Title: PainlevÃ© equations and Hitchin systems in four-dimensional N = 2 theories

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Abstract: <p>PainlevÃ© equations can be obtained both from time-dependent classical Hamiltonian systems and from isomonodromic deformation problems. These realizations lead to a precise matching between PainlevÃ© equations and Hitchin systems associated to four-dimensional N=2 SQCD as well as Argyres-Douglas theories. Long-time analysis of the PainlevÃ© Hamiltonians dynamics allows to extract the unrefined "instanton" partition function for these theories at all strong-coupling points</p>

Painlevé equations and Hitchin systems in four dimensional $\mathcal{N} = 2$ theories

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with G. Bonelli, O. Lisovyy, K. Maruyoshi, A. Tanzini

By reversing the reasoning, obtain Z_{Nek} at strong coupling:

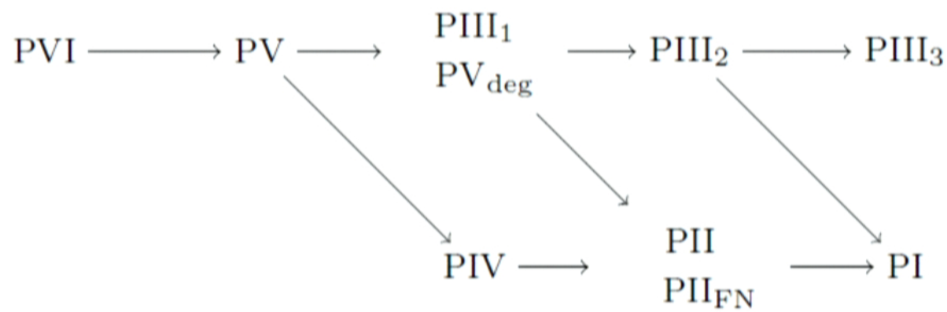
series solution around $t \sim \infty$ for (all) Painlevé τ-functions

$$\Downarrow$$

strong coupling ($\Lambda \sim \infty$) expansion of ("dual") partition function Z_{Nek} of four-dimensional $\mathcal{N} = 2$ $SU(2)$ SQCD and Argyres-Douglas theories

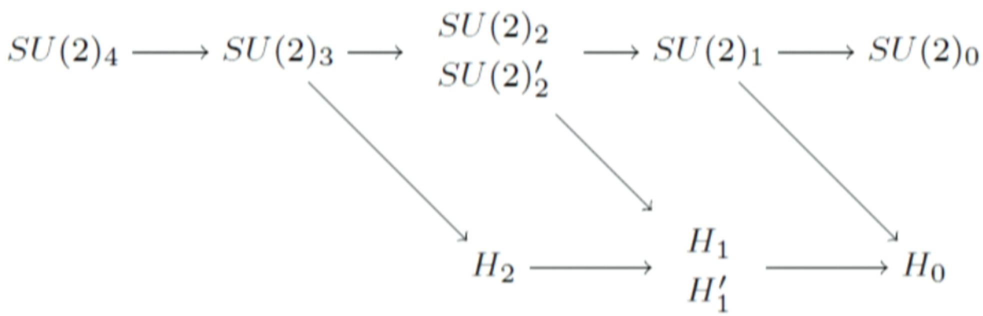
Painlevé equations

Later developments (Lax pair formulation, analysis space initial conditions) led to a refinement $\Longrightarrow$ 8 equations, related by *confluence*:

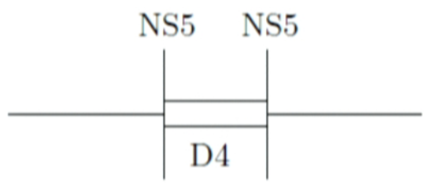


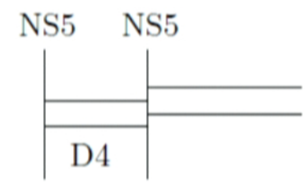
$\mathrm{PIII}_1/\mathrm{PV}_{\mathrm{deg}}$ and $\mathrm{PII}/\mathrm{PII}_{FN}$: same equation, different Lax pair (see later)

