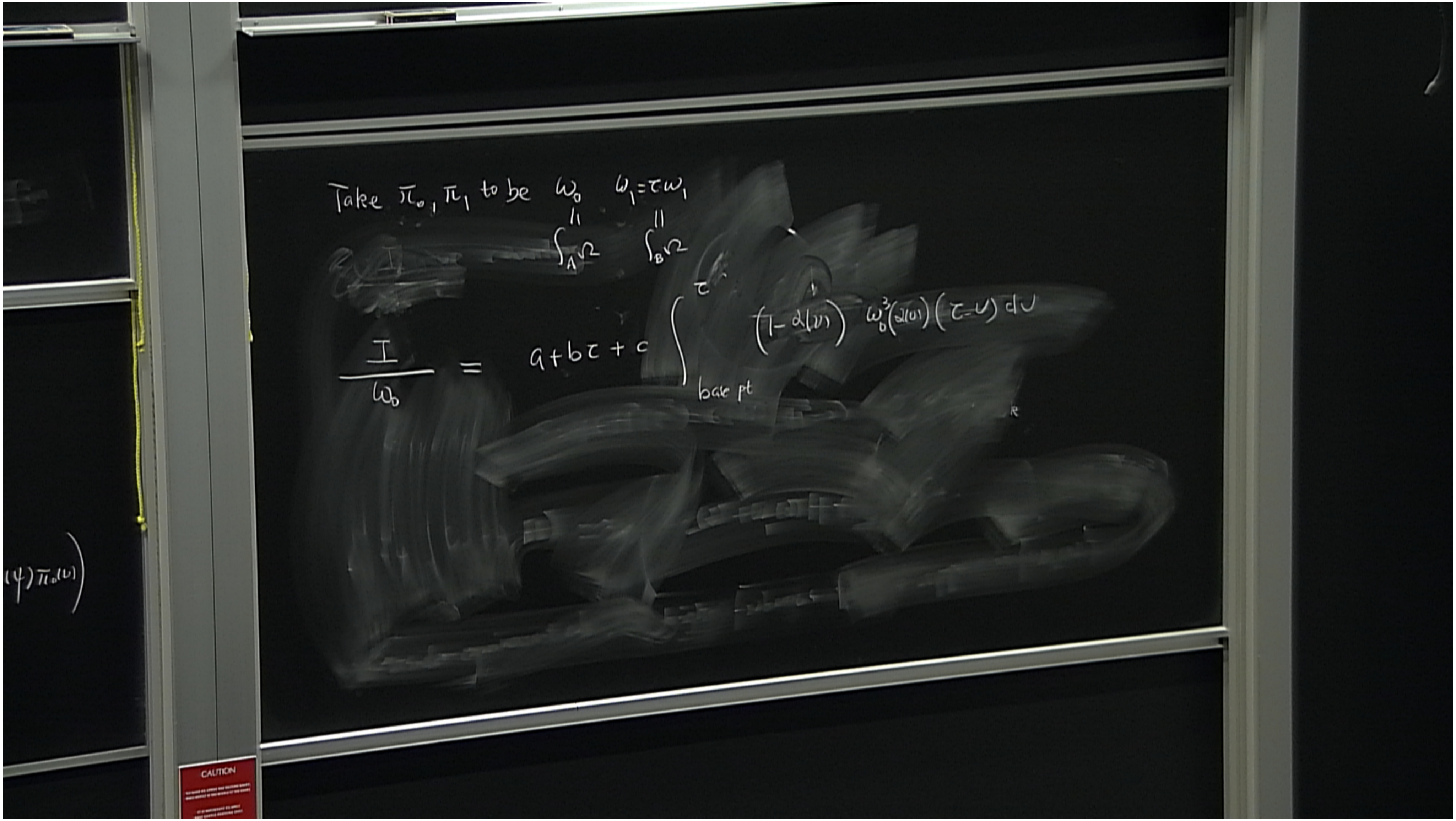

II
$\omega_2(\lambda) \sim (\log\lambda)^2 + \cdots$
$L_{PF}\ \omega_0(\lambda,\varepsilon) = \left(\frac{2}{3^3}\right)^2 \varepsilon^2$
$L_{PF}\ \omega_k(\lambda) = \frac{\left(\frac{2}{3^3}\right)^{k-2}}{(k-2)!}$
$L_{PF}\,\omega_0 = L_{PF}\,\omega_1 = 0$
$L_{PF}\,\omega_2 = C$
d to

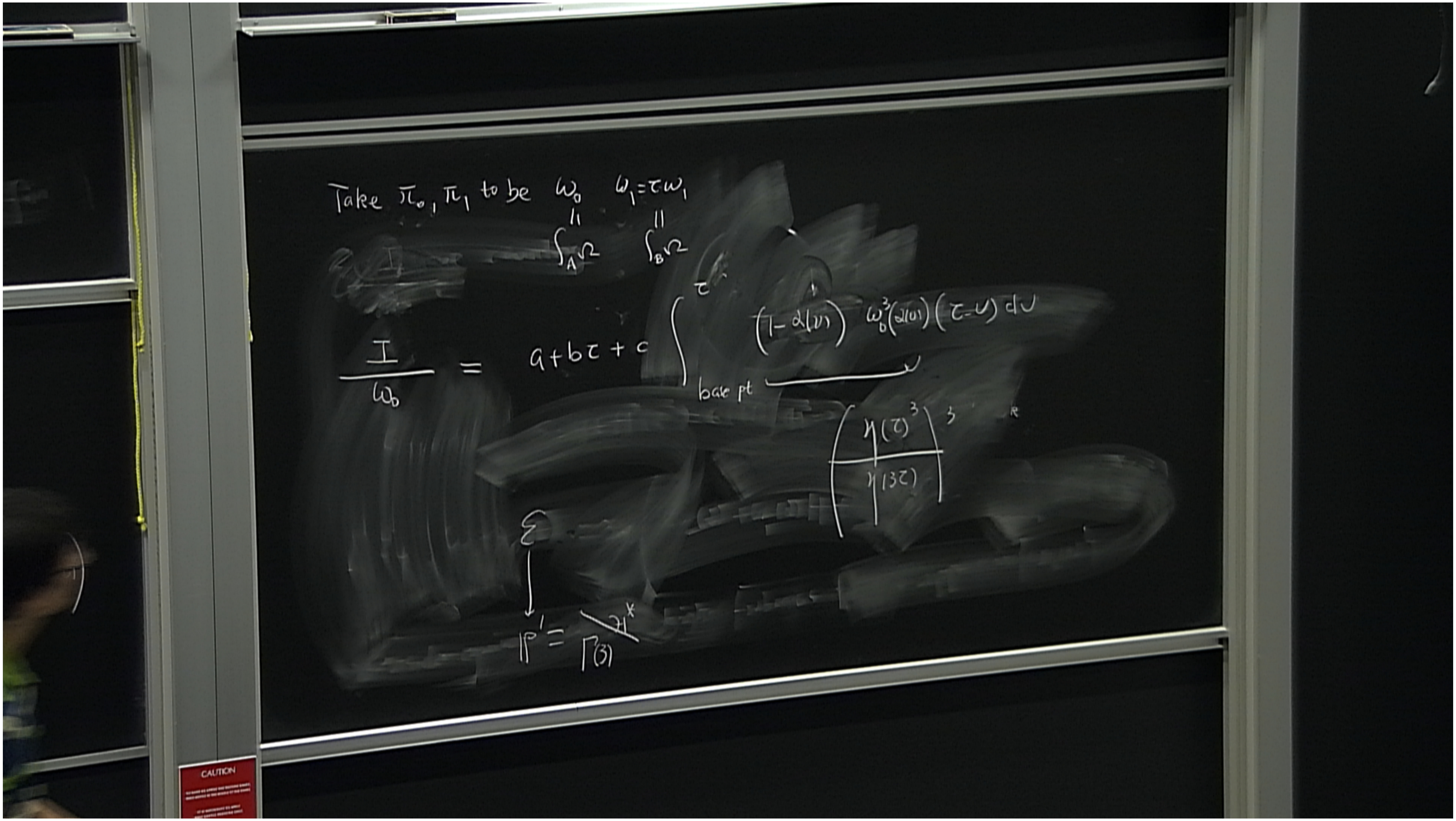

2. Curve counting
$\mathcal{E}_q$
$Q(H) = \Gamma$-class
Givental $\xi = H \in H^*(\mathbb{P}^2) = \mathbb{C}[H]/(H^3=0)$ $\sum q^d N^{GW}_{0,d,k}$
$[\mathcal{E}_q]\,\omega_0(z,H) = 3H\left(\omega_0(z) + \omega_1(z)H + \omega_2(z)H^2 + \cdots\right) \in H^*(\mathbb{P}^2)$
higher Frobenius functions appear in $\{\omega_k(z)\}$
Consider equivariant coh ,

