Title: Quantum logic is undecidable

Date: Oct 25, 2016 03:30 PM

URL: http://pirsa.org/16100035

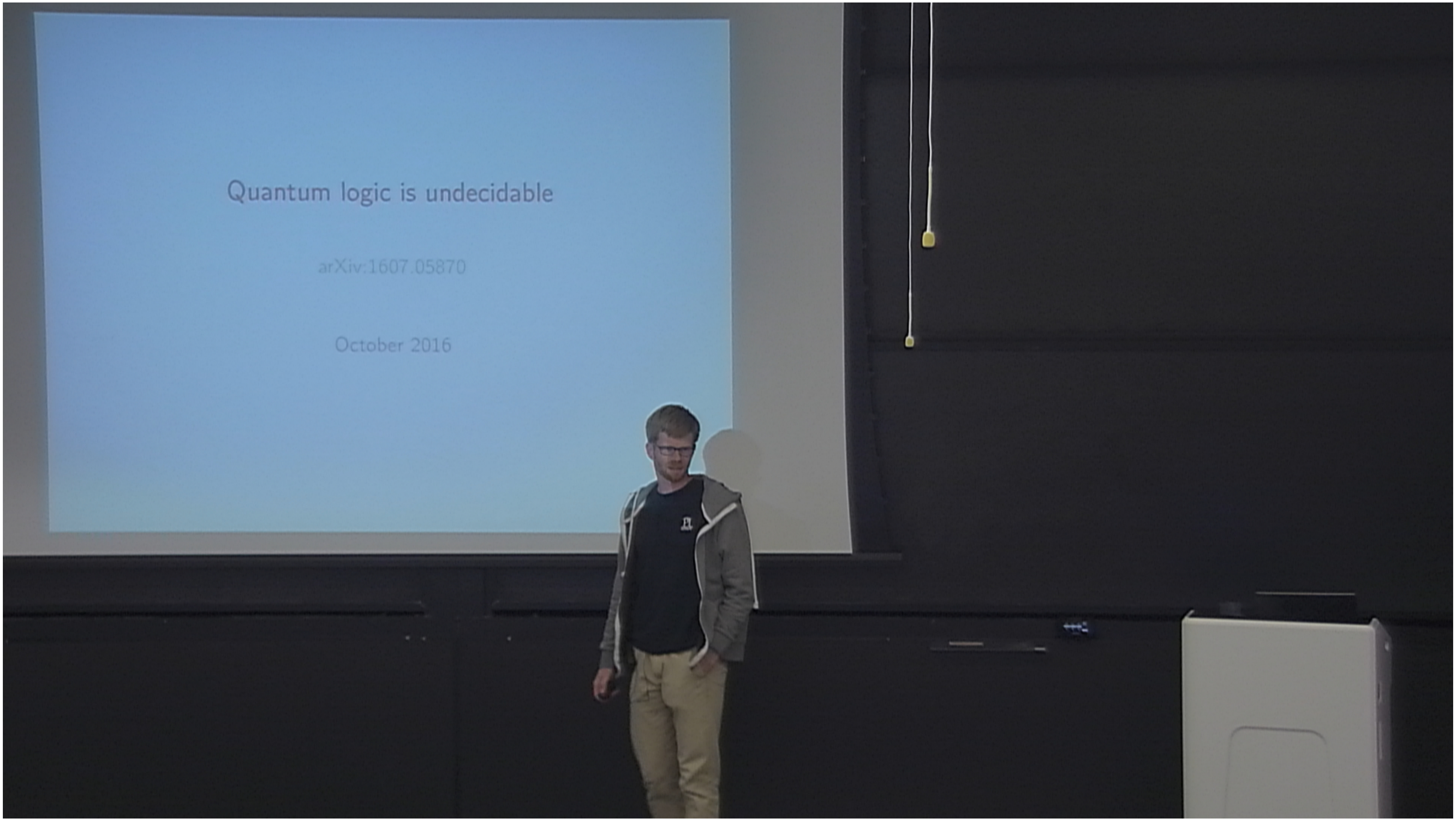
Abstract: <p>I will explain and prove the statement of the title. The proof relies on a recent result of Slofstra in combinatorial group theory and the hypergraph approach to contextuality.</p>

<p>Based on http://arxiv.org/abs/1607.05870.</p>

Quantum logic is undecidable

arXiv:1607.05870

October 2016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What is quantum logic?