First hint: there is a similar confluence diagram in 4d $\mathcal{N} = 2$ gauge theory



Sometimes, different class $\mathcal{S}$ and brane realizations; $SU(2)_2$ example:





Painlevé equations - first realization

Painlevé equations can be written as classical Hamiltonian systems

$$\frac{dq}{dt} = \frac{\partial \mathcal{H}_a(q,p;t)}{\partial p} \quad , \quad \frac{dp}{dt} = -\frac{\partial \mathcal{H}_a(q,p;t)}{\partial q}$$

with time-dependent Hamiltonian $\mathcal{H}_a(t)$ ($a = \text{I}, \ldots, \text{VI}$)

We can study time evolution of $\mathcal{H}_a(t) \Longrightarrow$ new 2° order, degree 2 ODE:

- σ-Painlevé equations: ODEs satisfied by

$$\boxed{\sigma_a(t) \propto \mathcal{H}_a(q(t), p(t); t)}$$

- τ-Painlevé equations: ODEs satisfied by $\tau_a(t)$

$$\boxed{\sigma_a(t) \propto \frac{d}{dt} \ln \tau_a(t)}$$

The $\tau_a(t)$ function is the one which usually enters in physical problems

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The $\tau_a(t)$ function is the one which usually enters in physical problems

Example - PII: $\ddot{q} = 2q^3 + tq + \alpha$

- $\sigma_{\mathrm{II}}(t)$ function / Hamiltonian:

$$\sigma_{\mathrm{II}}(t) = \mathcal{H}_{\mathrm{II}}(t) = \frac{1}{2}p^2 - \left(q^2 + \frac{t}{2}\right)p - \left(\alpha + \frac{1}{2}\right)q$$

- σ-PII equation:

$$\ddot{\sigma}_{\mathrm{II}}^2 = 2\dot{\sigma}_{\mathrm{II}}\left(\sigma_{\mathrm{II}} - t\dot{\sigma}_{\mathrm{II}}\right) - 4\dot{\sigma}_{\mathrm{II}}^3 + \frac{1}{4}\left(\alpha + \frac{1}{2}\right)^2$$

Example - PIII$_3$: $\ddot{q} = \frac{\dot{q}^2}{q} - \frac{\dot{q}}{t} + \frac{2q^2}{t^2} - \frac{2}{t}$

- $\sigma_{\mathrm{III}_3}(t)$ function / Hamiltonian:

$$\sigma_{\mathrm{III}_3}(t) = t\mathcal{H}_{\mathrm{III}_3}(t) = p^2q^2 - q - \frac{t}{q}$$

- σ-PIII$_3$ equation:

$$(t\ddot{\sigma}_{\mathrm{III}_3})^2 = 4(\dot{\sigma}_{\mathrm{III}_3})^2\left(\sigma_{\mathrm{III}_3} - t\dot{\sigma}_{\mathrm{III}_3}\right) - 4\dot{\sigma}_{\mathrm{III}_3}$$

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Painlevé equations also arise from isomonodromic deformations of systems of first order *linear* ODE with rational coefficients

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Consider a 2×2 system of linear ODE:

$$\frac{d}{dz}\Psi(z) = A(z)\Psi(z)$$

where $z \in C_{0,n}$ and the matrices $\Psi(z) \in GL(2,\mathbb{C})$, $A(z) \in sl(2,\mathbb{C})$

$$A(z) = \sum_{\nu=1}^{n} \frac{A^{(\nu)}(z)}{(z - z_\nu)^{r_\nu+1}} \quad \text{with} \quad A^{(\nu)}(z) = \sum_{i=0}^{r_\nu} A_i^{(\nu)}(z - z_\nu)^{r_\nu - i}$$

