Title: PSI 2016/2017 Statistical Mechanics - Lecture 8

Date: Oct 20, 2016 10:45 AM

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Abstract:

RENORMALIZATION

$$Z = \int D\varphi_< D\varphi_> \, e^{-S[\varphi_<,\varphi_>]} \equiv \int D\varphi_< \, e^{-\tilde{S}[\varphi_<]}$$

$$\Rightarrow \tilde{S}[\varphi_<] = \frac{1}{2}\int_k^{\Lambda/b} \left[\tilde{r}_2 k^2 + \tilde{r}_0\right] |\varphi_<(k)|^2 + \frac{\tilde{r}_4}{4!}\int_{k_1}^{\Lambda/b} \varphi_<(k_1)\cdots\varphi_<(k_4)\,(2\pi)^d$$

BRING TO ORIGINAL FORM

$$\tilde{k} = kb, \quad \tilde{\varphi}(\tilde{k}) = \varphi_< \, b^{-\frac{d+2-\eta}{2}}$$

- $(r,u) \to (\tilde{r},\tilde{u})$

$$\tilde{r}_2 = b^{\eta}, \quad \tilde{r} =$$

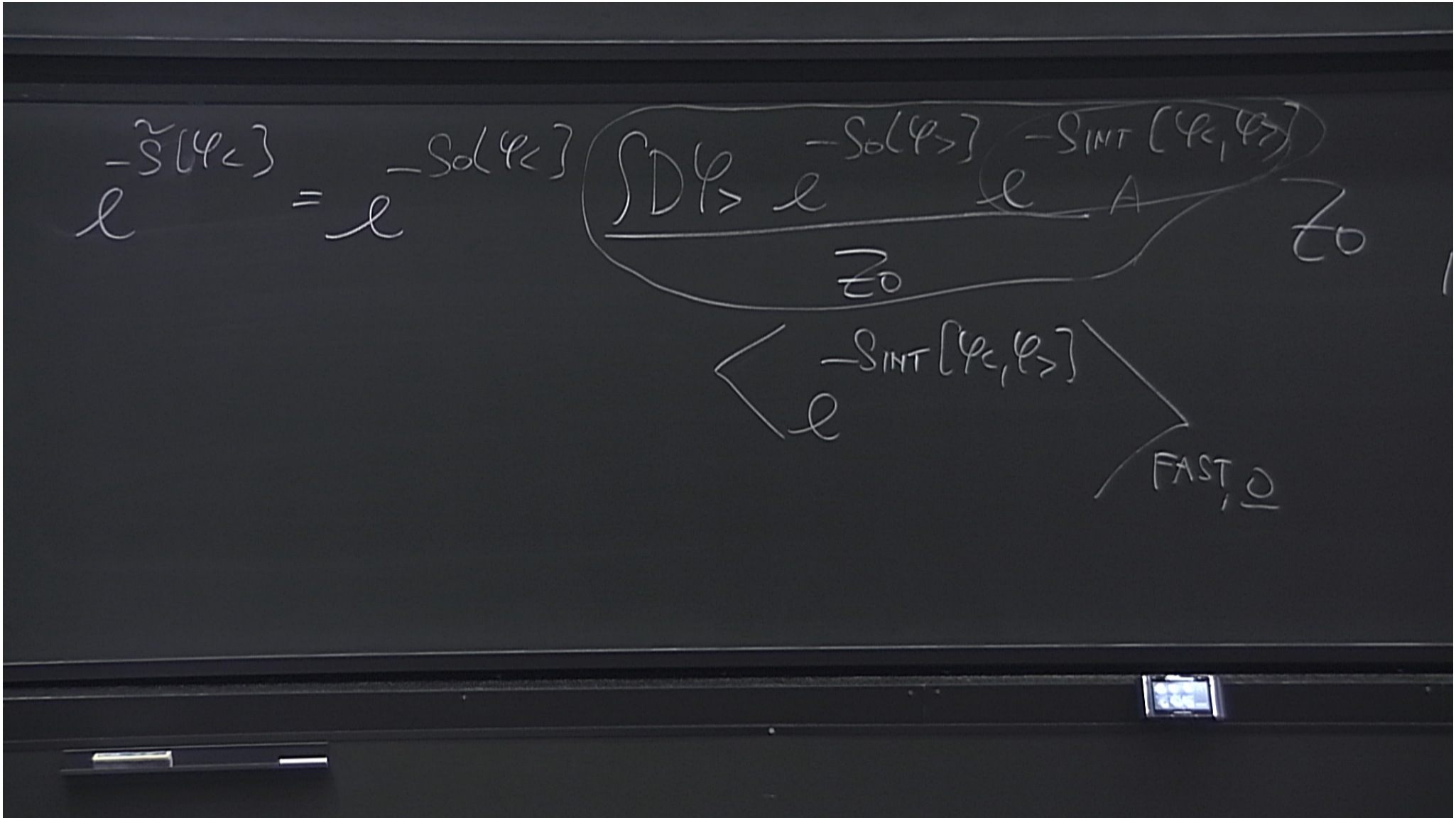
$$\equiv \int D\varphi_< \, e^{-\tilde{S}[\varphi_<]}$$

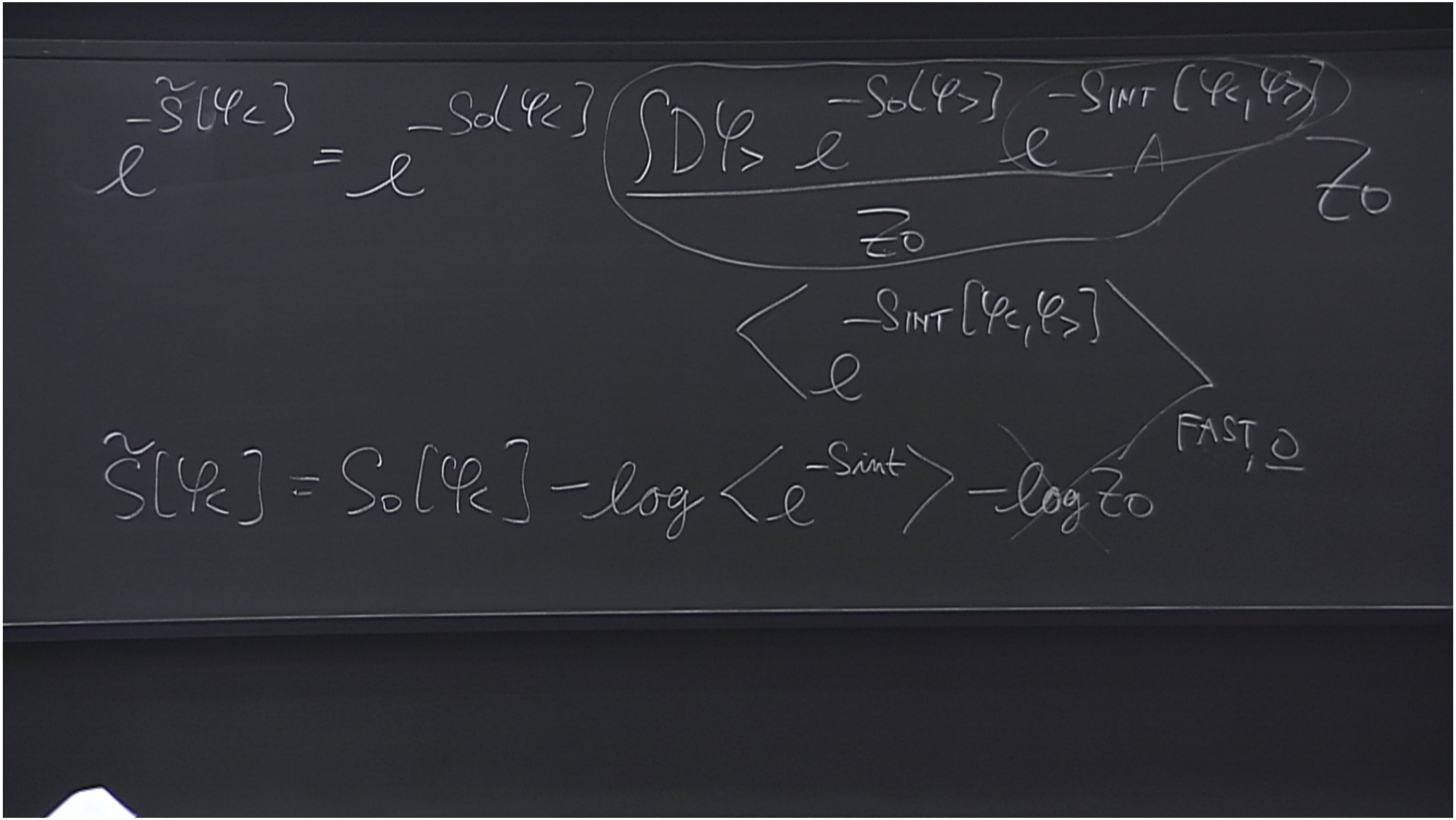
$$\|^2 + \frac{\tilde{\lambda}_4}{4!} \int_{k_1 \ldots k_4}^{\Lambda/b} \varphi_<(k_1) \ldots \varphi_<(k_4) (2\pi)^d \delta(k_1 + \ldots + k_4)$$

$$\frac{+2-\eta}{2}$$

- $(r, u) \to (\tilde{r}, \tilde{u})$

$$\tilde{r}_2 = b^\eta, \quad \tilde{r} = b^{2-\eta} \tilde{r}_0, \quad \tilde{u} = b^{4-2\eta-d} \tilde{r}_4$$


$e^{-\tilde{S}[\varphi_<]} = e^{-S_0[\varphi_<]} \int D\varphi_> e^{-S_0[\varphi_>]} e^{-S_{INT}[\varphi_<,\varphi_>]} A$
Z_0
Z_0
$\langle e^{-S_{INT}[\varphi_<,\varphi_>]} \rangle$
FAST, $\underline{Q}$


$e^{-\tilde{S}[\varphi_<]} = e^{-S_0[\varphi_<]} \frac{\int D\varphi_> e^{-S_0[\varphi_>]} e^{-S_{INT}[\varphi_<,\varphi_>]} A}{Z_0} Z_0$
$\langle e^{-S_{INT}[\varphi_<,\varphi_>]} \rangle$
FAST, Ω
$\tilde{S}[\varphi_<] = S_0[\varphi_<] - \log \langle e^{-S_{int}} \rangle - \log Z_0$

USE CUMULANT EXPANSION.

LET Ω BE A RANDOM VARIABLE

$$\Rightarrow \langle e^{\Omega} \rangle = e^{\langle \Omega \rangle + \frac{1}{2}(\langle \Omega^2 \rangle - \langle \Omega \rangle^2) + \ldots}$$

USE CUMULANT EXPANSION.

LET Ω BE A RANDOM VARIABLE

$$\Rightarrow \langle e^{\Omega} \rangle = e^{\langle \Omega \rangle + \frac{1}{2}\left(\langle \Omega^2 \rangle - \langle \Omega \rangle^2\right) + \ldots}$$

$$\tilde{S}[\varphi_c] = S_0[\varphi_c] + \langle S_{int} \rangle - \frac{1}{2}\left(\langle S_{int}^2 \rangle - \langle S_{int} \rangle^2\right)$$

USE CUMULANT EXPANSION.

