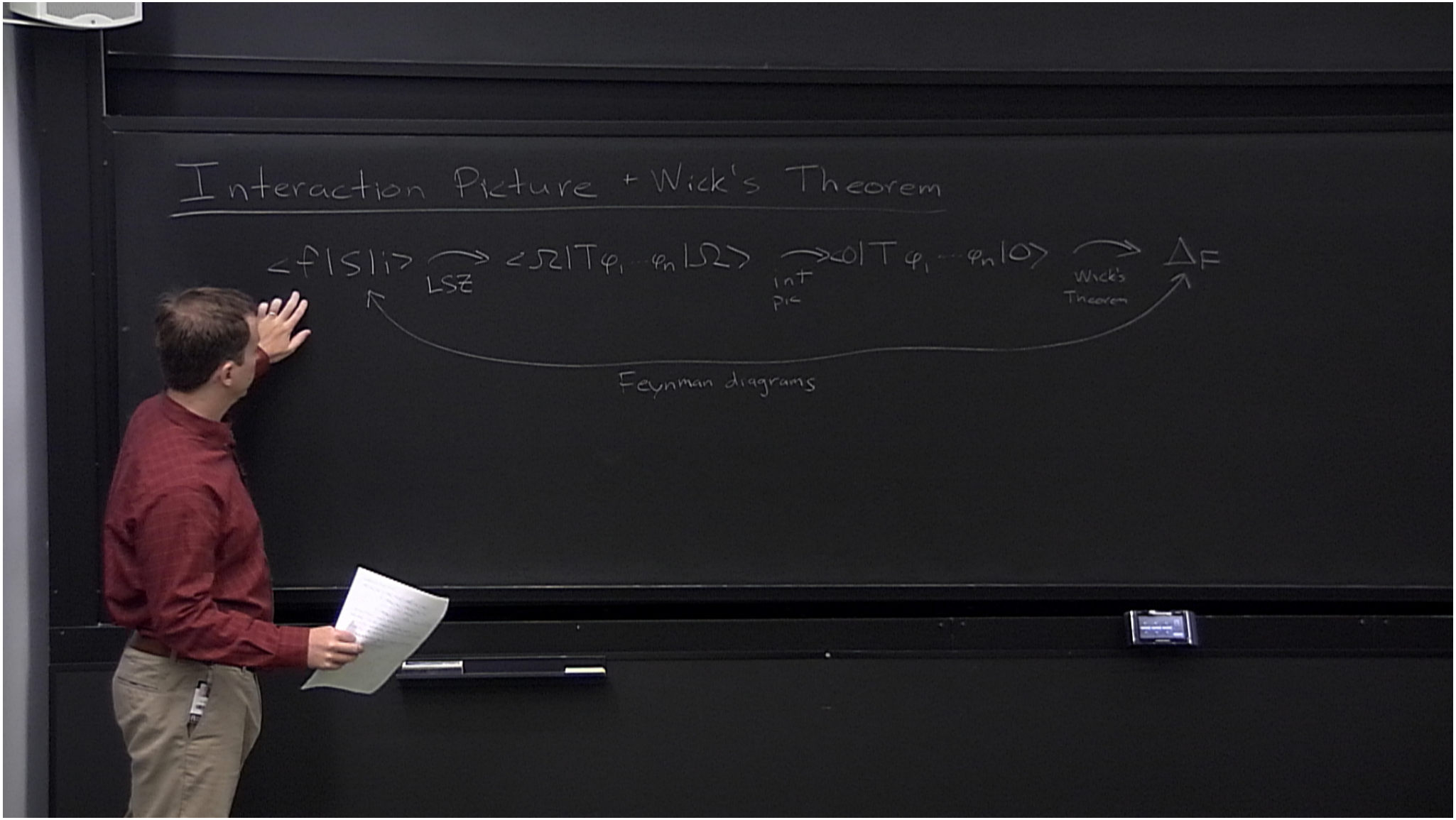
Title: PSI 2016/2017 Quantum Field Theory I - Lecture 4