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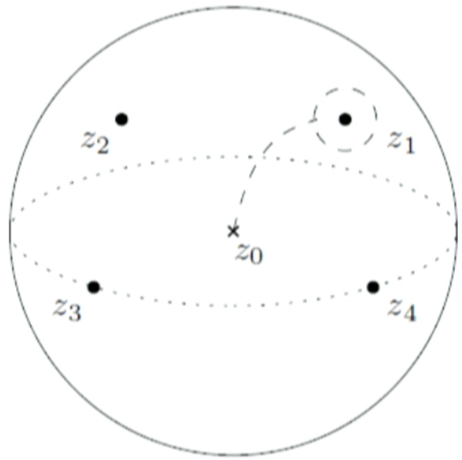
A point $z = z_*$ can be

- regular: $A(z)$ holomorphic at $z_* \implies \Psi$ holomorphic at z_*
- regular singularity: $z_* = z_\nu$, $r_\nu = 0 \implies \Psi$ branch point at z_*
- irregular singularity: $z_* = z_\nu$, $r_\nu \geqslant 1 \implies \Psi$ essential singularity at z_*

Near a regular singularity z_ν (take $A_0^{(\nu)}$ diagonal)

$$\Psi \sim \Psi^{(\nu)}(z) e^{A_0^{(\nu)} \ln(z - z_\nu)}$$



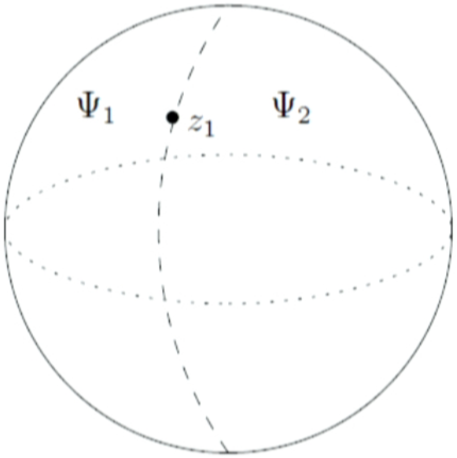
Moving around a regular singularity, Ψ acquires monodromy:

$$\boxed{\Psi \to \Psi M_\nu \,, \quad M_\nu \text{ monodromy matrices}}$$

Near an irregular singularity z_ν (take all $A_i^{(\nu)}$ diagonal)

$$\Psi \sim \Psi_k^{(\nu)}(z) e^{A_0^{(\nu)} \ln(z-z_\nu)} e^{\sum_{m=1}^{r_\nu} \frac{1}{m} A_m^{(\nu)} (z-z_\nu)^{-m}}$$

Stokes phenomenon: asymptotics Ψ depends on the region $\Omega_k \in C_{0,n}$



Crossing a Stokes line, Ψ jumps:

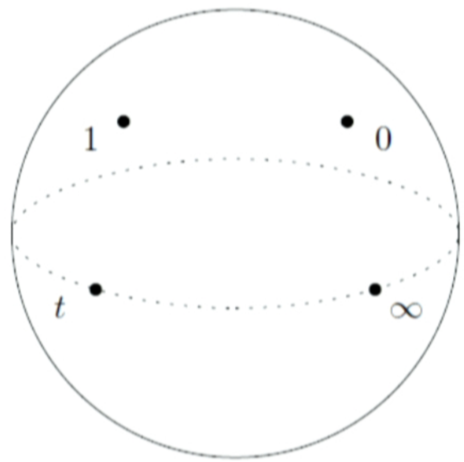
$$\boxed{\Psi_{k+1} \to \Psi_k S_k^{(\nu)} \,, \quad S_k^{(\nu)} \text{ Stokes matrices}}$$

How do Painlevé equations arise in this setting?