LET Ω BE A RANDOM VARIABLE

$$\Rightarrow \langle e^{\Omega} \rangle = e^{\langle \Omega \rangle + \frac{1}{2}\left(\langle \Omega^2 \rangle - \langle \Omega \rangle^2\right) + \ldots}$$

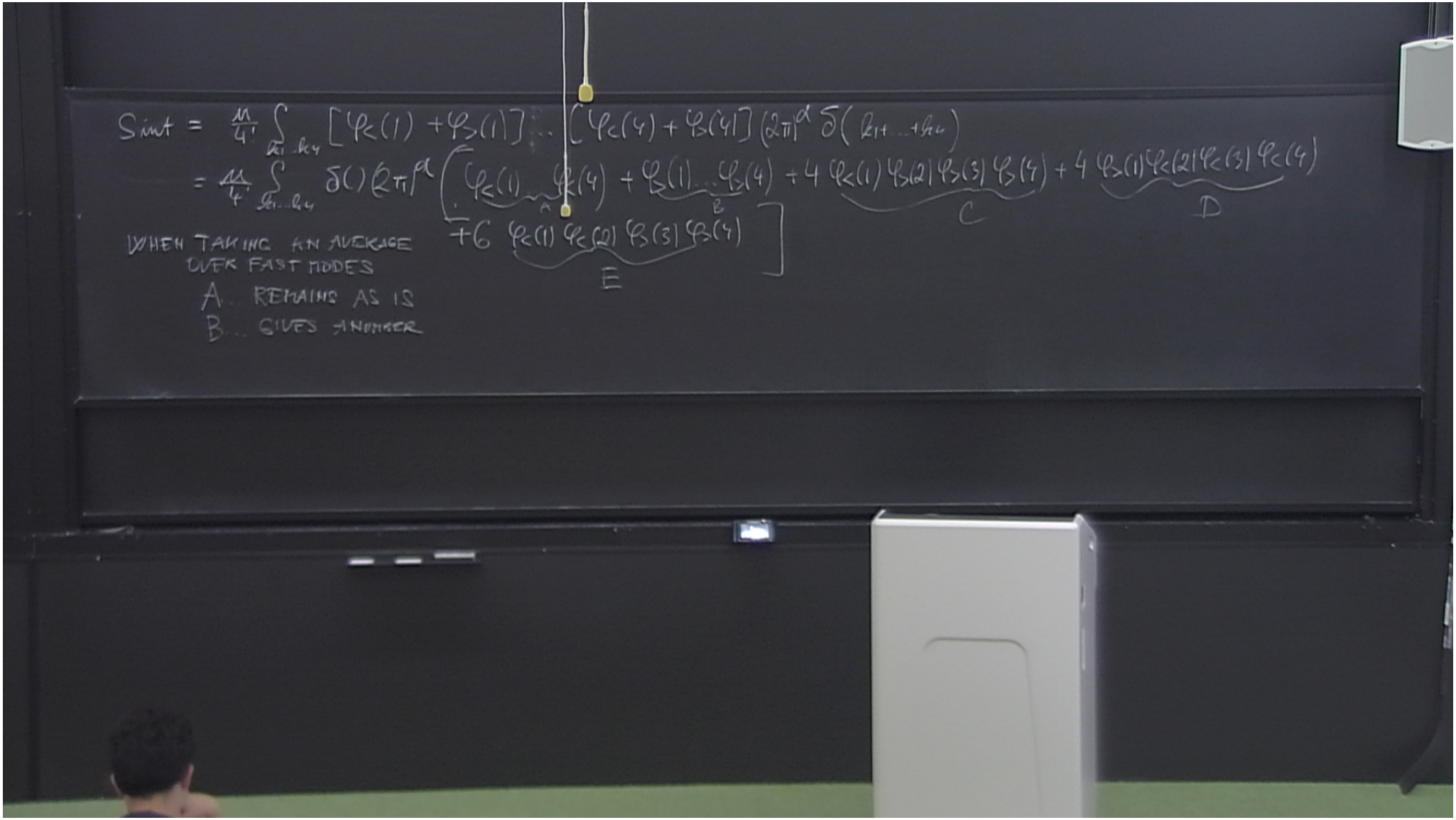
$$\tilde{S}[\varphi_c] = S_0[\varphi_c] + \langle S_{int} \rangle - \frac{1}{2}\left(\langle S_{int}^2 \rangle - \langle S_{int} \rangle^2\right)$$

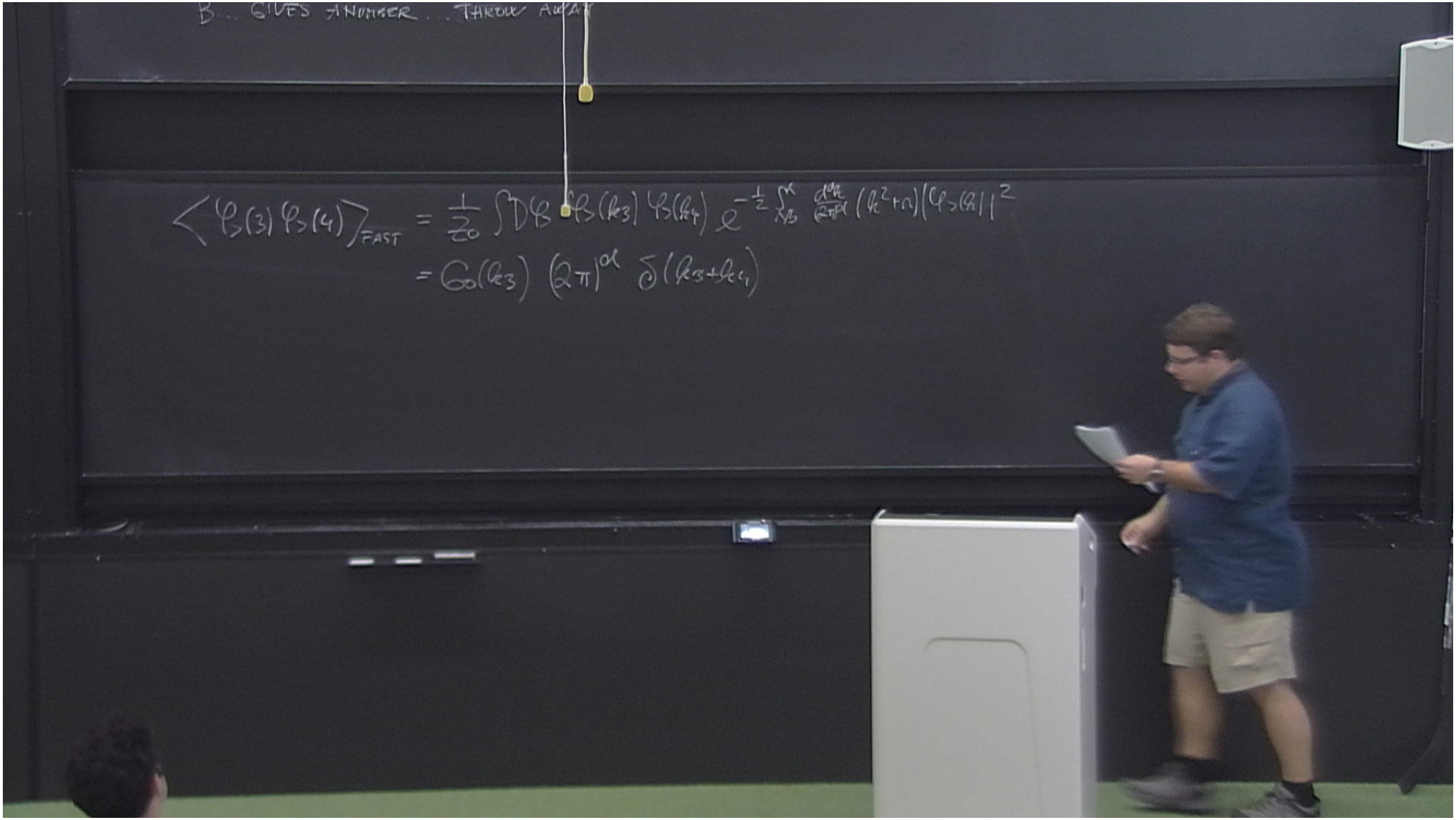
USE CUMULANT EXPANSION.

LET Ω BE A RANDOM VARIABLE

$$\Rightarrow \boxed{\langle e^{\Omega} \rangle = e^{\langle \Omega \rangle + \frac{1}{2}\left(\langle \Omega^2 \rangle - \langle \Omega \rangle^2\right) + \ldots}}$$

$$\boxed{\tilde{S}[\varphi_c] = S_0[\varphi_c] + \langle S_{int} \rangle - \frac{1}{2}\left(\langle S_{int}^2 \rangle - \langle S_{int} \rangle^2\right)}$$


$S_{int} = \frac{\lambda\lambda}{4!} \int_{k_1 \ldots k_4} [\varphi_<(1) + \varphi_>(1)] \ldots [\varphi_<(4) + \varphi_>(4)] (2\pi)^d \delta(k_1 + \ldots + k_4)$
$= \frac{\lambda\lambda}{4!} \int_{k_1 \ldots k_4} \delta(\ldots)(2\pi)^d \Big(\underbrace{\varphi_<(1) \ldots \varphi_<(4)}_{A} + \underbrace{\varphi_>(1) \ldots \varphi_>(4)}_{B} + 4 \underbrace{\varphi_<(1) \varphi_>(2) \varphi_>(3) \varphi_>(4)}_{C} + 4 \underbrace{\varphi_>(1) \varphi_<(2) \varphi_<(3) \varphi_<(4)}_{D}$
$+ 6 \underbrace{\varphi_<(1) \varphi_<(2) \varphi_>(3) \varphi_>(4)}_{E} \Big]$
WHEN TAKING AN AVERAGE OVER FAST MODES
A .. REMAINS AS IS
B .. GIVES A NUMBER

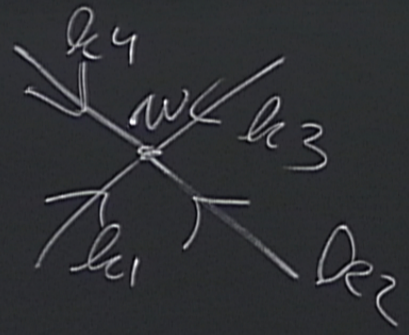

B.. GIVES ANUMBER.. THROW AWAY
$\langle \varphi_>(3)\varphi_>(4)\rangle_{FAST} = \frac{1}{Z_0}\int D\beta\, \varphi_>(k_3)\varphi_>(k_4)\, e^{-\frac{1}{2}\int_{\Lambda/b}^{\Lambda}\frac{d^dk}{(2\pi)^d}(k^2+m)|\varphi_>(k)|^2}$
$= G_0(k_3)\,(2\pi)^d\,\delta(k_3+k_4)$

DIAGRAMATIC NOTATION:

