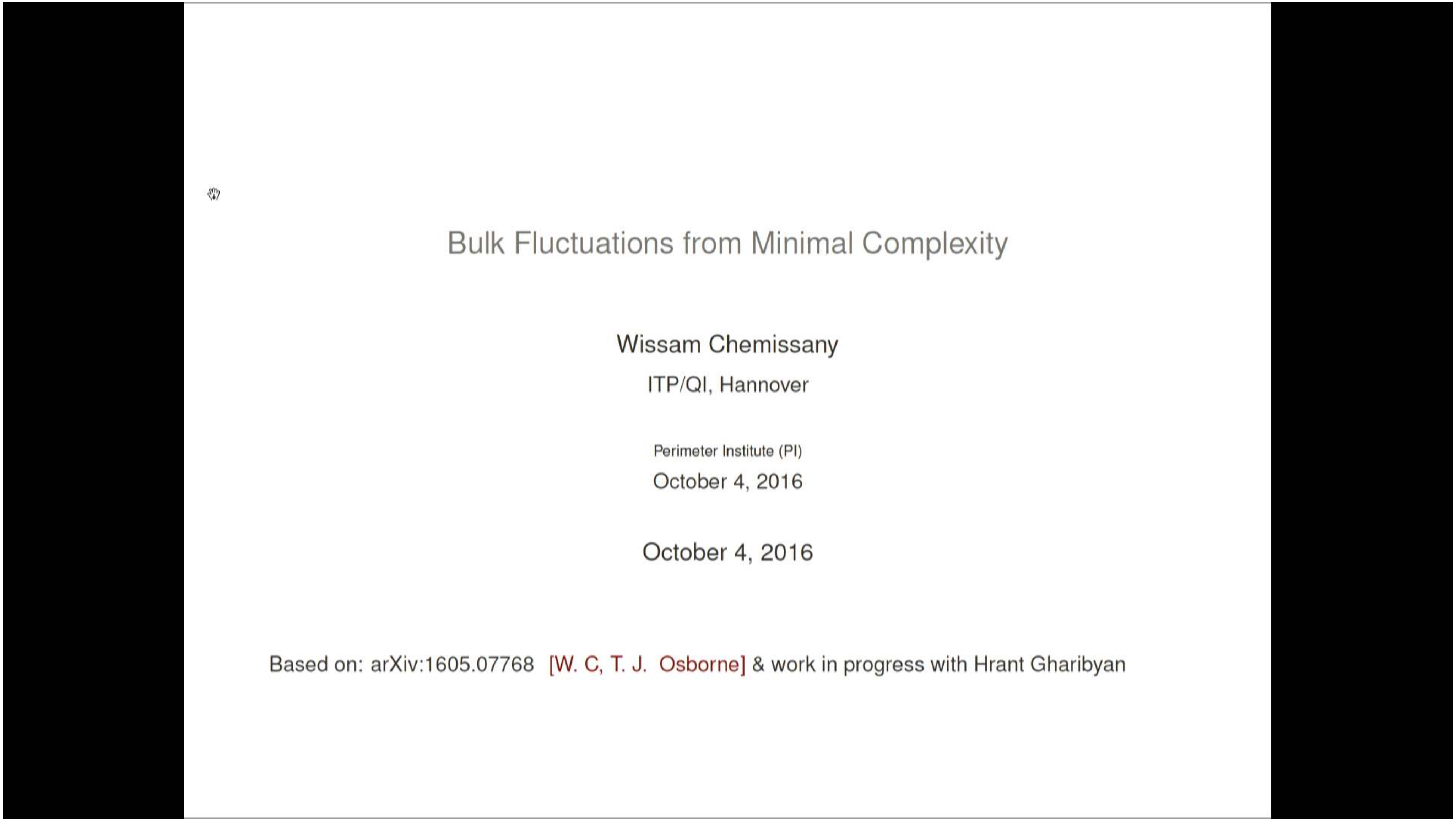
Title: Complexity and Holographic Fluctuations

Date: Oct 04, 2016 03:30 PM

URL: http://pirsa.org/16100037

Abstract: <p>I discuss, from a quantum information perspective, recent proposals of Maldacena, Ryu, Takayanagi, van Raamsdonk, Swingle, and Susskind that spacetime is an emergent property of the quantum entanglement of an associated boundary quantum system. I review the idea that the informational principle of minimal complexity determines a dual holographic bulk spacetime from a minimal quantum circuit U preparing a given boundary state from a trivial reference state. I describe how this idea may be extended to determine the relationship between the fluctuations of the bulk holographic geometry and the fluctuations of the boundary low-energy subspace. In this way we obtain, for every quantum system, an Einstein-like equation of motion for what might be interpreted as a bulk gravity theory dual to the boundary system. If time permits I will comment on the link to Brownian quantum circuits and tensor networks. </p>



Bulk Fluctuations from Minimal Complexity

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October 4, 2016

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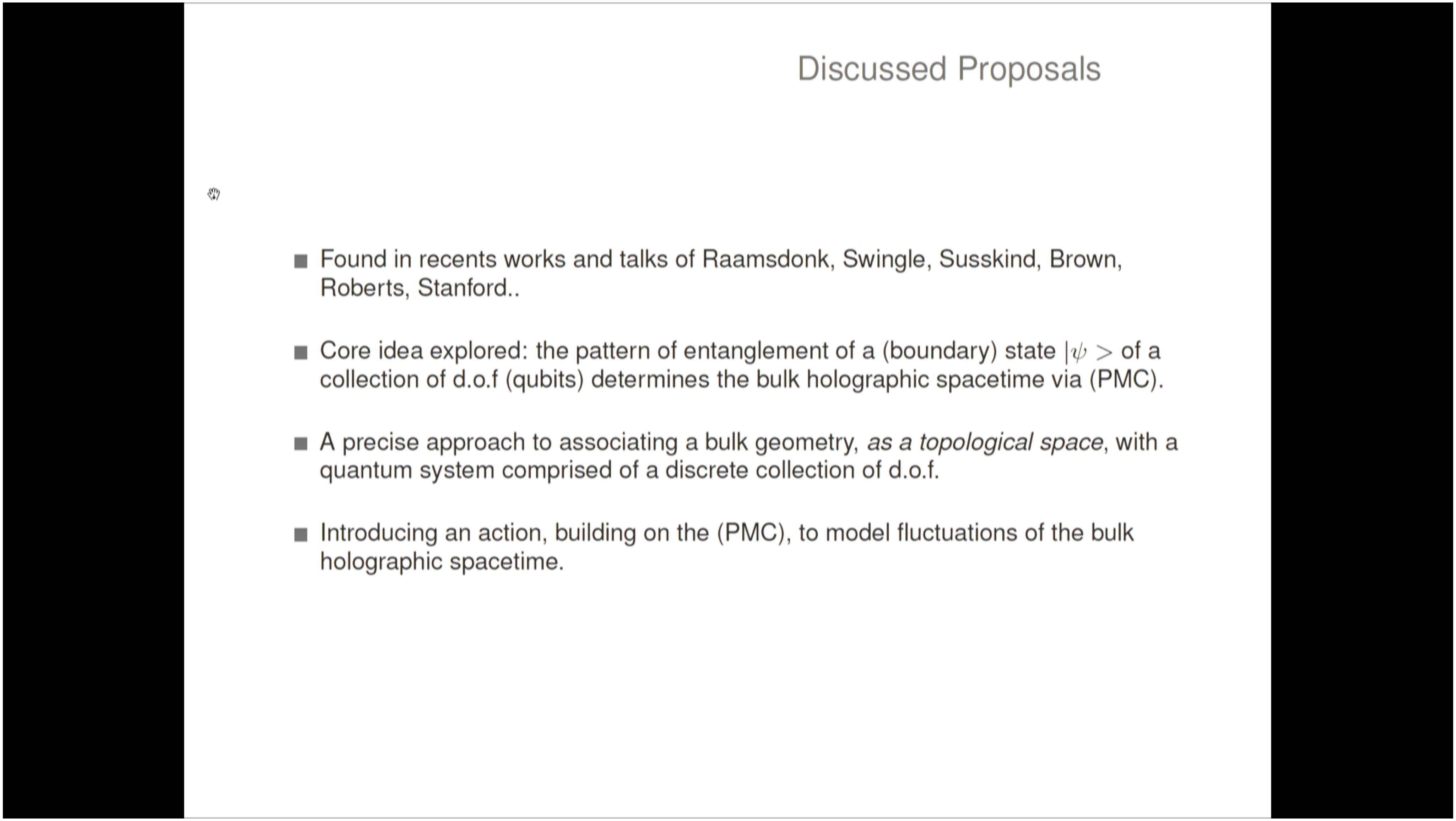
Based on: arXiv:1605.07768 [W. C, T. J. Osborne] & work in progress with Hrant Gharibyan

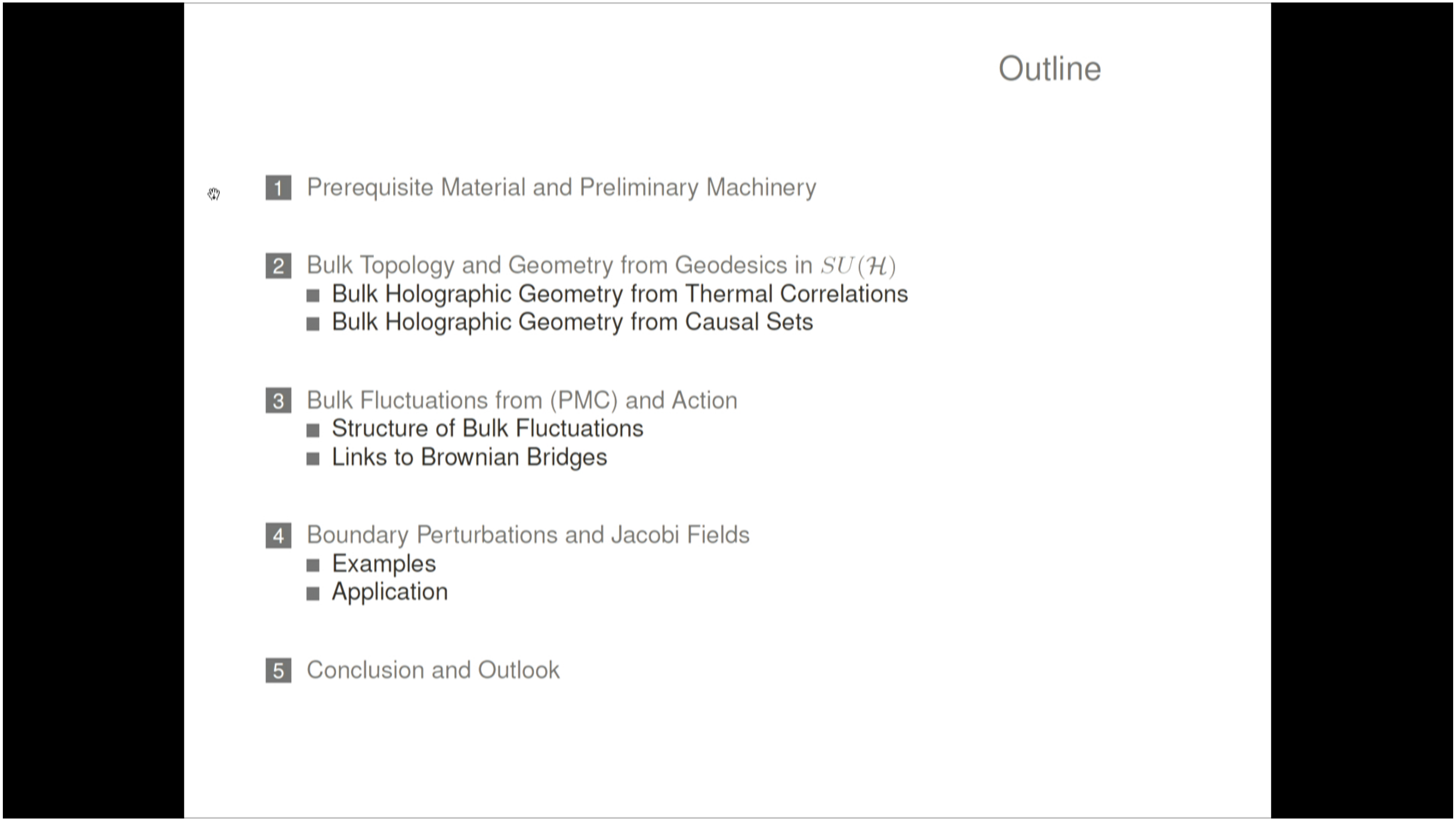
Maximizing the usefulness of the principle of minimal complexity

- The main motivation of this work is to discuss, from a QI perspective, recent proposals that spacetime is an *emergent* property of quantum entanglement.
- A proposal has been put forward to determine a dual holographic bulk spacetime from the informational principle of minimal complexity (PMC).
- The main ingredient of this proposal is a minimal quantum circuit U preparing a given boundary state from a trivial reference state.
- The powerfulness of (PMC) allows to determine the relationship between the *fluctuations* of the bulk geometry and the fluctuations of the boundary low-energy subspace.
- For every quantum system, one obtains an Einstein-like equation of motion for what might be interpreted a bulk gravity theory dual to the boundary system.