- Choice of $A(z) \Longrightarrow$ fixed M_ν / $S_k^{(\nu)}$ matrices
- Choice of M_ν / $S_k^{(\nu)} \Longrightarrow$ many-parameters family of $A(z;\vec{t})$

Isomonodromic deformations of $A(z;\vec{t})$: deformations $\vec{t}$ preserving $M_\nu/S_k^{(\nu)}$

Example: $A(z;t)$ with 4 regular singularities at $0, 1, t, \infty$



Isomonodromic deformation: variations of t which do not change M_ν

1-parameter case: deformation by t of $A(z;t)$ is isomonodromic if

$$\frac{d}{dt}\Psi(z;t) = B(z;t)\Psi(z;t)$$

where $B(z;t) \in sl(2,\mathbb{C})$ is determined in terms of elements of $A(z;t)$

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All in all, we have an overdetermined system

$$\begin{cases} \partial_z \Psi(z;t) = A(z;t)\Psi(z;t) \\ \partial_t \Psi(z;t) = B(z;t)\Psi(z;t) \end{cases} \qquad A(z;t), B(z;t) \text{ Lax pair}$$

Compatibility condition $\Psi_{zt} = \Psi_{tz}$ equivalent to

$$\partial_t A(z;t) = \partial_z B(z;t) + [B(z;t), A(z;t)]$$

Gives a set of equations which reduce to a Painlevé equation

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PVI: 4 regular singularities; PV: 2 regular, 1 irregular; ...

Example: Lax pair for PII

$$A = A_0 + zA_1 + z^2 A_2 = \begin{pmatrix} z^2 + p + t/2 & u(z-q) \\ -\frac{2}{u}(pz + \theta + pq) & -(z^2 + p + t/2) \end{pmatrix}$$

$$B = B_0 + zB_1 = \begin{pmatrix} z/2 & u/2 \\ -p/u & -z/2 \end{pmatrix}$$

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$$\dot{u} = -qu \quad , \quad \dot{q} = p + q^2 + t/2 \quad , \quad \dot{p} = -2pq - \theta$$

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Remark: the trace

$$\frac{1}{2}\mathrm{Tr}\, A^2 = z^4 + tz^2 - 2\theta z + 2\sigma_{II}(t) + \frac{t^2}{4}$$

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As $\kappa \to 0$ study system near time T_0

- Connection $A \implies$ one-form $A \in \Omega^{(1,0)}(C_{0,n}, sl(2,\mathbb{C}))$
- Higgs bundle, spectral curve

$$\det(y - A) = 0 \quad \implies \quad y^2 = \frac{1}{2}\mathrm{Tr}A^2$$

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$\kappa \to 0$: Painlevé connection $\kappa\partial_z - A \Longrightarrow$ Painlevé Higgs field A;
$\kappa \neq 0$: Higgs field $\Longrightarrow$ connection via isospectral/Whitham deformations;
hints towards relation to Hitchin systems

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Physics background

How do Painlevé equations appear in 4d $\mathcal{N}=2$ gauge theories?

Study Coulomb branch $\mathcal{B} \Longrightarrow$ Hitchin system (torus fibration over $\mathcal{B}$)

$$6\text{d } \mathcal{N}=(2,0)\ A_{N-1} \text{ theory on } \mathbb{R}^3 \times S^1_R \times C_{g,n}$$

$$\Big\Downarrow \text{ on } C_{g,n} \text{ (twisted)}$$

$$4\text{d } \mathcal{N}=2 \text{ theory on } \mathbb{R}^3 \times S^1_R$$

$$\Big\Downarrow \text{ on } S^1_R$$

$$3\text{d } \mathcal{N}=4 \text{ theory on } \mathbb{R}^3;$$

$$\sigma\text{-model with hyperkähler target } \mathcal{M}_H$$

To see this, reverse the order of compactification [GMN]:

$$6\text{d } \mathcal{N} = (2,0)\ A_{N-1} \text{ theory on } \mathbb{R}^3 \times S^1_R \times C_{g,n}$$

$$\Big\Downarrow \text{ on } S^1_R$$

$$5\text{d } \mathcal{N} = 2\ SU(N) \text{ SYM theory on } \mathbb{R}^3 \times C_{g,n}$$

$$\Big\Downarrow \text{ on } C_{g,n} \text{ (twisted)}$$

$$3\text{d } \mathcal{N} = 4 \text{ theory on } \mathbb{R}^3;$$

$$\sigma\text{-model with hyperkähler target } \mathcal{M}_H$$

In this setting

$$\begin{aligned} \mathcal{M}_H &= \{\text{5d BPS configurations which are Poincaré invariant in } \mathbb{R}^3\} \\ &= \{\text{moduli space of solutions Hitchin equations associated to } C_{g,n}\} \end{aligned}$$