- Idea: Quantum weirdness is an illusion due to reasoning in Boolean logic, which is inadequate at the quantum level.
- Example: in Boolean logic, $\wedge$ distributes over $\vee$,

$$P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R),$$

where P, Q and R propositions. Not so in quantum logic!

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- Quantum propositions are projection operators on Hilbert space, or equivalently closed subspaces.
- Connectives of quantum logic:
  - $\wedge$ is intersection of subspaces,
  - $\vee$ is the closed linear span,
  - Negation is the orthogonal complement.
- Example where the above distributivity fails?

The laws of quantum logic

So which laws of logic are valid quantumly?

- Some rules of Boolean logic still apply, e.g.

$$P \vee P^{\perp} = 1, \qquad P \wedge P^{\perp} = 0.$$

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$$P \vee (P^{\perp} \wedge Q) = Q.$$

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- These are some particular laws. Is it possible to classify all of them?
- More precise question: what is the **complexity** of telling whether a given candidate law is valid or not?

Complexity of quantum logic

Theorem

There is no algorithm to decide whether an implication of the form

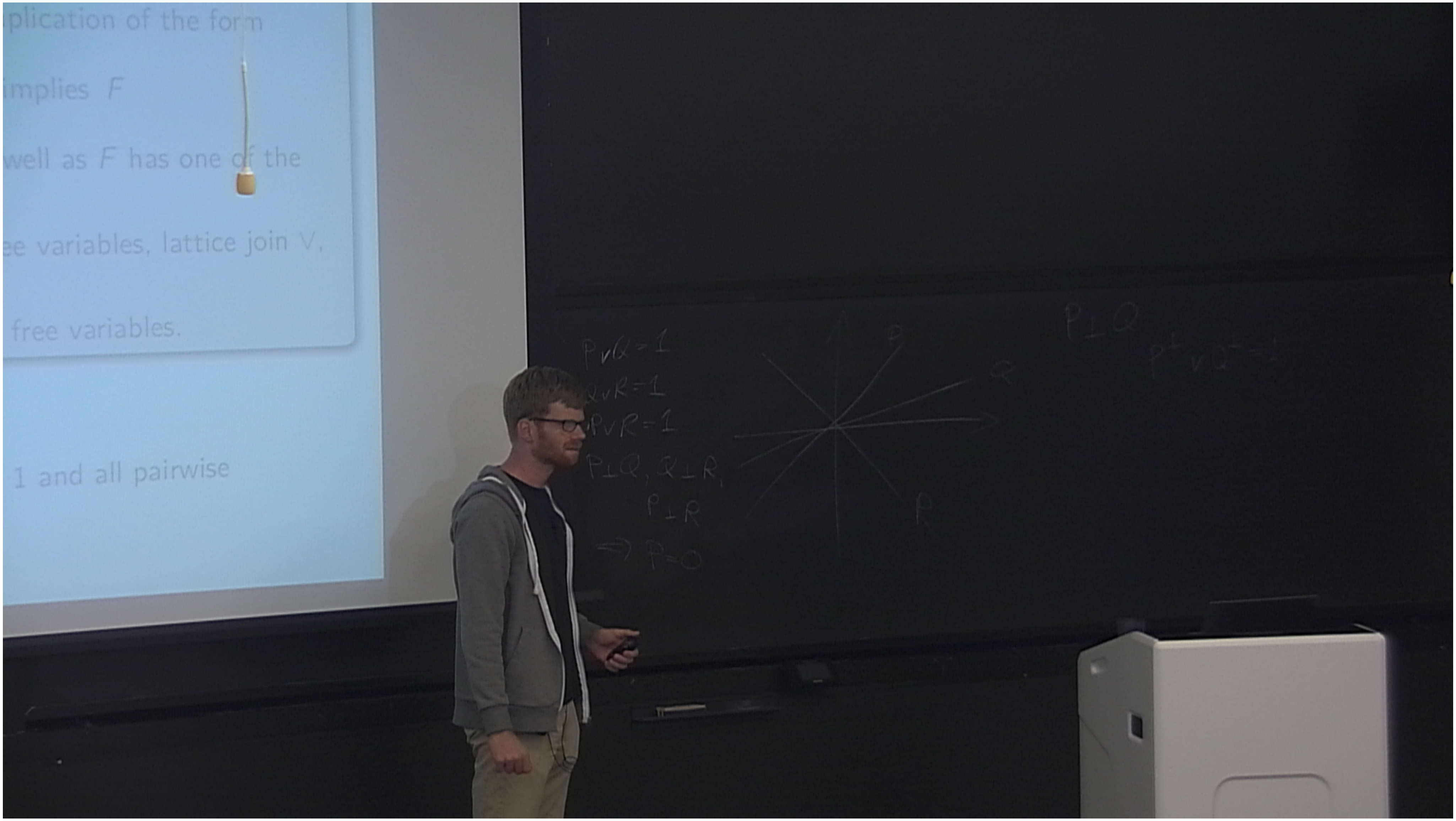
$$(E_1 \text{ and } E_2 \text{ and } \ldots \text{ and } E_k) \text{ implies } F$$