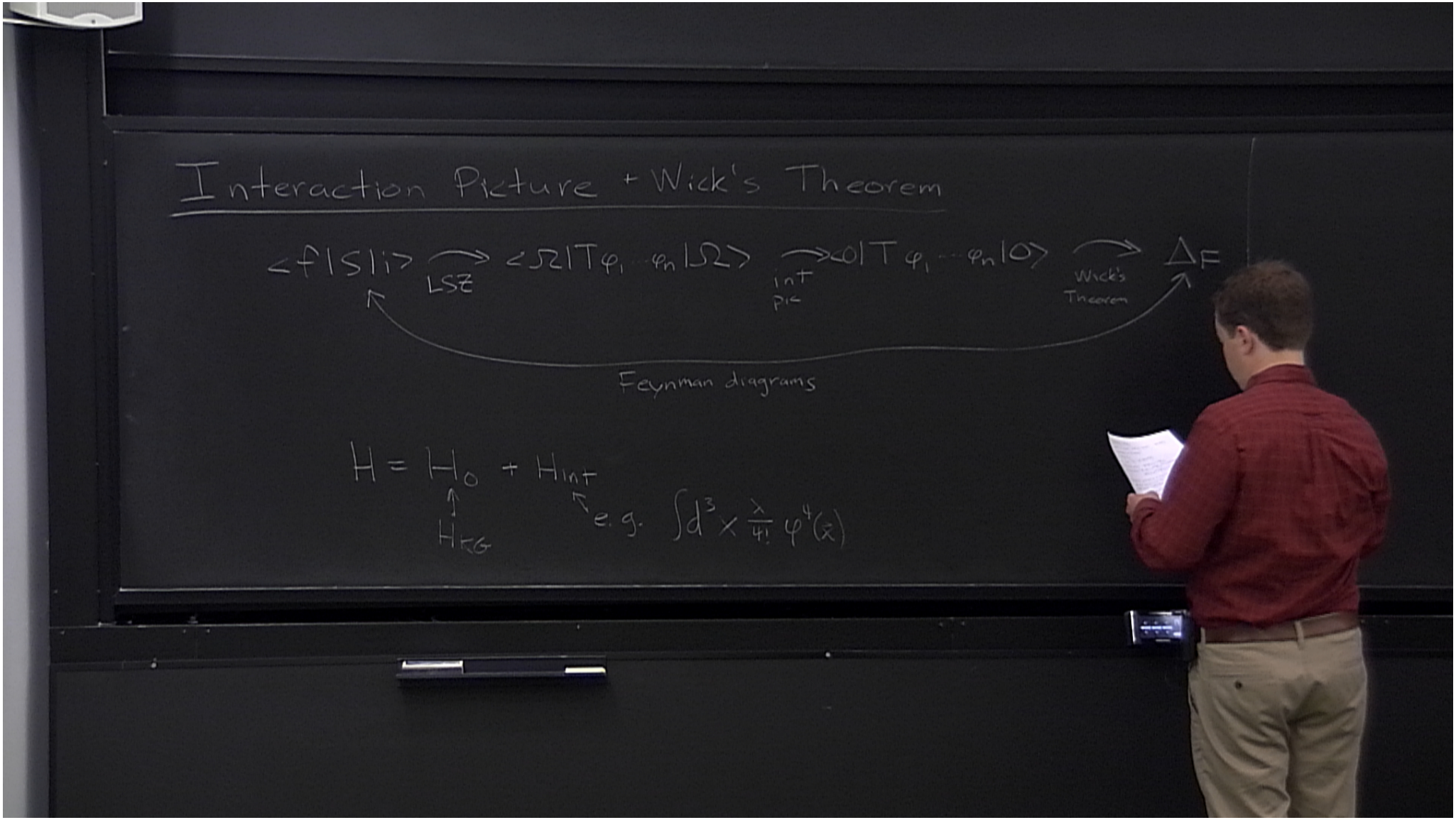
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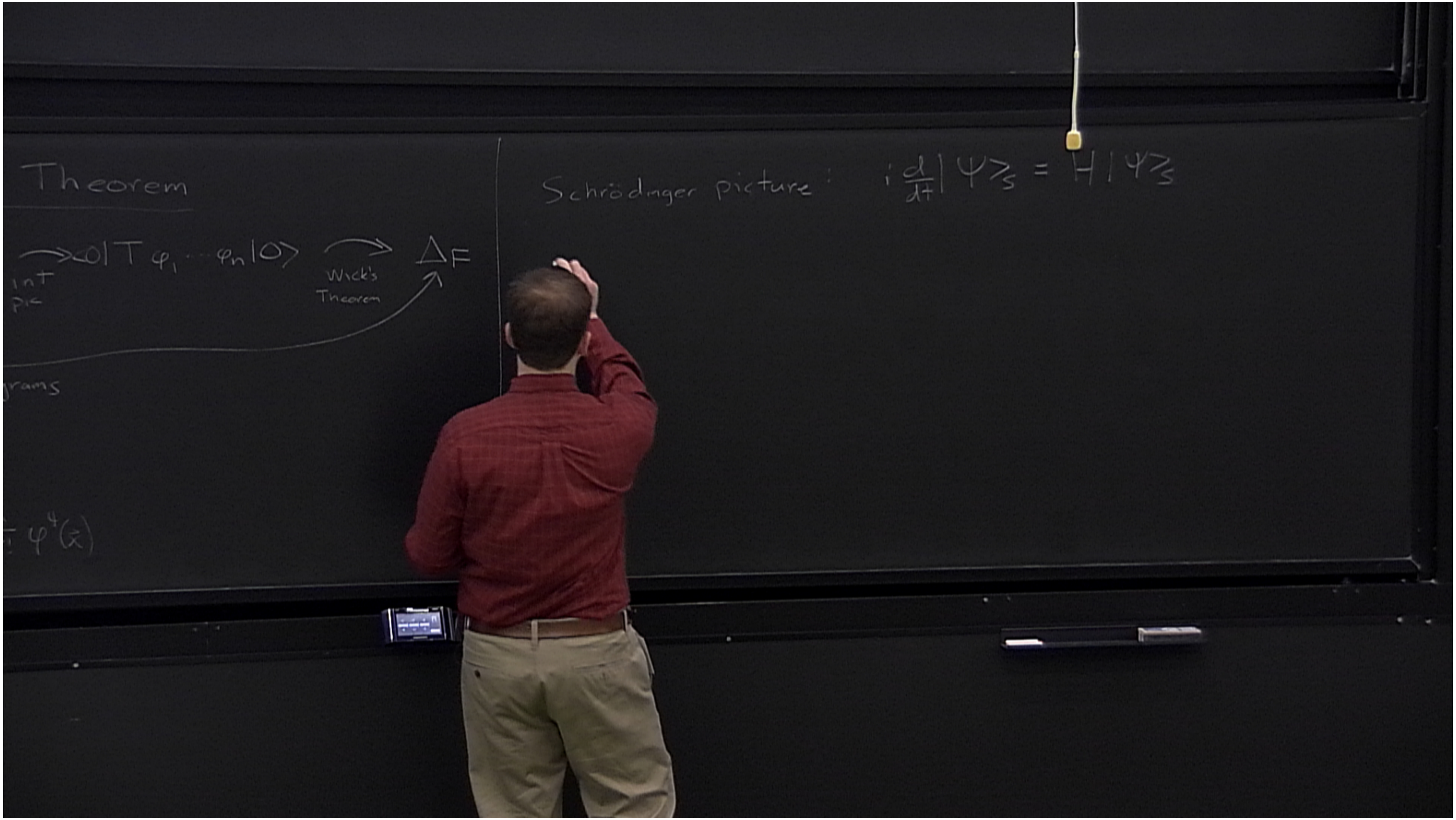
URL: http://pirsa.org/16100004