More in detail, $\mathcal{M}_H$ is the moduli space of solutions of the equations

$$\left.\begin{aligned} F_{z\bar{z}} + R^2[\varphi_z, \bar{\varphi}_{\bar{z}}] &= 0 \\ D_{\bar{z}}\varphi_z &= 0 \\ D_z\bar{\varphi}_{\bar{z}} &= 0 \end{aligned}\right. \quad \text{mod } G$$

with prescribed singular behaviour at the punctures of $C_{g,n}$

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- z: complex coordinate on $C_{g,n}$
- A_z: reduction to $C_{g,n}$ of the 5d vector field ($F_{z\bar{z}}$ field strength);
- φ_z: complex adjoint scalar from the 5d ones ((1,0)-form after twist);
- two types of singular behaviour at the puncture $z = z_*$:

$$\varphi_z \sim \frac{1}{z - z_*} \text{ (regular)} \quad \text{or} \quad \varphi_z \sim \frac{1}{(z - z_*)^{1+r}} \text{ (irregular, } r \geqslant 1)$$

and similarly for the other fields

$\mathcal{M}_H$ hyperkähler $\Rightarrow$ complex structure determined by choice $\zeta \in \mathbb{CP}^1$

- $\zeta = 0$: $\mathcal{M}_H \longrightarrow$ moduli space Higgs bundles $(D_{\bar{z}}, R\varphi_z)$
- $\zeta \in \mathbb{C}^\times$: $\mathcal{M}_H \longrightarrow$ moduli space flat connections $\nabla = \frac{R}{\zeta}\varphi + D + R\zeta\bar{\varphi}$
- $\zeta = \infty$: $\mathcal{M}_H \longrightarrow$ moduli space anti-Higgs bundles $(D_z, R\varphi_{\bar{z}})$

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Higgs bundle $\Rightarrow$ integrable system of 4d $\mathcal{N} = 2$ theory [Donagi-Witten]:

- Hamiltonians $\longleftrightarrow$ 4d Coulomb branch moduli u^i;
- spectral curve $\longleftrightarrow$ Seiberg-Witten curve (curve in $T^*C_{g,n}$)

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Painlevé Higgs bundles $(\kappa = 0) \Longleftrightarrow$ Higgs bundles SQCD, Argyres-Douglas

- Painlevé spectral curve $\longleftrightarrow$ Seiberg-Witten curve

$$\boxed{\det(y - A) = 0 \quad \Longrightarrow \quad \det(y - \varphi_z) = 0}$$

- Painlevé Hamiltonian $\mathcal{H} \longleftrightarrow$ 4d Coulomb branch modulus u

Painlevé connections ($\kappa \neq 0$) $\Longleftrightarrow$ oper limit connection $\nabla = \frac{R}{\zeta}\varphi + D + R\zeta\bar{\varphi}$

$$\nabla \underset{oper}{\longrightarrow} \boxed{\hbar\partial_z - \varphi_z \quad \Longleftrightarrow \quad \kappa\partial_z - A}$$

Oper limit: $R \to 0$, $\zeta \to 0$ keeping $\zeta/R = \hbar$ fixed

Painlevé isomonodromic problem	$\mathcal{N}=2$ **theory Hitchin system**
Painlevé connection $\kappa\partial_z - A$	oper $\hbar\partial_z - \varphi_z$
overall scale κ	oper parameter $\hbar$
Painlevé time t	gauge coupling Λ
Painlevé σ-function (Hamiltonian)	Coulomb branch parameter u
Painlevé free parameters	masses $\mathcal{N}=2$ theory