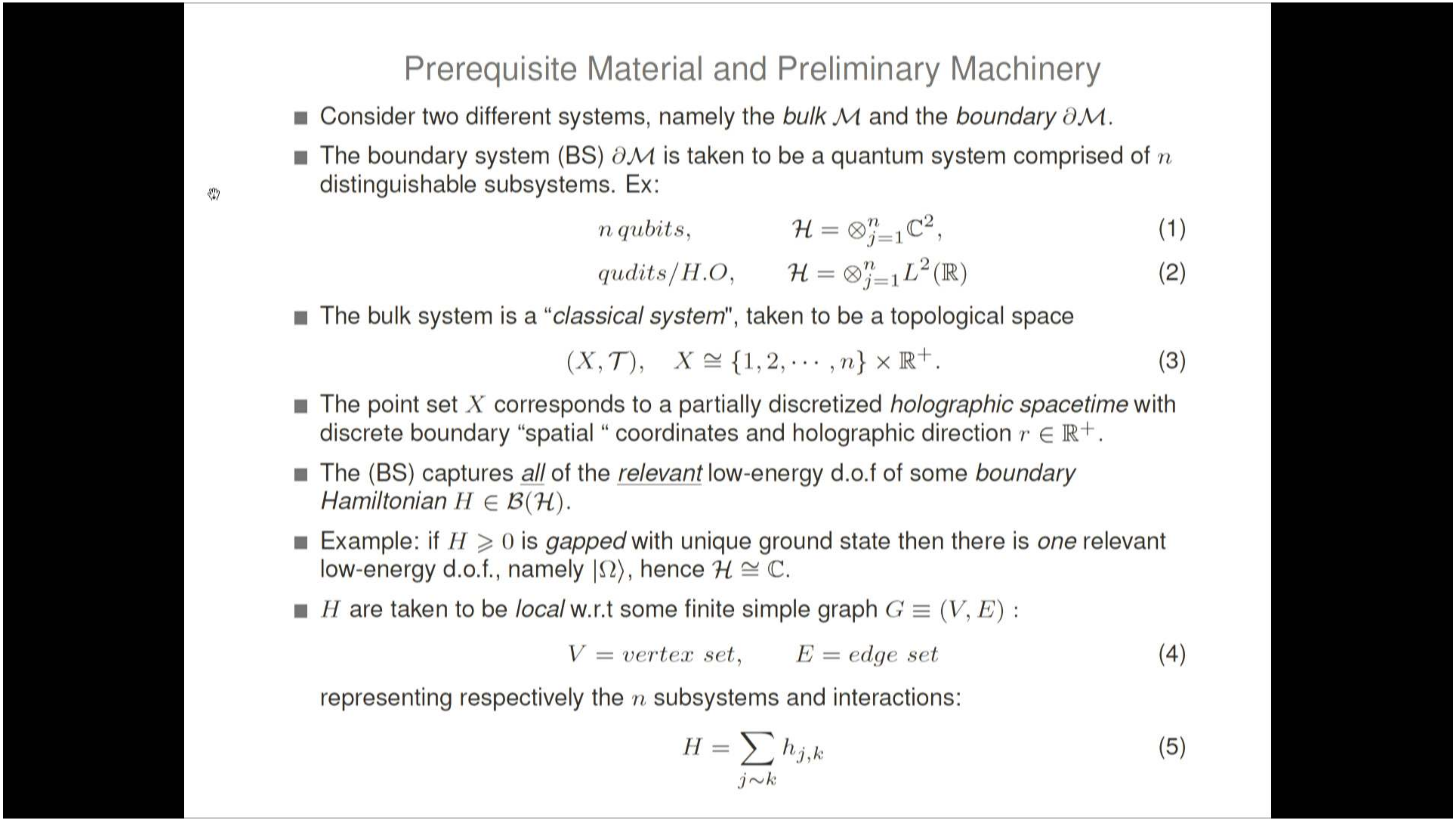
Discussed Proposals

- Found in recents works and talks of Raamsdonk, Swingle, Susskind, Brown, Roberts, Stanford..
- Core idea explored: the pattern of entanglement of a (boundary) state $|\psi >$ of a collection of d.o.f (qubits) determines the bulk holographic spacetime via (PMC).
- A precise approach to associating a bulk geometry, *as a topological space*, with a quantum system comprised of a discrete collection of d.o.f.
- Introducing an action, building on the (PMC), to model fluctuations of the bulk holographic spacetime.



Outline

1. Prerequisite Material and Preliminary Machinery
2. Bulk Topology and Geometry from Geodesics in $SU(\mathcal{H})$
   - Bulk Holographic Geometry from Thermal Correlations
   - Bulk Holographic Geometry from Causal Sets
3. Bulk Fluctuations from (PMC) and Action
   - Structure of Bulk Fluctuations
   - Links to Brownian Bridges
4. Boundary Perturbations and Jacobi Fields
   - Examples
   - Application
5. Conclusion and Outlook



Prerequisite Material and Preliminary Machinery

- Consider two different systems, namely the *bulk* $\mathcal{M}$ and the *boundary* $\partial\mathcal{M}$.
- The boundary system (BS) $\partial\mathcal{M}$ is taken to be a quantum system comprised of n distinguishable subsystems. Ex:

$$n\,qubits, \qquad \mathcal{H} = \otimes_{j=1}^{n}\mathbb{C}^2, \tag{1}$$

$$qudits/H.O, \qquad \mathcal{H} = \otimes_{j=1}^{n}L^2(\mathbb{R}) \tag{2}$$

- The bulk system is a "*classical system*", taken to be a topological space

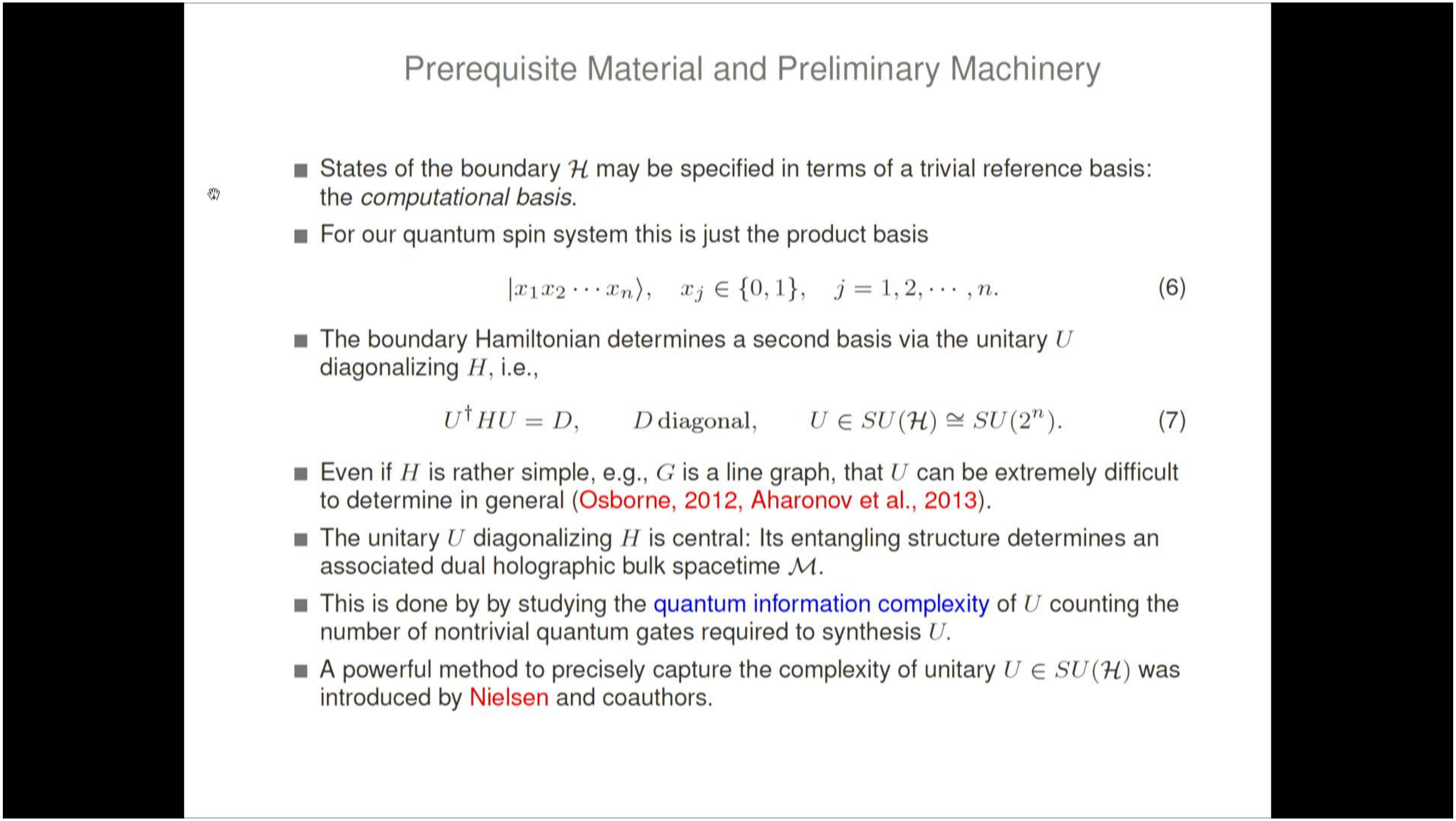
$$(X,\mathcal{T}), \quad X \cong \{1,2,\cdots,n\}\times\mathbb{R}^+. \tag{3}$$

- The point set X corresponds to a partially discretized *holographic spacetime* with discrete boundary "spatial " coordinates and holographic direction $r \in \mathbb{R}^+$.
- The (BS) captures *all* of the *relevant* low-energy d.o.f of some *boundary Hamiltonian* $H \in \mathcal{B}(\mathcal{H})$.
- Example: if $H \geqslant 0$ is *gapped* with unique ground state then there is *one* relevant low-energy d.o.f., namely $|\Omega\rangle$, hence $\mathcal{H} \cong \mathbb{C}$.
- H are taken to be *local* w.r.t some finite simple graph $G \equiv (V,E)$:

$$V = vertex\ set, \qquad E = edge\ set \tag{4}$$

representing respectively the n subsystems and interactions:

$$H = \sum_{j\sim k} h_{j,k} \tag{5}$$



Prerequisite Material and Preliminary Machinery

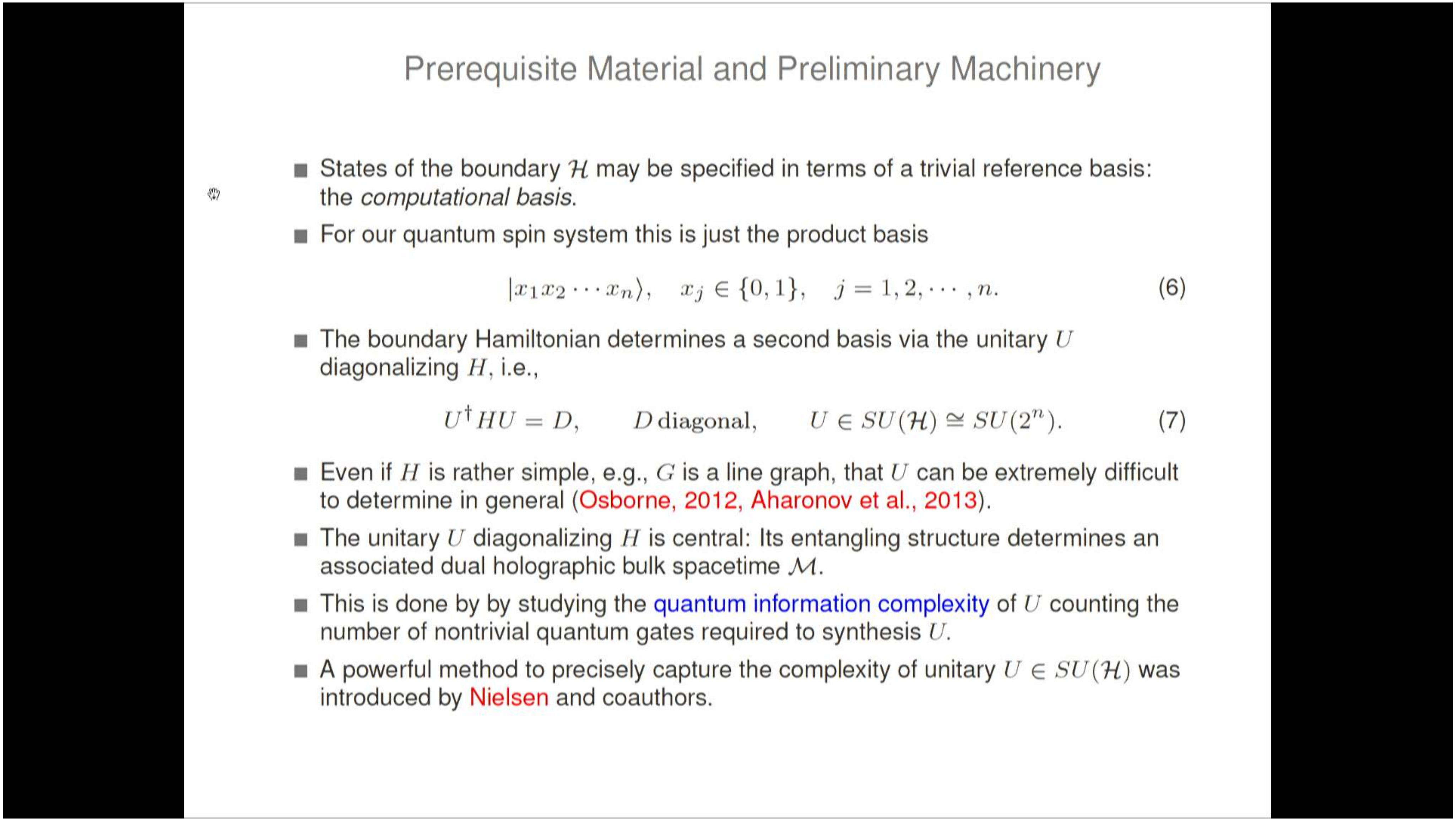
- States of the boundary $\mathcal{H}$ may be specified in terms of a trivial reference basis: the *computational basis*.
- For our quantum spin system this is just the product basis

$$|x_1 x_2 \cdots x_n\rangle, \quad x_j \in \{0,1\}, \quad j = 1, 2, \cdots, n. \tag{6}$$

- The boundary Hamiltonian determines a second basis via the unitary U diagonalizing H, i.e.,

$$U^\dagger H U = D, \qquad D\,\text{diagonal}, \qquad U \in SU(\mathcal{H}) \cong SU(2^n). \tag{7}$$

- Even if H is rather simple, e.g., G is a line graph, that U can be extremely difficult to determine in general (Osborne, 2012, Aharonov et al., 2013).
- The unitary U diagonalizing H is central: Its entangling structure determines an associated dual holographic bulk spacetime $\mathcal{M}$.
- This is done by by studying the quantum information complexity of U counting the number of nontrivial quantum gates required to synthesis U.
- A powerful method to precisely capture the complexity of unitary $U \in SU(\mathcal{H})$ was introduced by Nielsen and coauthors.