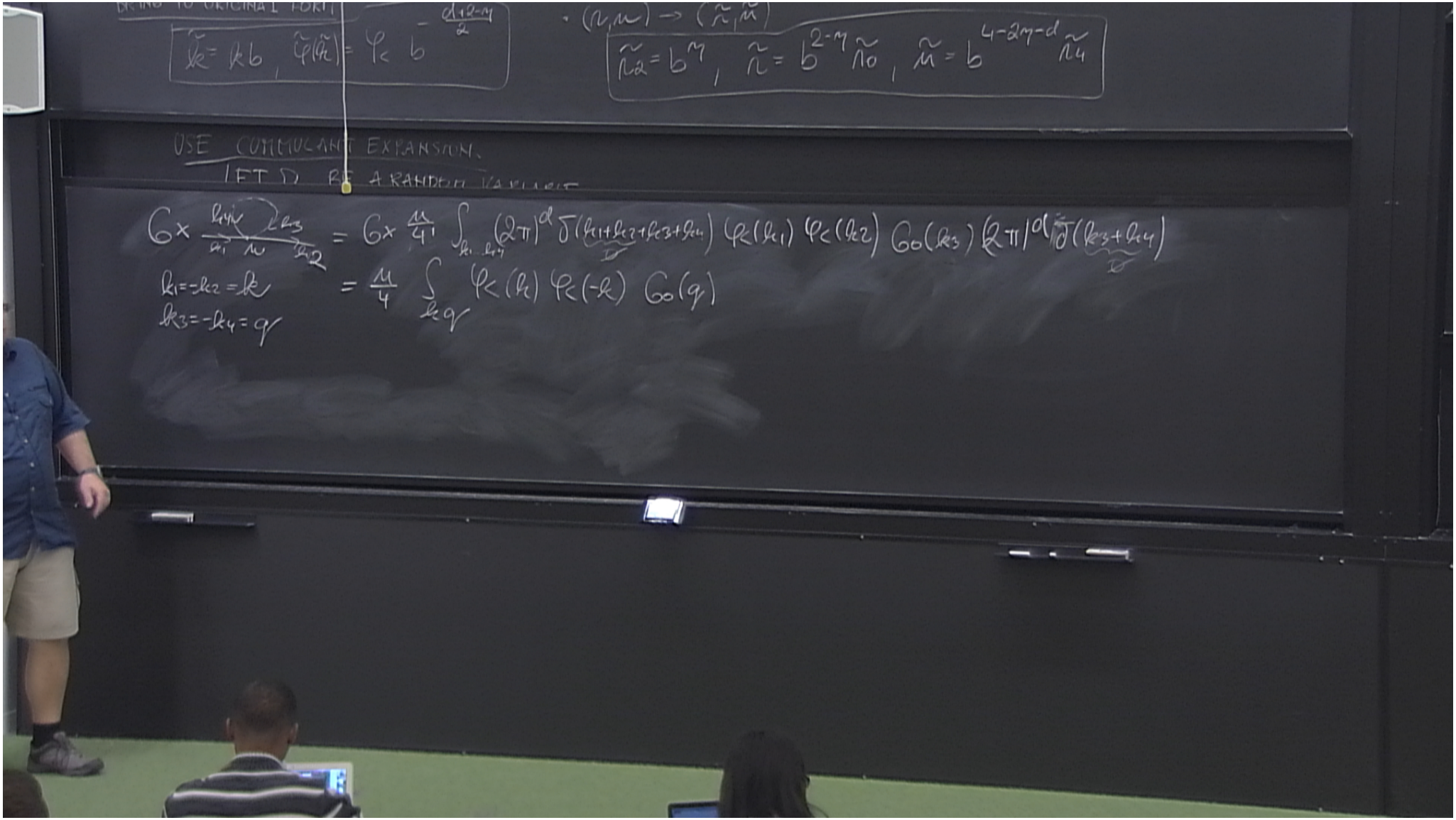
FAST ——— .

SLOW ———

VORTEX



$$= \frac{u}{4!} (2\pi)^d \delta(k_1 + k_2 + q_3 + q_4)$$


$\tilde{k} = kb$, $\tilde{\varphi}(\tilde{k}) = \varphi_< b^{-\frac{d+2-\eta}{2}}$
$(r, u) \to (\tilde{r}, \tilde{u})$
$\tilde{r}_2 = b^{\eta}$, $\tilde{r} = b^{2-\eta} r_0$, $\tilde{u} = b^{4-2\eta-d} \tilde{u}_4$
USE CUMULANT EXPANSION.
$G \times$ (diagram: k_1, k_3, $r_0 u$, k_2) $= G \times \frac{u}{4!} \int_{k_1 \ldots k_4} (2\pi)^d \delta(k_1+k_2+k_3+k_4) \varphi_<(k_1) \varphi_<(k_2) G_0(k_3) (2\pi)^d \delta(k_3+k_4)$
$k_1 = -k_2 = k$
$k_3 = -k_4 = q$
$= \frac{u}{4} \int_{kq} \varphi_<(k) \varphi_<(-k) G_0(q)$

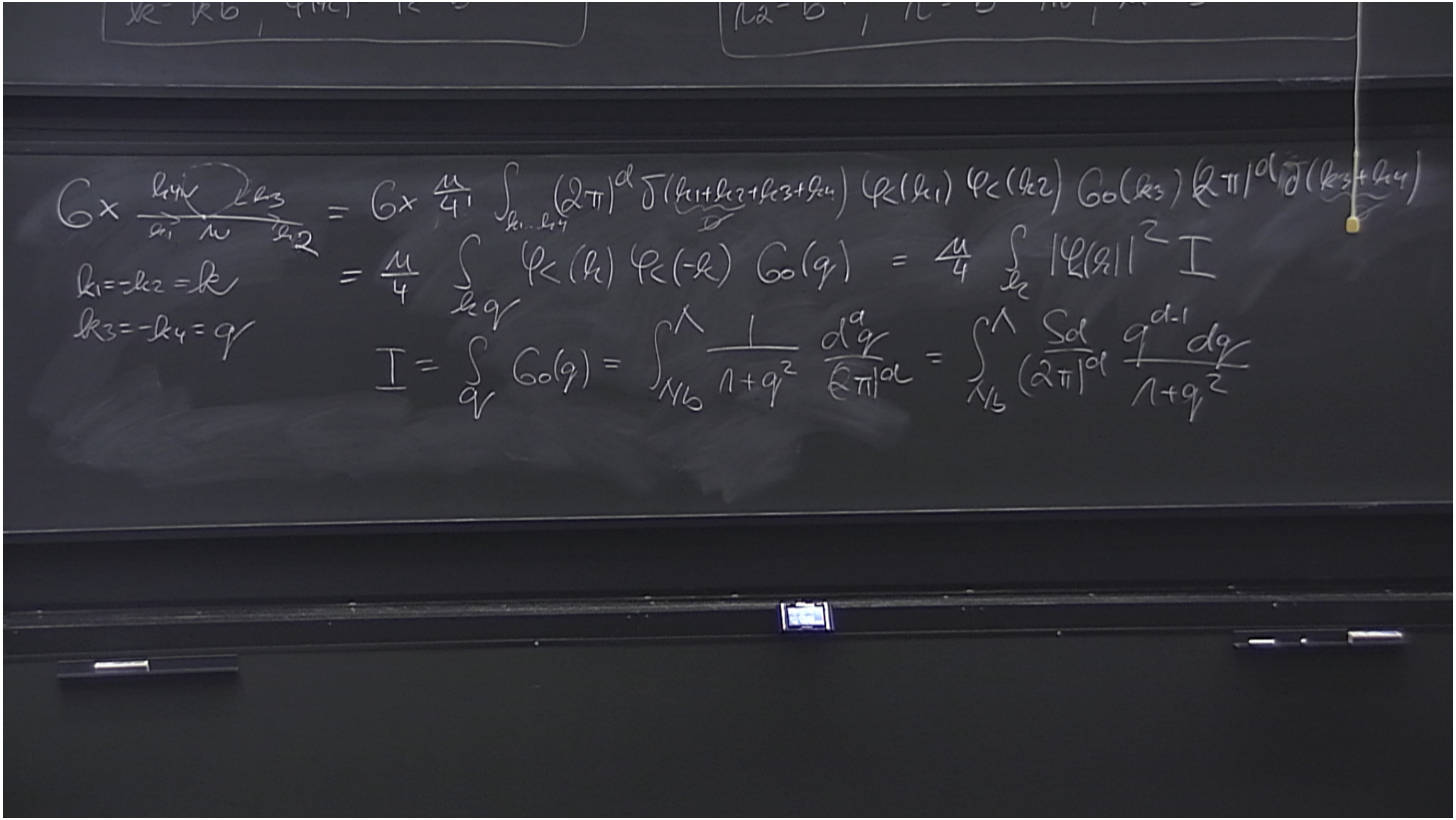
$$G \times \quad = G \times \frac{u}{4!} \int_{k_1 \ldots k_4} (2\pi)^d \delta(k_1+k_2+k_3+k_4)\, \varphi_<(k_1)\, \varphi_<(k_2)\, G_0(k_3)\, (2\pi)^d \delta(k_3+k_4)$$

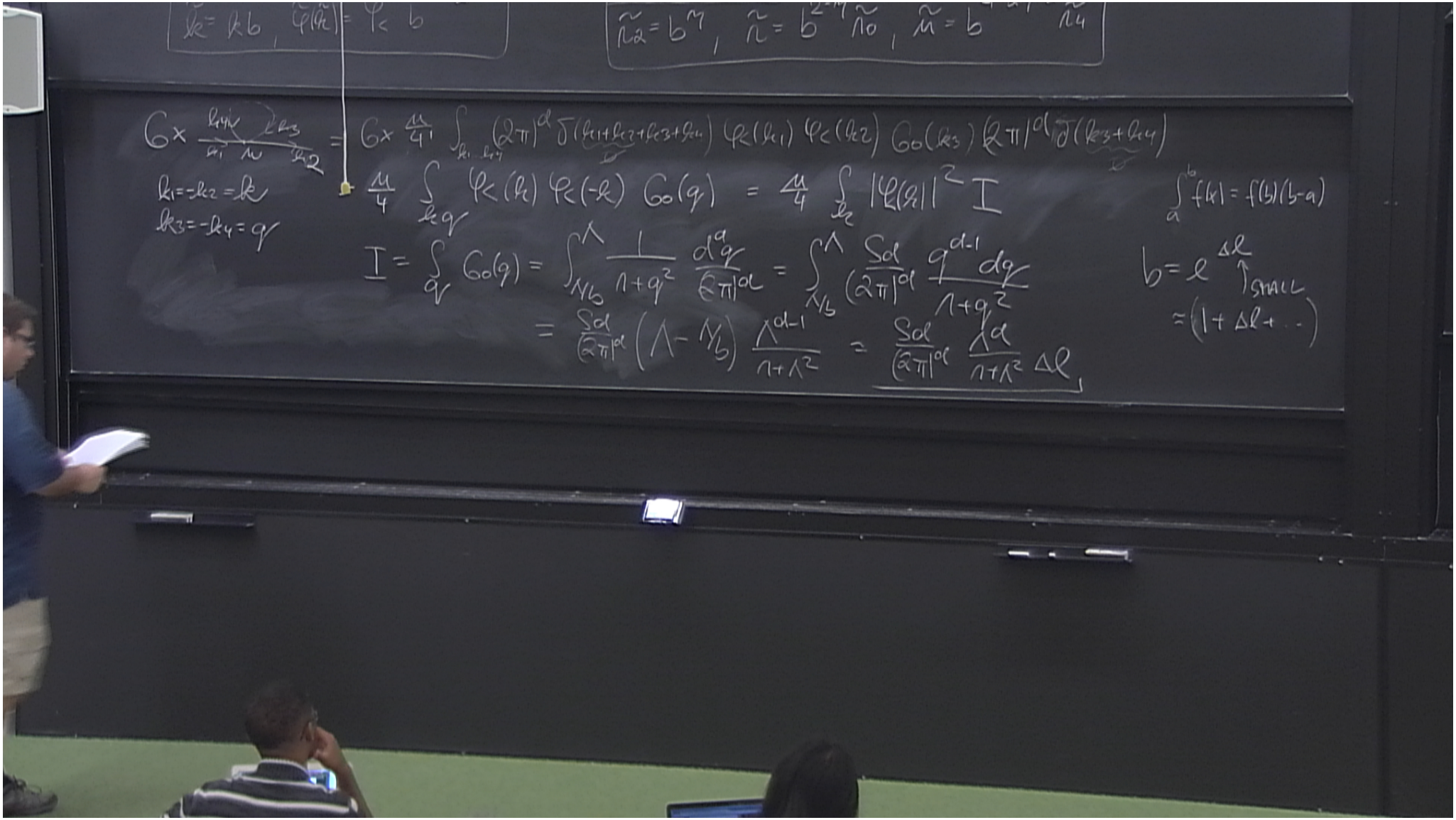
$$= \frac{u}{4} \int_{kq} \varphi_<(k)\, \varphi_<(-k)\, G_0(q) = \frac{u}{4} \int_k |\varphi_<(k)|^2 I$$