Abstract:


Interaction Picture + Wick's Theorem


Interaction Picture + Wick's Theorem
⟨f|S|i⟩
LSZ
⟨Ω|T φ1 ··· φn|Ω⟩
int pic
⟨0|T φ1 ··· φn|0⟩
Wick's Theorem
ΔF
Feynman diagrams


Interaction Picture + Wick's Theorem
<f|S|i>
LSZ
<Ω|T φ1 ··· φn|Ω>
int pic
<0|T φ1 ··· φn|0>
Wick's Theorem
ΔF
Feynman diagrams
H = H0 + Hint
H0 = HKG
e.g. ∫d³x λ/4! φ⁴(x)


Theorem
Schrödinger picture
Wick's Theorem

Theorem

$\langle 0|T\,\varphi_1 \cdots \varphi_n|0\rangle$ → Wick's Theorem → Δ_F

...ams

$\varphi^4(\vec{x})$

Schrödinger picture: $i\frac{d}{dt}|\psi\rangle_S = H|\psi\rangle_S$

Heisenberg picture: $\mathcal{O}_H(t) = e^{iHt}\mathcal{O}_S e^{-iHt}$

$= U^\dagger(t,0)\,\mathcal{O}_S\,U(t,0)$

$U(t,t') = e^{-iH(t-t')}$

$H = H_0 + H_{int}$

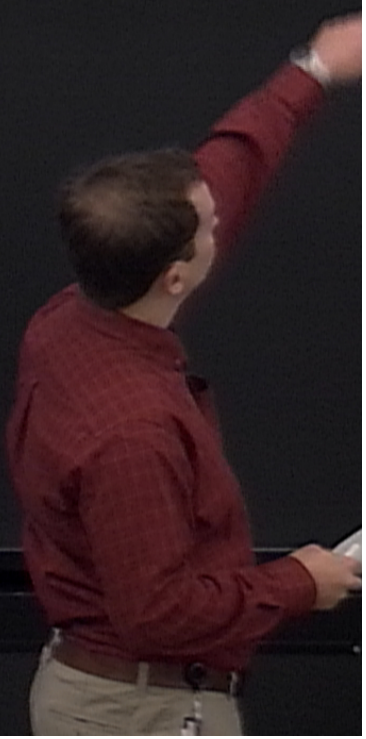
H_0: H_{KG}

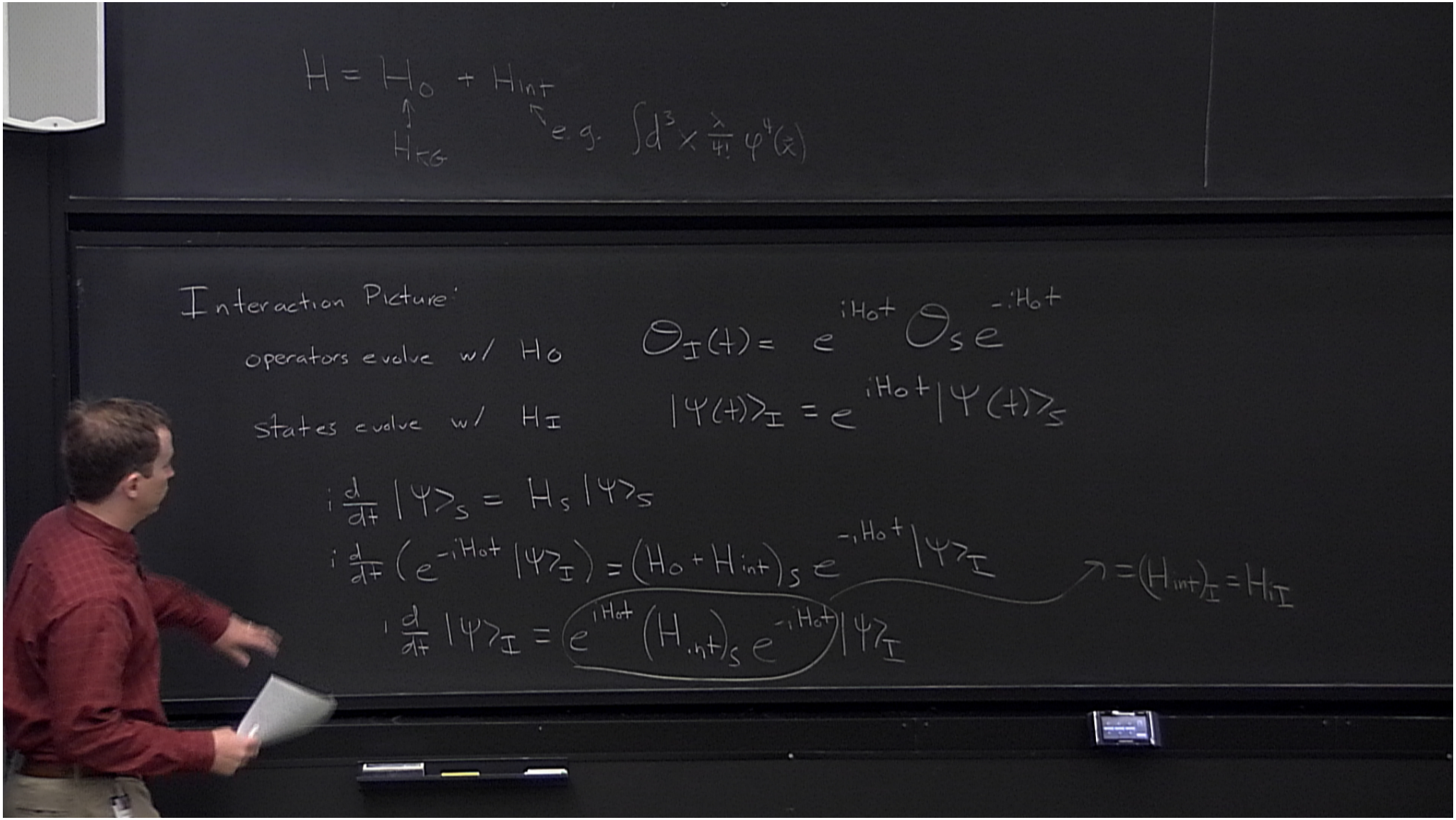
H_{int}: e.g. $\int d^3x \frac{\lambda}{4!}\varphi^4(\vec{x})$