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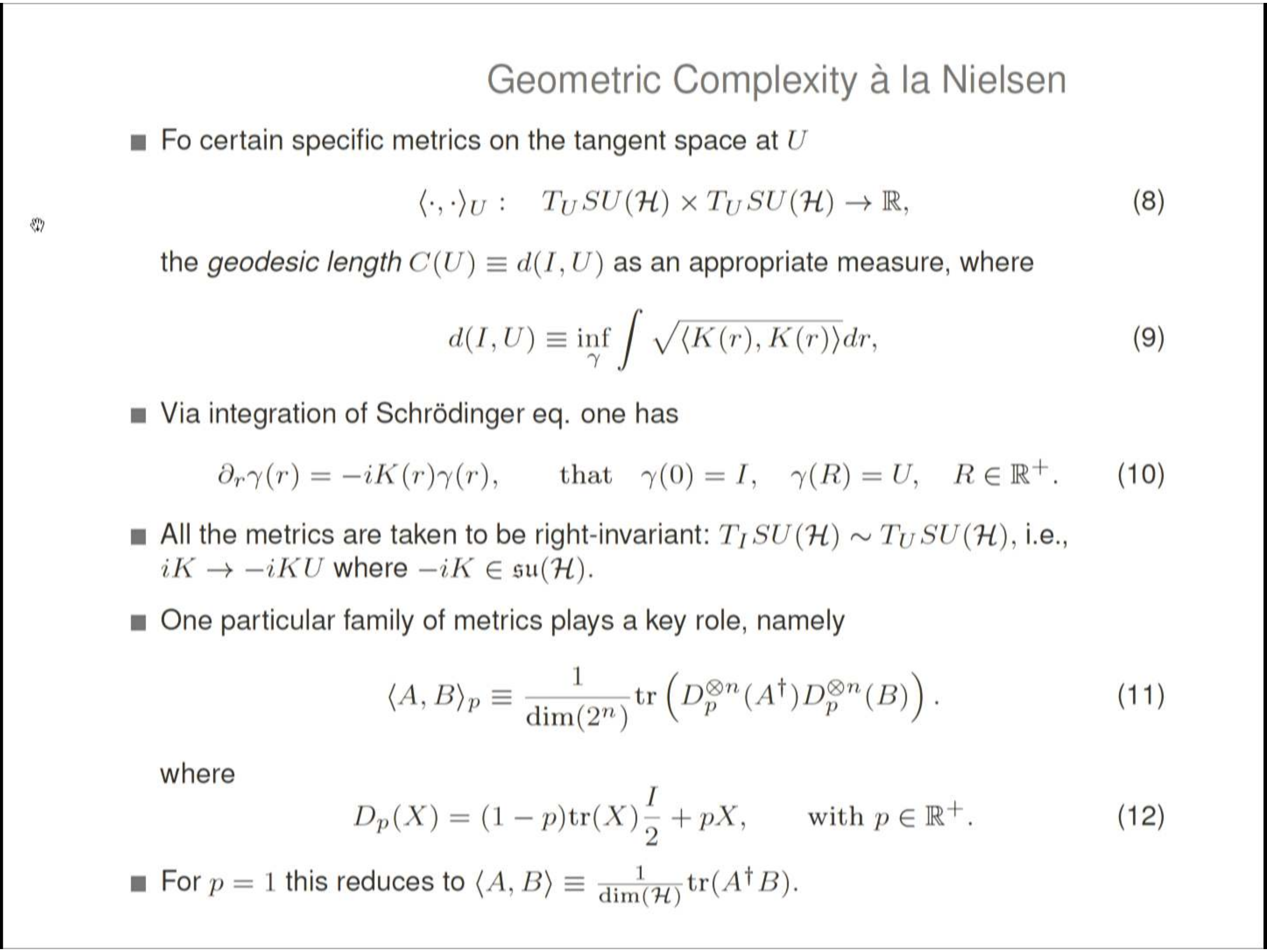


Geometric Complexity à la Nielsen

- Fo certain specific metrics on the tangent space at U

$$\langle \cdot , \cdot \rangle_U : \quad T_U SU(\mathcal{H}) \times T_U SU(\mathcal{H}) \to \mathbb{R}, \tag{8}$$

the *geodesic length* $C(U) \equiv d(I, U)$ as an appropriate measure, where

$$d(I, U) \equiv \inf_{\gamma} \int \sqrt{\langle K(r), K(r) \rangle} dr, \tag{9}$$

- Via integration of Schrödinger eq. one has

$$\partial_r \gamma(r) = -iK(r)\gamma(r), \qquad \text{that} \quad \gamma(0) = I, \quad \gamma(R) = U, \quad R \in \mathbb{R}^+. \tag{10}$$

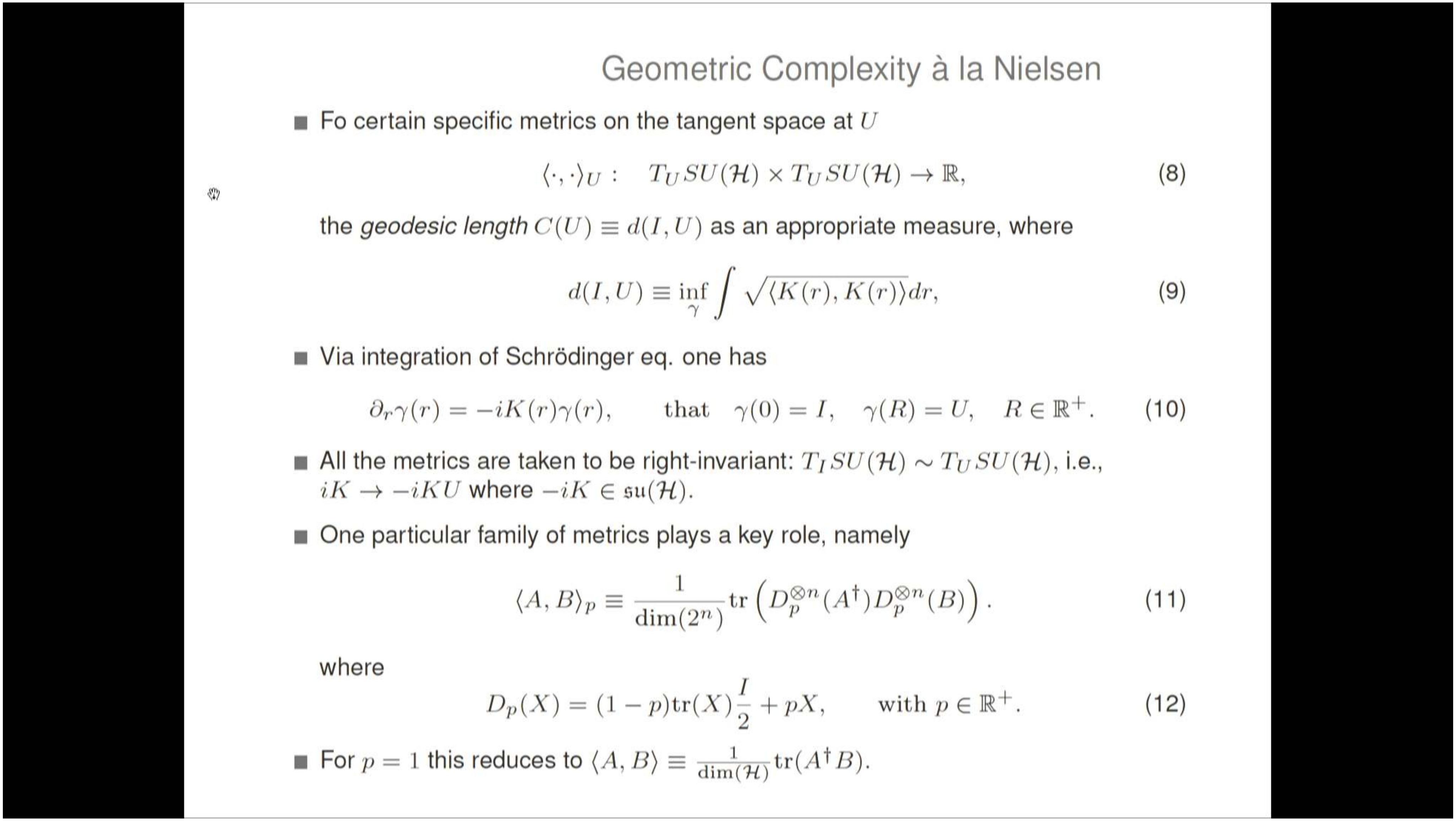
- All the metrics are taken to be right-invariant: $T_I SU(\mathcal{H}) \sim T_U SU(\mathcal{H})$, i.e., $iK \to -iKU$ where $-iK \in \mathfrak{su}(\mathcal{H})$.
- One particular family of metrics plays a key role, namely

$$\langle A, B \rangle_p \equiv \frac{1}{\dim(2^n)} \mathrm{tr} \left(D_p^{\otimes n}(A^\dagger) D_p^{\otimes n}(B) \right). \tag{11}$$

where

$$D_p(X) = (1-p)\mathrm{tr}(X)\frac{I}{2} + pX, \qquad \text{with } p \in \mathbb{R}^+. \tag{12}$$

- For $p = 1$ this reduces to $\langle A, B \rangle \equiv \frac{1}{\dim(\mathcal{H})} \mathrm{tr}(A^\dagger B)$.



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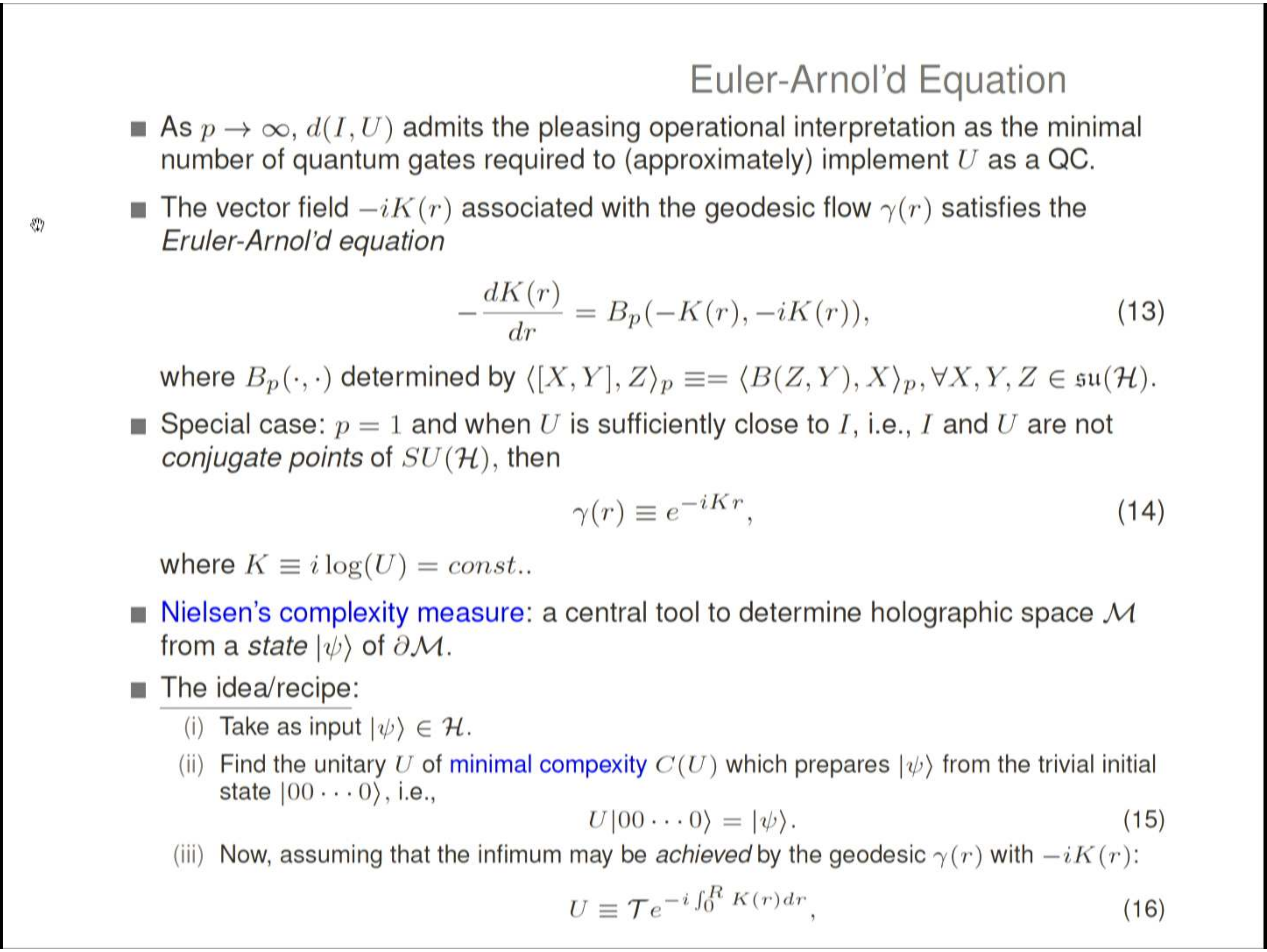


Euler-Arnol'd Equation

- As $p \to \infty$, $d(I,U)$ admits the pleasing operational interpretation as the minimal number of quantum gates required to (approximately) implement U as a QC.
- The vector field $-iK(r)$ associated with the geodesic flow $\gamma(r)$ satisfies the *Eruler-Arnol'd equation*

  $$-\frac{dK(r)}{dr} = B_p(-K(r), -iK(r)), \tag{13}$$

  where $B_p(\cdot,\cdot)$ determined by $\langle [X,Y],Z\rangle_p \equiv= \langle B(Z,Y),X\rangle_p, \forall X,Y,Z \in \mathfrak{su}(\mathcal{H})$.
- Special case: $p = 1$ and when U is sufficiently close to I, i.e., I and U are not *conjugate points* of $SU(\mathcal{H})$, then

  $$\gamma(r) \equiv e^{-iKr}, \tag{14}$$

  where $K \equiv i\log(U) = const.$.
- Nielsen's complexity measure: a central tool to determine holographic space $\mathcal{M}$ from a *state* $|\psi\rangle$ of $\partial\mathcal{M}$.
- The idea/recipe:
  - (i) Take as input $|\psi\rangle \in \mathcal{H}$.
  - (ii) Find the unitary U of minimal compexity $C(U)$ which prepares $|\psi\rangle$ from the trivial initial state $|00\cdots 0\rangle$, i.e.,

    $$U|00\cdots 0\rangle = |\psi\rangle. \tag{15}$$
  - (iii) Now, assuming that the infimum may be *achieved* by the geodesic $\gamma(r)$ with $-iK(r)$:

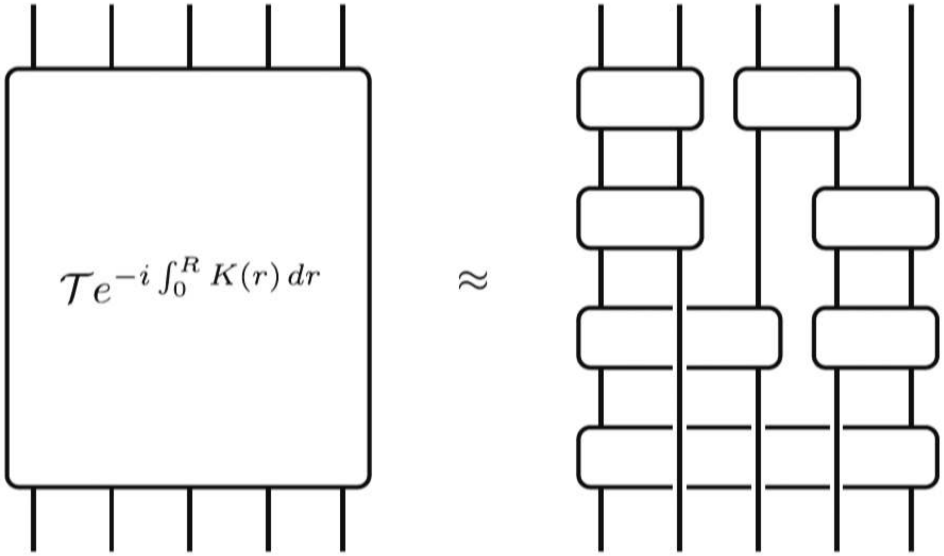
    $$U \equiv \mathcal{T}e^{-i\int_0^R K(r)dr}, \tag{16}$$