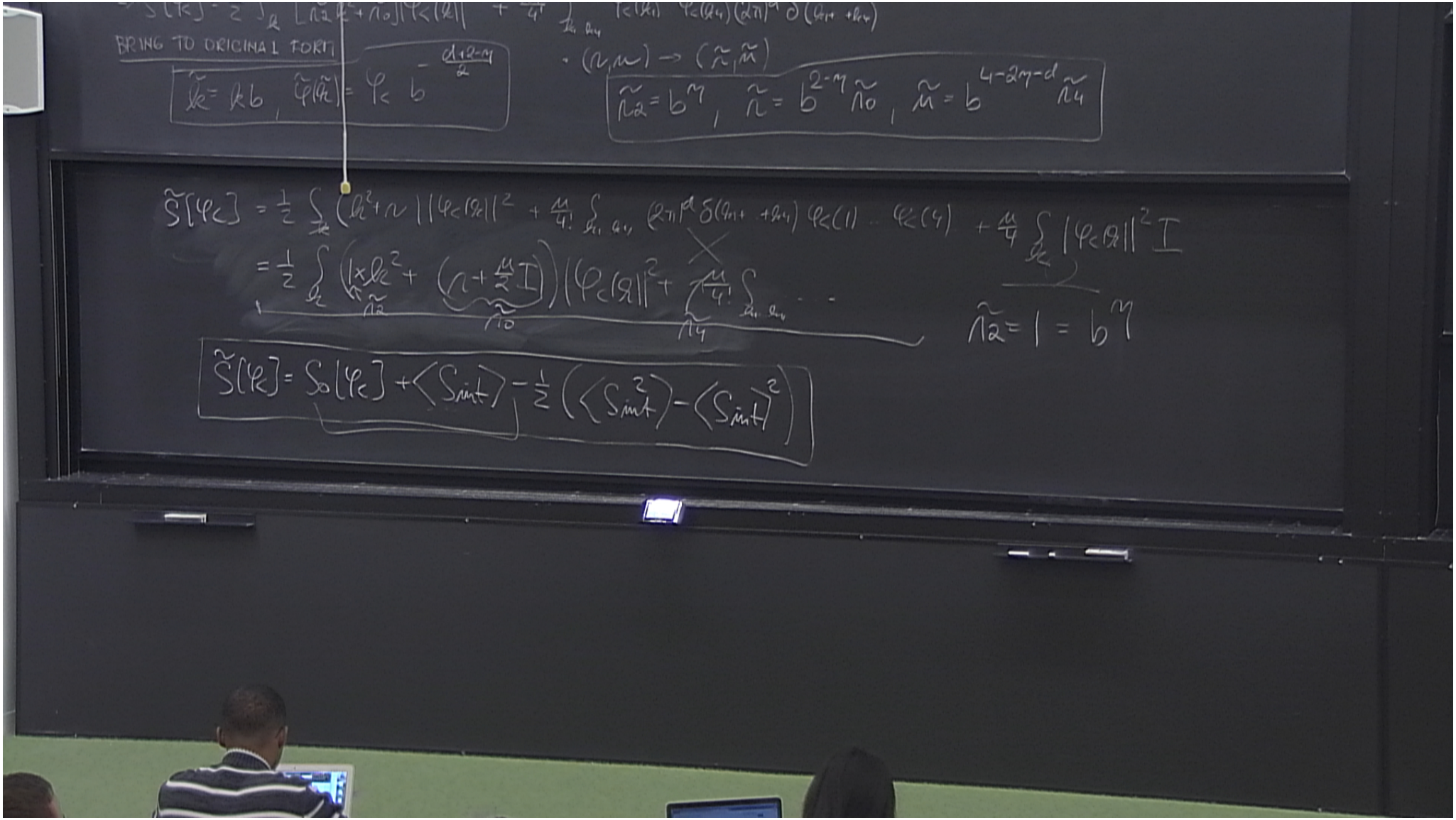
$k_1 = -k_2 = k$

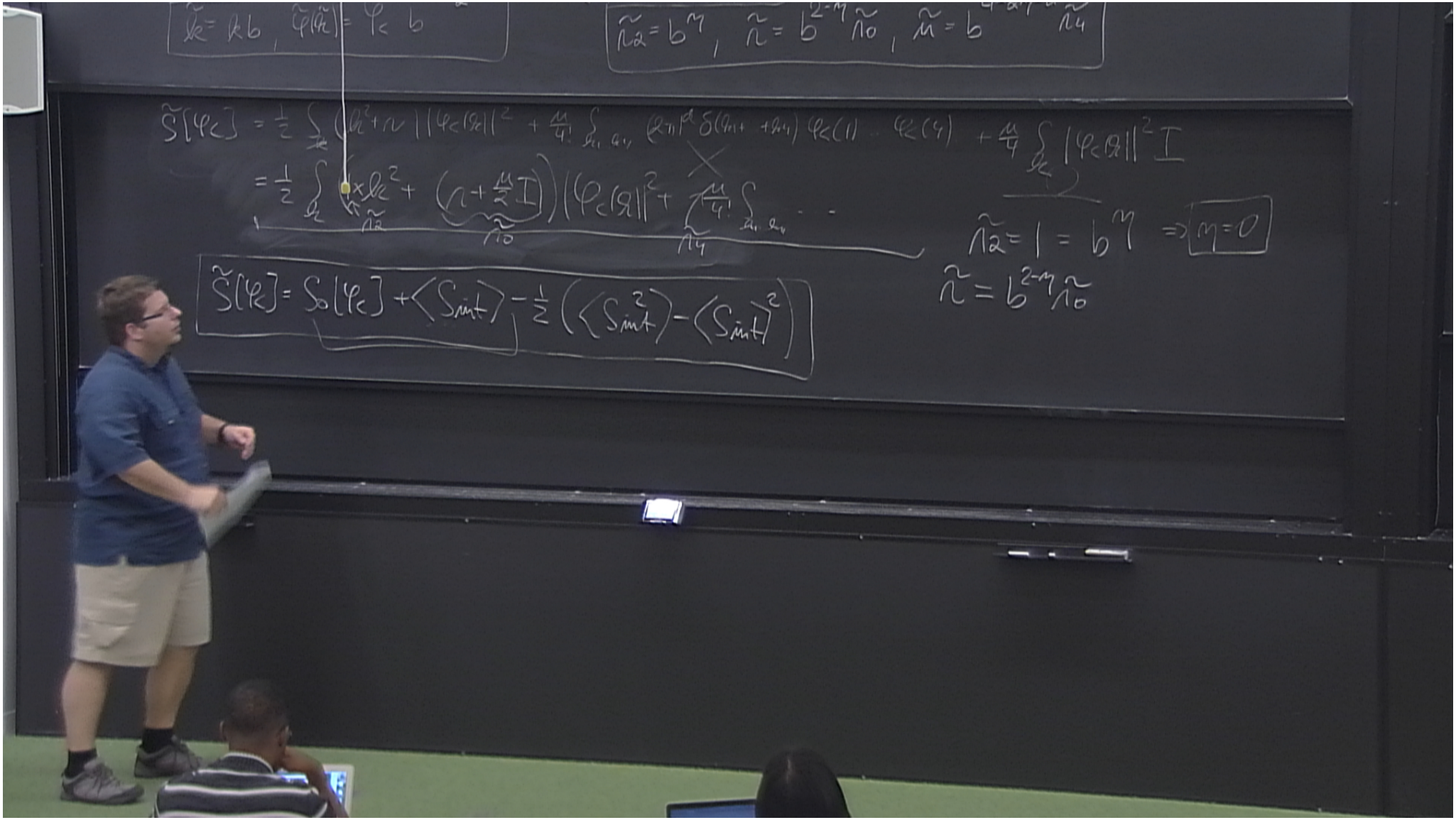
$k_3 = -k_4 = q$

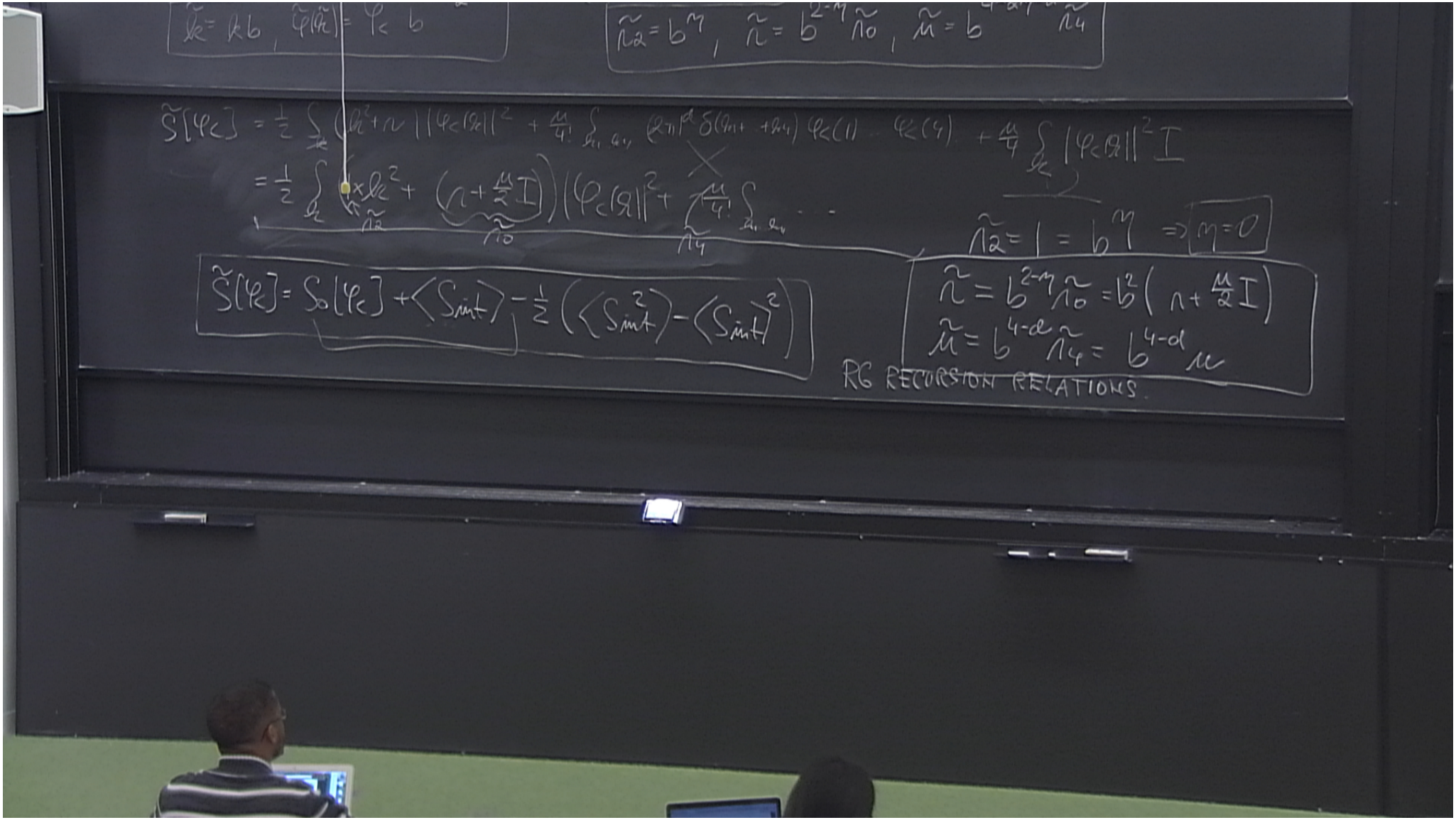
$$I = \int_q G_0(q) = \int_{\Lambda/b}^{\Lambda} \frac{1}{1+q^2} \frac{d^d q}{(2\pi)^d} = \int_{\Lambda/b}^{\Lambda}$$