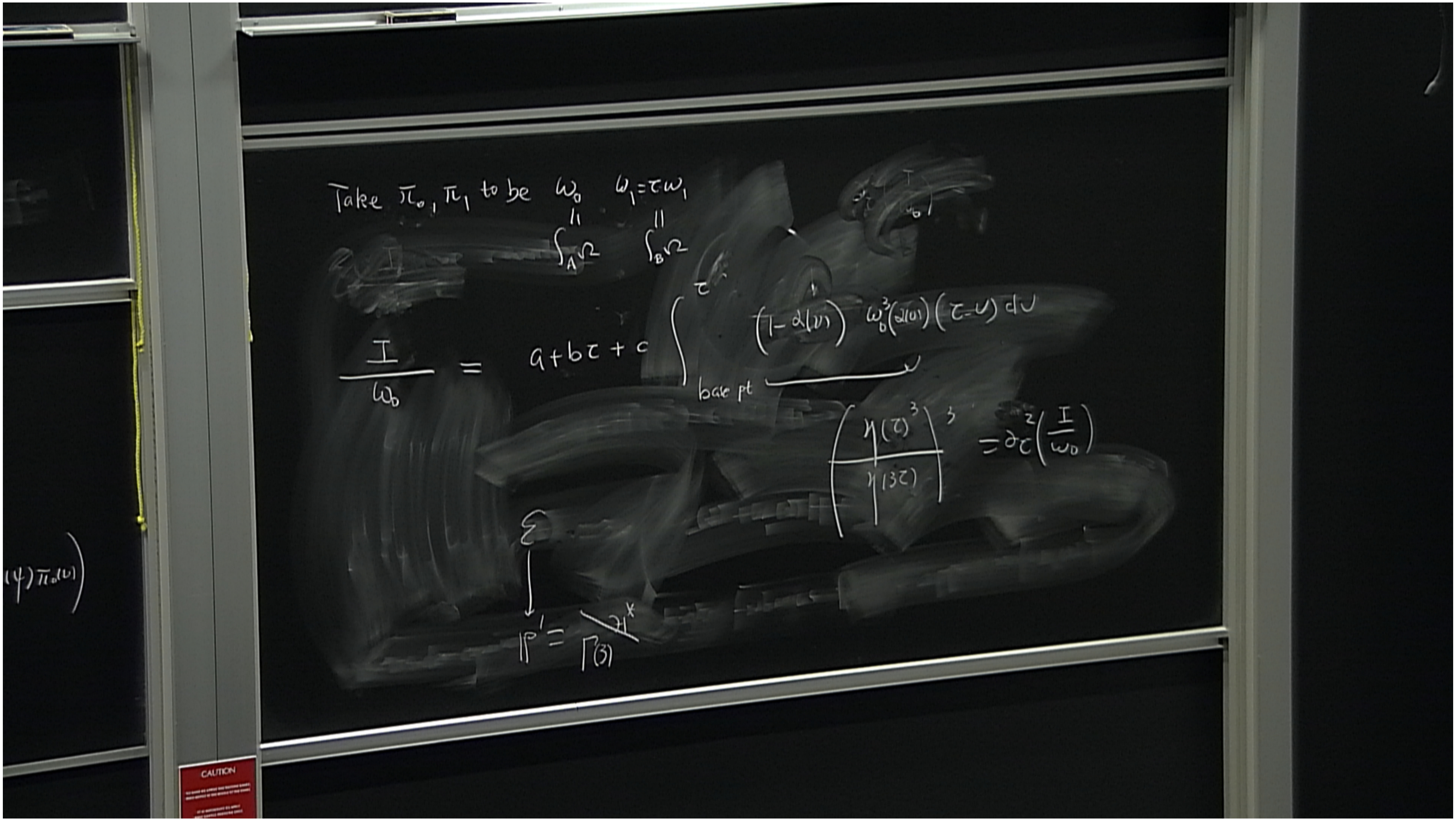

Wronskian method.
L_PF ω_2 = c
Take π_0(z) π_1(z) sols to L_PF

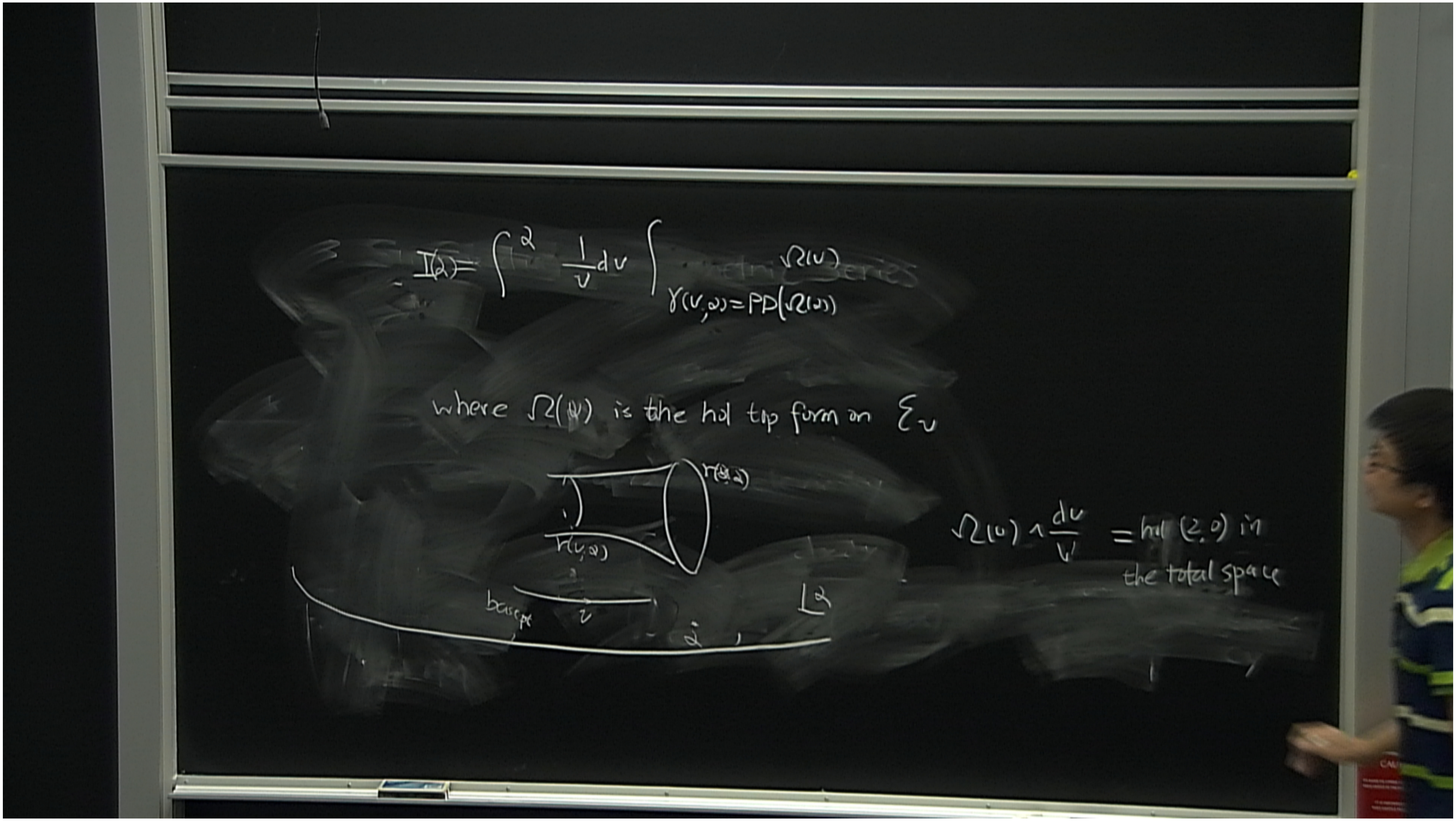


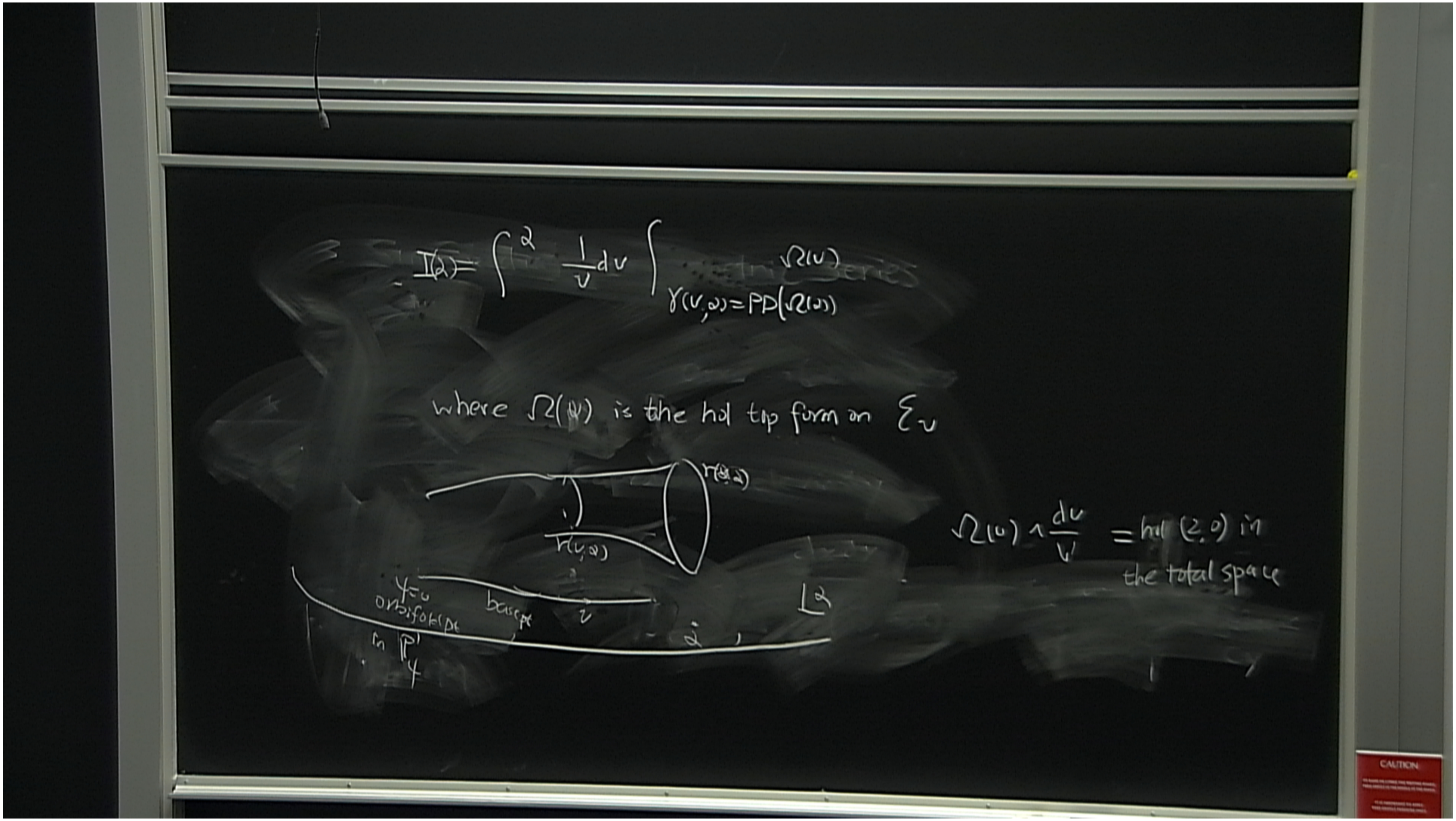
Wronskian method
L_PF w_2 = C
Take π_0(ψ) π_1(ψ) sols to L_PF f = 0
W(ψ) = π_0'(ψ) π_1(ψ) − π_1'(ψ) π_0(ψ)
Sol to inhomogeneous PF eq^n
I = aπ_0 + bπ_1 + c ∫_base^ψ 1/(coeff of ∂_ψ^2) 1/W(v) (π_0(ψ) π_1(v) − π_1(ψ) π_0(v))
CAUTION