Quantum Circuit $V \approx U$

This may be approximated by discretization: find a *quantum circuit*

$$V \equiv V_T V_{T-1} \cdots V_1, \qquad V_j, \quad j = 1, 2, \cdots T, \tag{17}$$

are 1 or 2-qubit *quantum gates* such that $V \approx U$:



Bulk Topology and Geometry from Geodesics in $SU(\mathcal{H})$

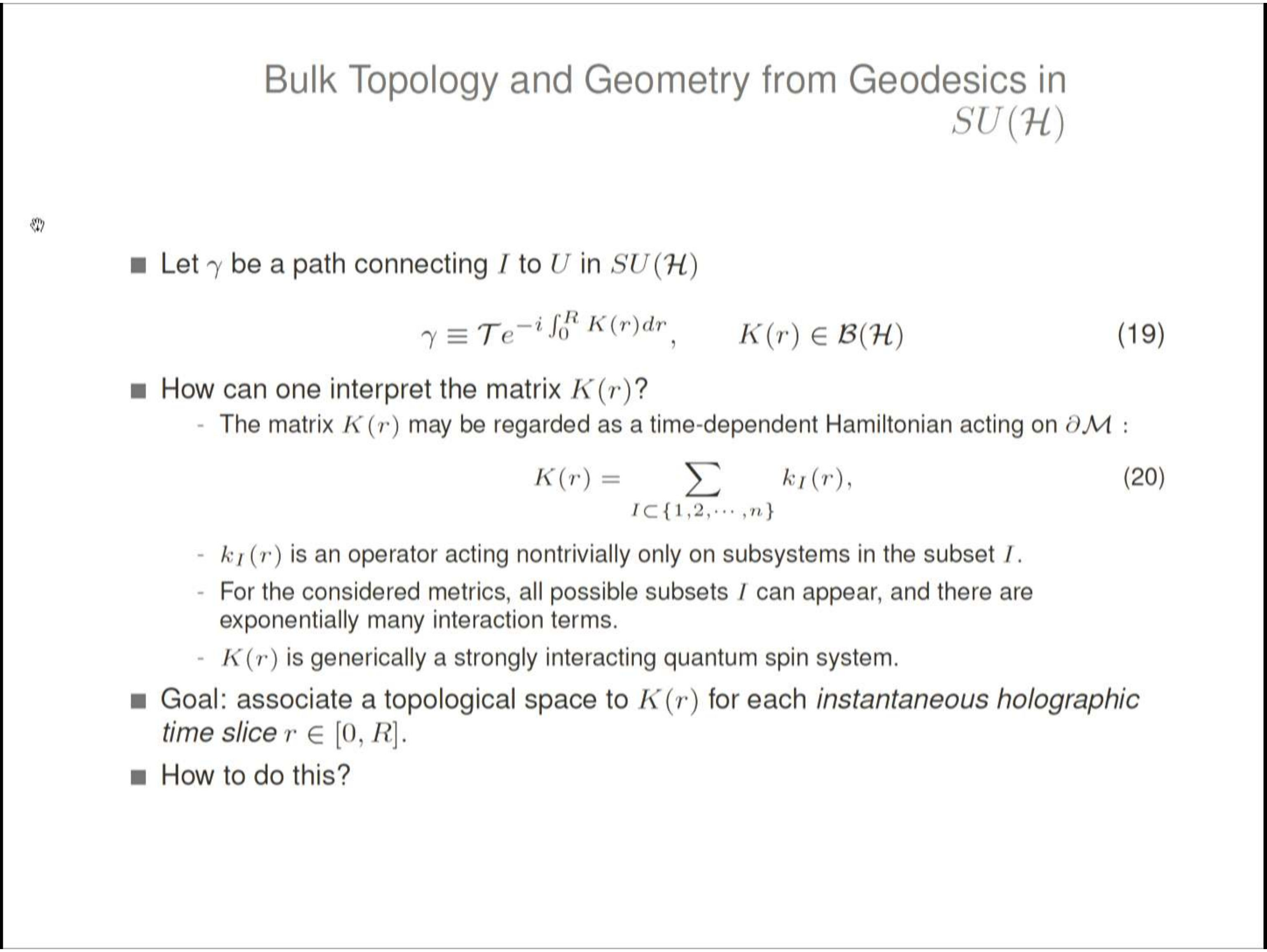
- Let γ be a path connecting I to U in $SU(\mathcal{H})$

$$\gamma \equiv \mathcal{T} e^{-i \int_0^R K(r) dr}, \qquad K(r) \in \mathcal{B}(\mathcal{H}) \tag{19}$$

- How can one interpret the matrix $K(r)$?
  - The matrix $K(r)$ may be regarded as a time-dependent Hamiltonian acting on $\partial\mathcal{M}$:

$$K(r) = \sum_{I \subset \{1,2,\cdots,n\}} k_I(r), \tag{20}$$

  - $k_I(r)$ is an operator acting nontrivially only on subsystems in the subset I.
  - For the considered metrics, all possible subsets I can appear, and there are exponentially many interaction terms.
  - $K(r)$ is generically a strongly interacting quantum spin system.
- Goal: associate a topological space to $K(r)$ for each *instantaneous holographic time slice* $r \in [0, R]$.
- How to do this?



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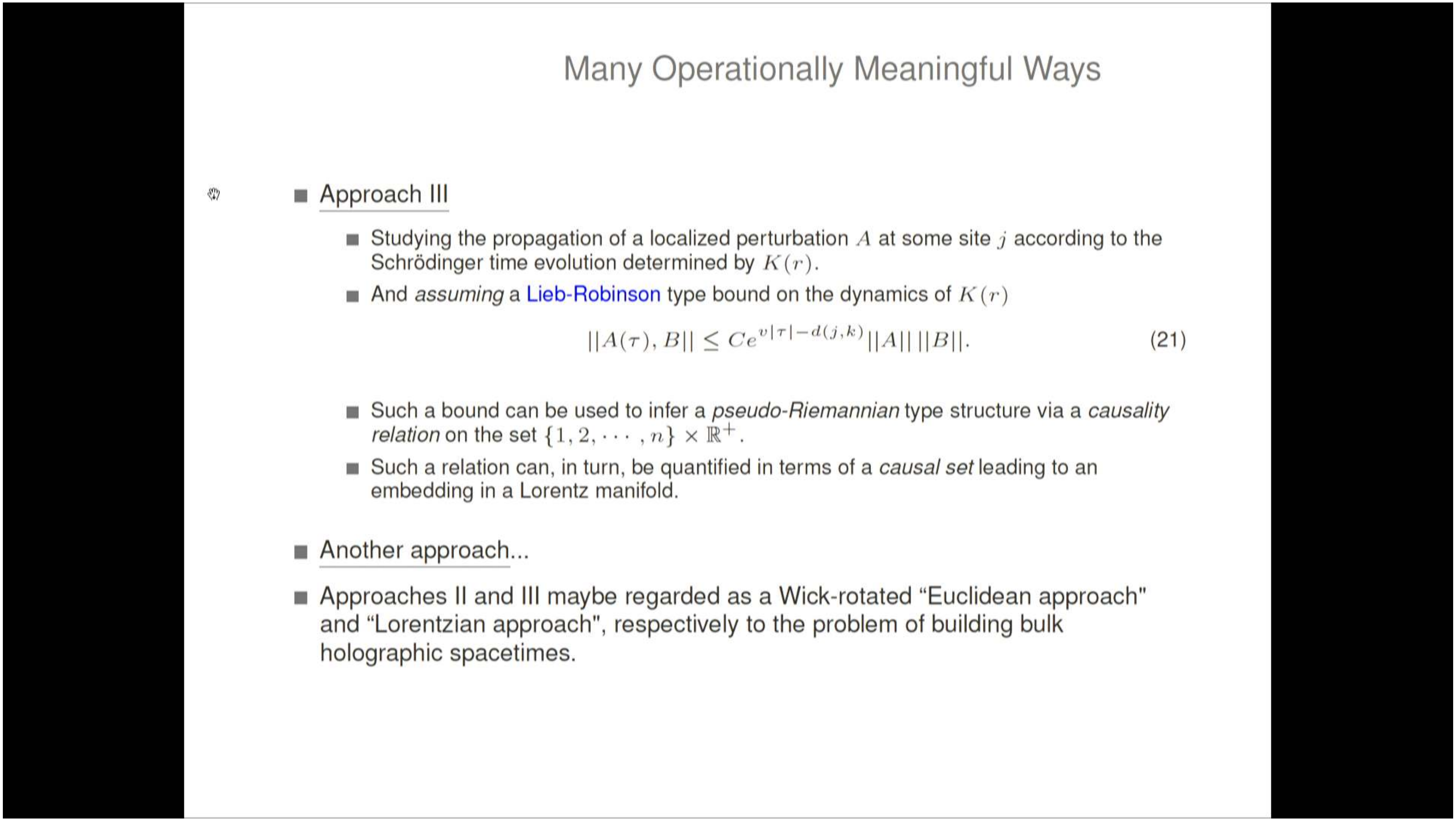
Many Operationally Meaningful Ways

- It depends on the physical questions one asks!
- Approach I
  - To interpret $K(r)$ as a *free-particle Hamiltonian* for some possibly very complicated configuration space $\mathcal{X}$.
  - Building $\mathcal{X}$ by matching the dispersion relation of the localized excitations of $K(r)$ to that of free-particle Hamiltonian on $\mathcal{X}$.
- Approach II
  - To study the response of high temperature states $\rho_\beta(r)$, with β small to localized perturbations A and B at different sites:
  - At zero inverse temperature $\beta = 0$ all perturbations on different sites will be completely uncorrelated.
  - However, when β is small there are residual correlations between nearby sites allowing us to say when two sites are close.
  - While somewhat indirect, this approach has the considerable upside that it immediately leads to a positive-definite metric.



Many Operationally Meaningful Ways

- Approach III
  - Studying the propagation of a localized perturbation A at some site j according to the Schrödinger time evolution determined by $K(r)$.
  - And *assuming* a Lieb-Robinson type bound on the dynamics of $K(r)$

$$||A(\tau), B|| \leq Ce^{v|\tau|-d(j,k)}||A|| \, ||B||. \tag{21}$$

  - Such a bound can be used to infer a *pseudo-Riemannian* type structure via a *causality relation* on the set $\{1, 2, \cdots, n\} \times \mathbb{R}^+$.
  - Such a relation can, in turn, be quantified in terms of a *causal set* leading to an embedding in a Lorentz manifold.
- Another approach...
- Approaches II and III maybe regarded as a Wick-rotated “Euclidean approach" and “Lorentzian approach", respectively to the problem of building bulk holographic spacetimes.

Bulk Holographic Geometry from Thermal Correlations

- A quantum system of n quantum spins $\{1, 2, \cdots, n\}$ with Hamiltonian $K(r)$ is brought into thermal equilibrium at β.
- The state of the system is described by the Gibbs ensemble

$$\rho_\beta = \frac{e^{-\beta K(r)}}{\mathrm{tr}(e^{-\beta K(r)})} \tag{22}$$

- Consider the effect of a small perturbation $A \in \mathfrak{su}(\mathcal{H})$ localized at site j and B at site k.
- The resulting system state is

$$\rho_\beta(r) + \epsilon X \approx \frac{e^{-\beta K(r) + i\epsilon A}}{\mathrm{tr}(e^{-\beta K(r)})}, \qquad \rho_\beta(r) + \epsilon Y \approx \frac{e^{-\beta K(r) + i\epsilon B}}{\mathrm{tr}(e^{-\beta K(r)})}. \tag{23}$$

- How *distinguishable* is the perturbed state $\rho_\beta(r) + \epsilon X$ from the state $\rho_\beta(r) + \epsilon Y$?
- A at site j is *close*, or *adjacent*, to B local to site k if the states $\rho_\beta(r) + \epsilon X$ and $\rho_\beta(r) + \epsilon Y$ are *not completely distinguishable*.

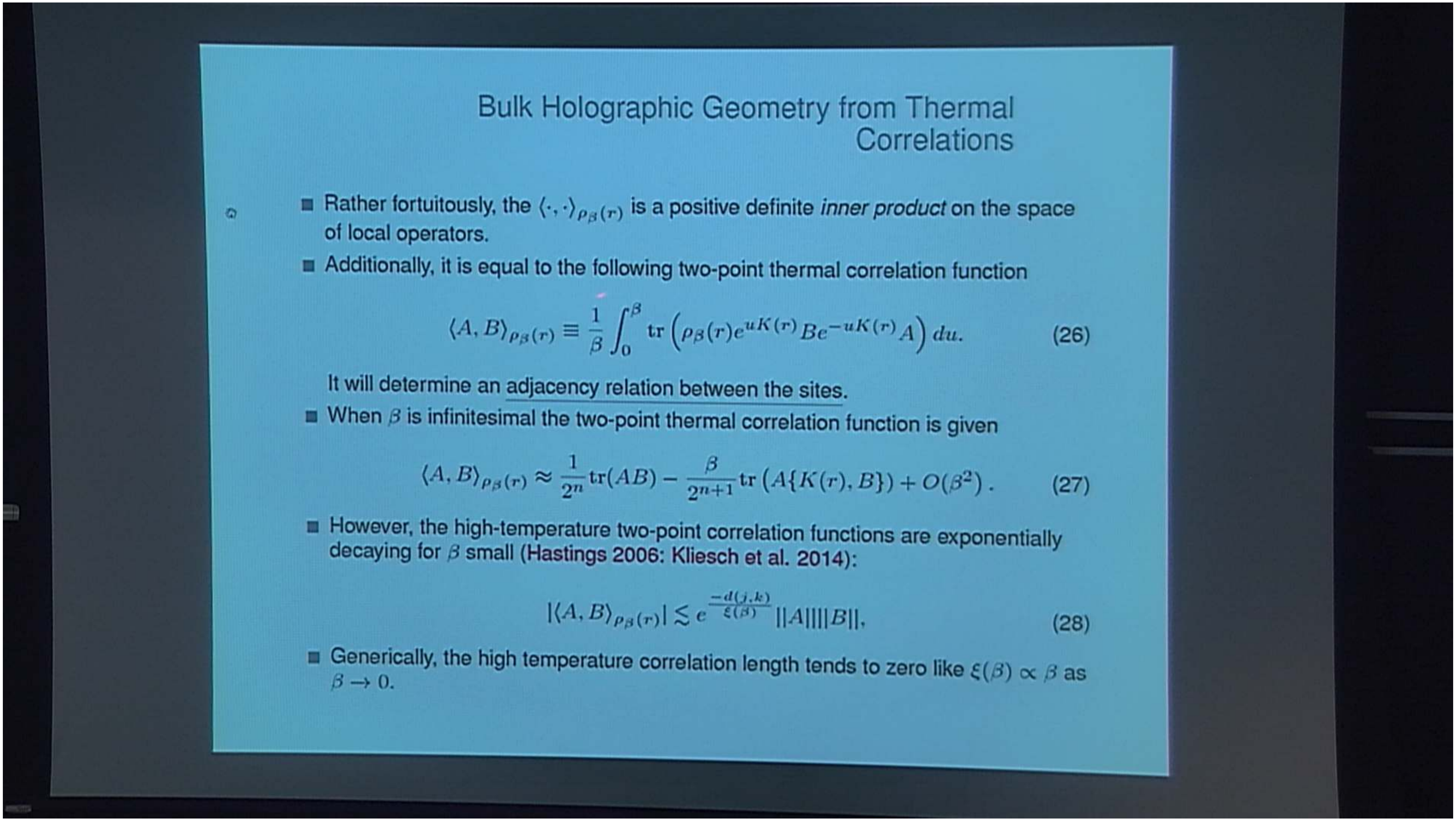
Bulk Holographic Geometry from Thermal Correlations

- Does this notion correspond to a topological/geometrical conception of closeness?
- Near the infinite-temperature fixed point $\rho \propto I$, all the correlations are disordered by thermal fluctuations.
- The effects of a local perturbation are delocalized only in a small surrounding region determined by the high-temperature correlation length depending on β.
- If $\rho_\beta(r) + \epsilon X$ and $\rho_\beta(r) + \epsilon Y$ are independent fluctuations (uncorrelated), A is *far* from B.
- This region, in turn, determines the desired adjacency for the site j and k, which supplies us with a metric quantity.
- Distinguishability, as measured by the relative entropy $S(\cdot||\cdot)$, of $\rho_\beta(r) + \epsilon X$ and $\rho_\beta(r) + \epsilon Y$ is quantified to $O(\epsilon)$ by

$$\langle A, B\rangle_{\rho_\beta(r)} \equiv -\frac{\partial^2}{\partial x \partial y} F(x,y)|_{x=y=0}, \qquad (24)$$

$$F(x,y) = -\frac{1}{\beta}\log\left(\mathrm{tr}\left(e^{-\beta K(r) + ixA + iyB}\right)\right) \qquad (25)$$

  is the *free energy* (Bény & Osborne, 2015).
- This idea has also been exploited in various incarnations (Ryu & Takayanagi 2012; Qi 2013).