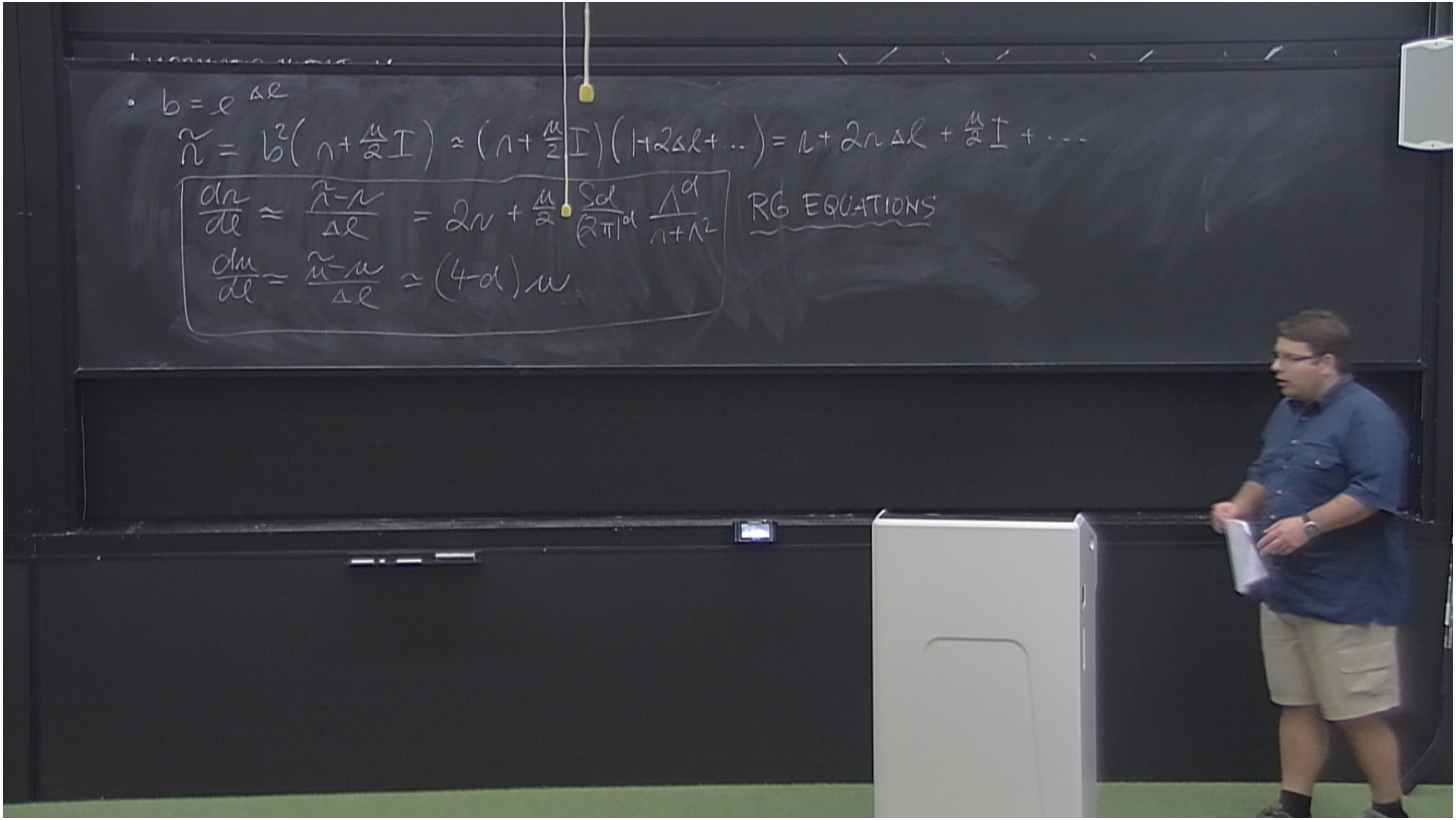

$G \times$ (one-loop diagram: k_4, k_3, k_1, Λ, k_2) $= G \times \frac{\mu}{4!} \int_{k_1 \dots k_4} (2\pi)^d \delta(k_1+k_2+k_3+k_4)\, \varphi_<(k_1)\, \varphi_<(k_2)\, G_0(k_3)\, (2\pi)^d \delta(k_3+k_4)$
$k_1 = -k_2 = k$
$k_3 = -k_4 = q$
$= \frac{\mu}{4} \int_{kq} \varphi_<(k)\, \varphi_<(-k)\, G_0(q) = \frac{\mu}{4} \int_k |\varphi_<(k)|^2\, I$
$I = \int_q G_0(q) = \int_{\Lambda/b}^{\Lambda} \frac{1}{1+q^2} \frac{d^d q}{(2\pi)^d} = \int_{\Lambda/b}^{\Lambda} \frac{S_d}{(2\pi)^d} \frac{q^{d-1}\, dq}{1+q^2}$

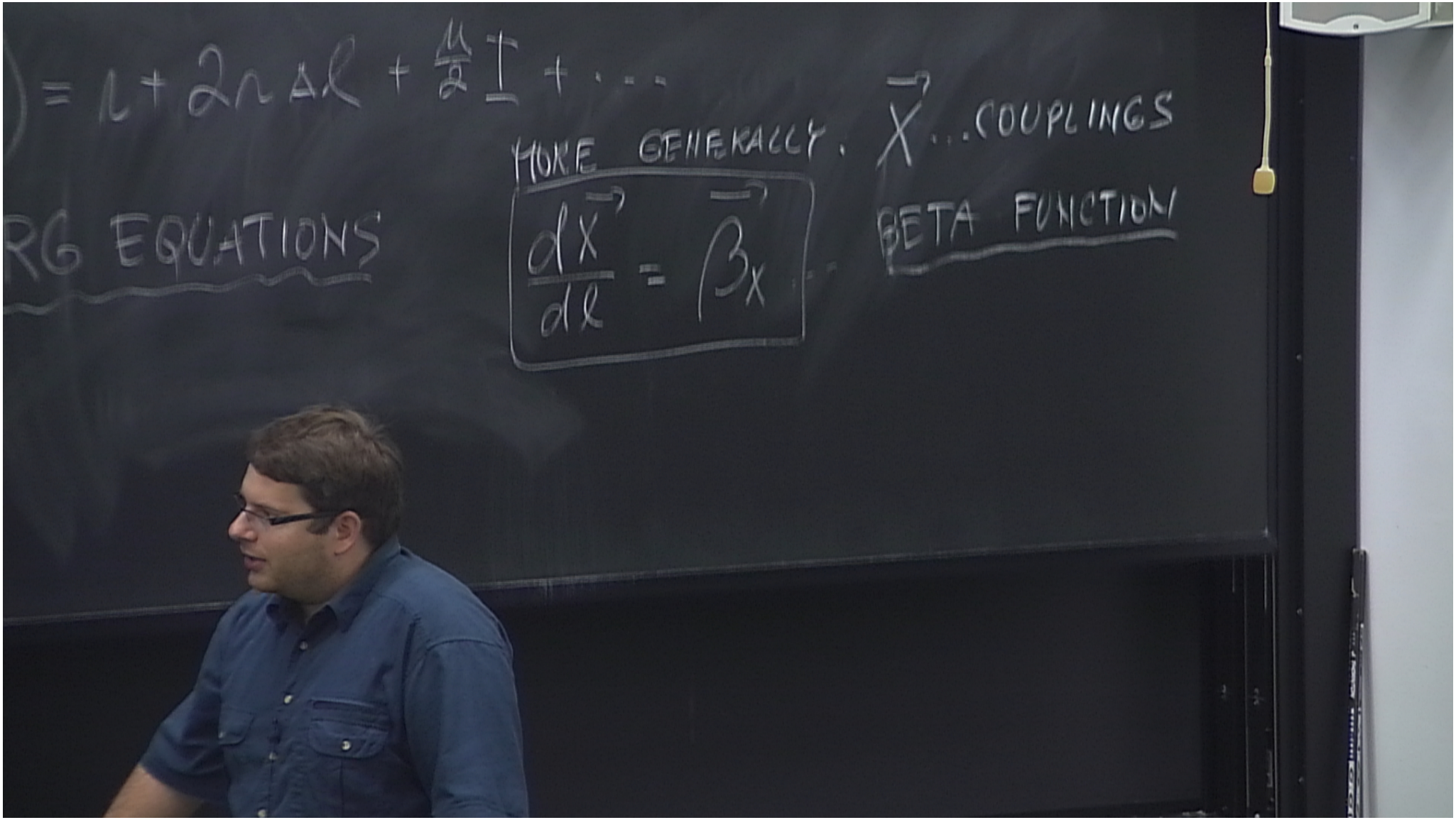

$k = k b$, $\varphi(k) = \varphi_< b$
$\tilde{r}_2 = b^{r}$, $\tilde{r} = b^{z-r} \tilde{r}_0$, $\tilde{u} = b^{r} u_4$
$k_1 = -k_2 = k$
$k_3 = -k_4 = q$
$\int_a^b f(x) = f(b)(b-a)$
$b = e^{\Delta\ell}$ SMALL
$\approx (1 + \Delta\ell + \dots)$

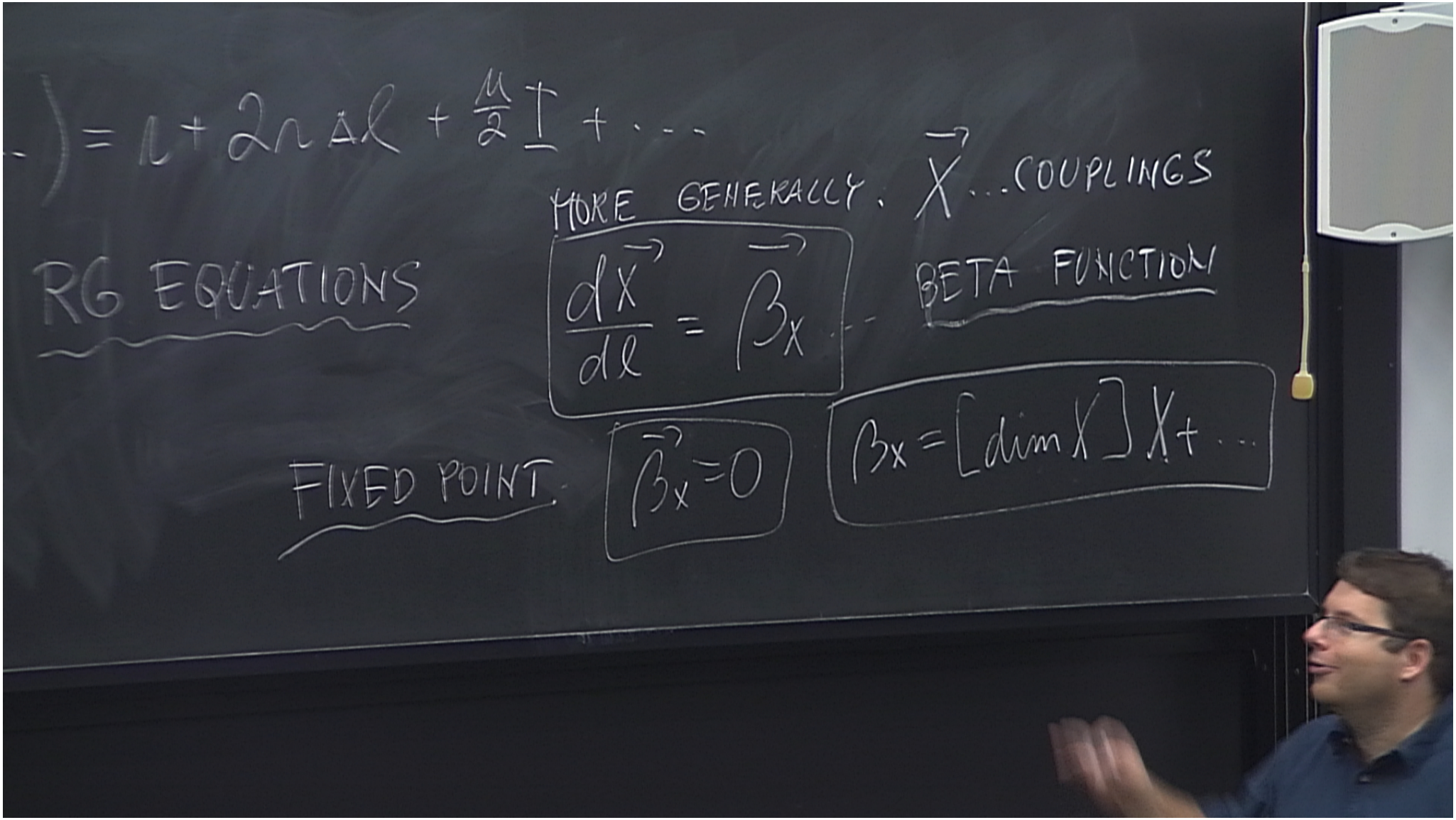

BRING TO ORIGINAL FORM
$\tilde{k} = kb, \quad \tilde{\varphi}(\tilde{k}) = \varphi_< b^{-\frac{d+2-\eta}{2}}$
$(r,u) \to (\tilde{r},\tilde{u})$
$\tilde{r}_2 = b^2, \quad \tilde{r} = b^{2-\eta}\tilde{r}_0, \quad \tilde{u} = b^{4-2\eta-d}\tilde{r}_4$
$\tilde{S}[\varphi_<] = \frac{1}{2}\int_k (k^2 + r)|\varphi_<(k)|^2 + \frac{u}{4!}\int_{q_1 \dots q_4} (2\pi)^d \delta(q_{1\dots4}) \varphi_<(1)\dots\varphi_<(4) + \frac{u}{4}\int_k |\varphi_<(k)|^2 I$
$= \frac{1}{2}\int_k \left(k^2 + \left(r + \frac{u}{2}I\right)\right)|\varphi_<(k)|^2 + \frac{u}{4!}\int_{q_1 \dots q_4} \dots$
$\tilde{r}_2 = 1 = b^\eta$
$\tilde{S}[\varphi_<] = S_0[\varphi_<] + \langle S_{int} \rangle - \frac{1}{2}\left(\langle S_{int}^2 \rangle - \langle S_{int} \rangle^2\right)$