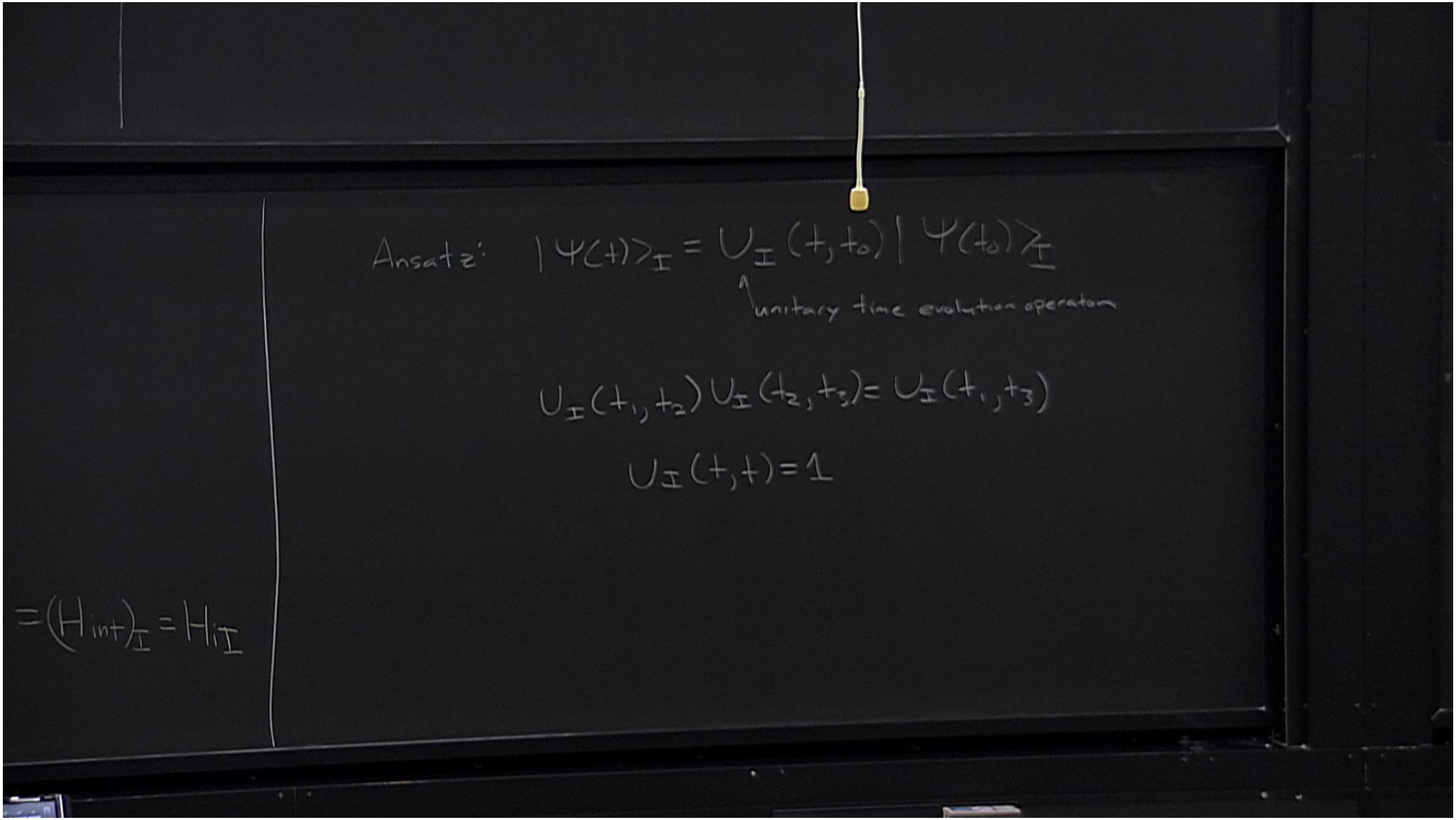
Interaction Picture:

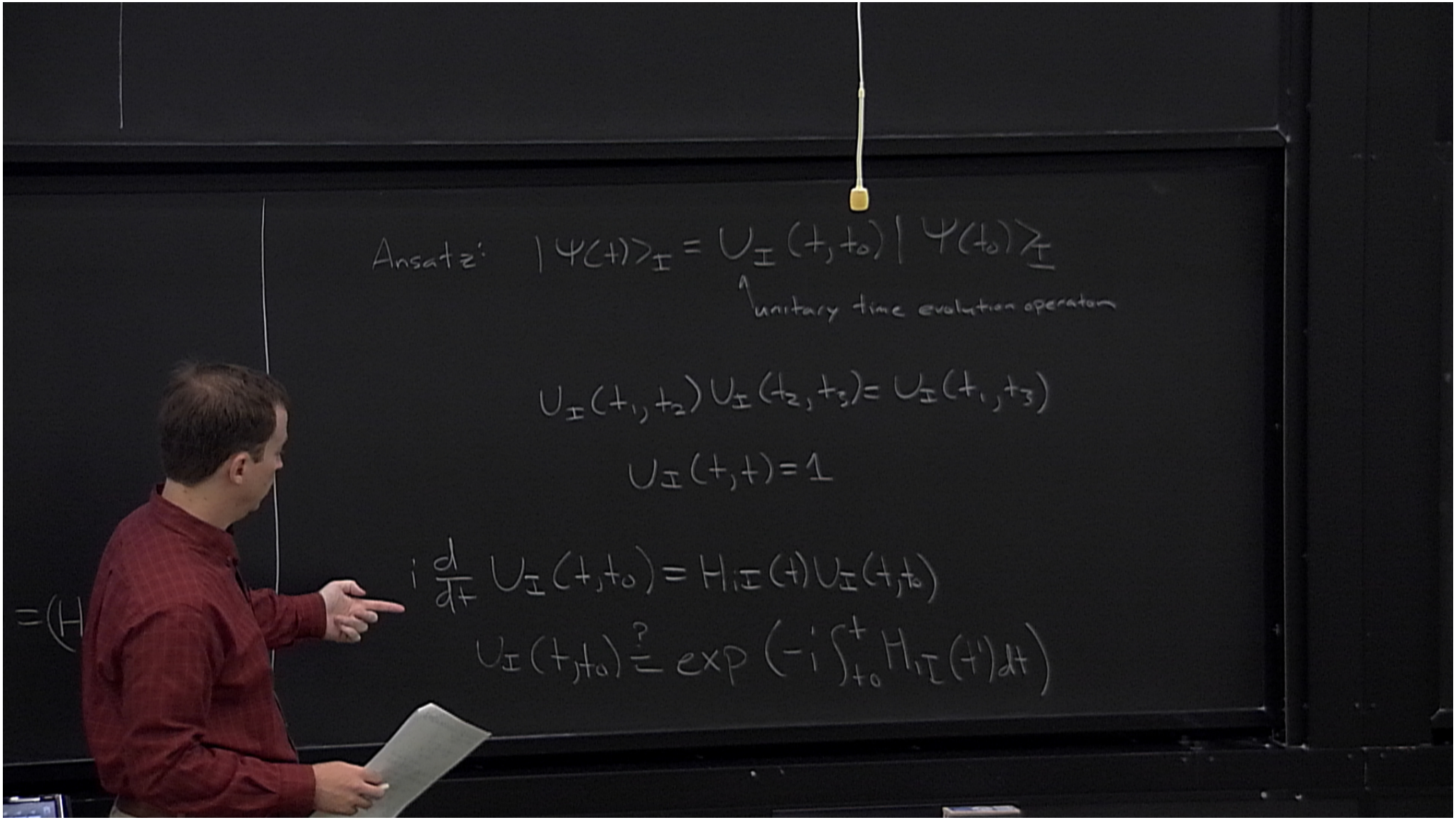
operators evolve w/ H_0 $\quad \mathcal{O}_I(t) = e^{iH_0t}\mathcal{O}_S e^{-iH_0t}$