Bulk Holographic Geometry from Thermal Correlations

- Rather fortuitously, the $\langle \cdot, \cdot \rangle_{\rho_\beta(r)}$ is a positive definite *inner product* on the space of local operators.
- Additionally, it is equal to the following two-point thermal correlation function

$$\langle A, B \rangle_{\rho_\beta(r)} \equiv \frac{1}{\beta} \int_0^\beta \mathrm{tr}\left(\rho_\beta(r) e^{uK(r)} B e^{-uK(r)} A \right) du. \qquad (26)$$

It will determine an adjacency relation between the sites.

- When β is infinitesimal the two-point thermal correlation function is given

$$\langle A, B \rangle_{\rho_\beta(r)} \approx \frac{1}{2^n} \mathrm{tr}(AB) - \frac{\beta}{2^{n+1}} \mathrm{tr}\left(A\{K(r), B\} \right) + O(\beta^2)\,. \qquad (27)$$

- However, the high-temperature two-point correlation functions are exponentially decaying for β small (Hastings 2006; Kliesch et al. 2014):

$$|\langle A, B \rangle_{\rho_\beta(r)}| \lesssim e^{\frac{-d(j,k)}{\xi(\beta)}} ||A|| ||B||, \qquad (28)$$

- Generically, the high temperature correlation length tends to zero like $\xi(\beta) \propto \beta$ as $\beta \to 0$.

Bulk Holographic Geometry from Thermal Correlations

- Thus, if $\langle A, B\rangle_{\rho_\beta(r)}$ is nonzero for β infinitesimal when $j \neq k$ this means that $d(j,k)$ must be arbitrarily small, i.e., j and k are *adjacent*.
- Our task is thus to extract a distance measure or metric, $d(j,k)$ from $\langle A, B\rangle_{\rho_\beta(r)}$.
- <u>How to do this?</u>
  - One direct way is simply to take a $\log$

$$d(j,k) \overset{!}{\equiv} \sup_{A,B} -\beta \log \frac{|\langle A, B\rangle_{\rho_\beta(r)}|}{||A||||B||}, \qquad j \neq k, \tag{29}$$

  being similar to (Qi 2013).
  - It is not clear if $d(j,k)$ so defined satisfies the triangle inequality

$$d(j,l) \leq d(j,k) + d(k,l). \tag{30}$$

  - A way around this is use $d(j,k)$ *only* to define an *adjacency relation* between pair of spins (j,k).
  - Then use the adjacency relation to build a metric. What does this mean?
  - First set up the adjacency matrix

$$A_{j,k} = \sup_{A,B} -\beta \log \frac{|\langle A, B\rangle_{\rho_\beta(r)}|}{||A||||B||}, \qquad j \neq k. \tag{31}$$

Bulk Holographic Geometry from Thermal Correlations

- $A_{j,k}$ defines a weighted graph structure $G(V,E)$ on the vertex set

$$V = \{1, 2, \cdots, n\}. \tag{32}$$

  - For any pair of points j and k in G, the distance between j and k is defined as the length of the shortest path

$$p = (e_1, e_2, \cdots, e_m), \qquad \text{and } e_l = (x_l, y_l) \tag{33}$$

    are edges, between j and k.

  - This is guaranteed to obey the triangle inequality. Thus the metric is defined as

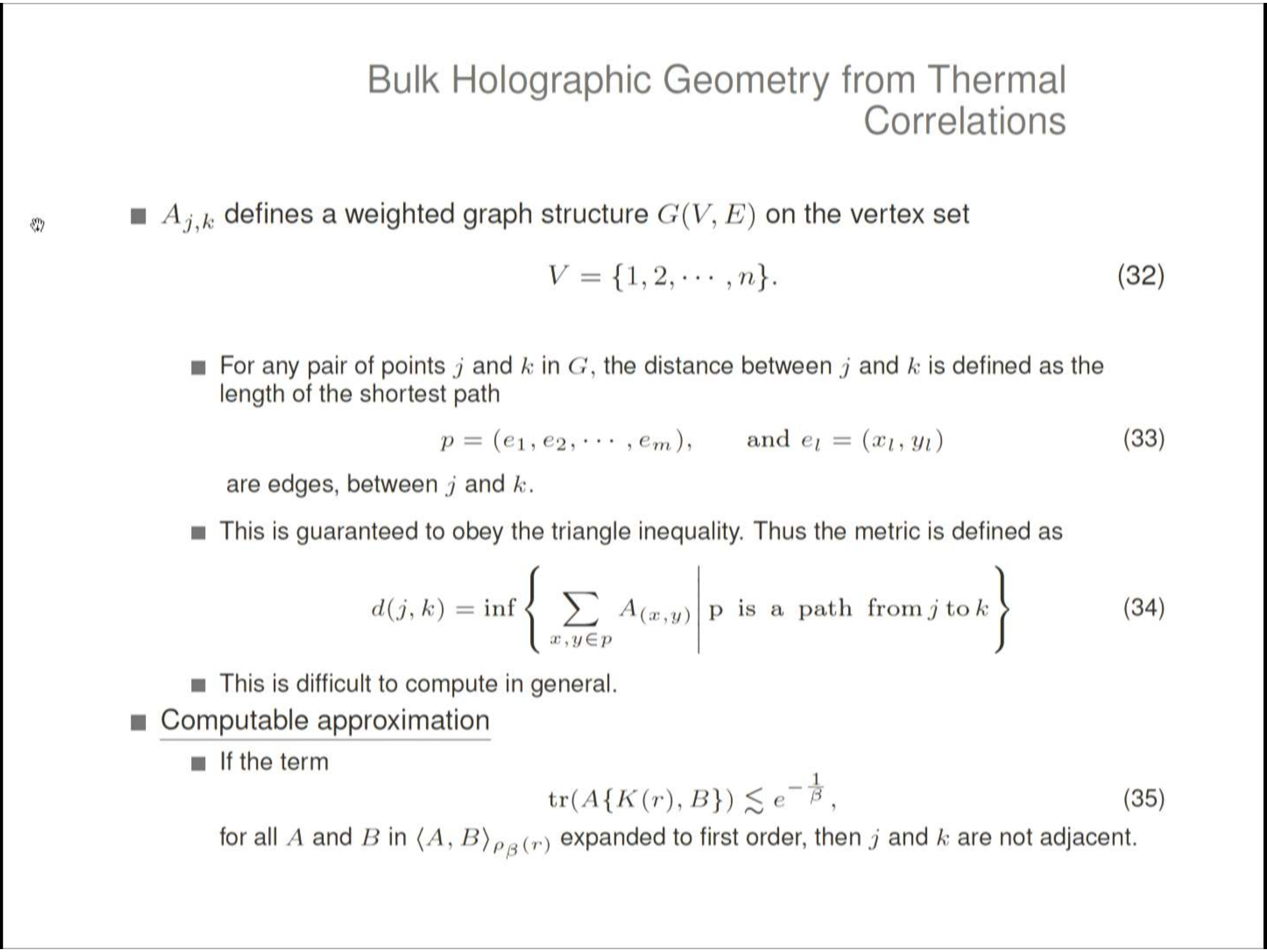
$$d(j,k) = \inf\left\{ \sum_{x,y\in p} A_{(x,y)} \middle| \text{ p is a path from } j \text{ to } k \right\} \tag{34}$$

  - This is difficult to compute in general.

- Computable approximation
  - If the term

$$\mathrm{tr}(A\{K(r), B\}) \lesssim e^{-\frac{1}{\beta}}, \tag{35}$$

    for all A and B in $\langle A, B\rangle_{\rho_\beta(r)}$ expanded to first order, then j and k are not adjacent.



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$$\mathrm{tr}(A\{K(r), B\}) \lesssim e^{-\frac{1}{\beta}}, \tag{35}$$

    for all A and B in $\langle A, B\rangle_{\rho_\beta(r)}$ expanded to first order, then j and k are not adjacent.

Bulk Holographic Geometry from Thermal Correlations

- Computable approximation
  - if, however, there are local operators A at j and B at k such that for β infinitesimal

    $$\langle A, B\rangle_{\rho_\beta(r)} \gg e^{-\frac{1}{\beta}}, \qquad (36)$$

    then j and k *are* adjacent.
- Restriction to hamiltonians $K(r)$ comprised of only one- and two-particle interaction terms $k_{j,k}(r)$ (case when $p \to \infty$).
- Then to the first order in β this is equivalent to asking if there are traceless A at j and B at k such that

  $$\mathrm{tr}(A\{K(r), B\}) \neq 0, \qquad (37)$$

  namely, j is adjacent to k if the two-particle interaction term $k_{j,k}(r)$ in $K(r)$ is nonzero.
- Physically this is equivalent to: j and k are adjacent if at time r an (infinitesimal) quantum gate was applied coupling j and k.
- When K is comprised of three-particle or higher interactions, one needs to go to higher orders in β to determine the adjacency.
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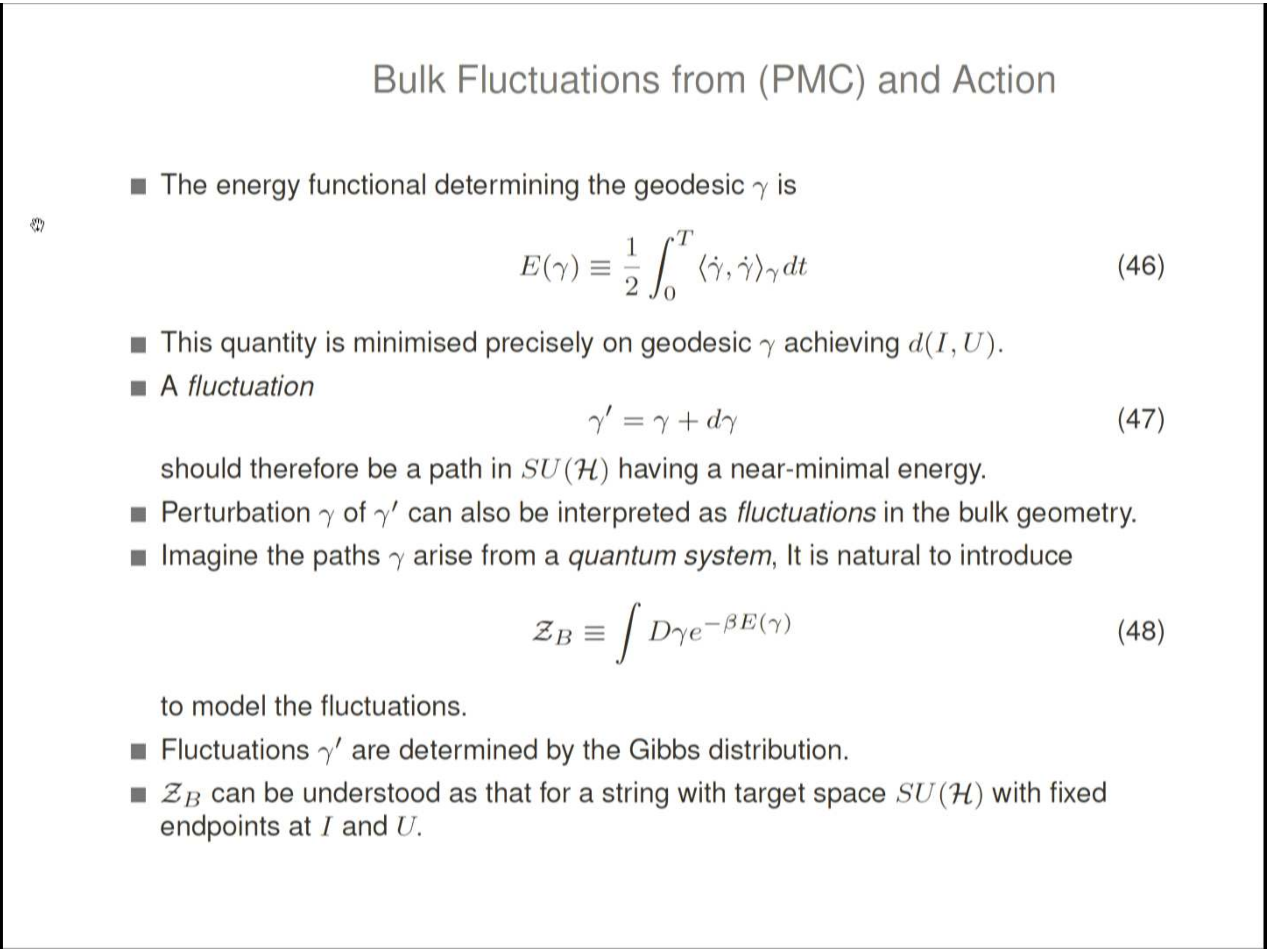
Bulk Holographic Geometry from Causal Sets

- The metric topology space does not capture an important aspect of quantum circuits comprised of local gates, namely, their *causal structure*.
- In every quantum circuit there is a kind of “light cone" of information propagation.
- A qubit j is said to be in the *past* of qubit k if there is a sequence of quantum gates in the circuit connecting j to k.
- Because the geodesics γ in $SU(\mathcal{H})$ obtained via (PMC) are generated by essentially local gates we should actually rather associate some kind of discretized *pseudo-Riemannian* manidold to the bulk holographic spacetime.
- In other words, it is rather more natural to think of $\mathcal{M}$ as a de Sitter-type (Bény 2013, Czech et al. 2015).
- One should regard the previous approach as the Wick-rotated Euclidean version of this approach.