holds for all Hilbert spaces, where each E_i as well as F has one of the following two forms:

- an equation phrased solely in terms of free variables, lattice join $\vee$, and 0;
- an orthogonality relation $\perp$ between two free variables.

Example:

- $P \vee Q = 1$ and $Q \vee R = 1$ and $R \vee P = 1$ and all pairwise orthogonalities implies $P = 0$.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Lemma

For projections $P_1, \ldots, P_n$, the following are equivalent:

- $\sum_i P_i = 1$,
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We denote this by $OC(P_1, \ldots, P_n)$. It is a conjunction of premises of the above form.

Definition

A **hypergraph** (V, E) is a finite set V together with a collection of subsets $E \subseteq 2^V$ called **hyperedges**.

Decision Problem[1]

Given a hypergraph (V, E), is there a **quantum representation** consisting of projections $(P_v)_{v \in V}$ such that $OC(\bar{P}_e)$ for all e?

[1] Antonio Acín, Tobias Fritz, Anthony Leverrier and Ana Belén Sainz, A Combinatorial Approach to Nonlocality and Contextuality, arXiv:1212.4084.

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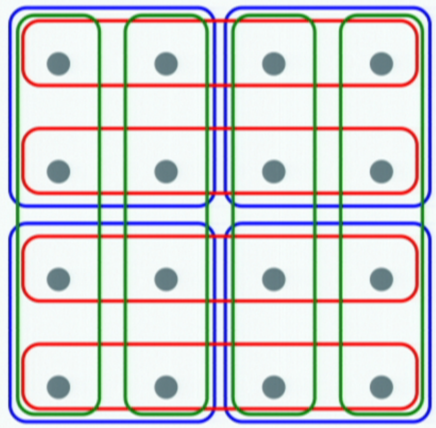

Next example: same, but with some nodes removed!

No algorithm is known, but proving undecidability seems hard.