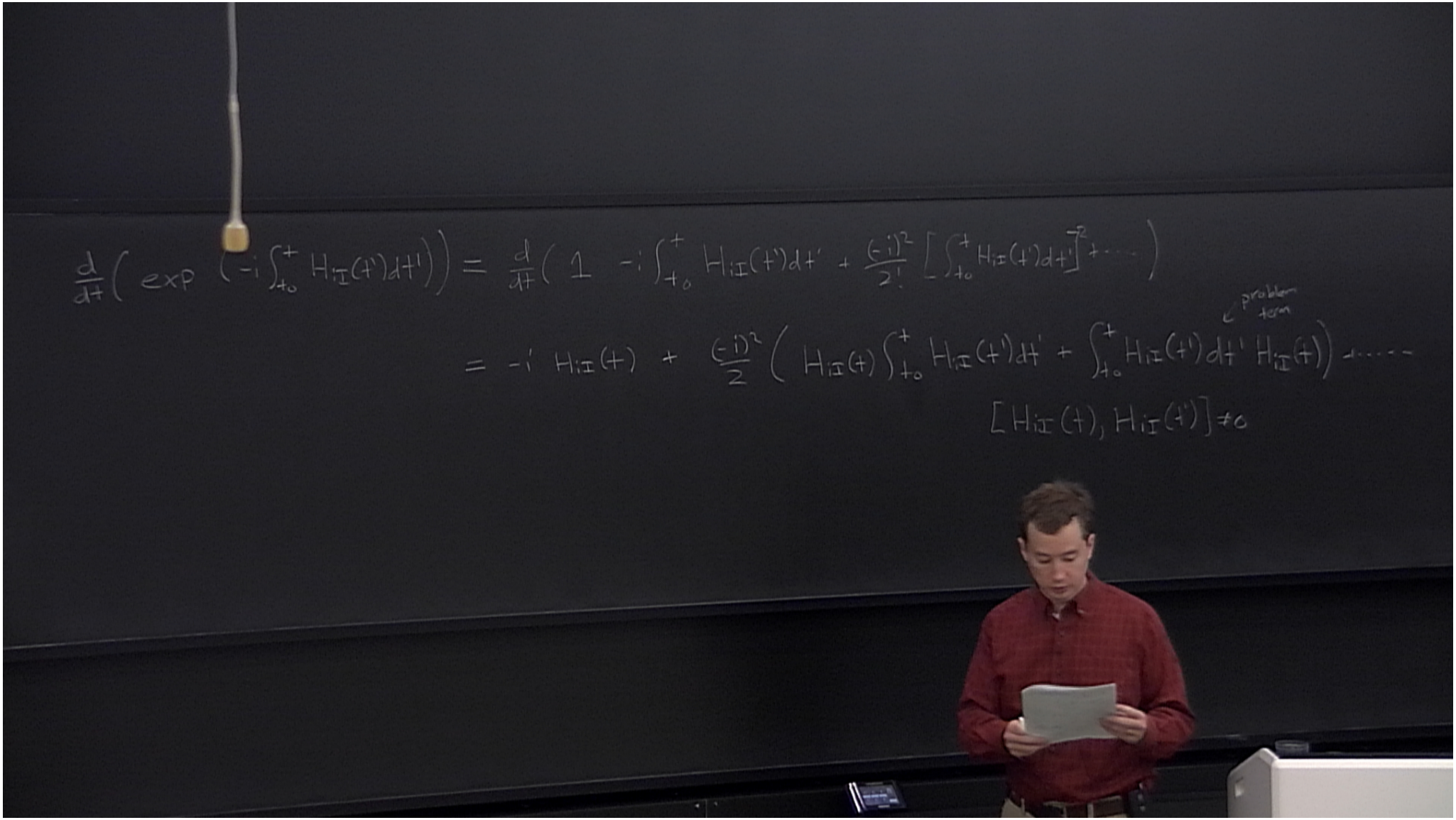
states evolve w/ H_I $\quad |\psi(t)\rangle_I = e^{iH_0t}|\psi(t)\rangle_S$