Bulk Fluctuations from (PMC) and Action

- The energy functional determining the geodesic γ is

$$E(\gamma) \equiv \frac{1}{2}\int_0^T \langle\dot{\gamma},\dot{\gamma}\rangle_\gamma dt \tag{46}$$

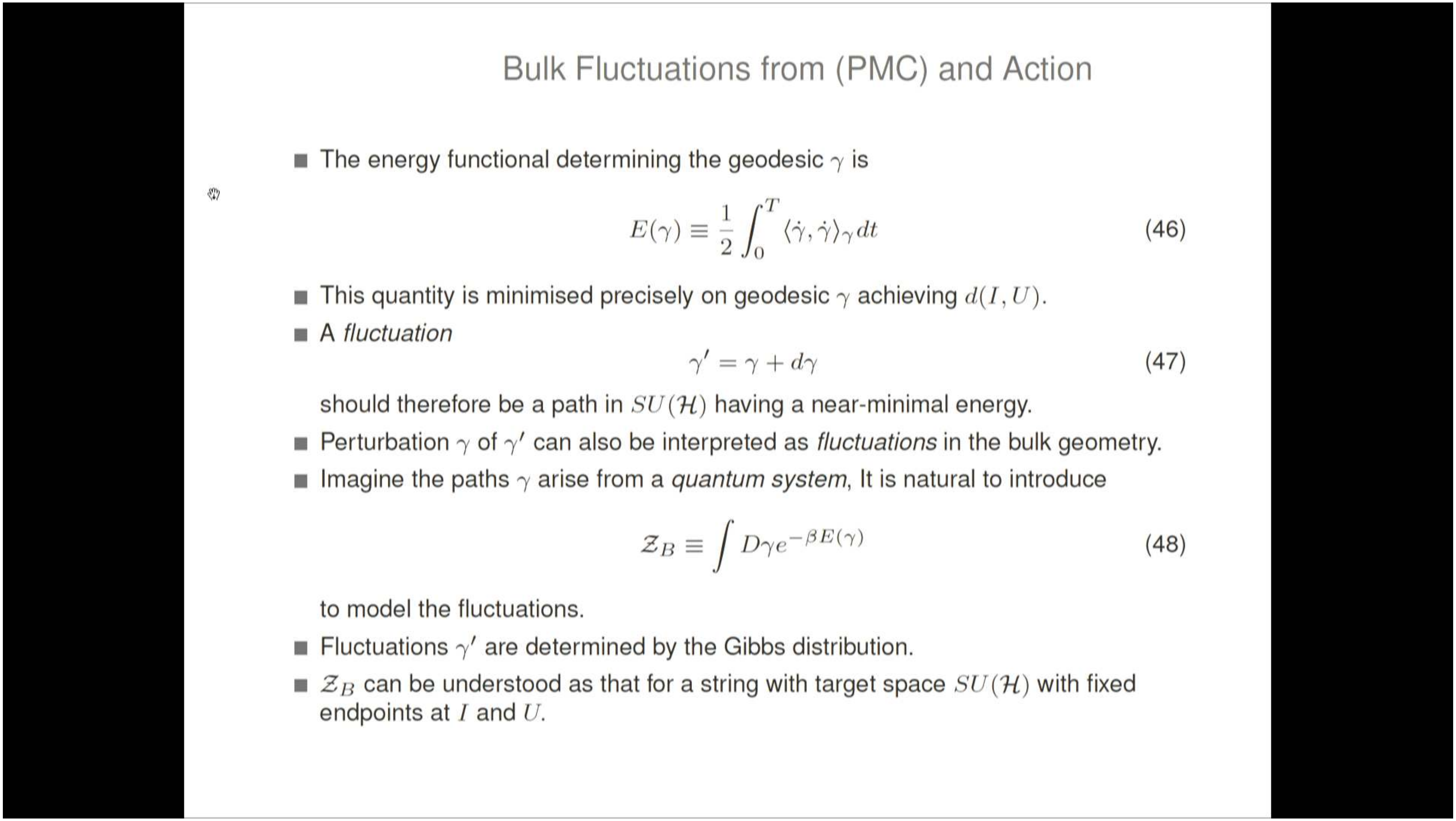
- This quantity is minimised precisely on geodesic γ achieving $d(I,U)$.
- A *fluctuation*

$$\gamma' = \gamma + d\gamma \tag{47}$$

  should therefore be a path in $SU(\mathcal{H})$ having a near-minimal energy.
- Perturbation γ of γ' can also be interpreted as *fluctuations* in the bulk geometry.
- Imagine the paths γ arise from a *quantum system*, It is natural to introduce

$$\mathcal{Z}_B \equiv \int D\gamma e^{-\beta E(\gamma)} \tag{48}$$

  to model the fluctuations.
- Fluctuations γ' are determined by the Gibbs distribution.
- $\mathcal{Z}_B$ can be understood as that for a string with target space $SU(\mathcal{H})$ with fixed endpoints at I and U.



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What is the structure of a fluctuation?

- The energy $E(\gamma)$ is only sensitive to the presence of *quantum gates* between pairs of spins.
- It is not sensitive to *which* spins j and k the gate is applied to.
- The structure of near-minimal fluctuations of a geodesic are equal to $\gamma(t)\ \forall t$ except at one instant $t = t_\omega$.
- At t_ω a unitary gate $V_{j,k}$ is applied to an arbitrary pair (j,k) followed immediately by $V^\dagger_{j,k}$.
- Such a geodesic corresponds to

$$Bulk\ holographic\ spacetime = minimal\ one\ except\ with\ a''wormhole''\ at\, t_\omega$$

- Such a wormhole immediately *evaporates*.
- The fluctuating bulk geometry determined by $\mathcal{Z}_B$ is comprised of spacetimes where wormholes are fluctuating in and out of existence between all pairs (j,k) of points.

Brownian Motions on $SU(\mathcal{H})$

- The path integral $\mathcal{Z}_B$ is remarkably simple; it is quadratic in $-iK(r)$.
- $D\gamma e^{-\beta E(\gamma)}$ may hence be understood as a Brownian measure on paths in $SU(\mathcal{H})$ generated by 2-local tangent vectors (Lashkari et al. 2011).
- In the $p \to \infty$ limit each path $\gamma(t)$ solves (SDE)

$$d\gamma(t) \propto i \sum_{j \neq k}^{n} \sum_{\alpha_k = 0}^{3} \sigma_j^{\alpha^j} \otimes \sigma_k^{\alpha_k} \gamma(t) dB_{\alpha_j \alpha_k}(t) - \frac{1}{2}\gamma(t)dt. \tag{49}$$

- What makes $\mathcal{Z}_B$ nontrivial is the constraint that the endpoints of the path are exactly I and U, turning $\mathcal{Z}_B$ into integral over Brownian Bridges (Lévy et al. 2015).
- Bulk fluctuations are interpreted as a very complicated random variable $g \equiv g(U)$ which depends in a rather nonlinear way on the realization U of the Brownian bridge.
- Comment: The proposal of $\mathcal{Z}_B$ essentially promotes the CA argument to a definition:
  The action $E(\gamma)$ is directly related to the complexity $d(I, U)$ in exactly the same way the energy of a geodesic is related to the geodesic length in Riemannian geometry, i.e., the minima of both quantities coincide.

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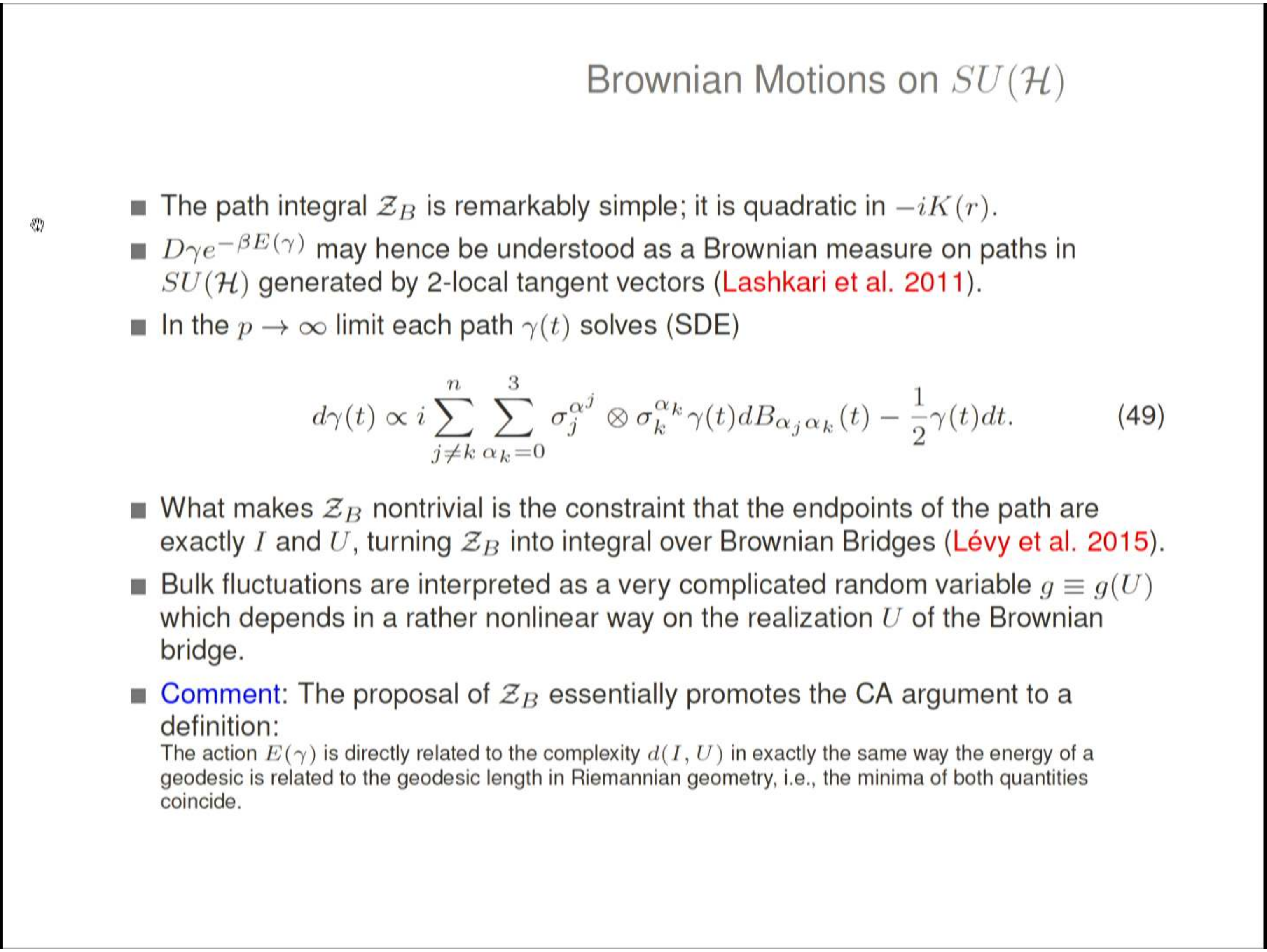

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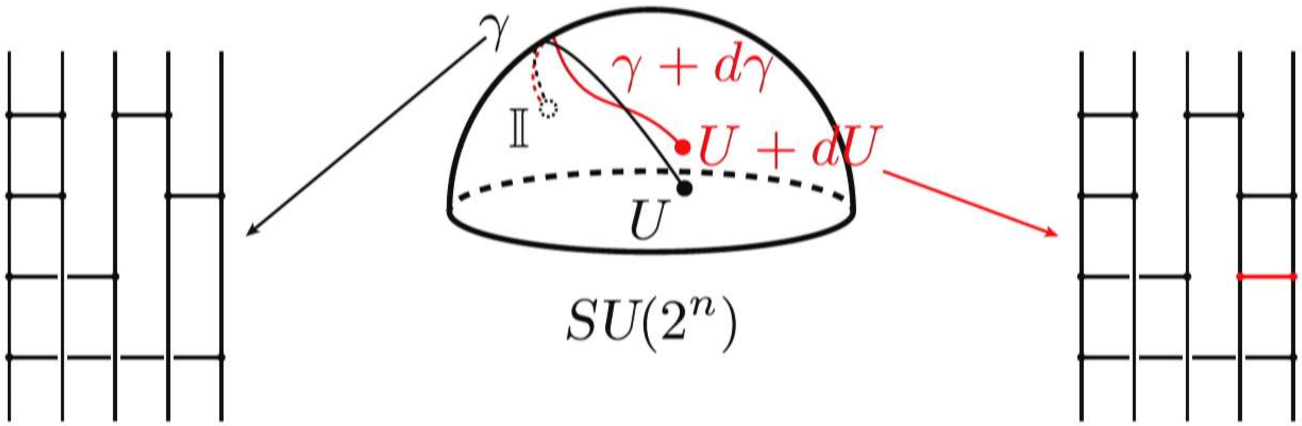
Boundary Perturbations and Jacobi Fields

- The (PMC) already determines an EOM constraining the structure of the induced bulk fluctuations.
- This equation could be understood as a kind of generalized Einstein equation.



- Model the perturbation of the unitary U, i.e., study perturbed unitaries:

$$U' = U + dU \tag{50}$$

- Two natural sources:
  - (i) Arising from the presence of *local external fields*, J

$$H(s, J) = H + s \sum_{j=1}^{n} \sum_{\alpha=1}^{3} J_{\alpha}^{j} \sigma_{j}^{\alpha}, \tag{51}$$

Jacobi Equation

- A shift in $\gamma(r)$ corresponds to a shift

$$\mathcal{M} \to \mathcal{M} + d\mathcal{M}, \tag{57}$$

  in the holographic space.
- Capturing the structure of the bulk holographic spacetime with a (metric) topology, we observe a shift in the topology $\mathcal{T}$ on the point set X.
- The first order shift $\partial_s \gamma(r,s)$ in $\gamma(r)$ satisfies the *Jacobi equation*

$$\begin{aligned}\partial_r^2 Y = {} & B_p(\partial_r Y + [X,Y], X) + B_p(X, \partial_r Y + [X,Y]) - [B_p(X,X), Y] + \\ & + [X, \partial_r Y],\end{aligned} \tag{58}$$

  $X \equiv (\partial_r \gamma)\gamma^{-1}$ and $Y \equiv (\partial_s \gamma)\gamma^{-1}$.
- The Jacobi equation may be naturally regarded as a kind of “Einstein equation” constraining the dynamics of the bulk geometrical fluctuations.
- The vector field Y capturing the bulk geometrical fluctuation $d\mathcal{M}$ is directly a function of the external boundary field J_α^j.
- This allows us to deduce a precise bulk/boundary correspondence. This observation is the main contribution of this part.