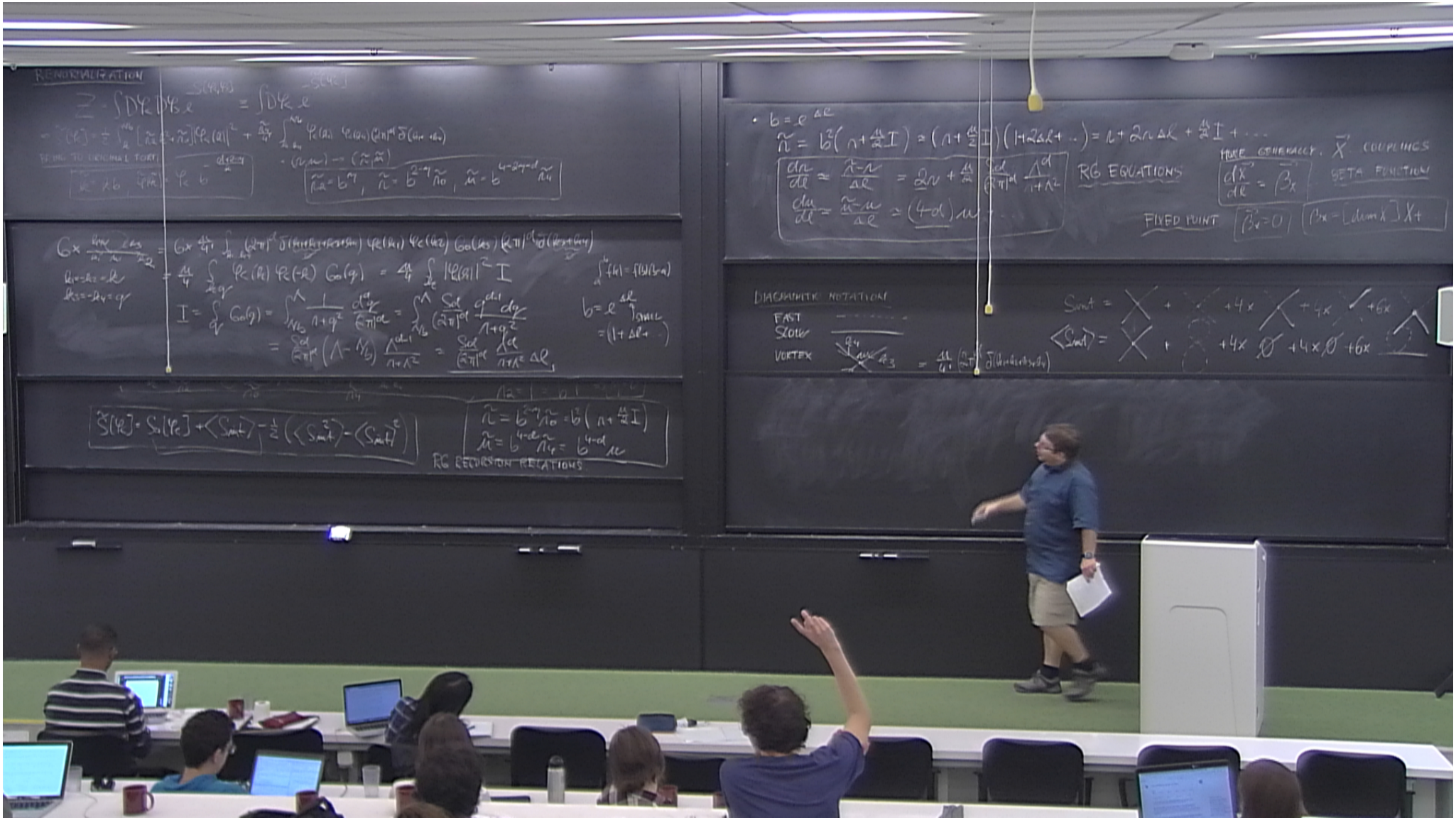

$\tilde{k} = kb, \quad \tilde{\varphi}(\tilde{k}) = \varphi_< b$
$\tilde{\Lambda}_2 = b^m, \quad \tilde{\Lambda} = b^{2-m}\tilde{\Lambda}_0, \quad \tilde{\mu} = b^{4-m}\tilde{\Lambda}_4$
$\tilde{S}[\varphi_<] = \frac{1}{2}\int_{k_<} (k^2 + \Lambda)\|\varphi_<(k)\|^2 + \frac{\mu}{4!}\int \delta(q_1+\dots+q_4)\,\varphi_<(1)\cdots\varphi_<(4) + \frac{\mu}{4}\int_{k_<}\|\varphi_<(k)\|^2 I$
$= \frac{1}{2}\int_{k_<} k^2 + \left(\Lambda + \frac{\mu}{2}I\right)\|\varphi_<(k)\|^2 + \frac{\mu}{4!}\int \cdots$
$\tilde{\Lambda}_2 \quad \tilde{\Lambda}_0 \quad \tilde{\Lambda}_4$
$\tilde{S}[\varphi_<] = S_0[\varphi_<] + \langle S_{int}\rangle - \frac{1}{2}\left(\langle S_{int}^2\rangle - \langle S_{int}\rangle^2\right)$
$\tilde{\Lambda}_2 = 1 = b^m \Rightarrow m = 0$
$\tilde{\Lambda} = b^{2-m}\tilde{\Lambda}_0$