Take π₀, π₁ to be ω₀ ω₁ = τω₁
∫_A Ω ∫_B Ω
I / ω₀ = a + bτ + c ∫_{base pt}^{τ} (1 − λ(υ)) ω₀³(λ(υ)) (τ − υ) dυ
CAUTION


Take π0, π1 to be ω0 = ∫A Ω   ω1 = τω1 = ∫B Ω
I / ω0 = a + bτ + c ∫ (base pt to τ) (1 − α(v)) ω0^3(α(v)) (τ − v) dv
(η(τ)^3 / η(3τ))^3
E
P′ = Γ(3) \ H*


Take π0, π1 to be ω0 ω1 = τω1
∫A Ω ∫B Ω
I/ω0 = a + bτ + c ∫ base pt τ (1 − α(v)) ω0³(α(v)) (τ − v) dv
(η(τ)³ / η(3τ))³ = ∂τ² (I/ω0)
ε
P' = Γ(3)


I(λ) = ∫^λ 1/v dv ∫_{γ(v,λ)=PD(Ω(λ))} Ω(v)
where Ω(v) is the hol top form on Σ_v
Ω(v) ∧ dv/v = hol (2,0) in the total space
basept


I(λ) = ∫^λ 1/v dv ∫_{γ(v,λ)=PD(Ω(λ))} Ω(v)
where Ω(v) is the hol top form on Σ_v
γ(λ,λ)
γ(v,λ)
basept
Ω(v) ∧ dv/v = hol (2,0) in the total space

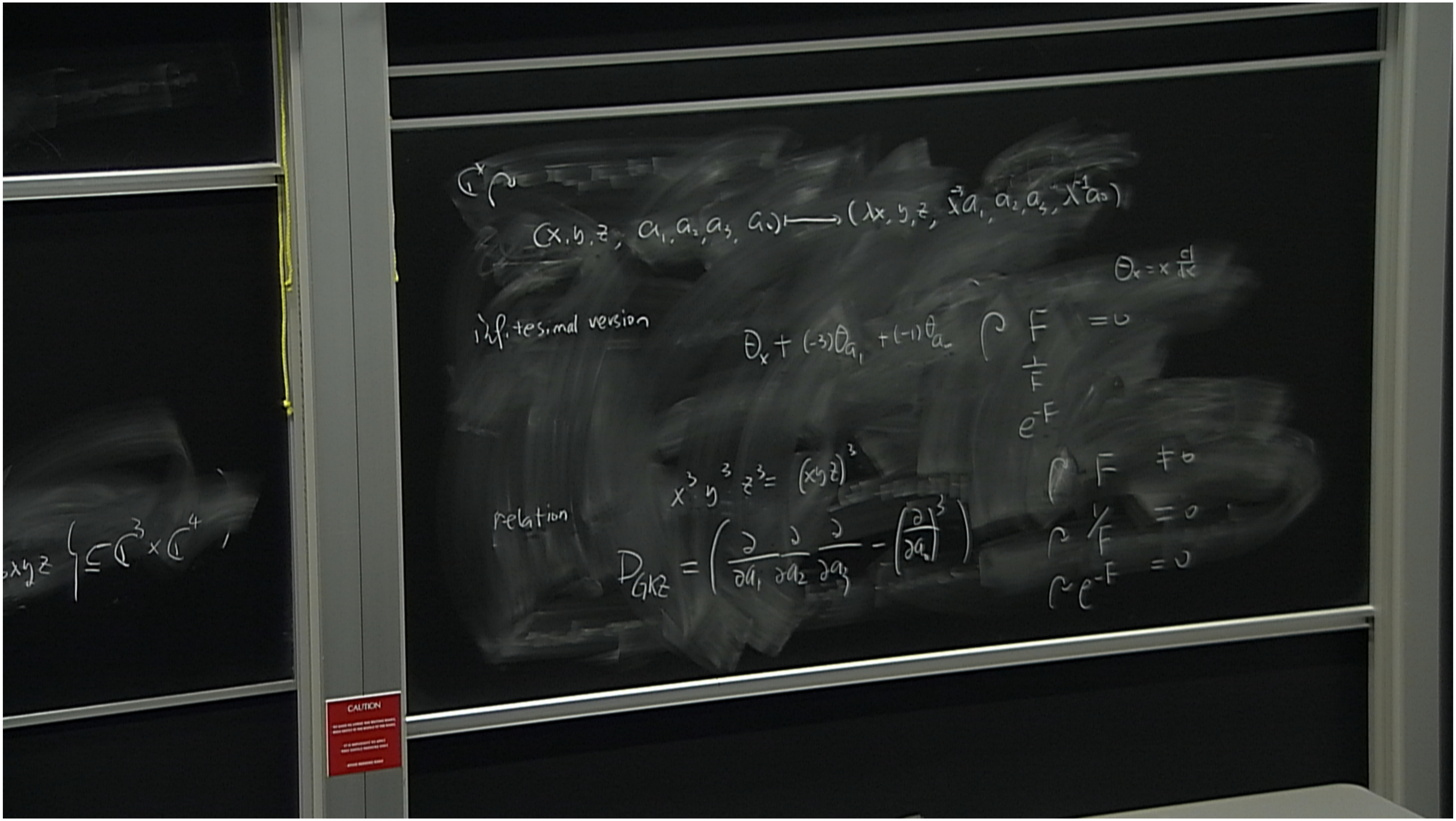
Take $\pi_0(\psi)$ $\pi_1(\psi)$ sols to $L_{PF}\,\Gamma = 0$