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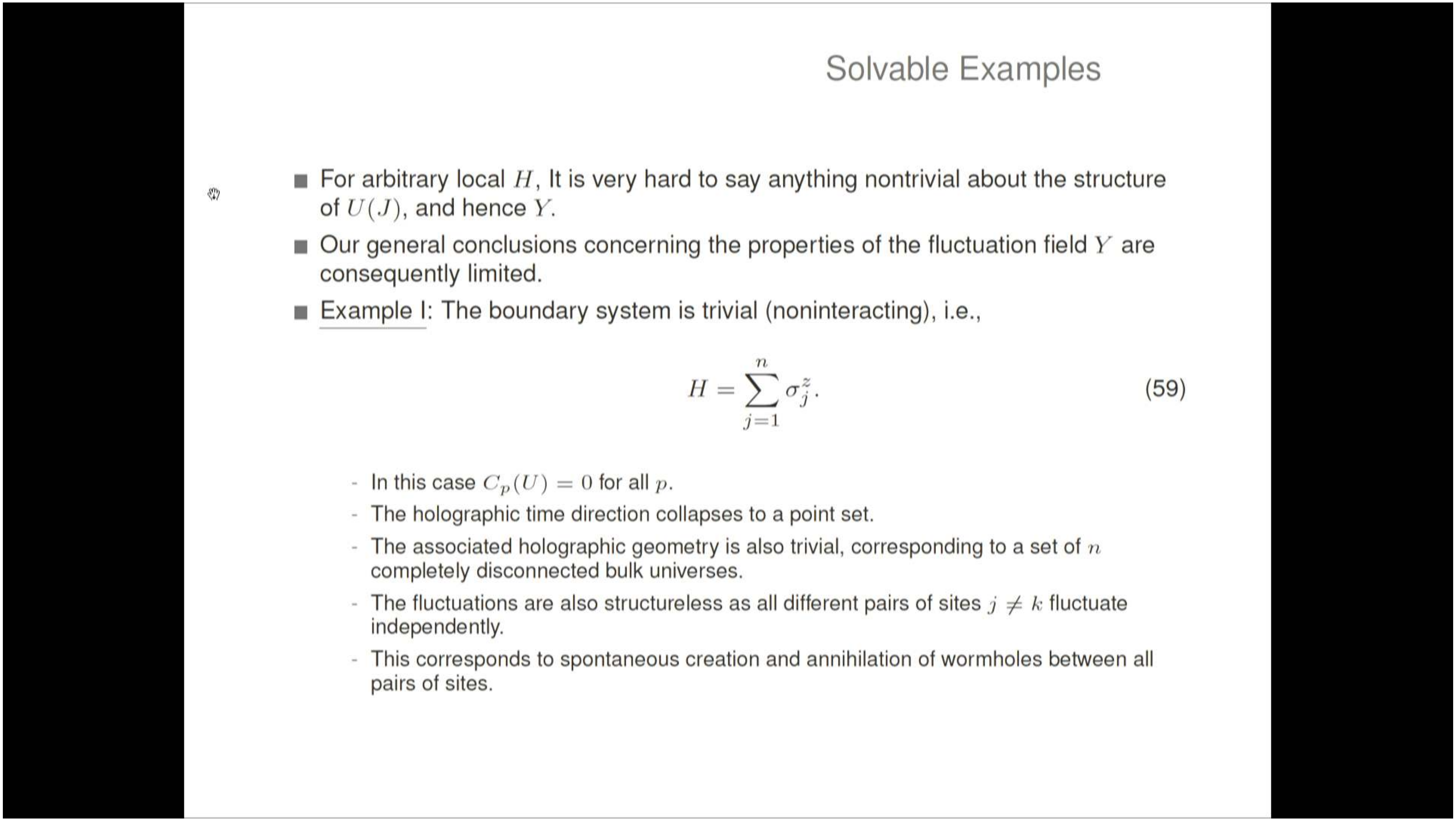
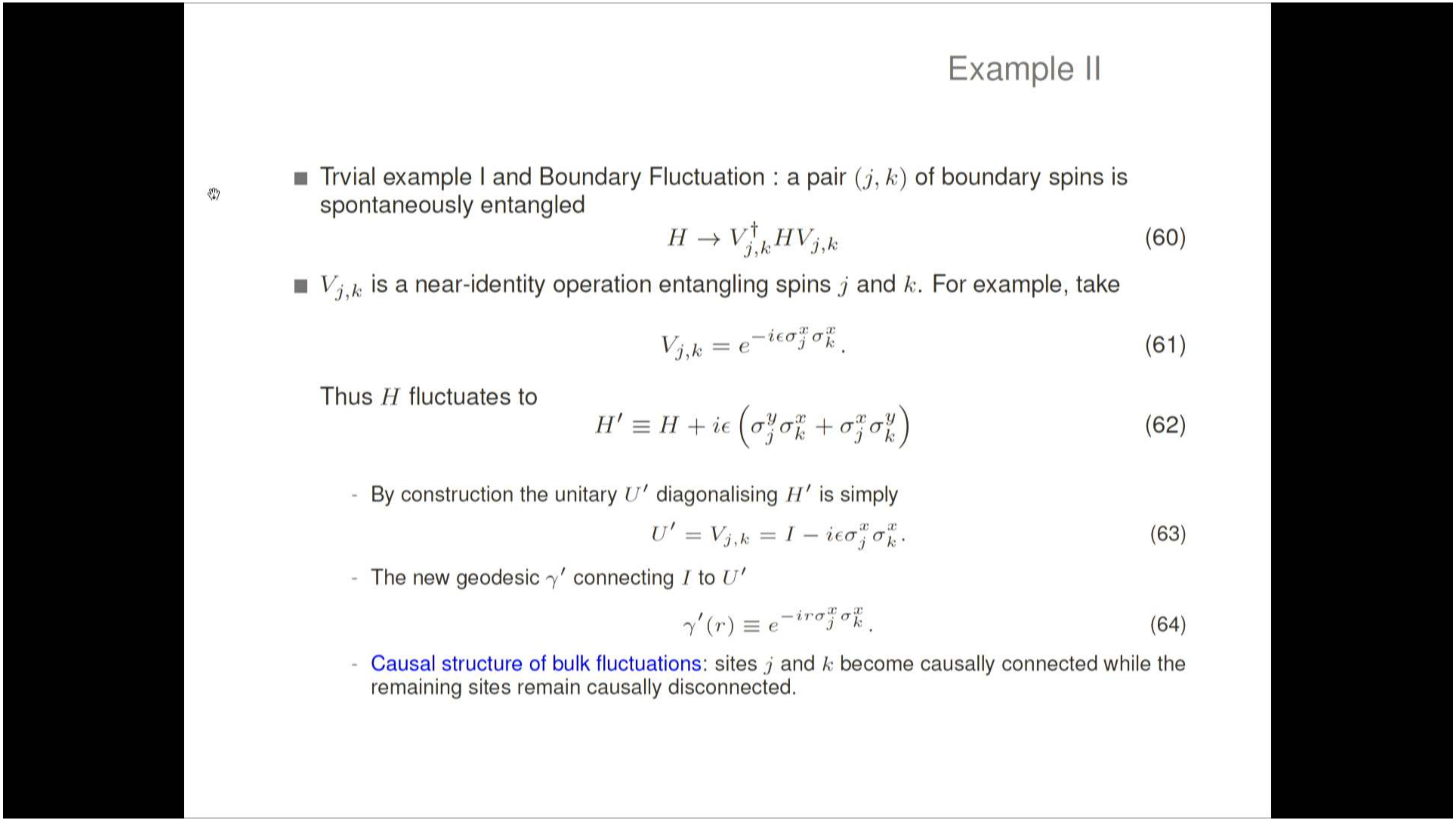


Solvable Examples

- For arbitrary local H, It is very hard to say anything nontrivial about the structure of $U(J)$, and hence Y.
- Our general conclusions concerning the properties of the fluctuation field Y are consequently limited.
- Example I: The boundary system is trivial (noninteracting), i.e.,

$$H = \sum_{j=1}^{n} \sigma_j^z . \tag{59}$$

  - In this case $C_p(U) = 0$ for all p.
  - The holographic time direction collapses to a point set.
  - The associated holographic geometry is also trivial, corresponding to a set of n completely disconnected bulk universes.
  - The fluctuations are also structureless as all different pairs of sites $j \neq k$ fluctuate independently.
  - This corresponds to spontaneous creation and annihilation of wormholes between all pairs of sites.



Example II

- Trvial example I and Boundary Fluctuation : a pair (j,k) of boundary spins is spontaneously entangled

$$H \to V_{j,k}^{\dagger} H V_{j,k} \tag{60}$$

- $V_{j,k}$ is a near-identity operation entangling spins j and k. For example, take

$$V_{j,k} = e^{-i\epsilon\sigma_j^x \sigma_k^x}. \tag{61}$$

Thus H fluctuates to

$$H' \equiv H + i\epsilon\left(\sigma_j^y \sigma_k^x + \sigma_j^x \sigma_k^y\right) \tag{62}$$

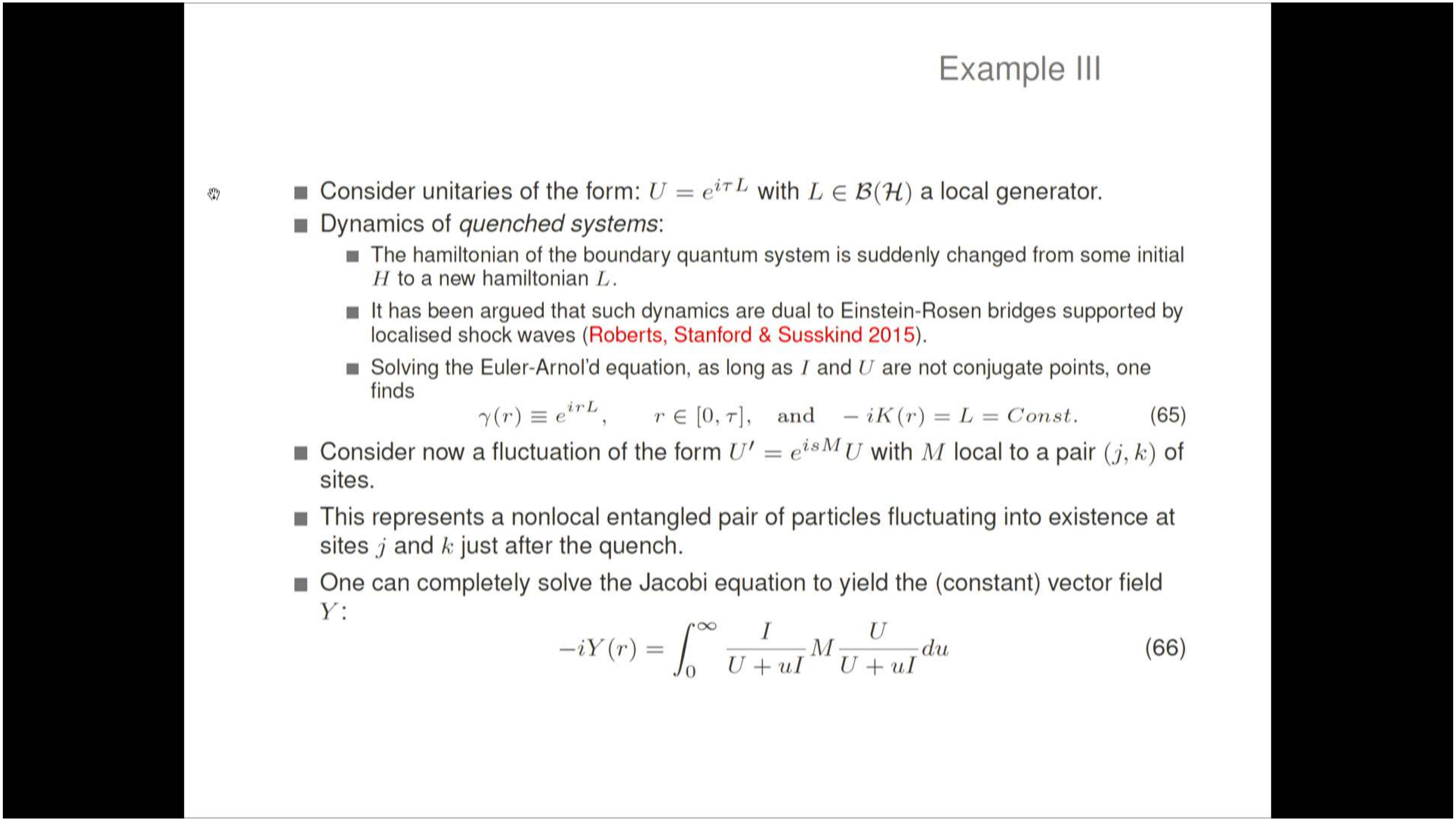
  - By construction the unitary U' diagonalising H' is simply

$$U' = V_{j,k} = I - i\epsilon\sigma_j^x \sigma_k^x. \tag{63}$$

  - The new geodesic γ' connecting I to U'

$$\gamma'(r) \equiv e^{-ir\sigma_j^x \sigma_k^x}. \tag{64}$$

  - Causal structure of bulk fluctuations: sites j and k become causally connected while the remaining sites remain causally disconnected.