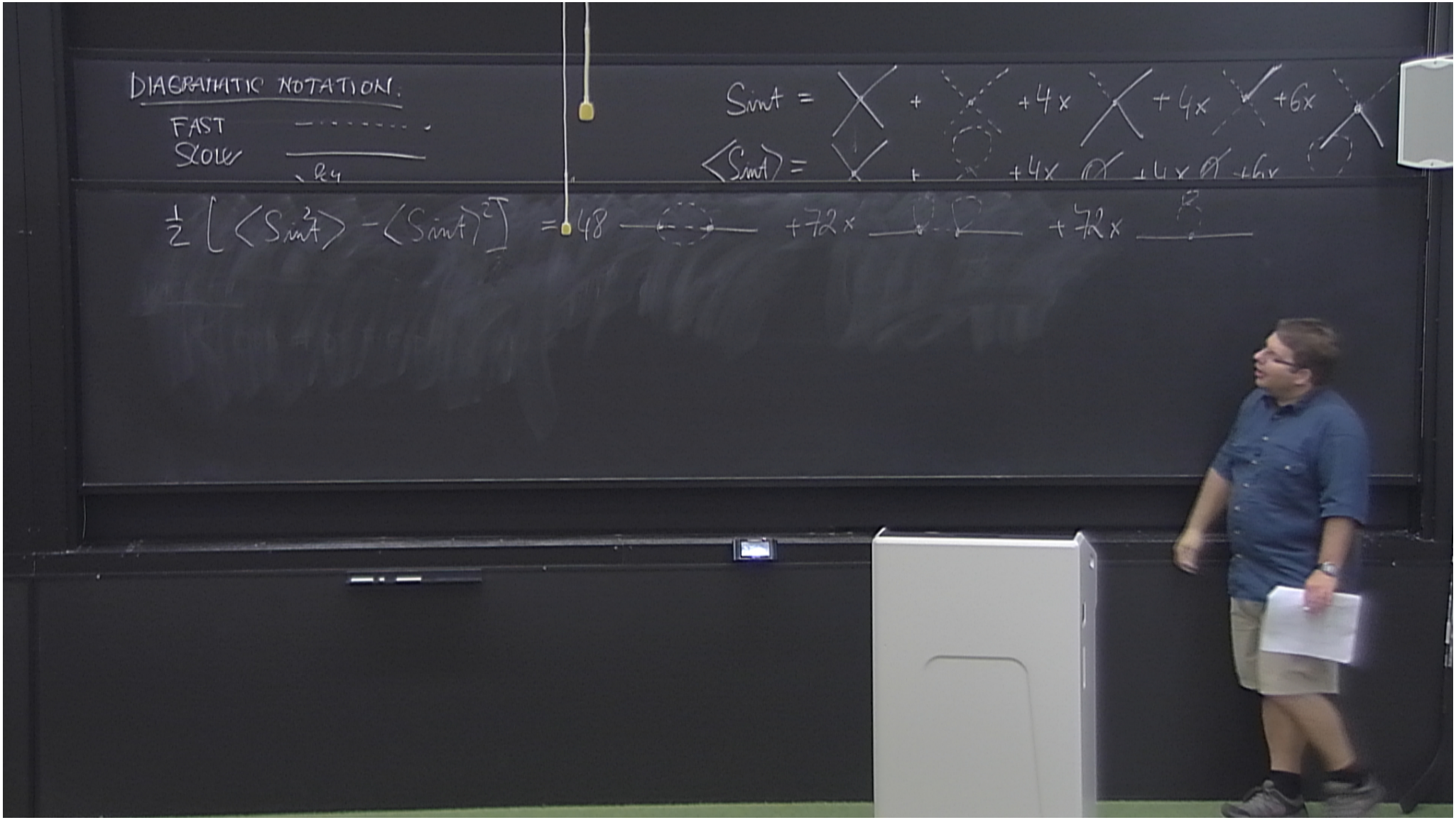

RG RECURSION RELATIONS

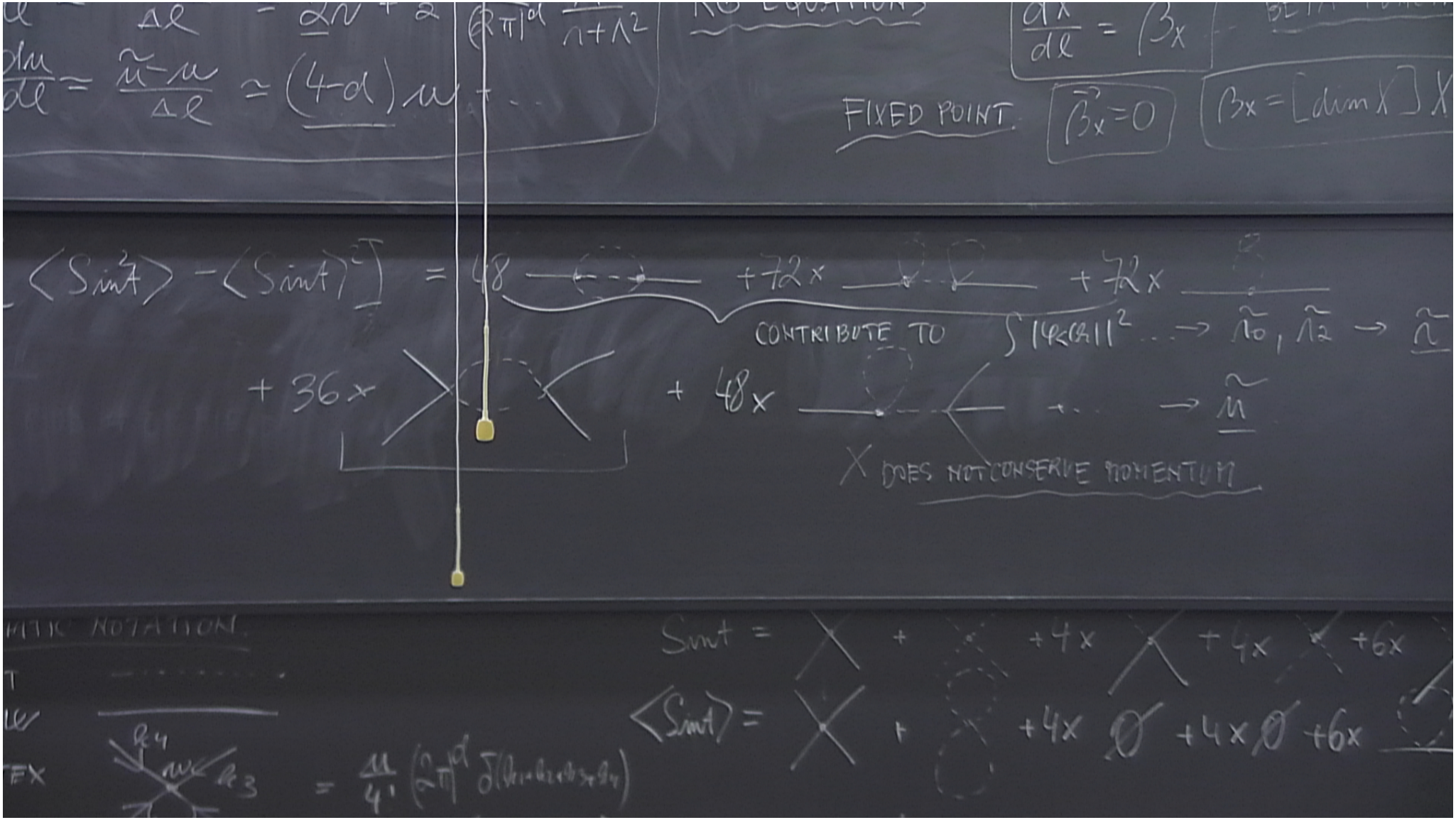

• $b = e^{\Delta \ell}$
$\tilde{r} = b^2\left(r + \frac{u}{2} I\right) \approx \left(r + \frac{u}{2} I\right)\left(1 + 2\Delta\ell + \ldots\right) = r + 2r\Delta\ell + \frac{u}{2} I + \cdots$
$\frac{dr}{d\ell} \approx \frac{\tilde{r} - r}{\Delta\ell} = 2r + \frac{u}{2} \frac{S_d}{(2\pi)^d} \frac{\Lambda^d}{1 + \Lambda^2}$
RG EQUATIONS
$\frac{du}{d\ell} = \frac{\tilde{u} - u}{\Delta\ell} \simeq (4 - d)u$

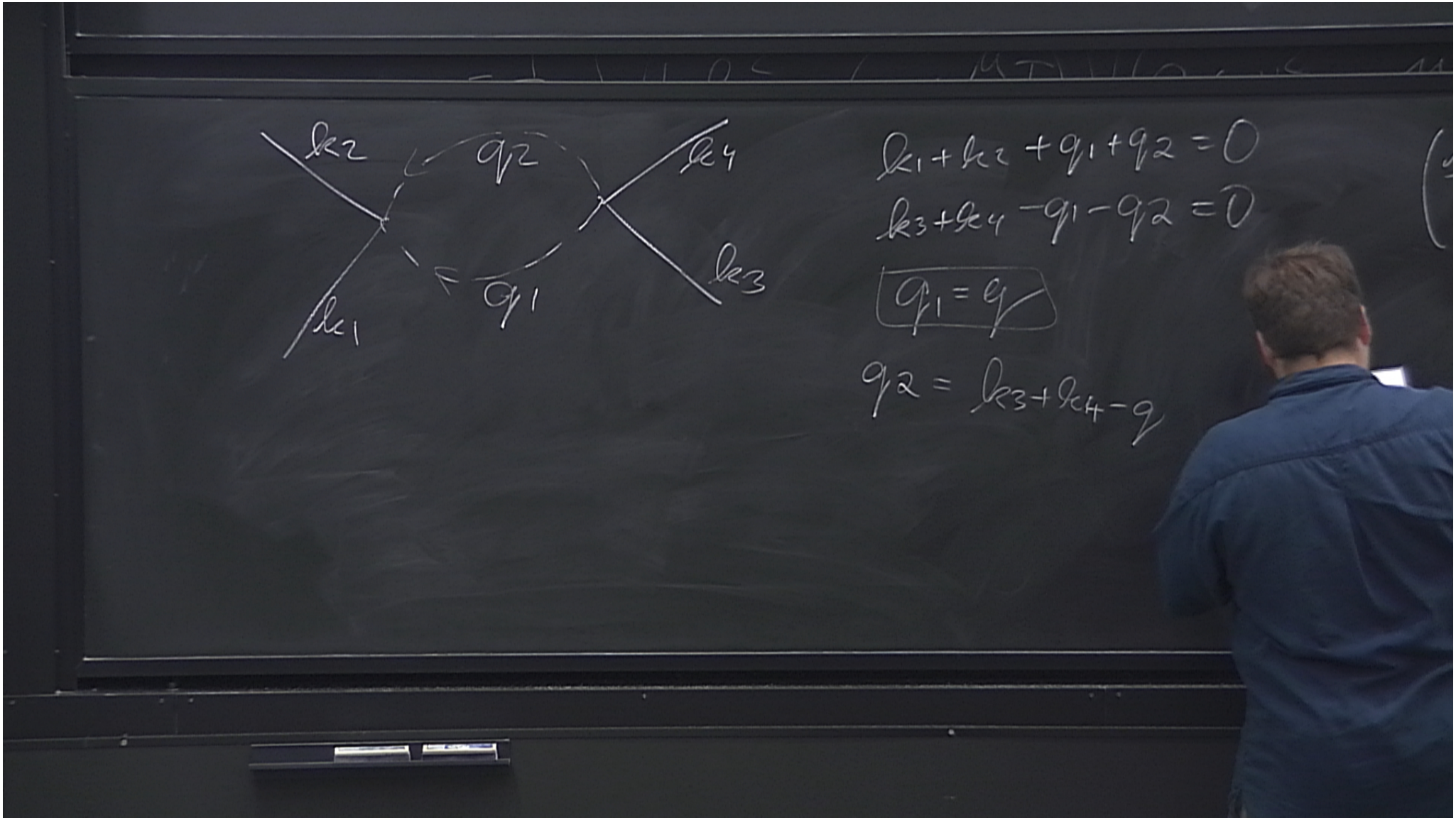

RG EQUATIONS
MORE GENERALLY, $\vec{X}$...COUPLINGS
$\frac{d\vec{X}}{dl} = \vec{\beta}_X$
BETA FUNCTION


RG EQUATIONS
MORE GENERALLY. $\vec{X}$ ... COUPLINGS
$\frac{d\vec{X}}{d\ell} = \vec{\beta}_X$
BETA FUNCTION
FIXED POINT
$\vec{\beta}_X = 0$
$\beta_X = [\dim X]\, X + \ldots$


RENORMALIZATION
RG EQUATIONS
FIXED POINT
RG RECURSION RELATIONS
DIAGRAMMATIC NOTATION
FAST
SLOW
VORTEX


DIAGRAMATIC NOTATION:
FAST
SLOW
S_{int} =
+4x
+4x
+6x
$\langle S_{int} \rangle$ =
+4x
+4x
+6x
$\frac{1}{2}\left[\langle S_{int}^2 \rangle - \langle S_{int} \rangle^2\right]$ = 48
+72x
+72x


FIXED POINT
CONTRIBUTE TO
DOES NOT CONSERVE MOMENTUM

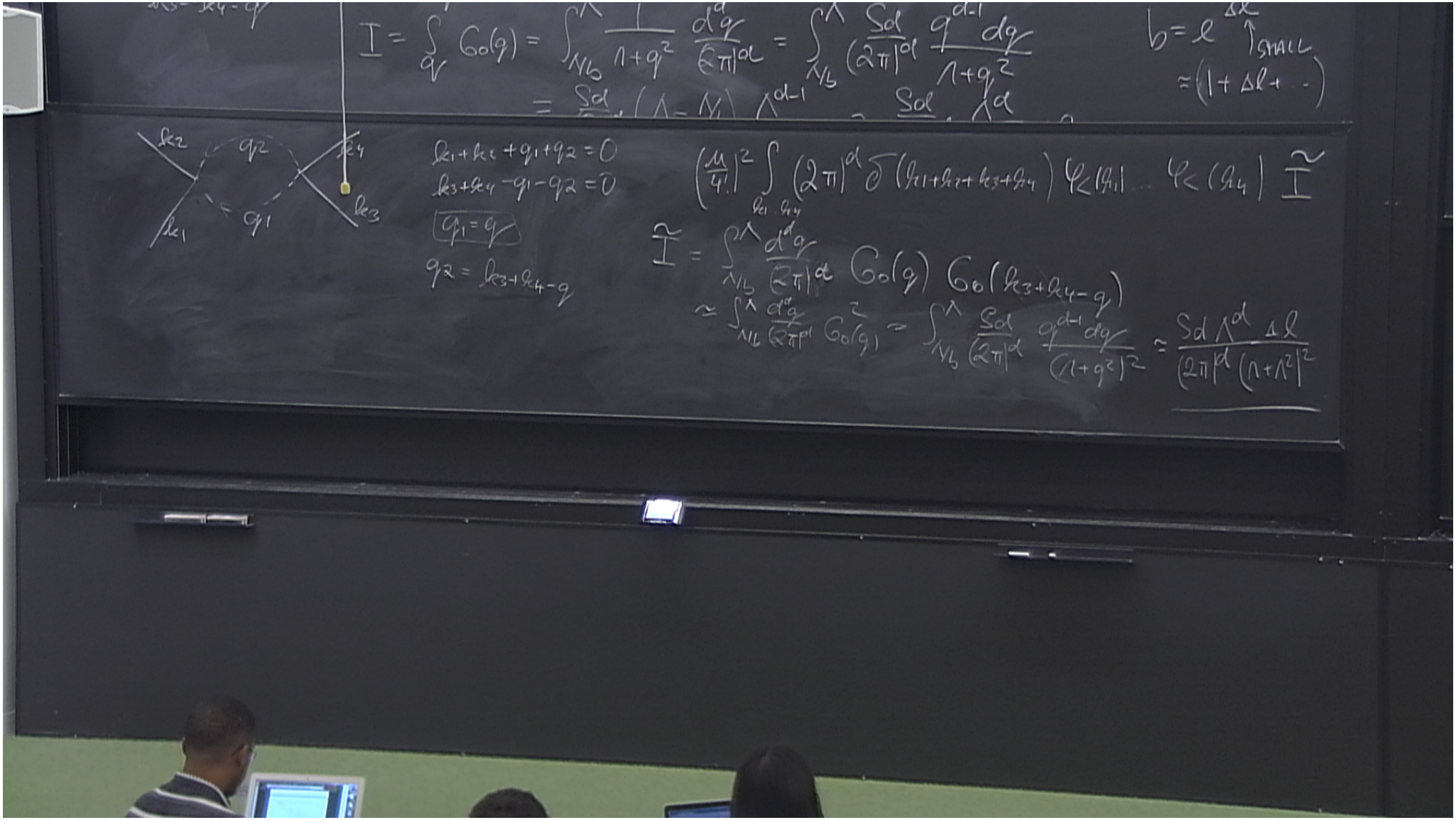

k_2
q_2
k_4
k_3
q_1
k_1
$k_1 + k_2 + q_1 + q_2 = 0$
$k_3 + k_4 - q_1 - q_2 = 0$
$q_1 = q$
$q_2 = k_3 + k_4 - q$