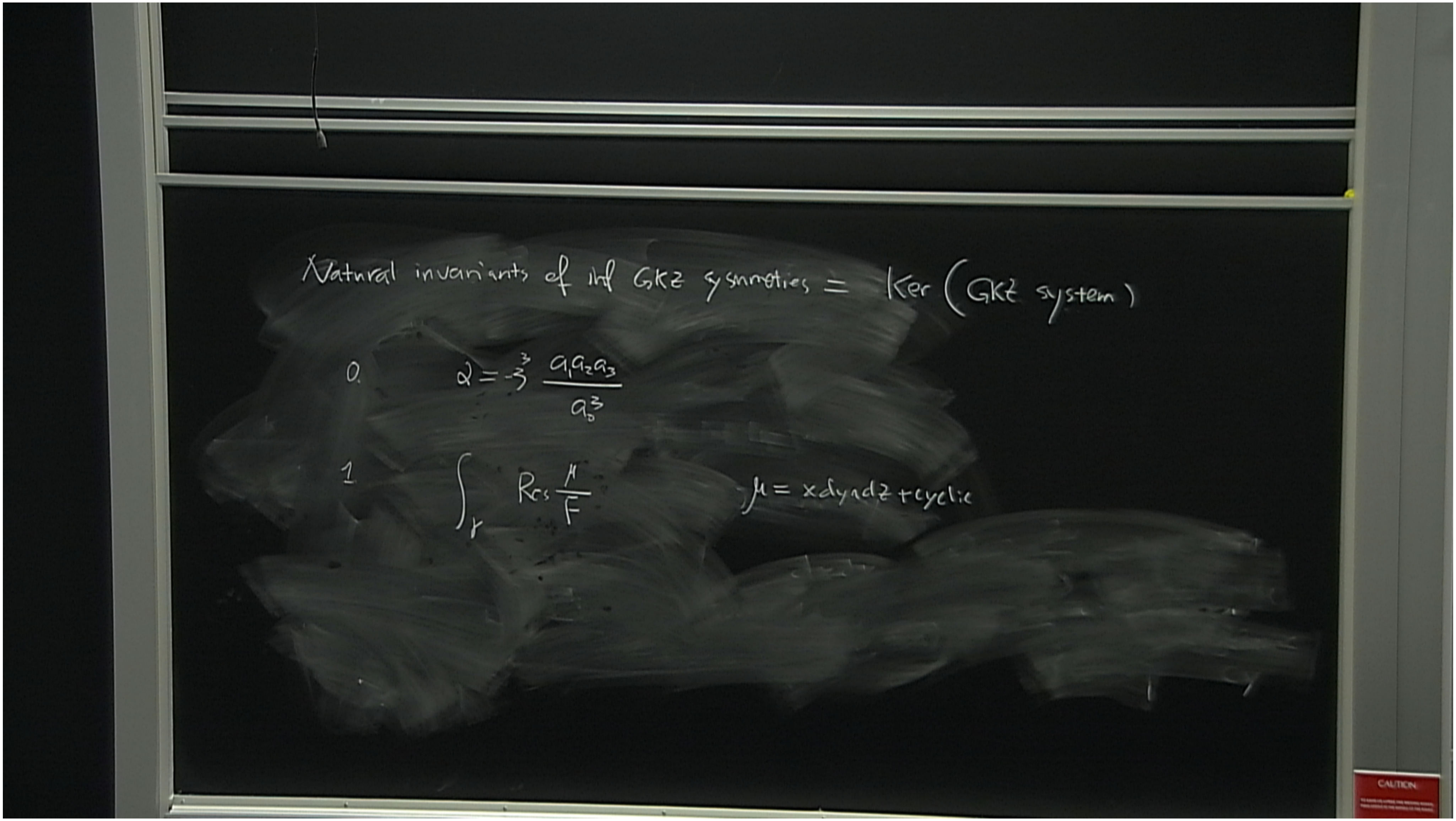
III. GKZ system

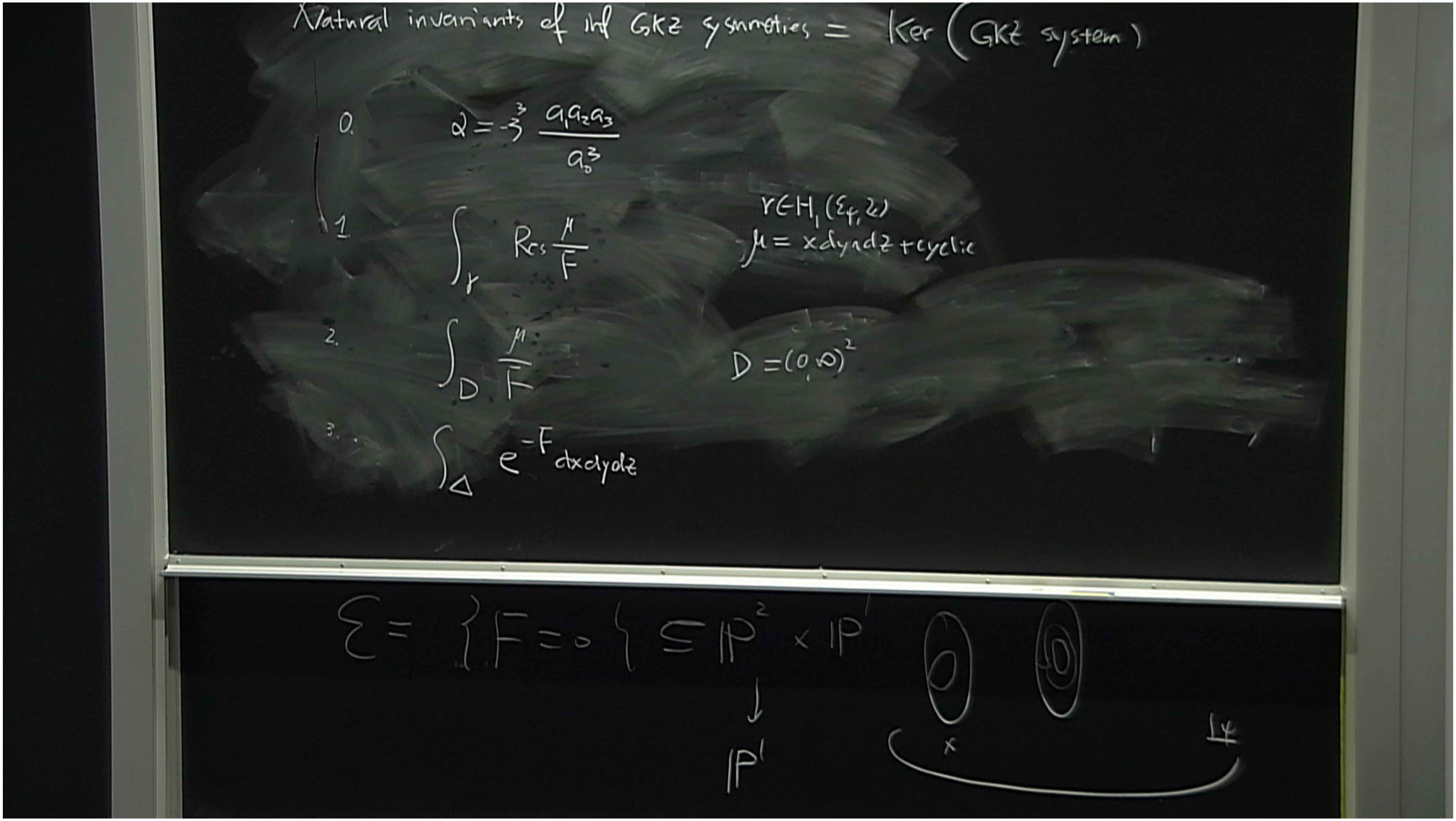
$\{F=0\}$ family of varieties

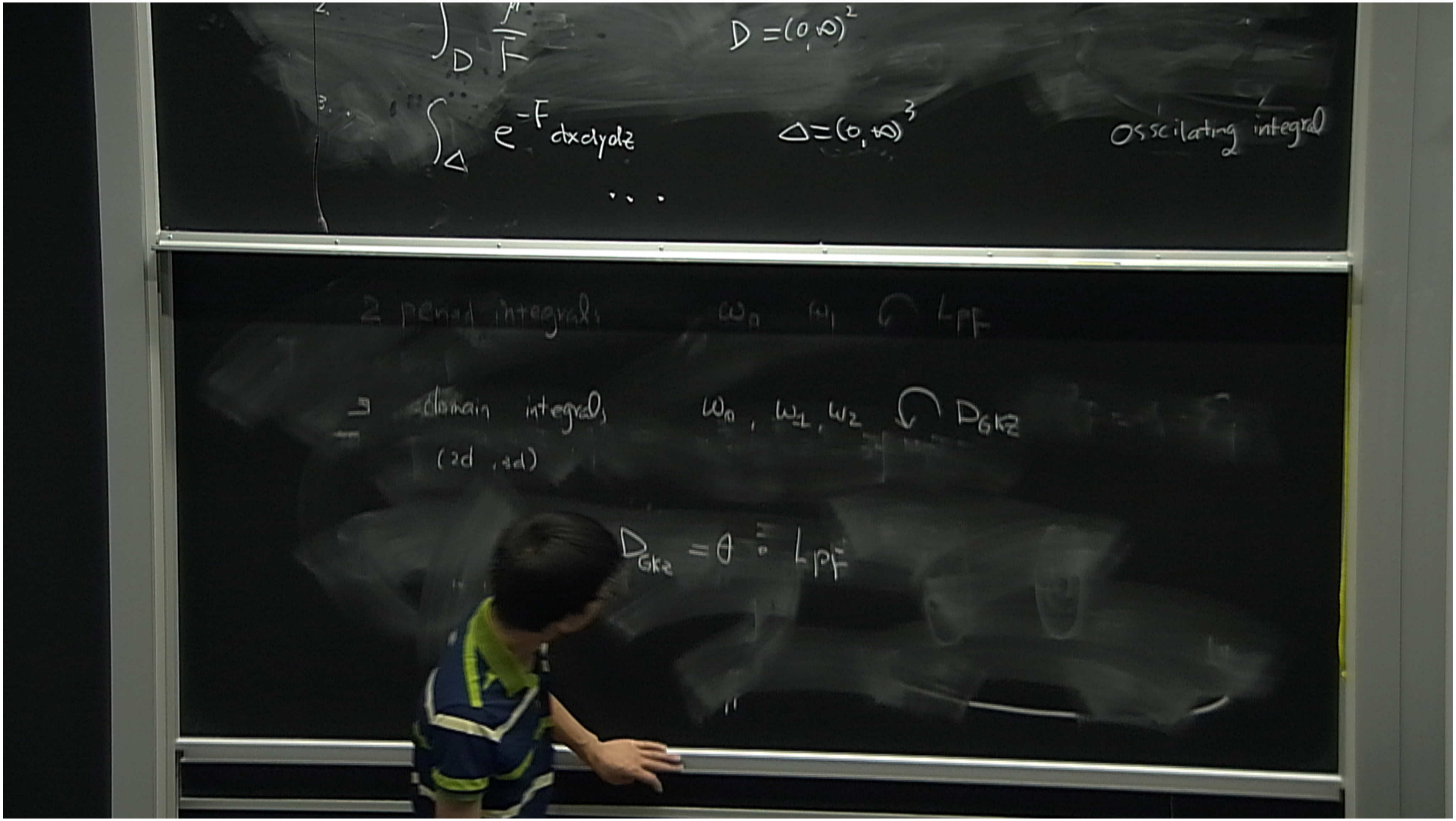
Consider a family of polynomials

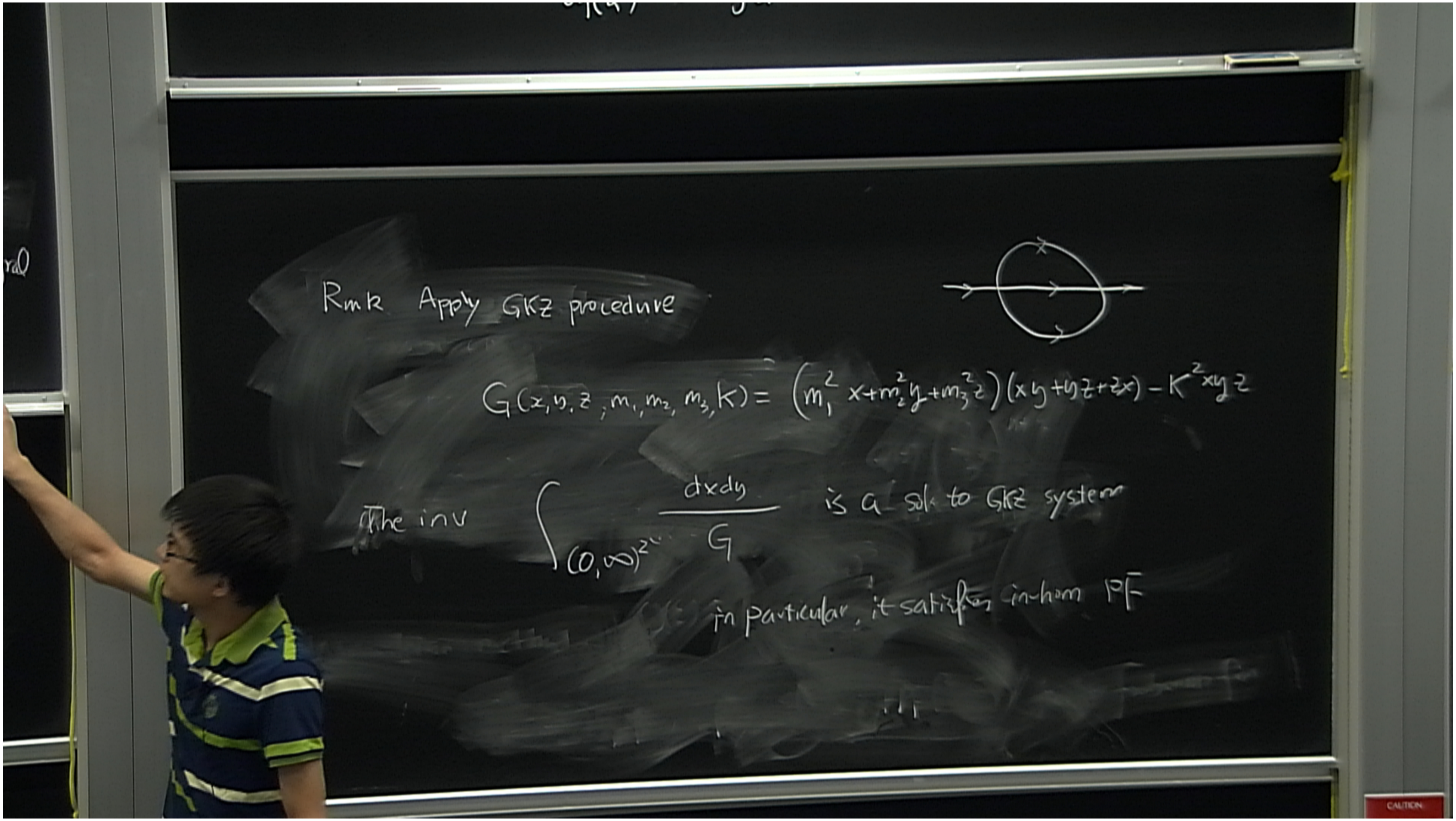
$$\{ F(x,y,z,a_1,a_2,a_3,a_0) = a_1x^3+a_2y^3+a_3z^3+a_0xyz \} \subseteq \mathbb{C}^3 \times \mathbb{C}^4 ,$$