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Painlevé free parameters	masses $\mathcal{N} = 2$ theory

Twist fields can be realized in terms of fermion bilinears

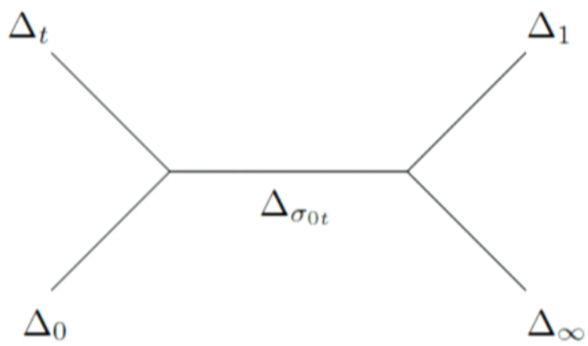
$$\mathcal{O}_{z_\nu} = \exp\left(\int_{\mathcal{C}_\nu} \mathrm{Tr}[A_0^{(\nu)} J(y)]dy\right) \;, \quad J_{\beta\alpha} = \overline{\psi}_\beta \psi_\alpha \;\; \widehat{sl}(2)_1 \text{ current}$$

with conformal dimension $\Delta_\nu = \theta_\nu^2 = \frac{1}{2}\mathrm{Tr}(A_0^{(\nu)})^2$, $\pm\theta_\nu$ eigenvalues $A_0^{(\nu)}$

PVI τ-function realized as

$$\boxed{\tau_{\rm VI}(t) = \langle \mathcal{O}_0 \mathcal{O}_t \mathcal{O}_1 \mathcal{O}_\infty \rangle}$$

Need to consider the $c = 1$ four-point conformal block



Remark: subtlety with dimension $\Delta_{\sigma_{0t}}$

- $\psi_\alpha(z)$: monodromy $M_t M_0$ around fields in the OPE $\mathcal{O}_0\mathcal{O}_t$
- Let $e^{\pm 2\pi i \sigma_{0t}}$ be eigenvalues of $M_t M_0$: σ_{0t} defined up to $n \in \mathbb{Z}$
- Expect infinitely many primaries in OPE $\mathcal{O}_0\mathcal{O}_t$ with $\Delta_{\sigma_{0t}} = (\sigma_{0t} + n)^2$

$\tau_{\mathrm{VI}}(t)$ will involve linear combination of conformal blocks

$$\tau_{\rm VI}(t) = \sum_{n\in\mathbb{Z}} e^{2\pi i n \eta_{0t}} Z_{\rm Nek}^{N_F=4}(\vec{\theta}, \sigma_{0t}+n; t)$$

Dictionary with $\mathcal{N}=2$ $SU(2)$ $N_F=4$ theory:

- $c=1 \Longrightarrow \epsilon_1 = -\epsilon_2 = \epsilon$ (overall scale, analogue of $\kappa/\hbar \in \mathbb{C}^\times$)
- conformal dimensions $\theta_\nu^2 \longleftrightarrow$ masses m_ν^2/ϵ^2
- first Painlevé integration constant $\sigma_{0t} \longleftrightarrow$ Coulomb parameter a/ϵ
- second Painlevé integration constant $\eta_{0t} \longleftrightarrow$ dual parameter a_D/ϵ
- time variable $t \sim 0 \longleftrightarrow$ instanton parameter Λ/ϵ (weak coupling)

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Following coalescence diagram, obtain small t series for $\tau_{\rm V}$, $\tau_{{\rm III}_1}$, $\tau_{{\rm III}_2}$, $\tau_{{\rm III}_3}$:

$$\boxed{\tau_{\rm V}, \tau_{{\rm III}_1}, \tau_{{\rm III}_2}, \tau_{{\rm III}_3}\ (t\sim 0)} \iff \boxed{SU(2)\ \ N_F = 3,2,1,0\ \ (\Lambda \sim 0)}$$

(Exists bosonized version in $c=1$ Liouville CFT [Iorgov-Lisovyy-Teschner])

What about $\tau_{\rm I}$, $\tau_{\rm II}$, $\tau_{\rm IV}$ functions? No instanton expansions...