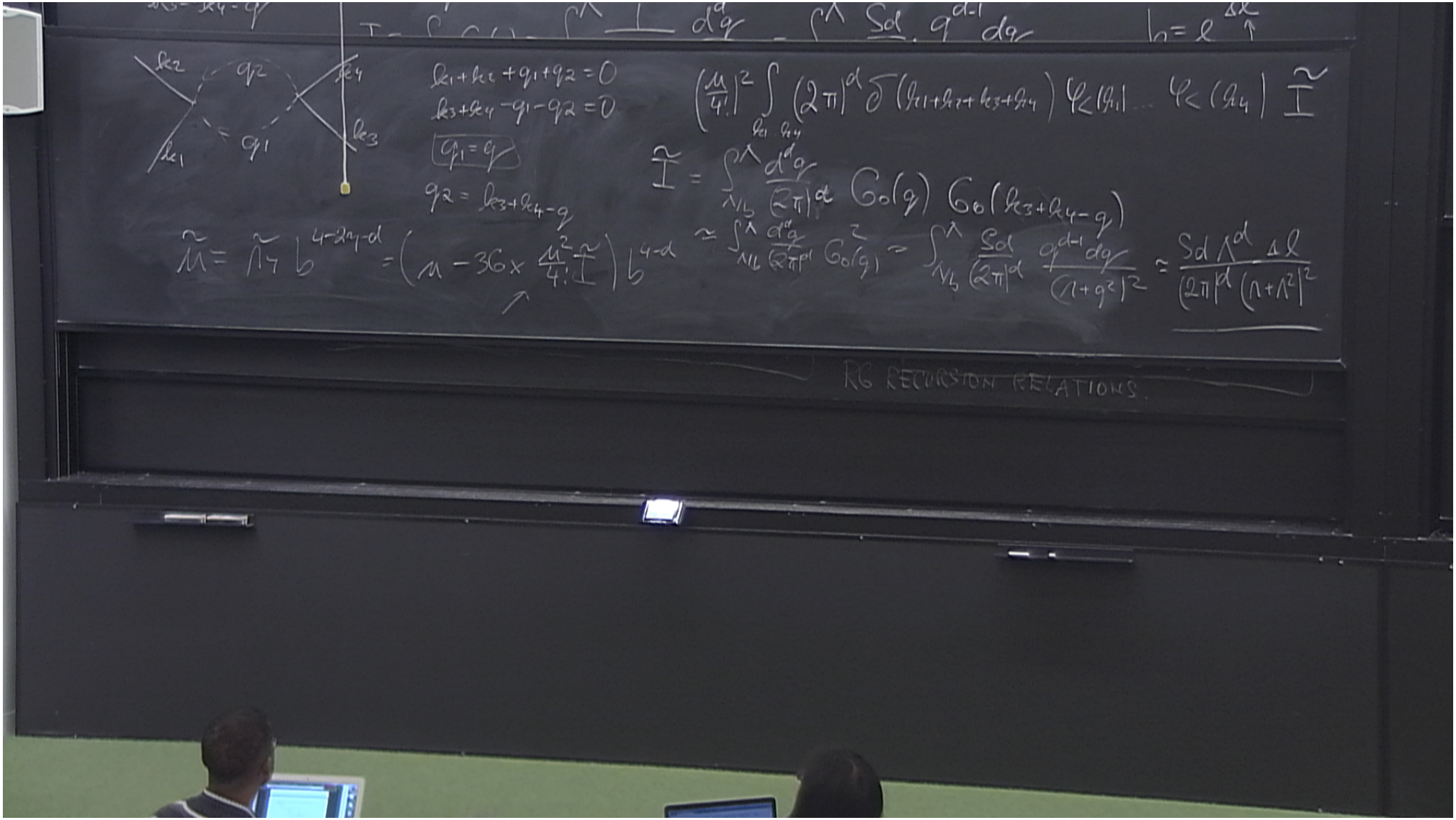
$$\left(\frac{u}{4!}\right)^2 \int_{q_{z1} \dots q_{z4}} (2\pi)^d \delta(q_{z1}+q_{z2}+k_3+q_{z4})\, \varphi_<(q_{z1}) \dots \varphi_<(q_{z4})\, \tilde{I}$$

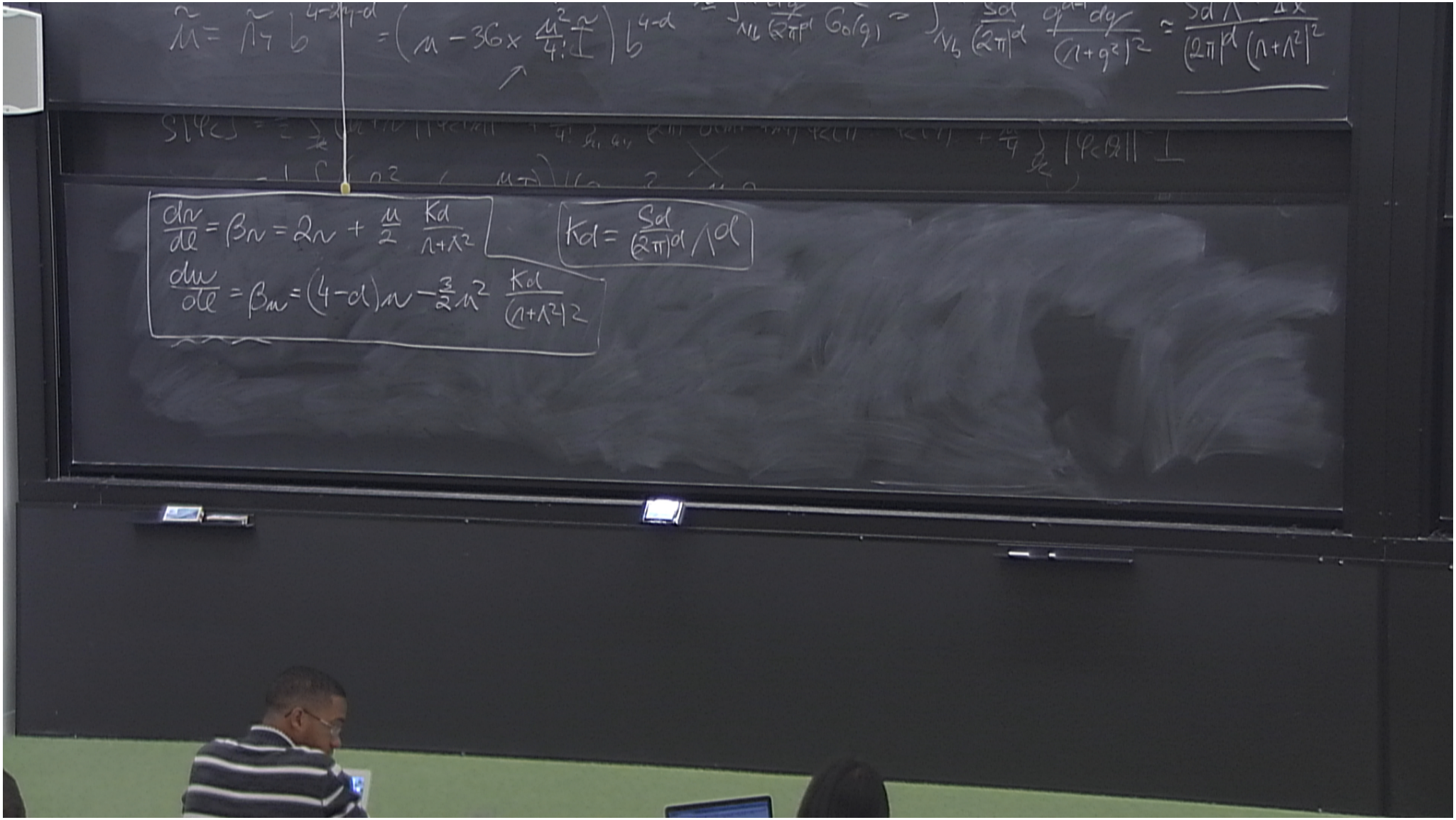
$$\tilde{I} = \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} G_0(q)$$

$$\left(\frac{u}{4!}\right)^2 \int_{k_1 \dots k_4} (2\pi)^d \delta(k_1+k_2+k_3+k_4)\, \varphi_<(k_1) \dots \varphi_<(k_4)\, \tilde{\tilde{I}}$$

$$\tilde{\tilde{I}} = \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d}\, G_0(q)\, G_0(k_3+k_4-q)$$


$I = \int_q G_0(q) = \int_{\Lambda/b}^{\Lambda} \frac{1}{1+q^2} \frac{d^d q}{(2\pi)^d} = \int_{\Lambda/b}^{\Lambda} \frac{S_d}{(2\pi)^d} \frac{q^{d-1} dq}{1+q^2}$
$= \frac{S_d}{d} (\Lambda - \Lambda/b) \Lambda^{d-1} \sim \frac{S_d}{d} \Lambda^d$
$b = e^{\Delta \ell}$ ↑ SMALL
$\approx (1 + \Delta \ell + \dots)$
k_2 q_2 k_4 q_1 k_1 k_3
$k_1 + k_2 + q_1 + q_2 = 0$
$k_3 + k_4 - q_1 - q_2 = 0$
$q_1 = q$
$q_2 = k_3 + k_4 - q$
$\left(\frac{u}{4!}\right)^2 \int_{k_1 \dots k_4} (2\pi)^d \delta(k_1 + k_2 + k_3 + k_4) \varphi(k_1) \dots \varphi(k_4) \tilde{I}$
$\tilde{I} = \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} G_0(q) G_0(k_3 + k_4 - q)$
$\approx \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} G_0(q)^2 = \int_{\Lambda/b}^{\Lambda} \frac{S_d}{(2\pi)^d} \frac{q^{d-1} dq}{(1+q^2)^2} \approx \frac{S_d \Lambda^d}{(2\pi)^d} \frac{\Delta \ell}{(1+\Lambda^2)^2}$


k_2
q_2
k_4
k_1
q_1
k_3
$k_{1}+k_{2}+q_{1}+q_{2}=0$
$k_{3}+k_{4}-q_{1}-q_{2}=0$
$q_1=q$
$q_2=k_3+k_4-q$
$\left(\frac{u}{4!}\right)^2 \int_{k_1 \dots k_4} (2\pi)^d \delta(k_1+k_2+k_3+k_4)\, \varphi_<(k_1) \dots \varphi_<(k_4)\, \tilde{I}$
$\tilde{I} = \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} G_0(q)\, G_0(k_3+k_4-q)$
$\simeq \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} G_0(q)^2 = \int_{\Lambda/b}^{\Lambda} \frac{S_d}{(2\pi)^d} \frac{q^{d-1} dq}{(1+q^2)^2} \simeq \frac{S_d \Lambda^d \Delta\ell}{(2\pi)^d (1+\Lambda^2)^2}$
$\tilde{u} = \tilde{\Lambda} b^{4-2\eta-d} = \left(u - 36 \times \frac{u^2}{4!} \tilde{I}\right) b^{4-d}$
RG RECURSION RELATIONS


$\frac{dv}{dl} = \beta_v = 2v + \frac{u}{2}\frac{K_d}{1+\lambda^2}$
$K_d = \frac{S_d}{(2\pi)^d}\Lambda^d$
$\frac{du}{dl} = \beta_u = (4-d)u - \frac{3}{2}u^2\frac{K_d}{(1+\lambda^2)^2}$