(x, y, z, a1, a2, a3, a0) ⟼ (λx, y, z, λ^-3 a1, a2, a3, λ^-1 a0)
Θx = x d/dx
infinitesimal version
Θx + (-3)Θa1 + (-1)Θa0
F = 0
1/F
e^-F
F ≠ 0
1/F = 0
e^-F = 0
x^3 y^3 z^3 = (xyz)^3
relation
D_GKZ = (∂/∂a1 ∂/∂a2 ∂/∂a3 − (∂/∂a0)^3)
⊆ ℂ^3 × ℂ^4

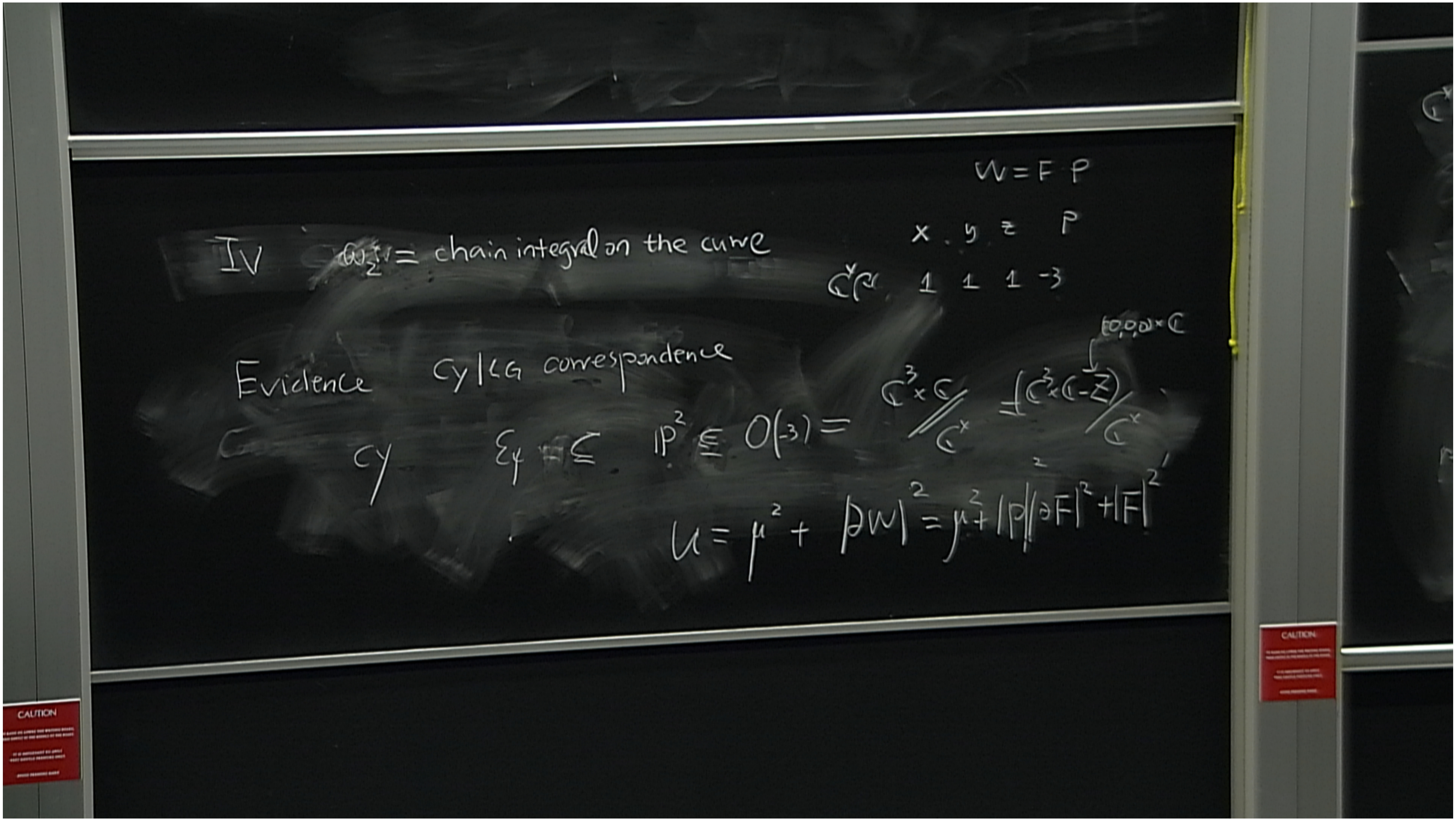

Natural invariants of inf GKZ symmetries = Ker (GKZ system)
0. α = -3^3 a1a2a3 / a0^3
1 ∫γ Res μ/F
μ = xdy∧dz + cyclic

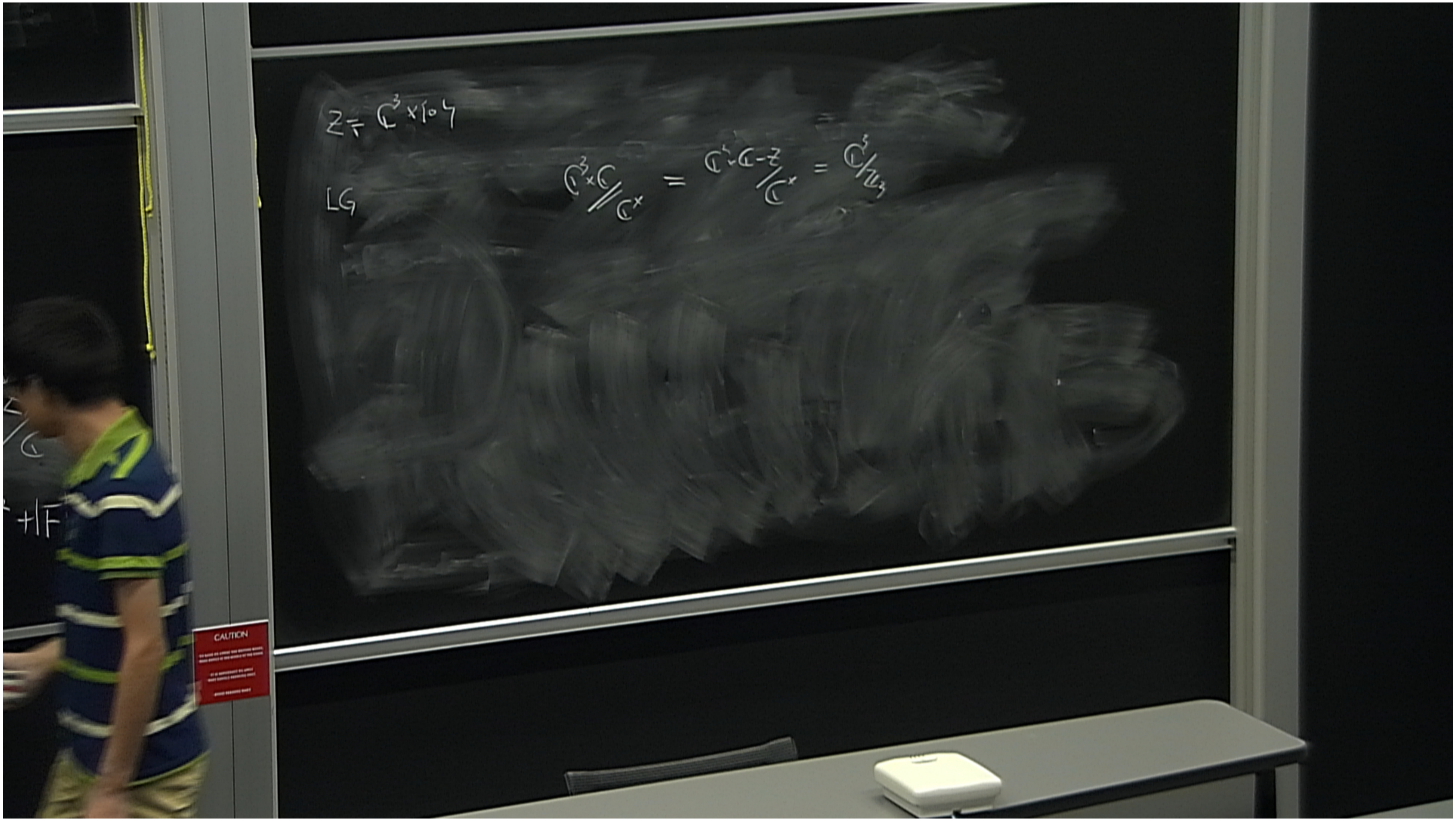

Natural invariants of inf GKZ symmetries = Ker (GKZ system)
0. $\alpha = -3^3 \frac{a_1 a_2 a_3}{a_0^3}$
1. $\int_\gamma \mathrm{Res} \frac{\mu}{F}$
$\gamma \in H_1(\mathcal{E}_\psi, \mathbb{Z})$
$\mu = x\, dy \wedge dz$ + cyclic
2. $\int_D \frac{\mu}{F}$
$D = (0,\infty)^2$
3. $\int_\Delta e^{-F} dx\, dy\, dz$
$\mathcal{E} = \{F = 0\} \subseteq \mathbb{P}^2 \times \mathbb{P}^1$
$\mathbb{P}^1$
x
ψ