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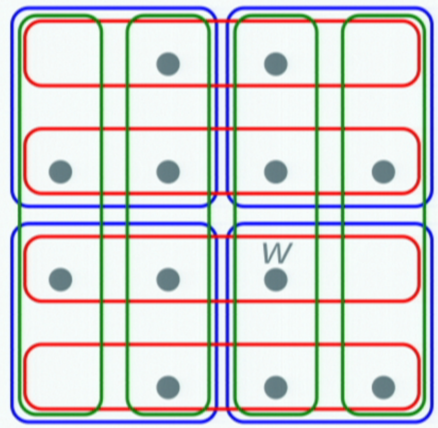
Decision Problem

Given a hypergraph (V, E) and $w \in V$, is $P_w = 0$ in every quantum representation of (V, E)?

Example:[2]



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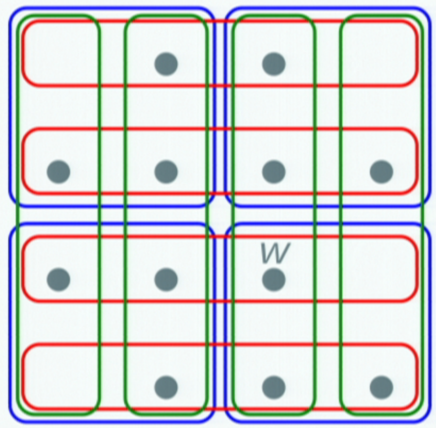

Our verdict is:

Main Theorem

This problem is undecidable.

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Hypergraph C*-algebras

For the proof, we will use that quantum representations of (V, E) are the same thing as representations of the **hypergraph C*-algebra** $C^*(V, E)$,

$$C^*(V,E) = \left\langle P_v,\ v \in V \;\middle|\; P_v^2 = P_v = P_v^*,\quad \sum_{v\in e} P_v = 1 \right\rangle$$

The decision problem then asks whether $P_w = 0$ in $C^*(V, E)$.

[3] Tobias Fritz, Tim Netzer, Andreas Thom, Can you compute the operator norm?, arXiv:1207.0975.

Another kind of algebraic structure plays a role in the proof:

Definition (Cleve, Liu, Slofstra[4] with minor modification)

[4] Richard Cleve, Li Liu, William Slofstra, Perfect Commuting-Operator Strategies for Linear System Games, arXiv:1606.02278.

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The **solution group** associated to a bipartite graph $G = I \cup T$ is the group with generators $(x_i)_{i \in I}$ and relations

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- $x_i^2 = 1$ for all $i \in I$,
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- $\prod_{i \in t} x_i = 1$ for all $t \in T$.

Decision Problem

Given (V, E) and $w \in V$, is $x_w = 1$ in the solution group $\Gamma(V, E)$?

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We now leverage Slofstra's result to prove our main theorem.

Lemma

A solution group C*-algebra $C^*(G)$ is computably isomorphic to a hypergraph C*-algebra $C^*(V, E)$ for a suitable (V, E).

Idea of proof:

- The x_i are ± 1-valued projective measurements.
- For every $t \in T$ there is a measurement corresponding to joint measurement of the $\{x_i : i \sim t\}$;
- The outcomes for which the parity of such a measurement is -1 are removed.
- This results in a contextuality scenario described by a hypergraph.

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The role of Hilbert space dimension

Final Remarks

- The undecidability relies crucially on the infinite-dimensionality on Hilbert space!
- The analogous decision problem in a fixed range of dimensions is decidable thanks to real quantifier elimination.

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[6] Leonard Lipshitz, The Undecidability of the Word Problems for Projective Geometries and Modular Lattices, jstor.org/stable/1996907.

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Final Remarks

- The undecidability relies crucially on the infinite-dimensionality on Hilbert space!
- The analogous decision problem in a fixed range of dimensions is decidable thanks to real quantifier elimination.
- For arbitrary finite Hilbert space dimension, undecidability of quantum logic was already known[6].

[6] Leonard Lipshitz, The Undecidability of the Word Problems for Projective Geometries and Modular Lattices, jstor.org/stable/1996907.