Example: τ-function for PII / Argyres-Douglas H_1

Expansion 1: $\arg t = \pi, \pm\frac{\pi}{3}$

$$\tau_{II}(t) = \sum_{n\in\mathbb{Z}} e^{2\pi i n\eta}\mathcal{G}(\sigma+n,s), \qquad 4t^3 = 9s^2$$

$$\mathcal{G}(\sigma,s) = e^{-\frac{3s^2}{32}+\sigma s} s^{-\frac{1}{12}-\frac{\sigma^2}{2}+\frac{\theta^2}{3}} 12^{-\frac{\sigma^2}{2}} G(1+\sigma)\left[1+\sum_{k=1}^{\infty}\frac{D_k(\sigma)}{s^k}\right]$$

$$D_1(\sigma) = \frac{\sigma(34\sigma^2-96\theta^2+31)}{72}, \quad D_2(\sigma) = \ldots$$

Expansion 2: $\arg t = 0, \pm\frac{2\pi}{3}$

$$\tau_{II}(t) = \sum_{n\in\mathbb{Z}} e^{2\pi i n\eta}\mathcal{G}(\sigma+n,s), \qquad 8t^3 = 9s^2$$

$$\mathcal{G}(\sigma,s) = e^{i\sigma s+\frac{i\pi\sigma^2}{2}} s^{-\sigma^2+\frac{\theta^2}{12}} 6^{-\sigma^2} G(1+\sigma+\frac{\theta}{2})G(1+\sigma-\frac{\theta}{2})\left[1+\sum_{k=1}^{\infty}\frac{D_k(\sigma)}{s^k}\right]$$

$$D_1(\sigma) = -\frac{i\sigma(68\sigma^2-9\theta^2+2)}{36}, \quad D_2(\sigma) = \ldots$$

Remark 1: expressions of the form

$$\sum_{n\in\mathbb{Z}} e^{2\pi i n\eta} Z_{\text{Nek}}(\vec{\theta}, \sigma + n; t)$$

already appeared in the physics literature:

- known as "dual" instanton partition function [Nekrasov-Okounkov]; related to free fermions, quantum cohomology instanton moduli space, Toda equation, Donaldson-Thomas invariants (but: a quantized: ?); actually, many "dual" partition functions
- partition function I-brane system: chiral free fermions on Riemann surface (intersection D4/D6 branes) [Dijkgraaf-Hollands-Sulkowski-Vafa]; related to Donaldson-Thomas invariants, bounded D0-D2-D4-D6 states, $\mathcal{D}$-modules
- ...

Remark 2: Verlinde loop algebra

Our expression for the τ-function is quasi-periodic in σ

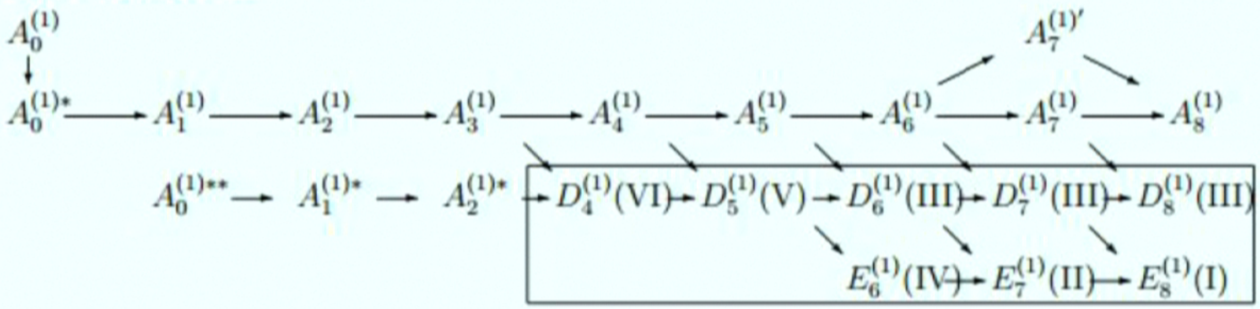
$$\tau(\sigma+1;t) = e^{-2\pi i\eta}\tau(\sigma;t)$$