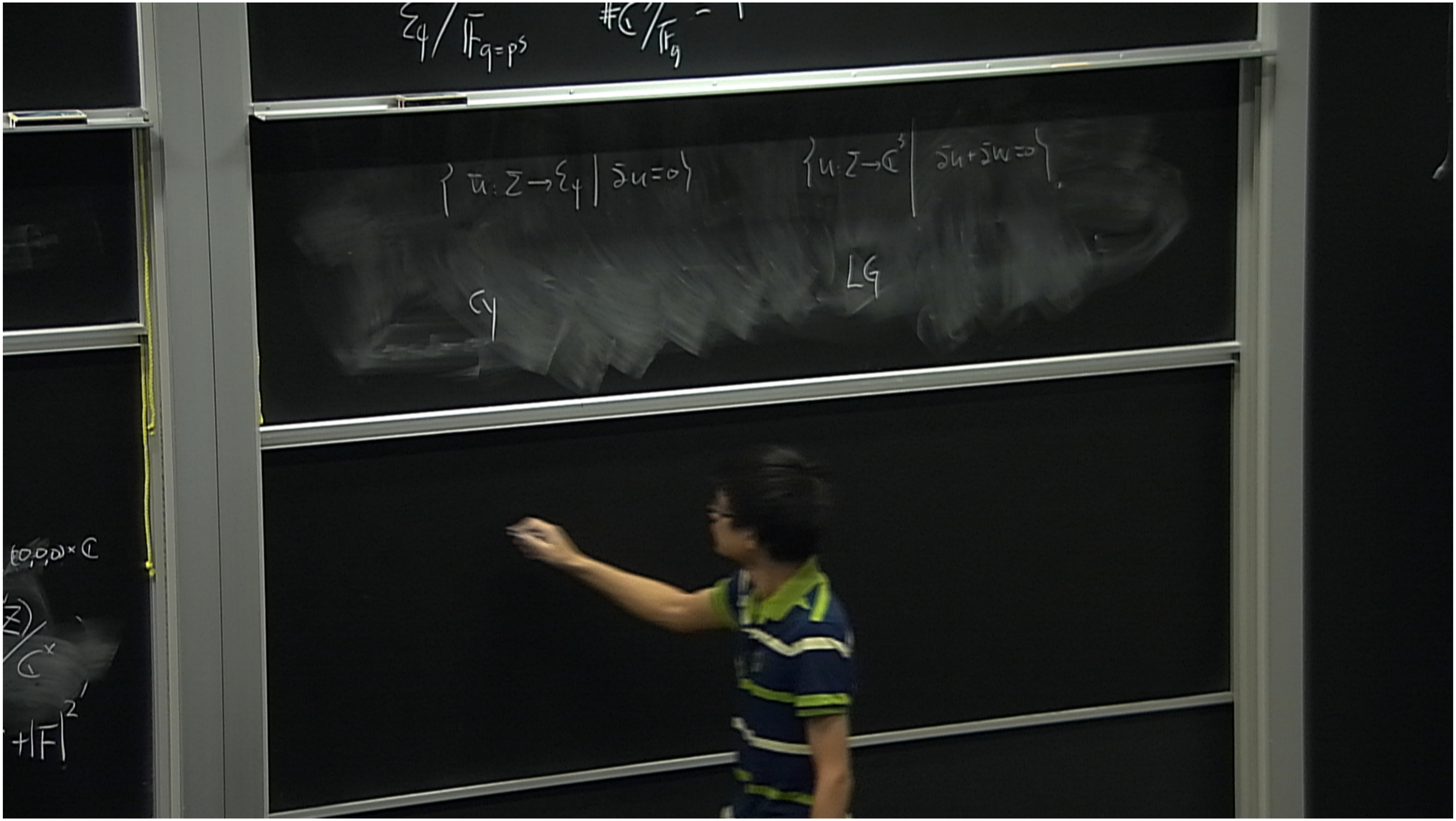

D =(0,∞)^2
e^-F dxdydz
Δ=(0,∞)^3
Osscilating integral
2 period integrals ω0 ω1 Lpf
3 domain integrals ω0, ω1, ω2 DGKZ
(2d, 3d)
DGKZ = θ Lpf