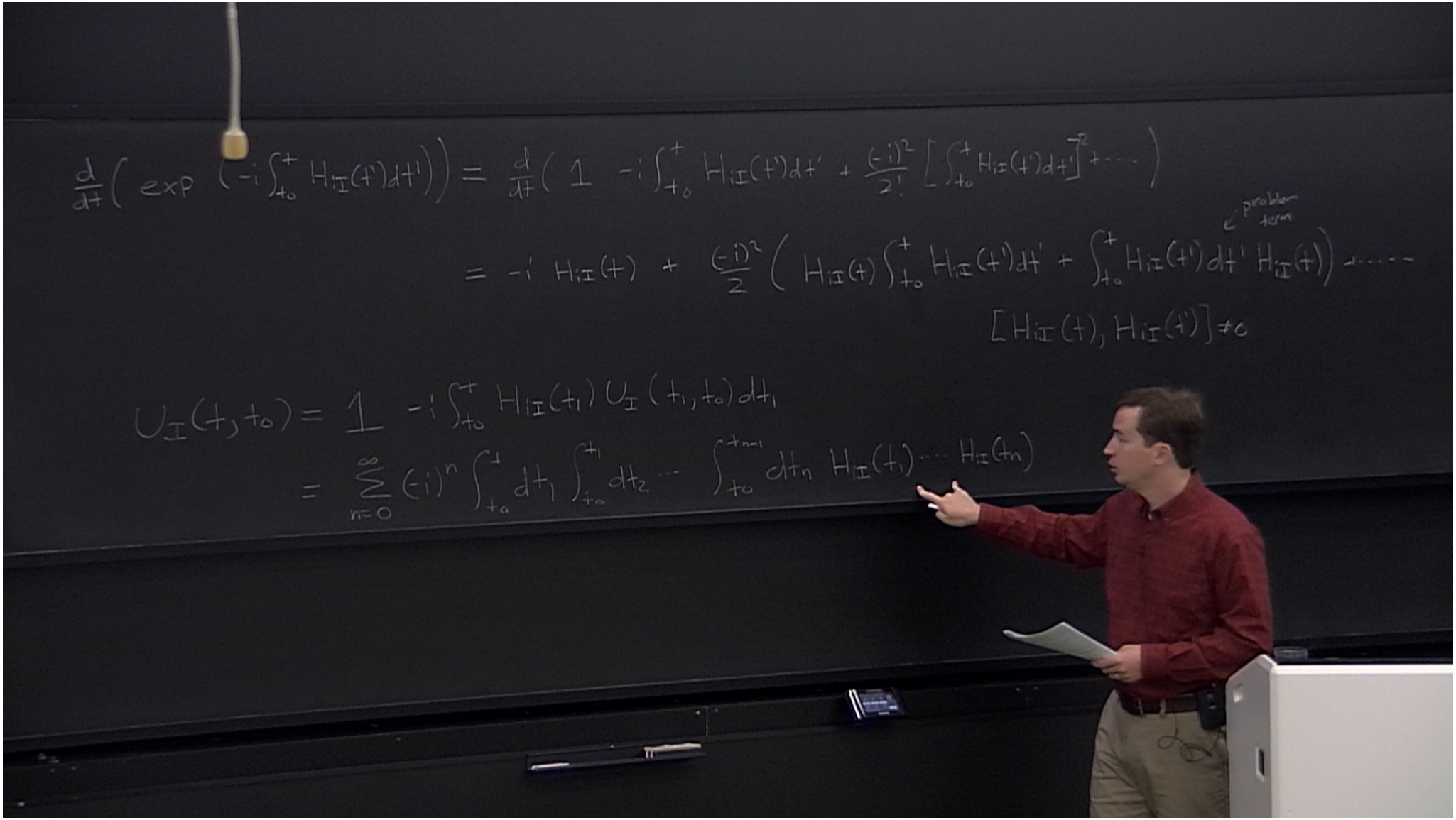



$H = H_0 + H_{int}$
H_{KG}
e.g. $\int d^3x \frac{\lambda}{4!}\varphi^4(\vec{x})$
Interaction Picture:
operators evolve w/ H_0
states evolve w/ H_I
$\mathcal{O}_I(t) = e^{iH_0t}\mathcal{O}_S e^{-iH_0t}$
$|\psi(t)\rangle_I = e^{iH_0t}|\psi(t)\rangle_S$
$i\frac{d}{dt}|\psi\rangle_S = H_S|\psi\rangle_S$
$i\frac{d}{dt}(e^{-iH_0t}|\psi\rangle_I) = (H_0 + H_{int})_S e^{-iH_0t}|\psi\rangle_I$
$= (H_{int})_I = H_I$
$i\frac{d}{dt}|\psi\rangle_I = e^{iH_0t}(H_{int})_S e^{-iH_0t}|\psi\rangle_I$


Ansatz: $|\psi(t)\rangle_I = U_I(t,t_0)\,|\psi(t_0)\rangle_I$
unitary time evolution operator
$U_I(t_1,t_2)\,U_I(t_2,t_3) = U_I(t_1,t_3)$
$U_I(t,t) = 1$
$= (H_{int})_I = H_I$


Ansatz: $|\psi(t)\rangle_I = U_I(t,t_0)\,|\psi(t_0)\rangle_I$
unitary time evolution operator
$U_I(t_1,t_2)\,U_I(t_2,t_3) = U_I(t_1,t_3)$
$U_I(t,t) = 1$
$i\frac{d}{dt}U_I(t,t_0) = H_{1,I}(t)\,U_I(t,t_0)$
$U_I(t,t_0) \stackrel{?}{=} \exp\left(-i\int_{t_0}^{t} H_{1,I}(t')\,dt'\right)$


$\frac{d}{dt}\left(\exp\left(-i\int_{t_0}^{t} H_{iI}(t')dt'\right)\right) = \frac{d}{dt}\left(1 - i\int_{t_0}^{t} H_{iI}(t')dt' + \frac{(-i)^2}{2!}\left[\int_{t_0}^{t} H_{iI}(t')dt'\right]^2 + \cdots\right)$
$= -i\,H_{iI}(t) + \frac{(-i)^2}{2}\left(H_{iI}(t)\int_{t_0}^{t} H_{iI}(t')dt' + \int_{t_0}^{t} H_{iI}(t')dt'\,H_{iI}(t)\right) - \cdots$
problem term
$[H_{iI}(t), H_{iI}(t')] \neq 0$


$\frac{d}{dt}\left(\exp\left(-i\int_{t_0}^{t} H_{II}(t')dt'\right)\right) = \frac{d}{dt}\left(1 - i\int_{t_0}^{t} H_{II}(t')dt' + \frac{(-i)^2}{2!}\left[\int_{t_0}^{t} H_{II}(t')dt'\right]^2 + \cdots\right)$
problem term
$= -i\,H_{II}(t) + \frac{(-i)^2}{2}\left(H_{II}(t)\int_{t_0}^{t} H_{II}(t')dt' + \int_{t_0}^{t} H_{II}(t')dt'\,H_{II}(t)\right) - \cdots$
$[H_{II}(t), H_{II}(t')] \neq 0$
$U_I(t,t_0) = 1 - i\int_{t_0}^{t} H_{II}(t_1)\,U_I(t_1,t_0)\,dt_1$
$= \sum_{n=0}^{\infty}(-i)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n\, H_{II}(t_1)\cdots H_{II}(t_n)$

$$\frac{d}{dt}\left(\exp\left(-i\int_{t_0}^{t} H_{I}(t')dt'\right)\right) = \frac{d}{dt}\left(1 - i\int_{t_0}^{t} H_{I}(t')dt' + \frac{(-i)^2}{2!}\left[\int_{t_0}^{t} H_{I}(t')dt'\right]^2 + \cdots\right)$$

$$= -i\,H_{I}(t) + \frac{(-i)^2}{2}\left(H_{I}(t)\int_{t_0}^{t} H_{I}(t')dt' + \int_{t_0}^{t} H_{I}(t')dt'\,H_{I}(t)\right) + \cdots$$

problem term

$$[H_{I}(t), H_{I}(t')] \neq 0$$

$$U_{I}(t,t_0) = 1 - i\int_{t_0}^{t} H_{I}(t_1)\,U_{I}(t_1,t_0)\,dt_1$$

$$= \sum_{n=0}^{\infty}(-i)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n\, H_{I}(t_1)\cdots H_{I}(t_n) \quad \text{time ordered} \quad t_1 \geq t_2 \geq t_3 \geq \cdots \geq t_n$$