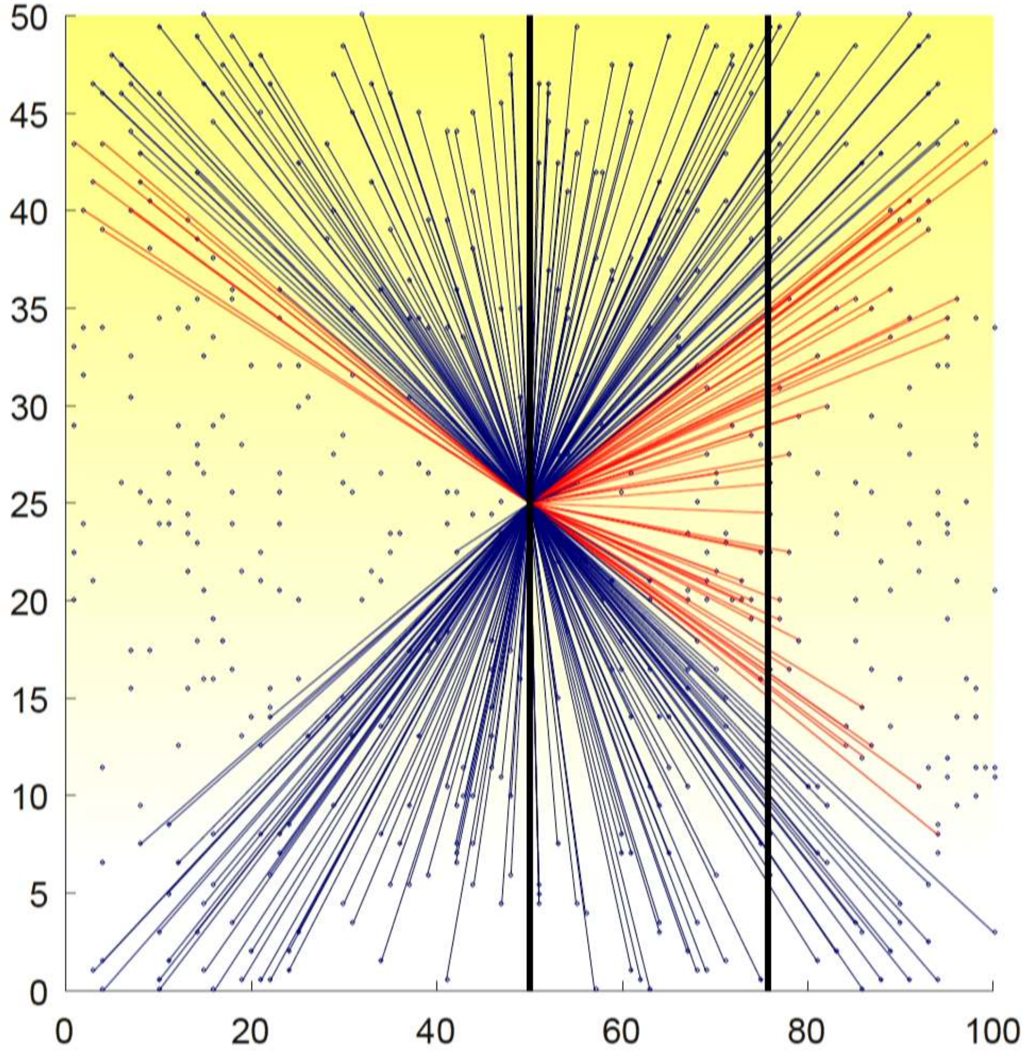
Example III

- Consider unitaries of the form: $U = e^{i\tau L}$ with $L \in \mathcal{B}(\mathcal{H})$ a local generator.
- Dynamics of *quenched systems*:
  - The hamiltonian of the boundary quantum system is suddenly changed from some initial H to a new hamiltonian L.
  - It has been argued that such dynamics are dual to Einstein-Rosen bridges supported by localised shock waves (Roberts, Stanford & Susskind 2015).
  - Solving the Euler-Arnol'd equation, as long as I and U are not conjugate points, one finds

$$\gamma(r) \equiv e^{irL}, \qquad r \in [0, \tau], \quad \text{and} \quad -iK(r) = L = Const. \tag{65}$$

- Consider now a fluctuation of the form $U' = e^{isM}U$ with M local to a pair (j, k) of sites.
- This represents a nonlocal entangled pair of particles fluctuating into existence at sites j and k just after the quench.
- One can completely solve the Jacobi equation to yield the (constant) vector field Y:

$$-iY(r) = \int_0^\infty \frac{I}{U + uI} M \frac{U}{U + uI} du \tag{66}$$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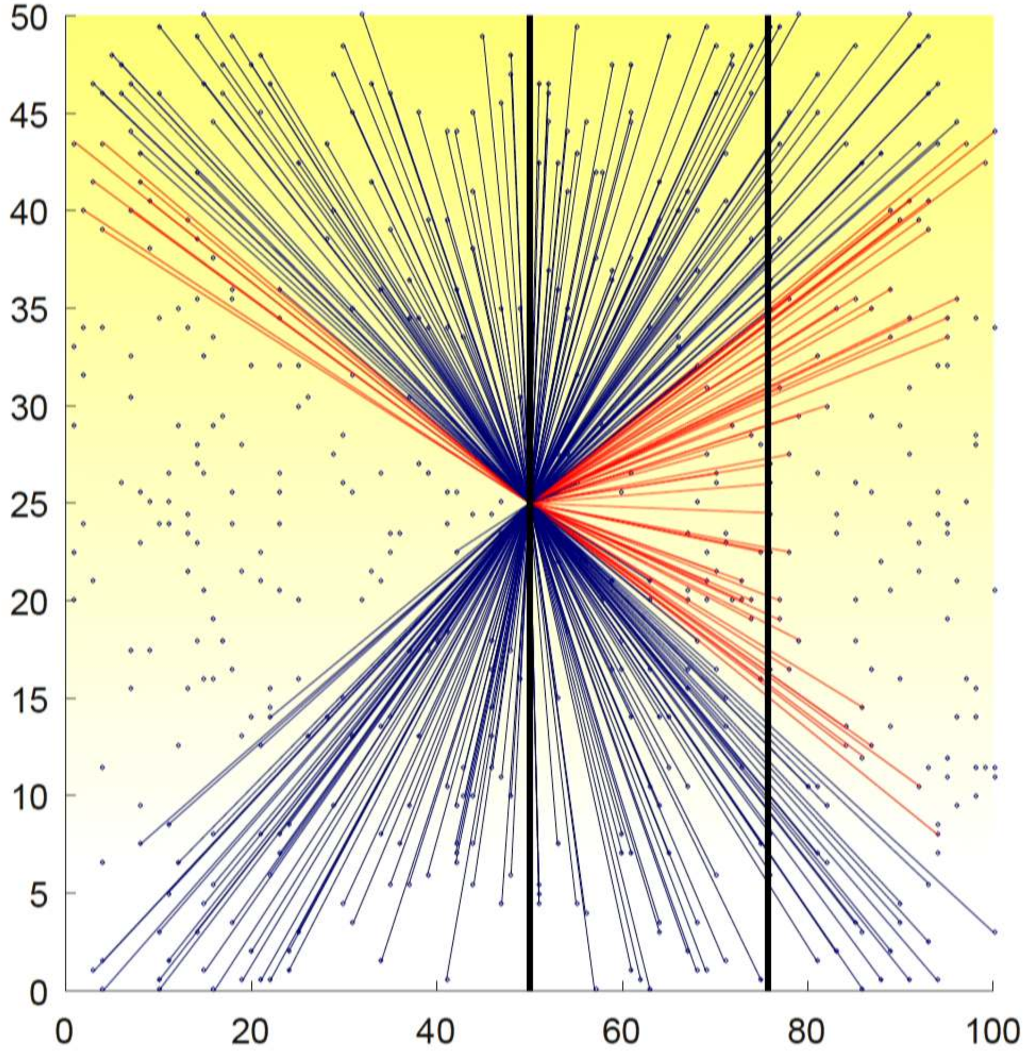
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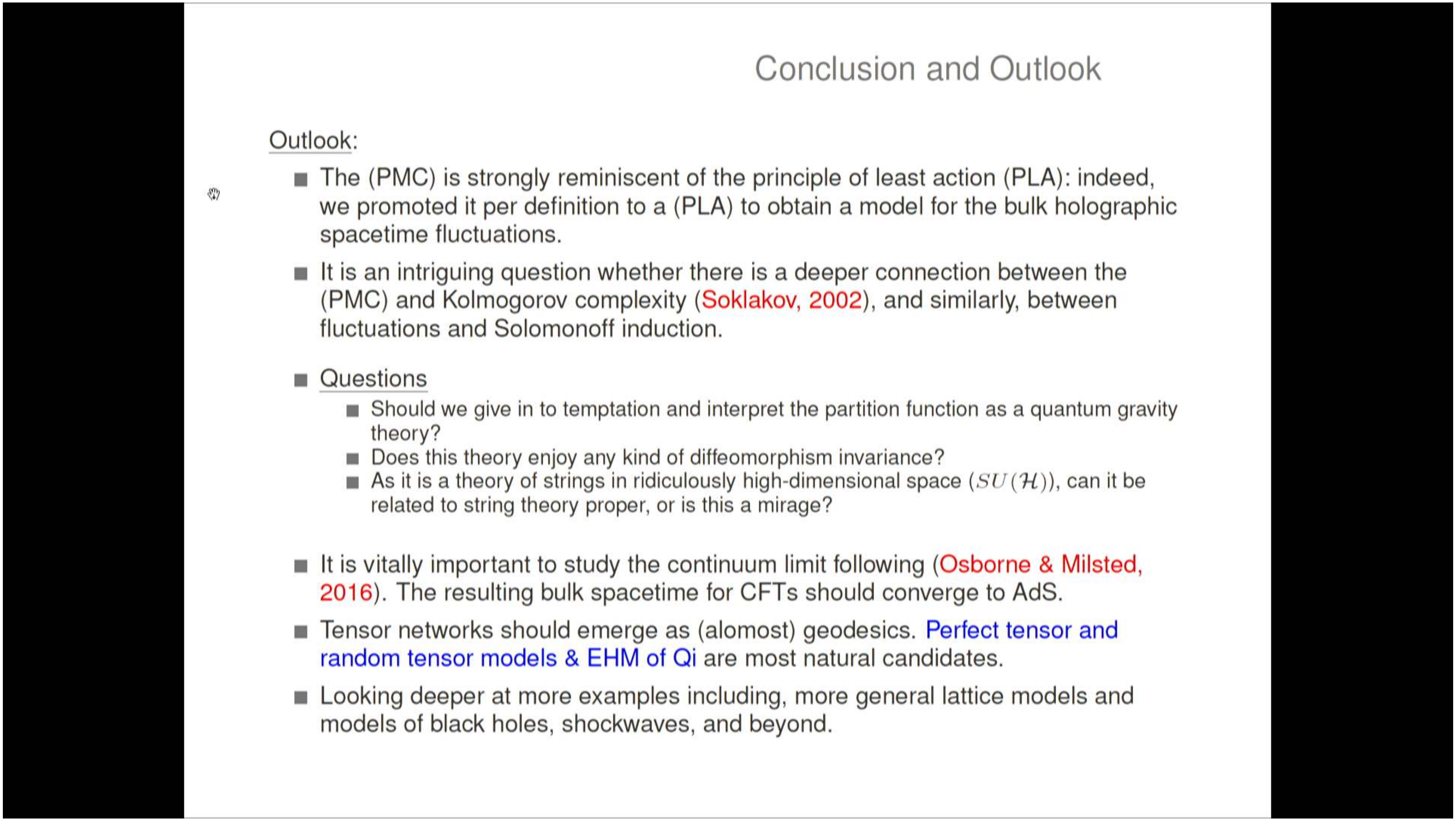
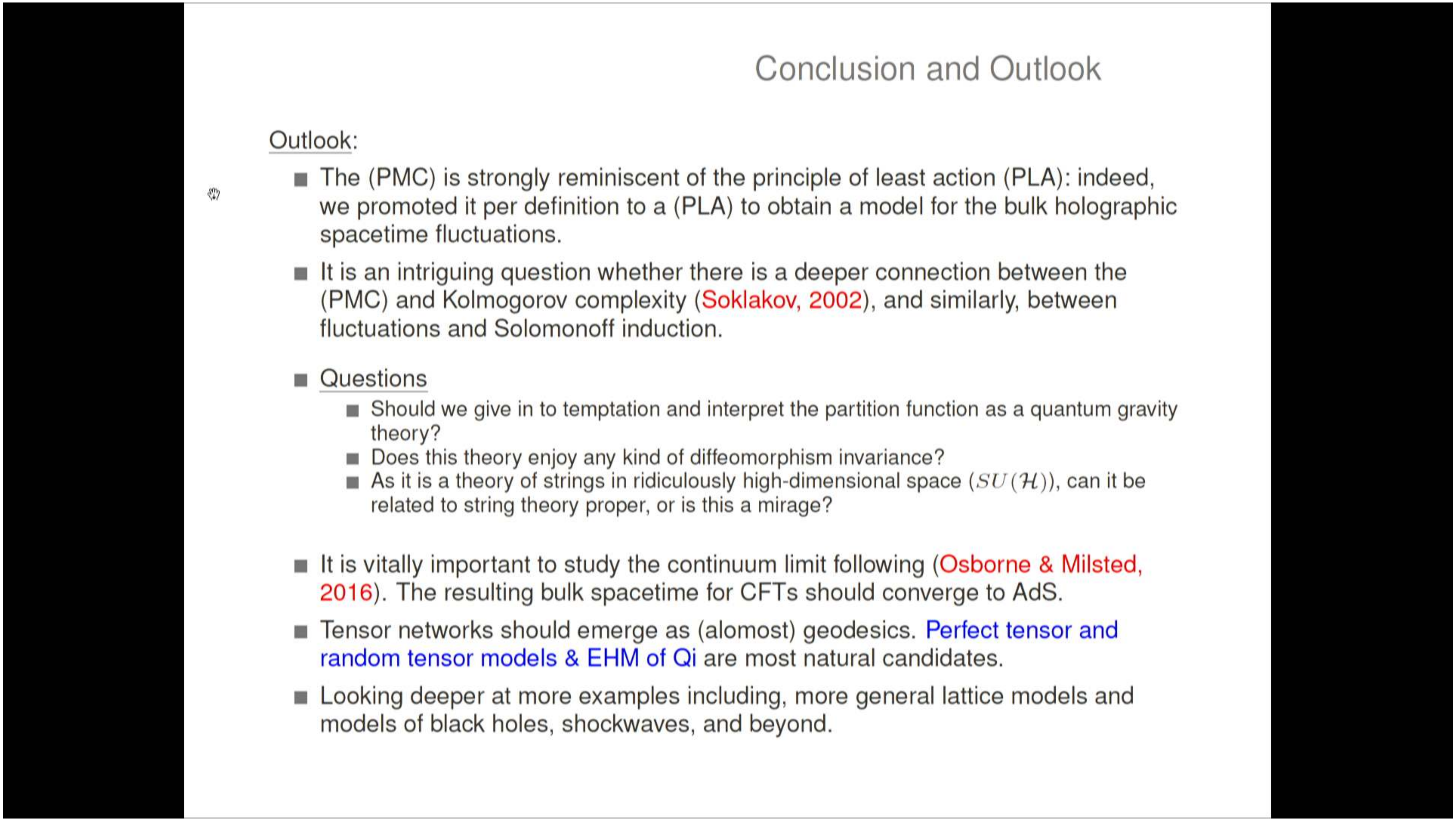


Conclusion and Outlook

Outlook:

- The (PMC) is strongly reminiscent of the principle of least action (PLA): indeed, we promoted it per definition to a (PLA) to obtain a model for the bulk holographic spacetime fluctuations.
- It is an intriguing question whether there is a deeper connection between the (PMC) and Kolmogorov complexity (Soklakov, 2002), and similarly, between fluctuations and Solomonoff induction.
- Questions
  - Should we give in to temptation and interpret the partition function as a quantum gravity theory?
  - Does this theory enjoy any kind of diffeomorphism invariance?
  - As it is a theory of strings in ridiculously high-dimensional space ($SU(\mathcal{H})$), can it be related to string theory proper, or is this a mirage?
- It is vitally important to study the continuum limit following (Osborne & Milsted, 2016). The resulting bulk spacetime for CFTs should converge to AdS.
- Tensor networks should emerge as (alomost) geodesics. Perfect tensor and random tensor models & EHM of Qi are most natural candidates.
- Looking deeper at more examples including, more general lattice models and models of black holes, shockwaves, and beyond.



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