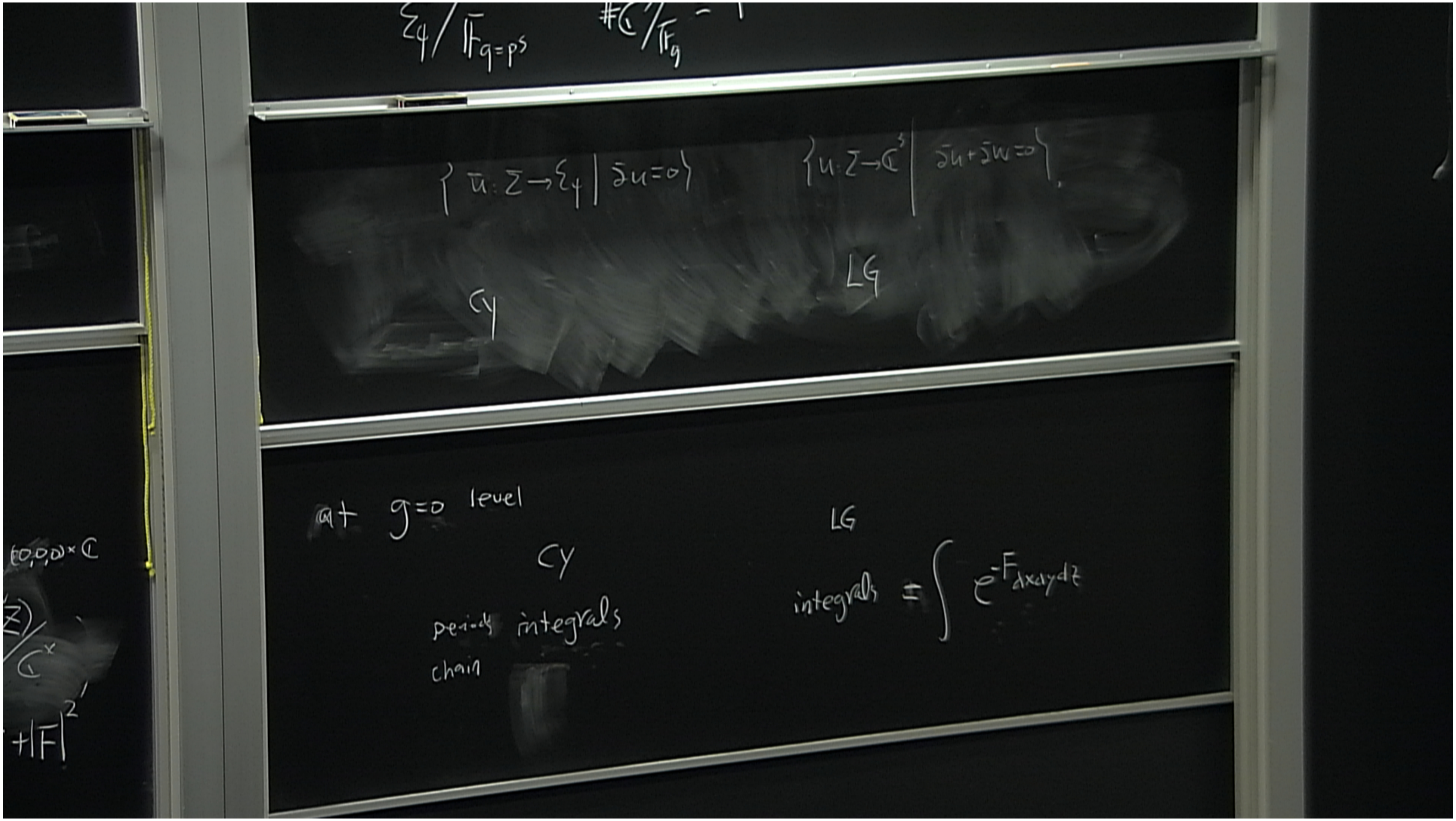


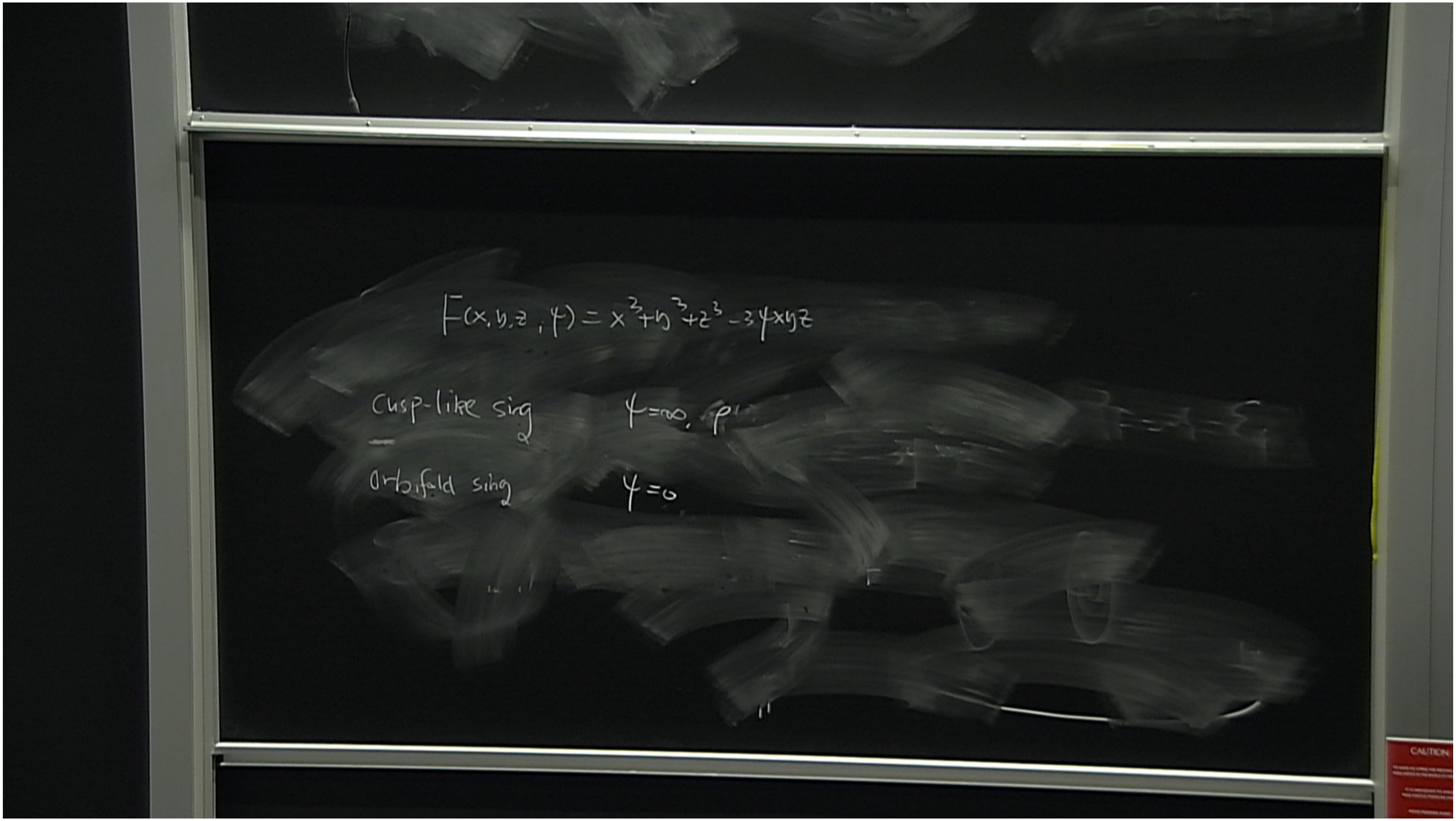
Rmk Apply GKZ procedure
G(x, y, z ; m1, m2, m3, K) = (m1² x + m2² y + m3² z)(xy + yz + zx) − K² xyz
The inv ∫(0,∞)² dxdy / G is a sol to GKZ system
in particular, it satisfies in-hom PF

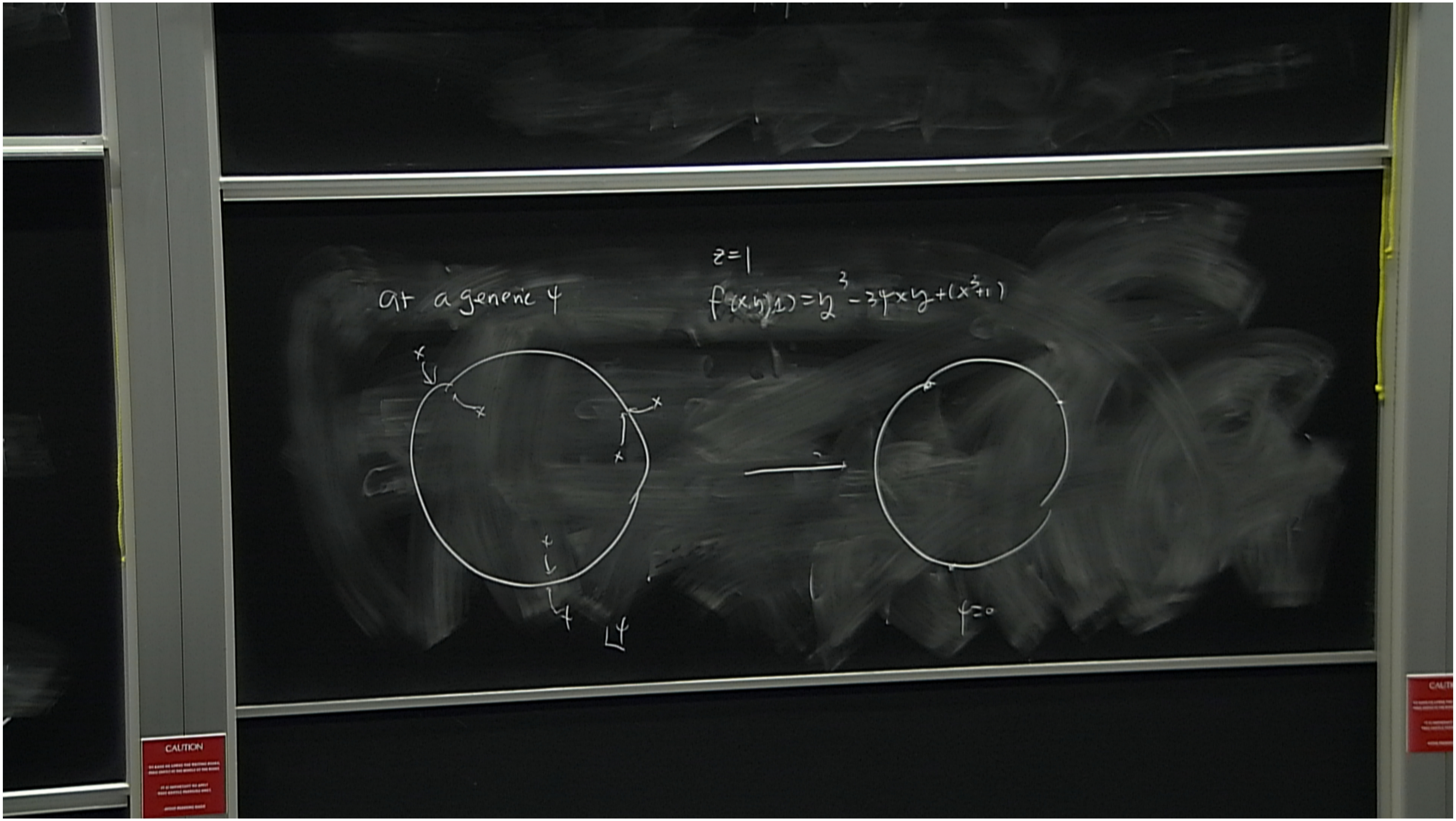

W = F P
x y z P
1 1 1 -3
IV
= chain integral on the curve
Evidence
CY|LG correspondence
CY
P² ⊆ O(-3) =
U = μ² + |∂W|² = μ² + |P|²|∂F|² + |F|²