$\Longrightarrow$ diagonalizes the operators $e^{\pm\partial_\sigma}$

Perspectives

Differential Painlevé equations are part of a more general story [Sakai]

The list of D



0) $A_0^{(1)}$ is a **boss** of Painlevé equations (**elliptic Painlevé**)

1) The A-series give **q-difference Painlevé equations**

2) $A_0^{(1)**}$, $A_1^{(1)*}$, $A_2^{(1)*}$: **difference Painlevé** (**Boalch**) corresponding to

$$[(111111, 222, 33)], \quad [(1111), (1111), (22)], \quad [(111), (111), (111)].$$

3) Eight in the box are **Painlevé differential equations** .

0),1) rank 1 6d $\mathcal{N} = (1, 0)$ and 5d $\mathcal{N} = 1$ $SU(2)$ $N_F = 0, \ldots, 7$

2) rank 1 4d Minahan-Nemeschansky (from a talk by [Ohyama])

5d: topological strings and spectral determinants

5d uplift: spectral determinant approach? [Grassi-Marino-Hatsuda]

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5d uplift: spectral determinant approach? [Grassi-Marino-Hatsuda]

- Given a toric Calabi-Yau 3-fold X, consider the mirror $\widetilde{X}$

$$W_X(e^x, e^p) = \mathcal{O}_X(x,p) - u = 0$$

- Quantize the mirror curve as $[\widehat{x}, \widehat{p}] = i\hbar$

$$\mathcal{O}_X(x,p) = u \quad \Longrightarrow \quad \widehat{\mathcal{O}}_X(\widehat{x}, \widehat{p})|\psi_n\rangle = u_n|\psi_n\rangle = e^{E_n}|\psi_n\rangle$$

Conjecture (tested numerically in many cases):

$$\Xi_X(u;\hbar) = \sum_{n\in\mathbb{Z}} \exp[J_X(E + 2\pi i n;\hbar)]$$

with grand potential $J_X = J_X^{\mathrm{np}} + J_X^{\mathrm{WKB}}$ built from topological string

- Non-perturbative (in $\hbar$) part $J_X^{\mathrm{np}} = F_{\mathrm{unref}}$ (from $\ln Z_{5d}$ at $\epsilon_1 + \epsilon_2 = 0$)
- WKB part J_X^{WKB} depends on F_{NS} (from $\ln Z_{5d}$ in NS limit $\epsilon_2 = 0$)

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Many similarities with our τ-functions

$$\tau(t) = \sum_{n\in\mathbb{Z}} e^{2\pi i n\eta} Z_{\text{Nek}}(\vec{\theta}, \sigma + n; t)$$

- τ-functions of Painlevé equations are Fredholm determinants
- A limit of $\Xi_{\mathbb{F}_0}$ removing F_{NS} reduces to τ_{III_3} [Bonelli-Grassi-Tanzini]
- η missing?

Hitchin systems 4d $\mathcal{N} = 2 \iff$ Painlevé isomonodromic problems:

- oper connection $\hbar\partial_z - \varphi_z \iff$ Painlevé connection $\kappa\partial_z - A$
- "dual" instanton partition function $\iff$ Painlevé τ-function

Time evolution $\tau(t)$ determined by isomonodromic deformation condition; used to extract strongly-coupled "$Z_{\text{inst}}^{c=1}$" of Argyres-Douglas and SQCD

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Future directions:

- Argyres-Douglas at superconformal point?
- flat sections: "dual" *ramified* partition function?
- $c \neq 1$: quantization of Painlevé Hamiltonian?
- q-difference Painlevé: relation to non-perturbative topological strings?

Thanks!