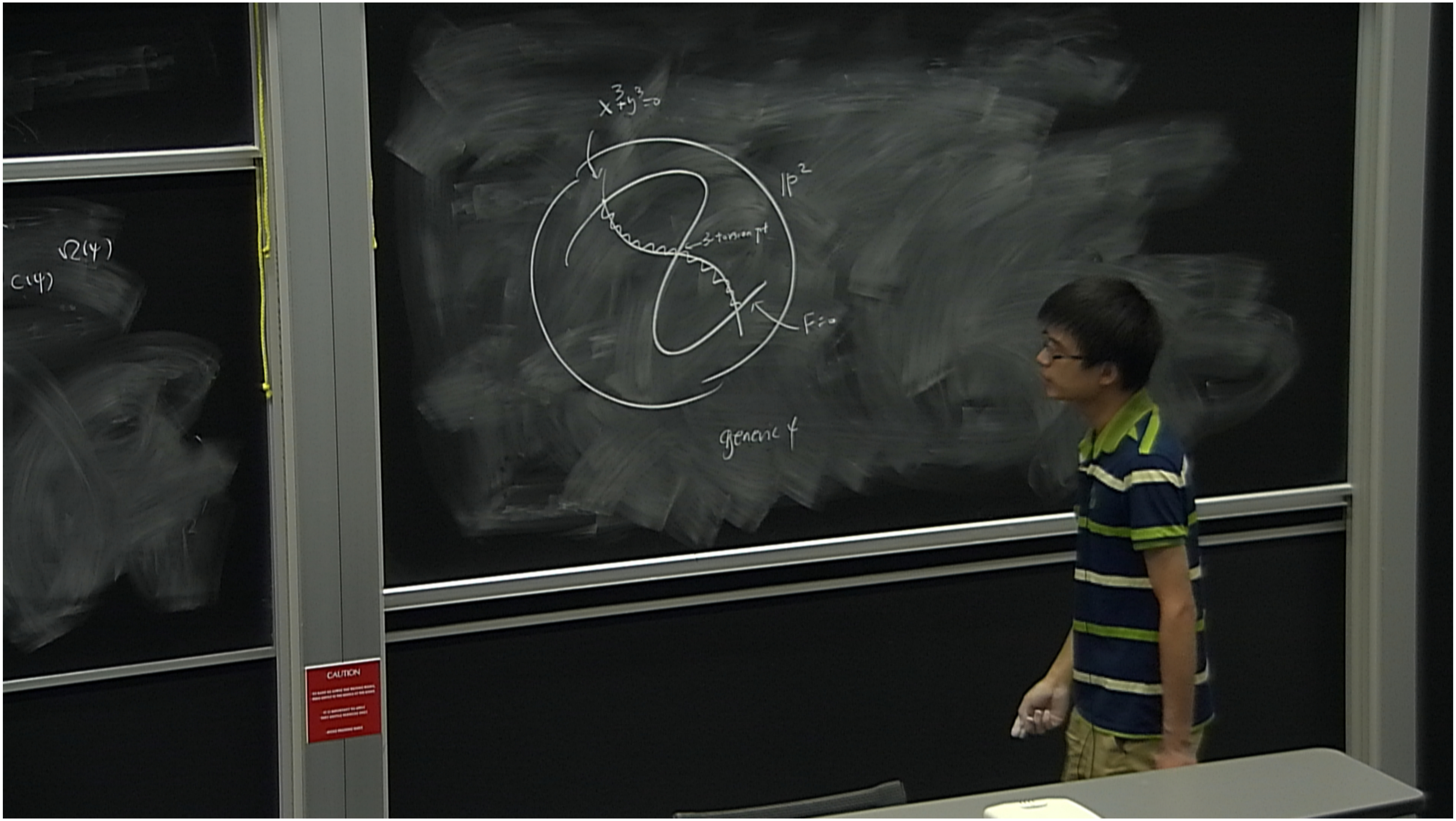

LG

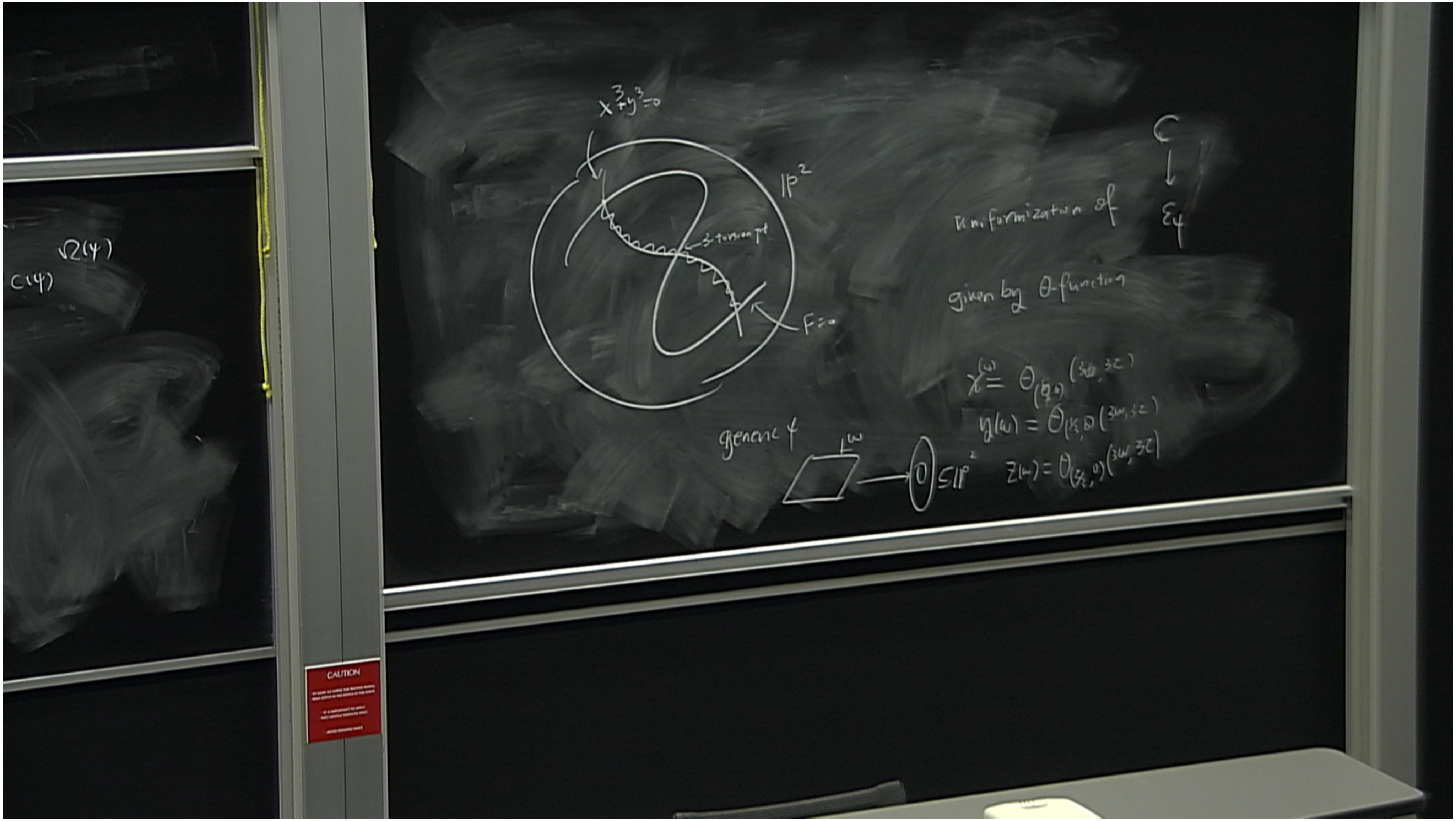

CY
LG


CY
LG
at g=0 level
LG
CY
integrals
period integrals
chain

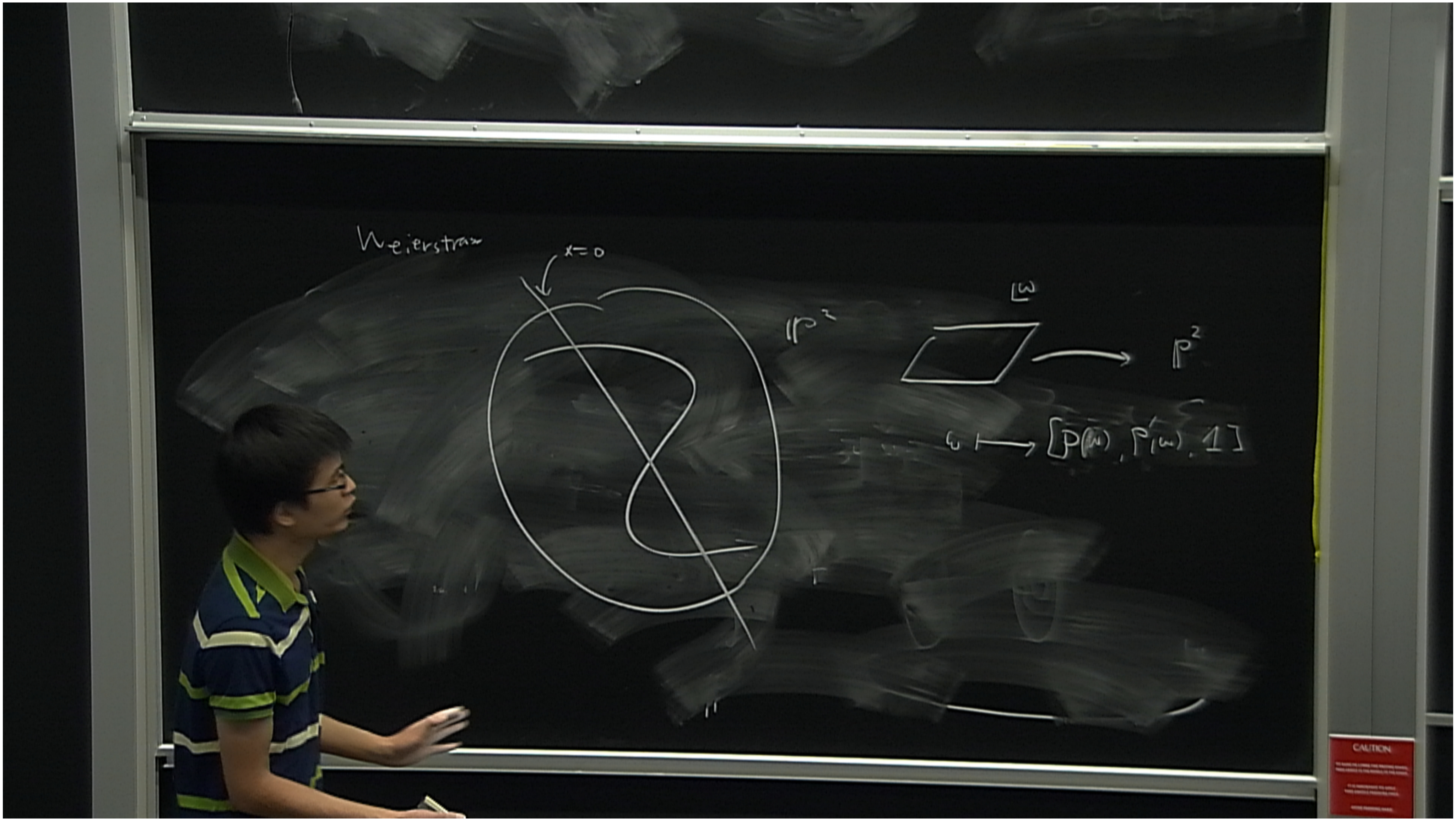

F(x,y,z,ψ) = x³+y³+z³−3ψxyz
Cusp-like sing
ψ=∞,
Orbifold sing
ψ=0


at a generic ψ
z=1
f(x,y,1)=y³-3ψxy+(x³+1)
ψ=0


$x^3+y^3=0$
3-torsion pt
$\mathbb{P}^2$
F=0
generic ψ
Ω(ψ)
C(ψ)
CAUTION



$x^3+y^3=0$
$\mathbb{P}^2$
3-torsion pt
F=0
Uniformization of
$\mathcal{C}$
$\mathcal{E}_\psi$
given by Θ-function
$x(\omega) = \Theta_{(\frac{1}{3},0)}(3\omega, 3\tau)$
$y(\omega) = \Theta_{(\frac{1}{3},0)}(3\omega, 3\tau)$
$z(\omega) = \Theta_{(\frac{2}{3},0)}(3\omega, 3\tau)$
generic ψ
$\subseteq \mathbb{P}^2$
Ω(ψ)
C(ψ)
CAUTION



Weierstrass
x=0
$\mathbb{P}^2$
$\mathbb{C}/\Lambda$
$\mathbb{P}^2$
$\omega \mapsto [\wp(\omega), \wp'(\omega), 1]$