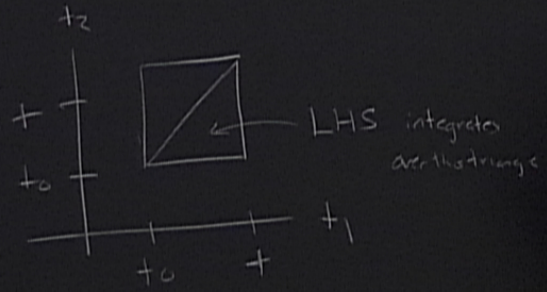
$= -i H_{iI}(t) + \frac{(-i)^2}{2}\left(H_{iI}(t)\int_{t_0}^{t} H_{iI}(t')dt' + \int_{t_0}^{t} H_{iI}(t')dt' H_{iI}(t)\right) + \cdots$

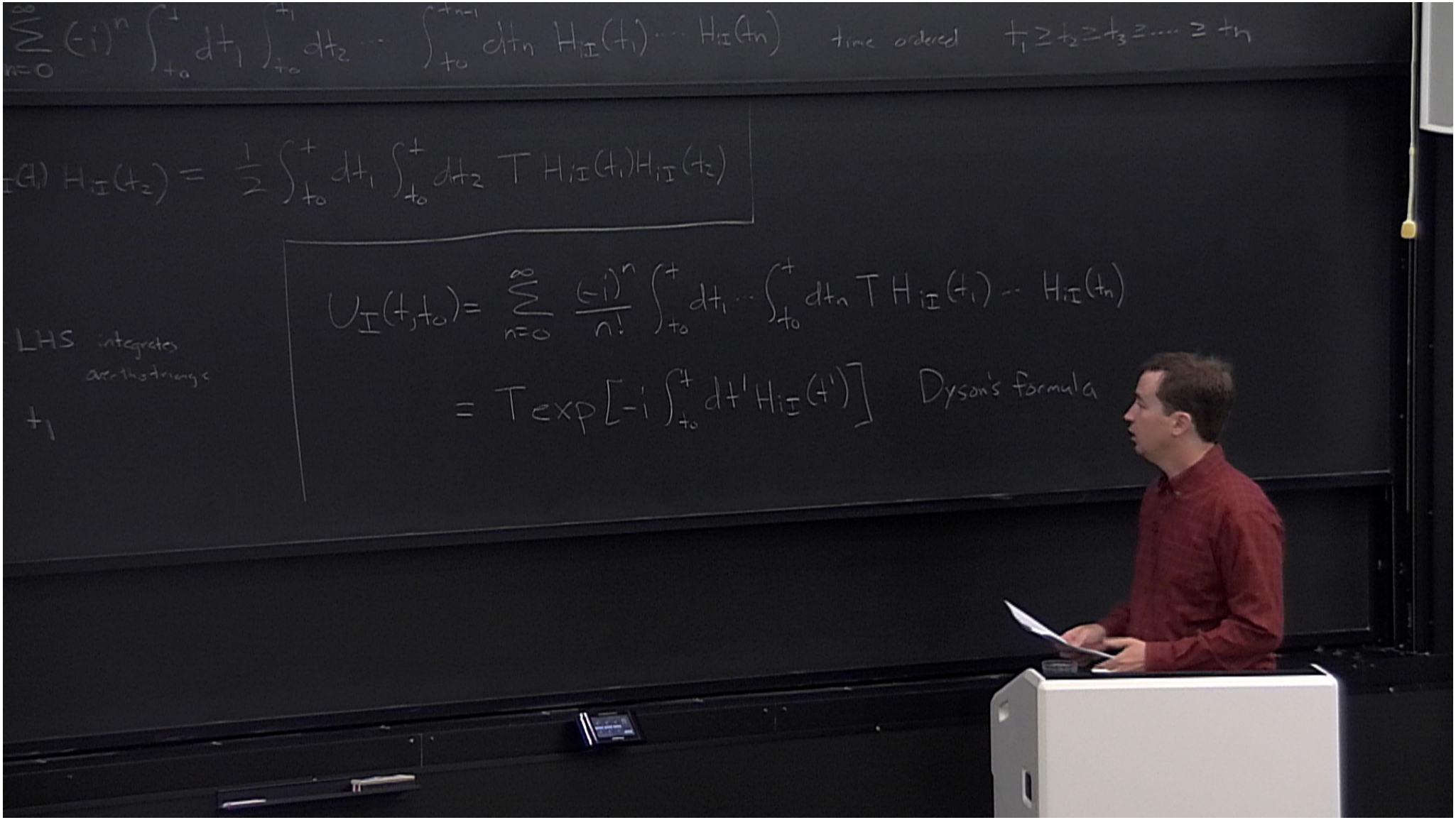
$[H_{iI}(t), H_{iI}(t')] \neq 0$

$U_I(t, t_0) = 1 - i\int_{t_0}^{t} H_{iI}(t_1)\, U_I(t_1, t_0)\, dt_1$

$= \sum_{n=0}^{\infty} (-i)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n\, H_I(t_1) \cdots H_I(t_n)$ time ordered $t_1 \geq t_2 \geq t_3 \geq \cdots \geq$

$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_{iI}(t_1)\, H_{iI}(t_2) =$




$\sum_{n=0}^{\infty} (-i)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \, H_{iI}(t_1) \cdots H_{iI}(t_n)$ time ordered $t_1 \geq t_2 \geq t_3 \geq \cdots \geq t_n$
$H_{iI}(t_1) H_{iI}(t_2) = \frac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \, T\, H_{iI}(t_1) H_{iI}(t_2)$
$U_I(t,t_0) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_n \, T\, H_{iI}(t_1) \cdots H_{iI}(t_n)$
$= T \exp\left[-i \int_{t_0}^{t} dt' H_{iI}(t')\right]$ Dyson's formula
LHS integrates over the triangle
t_1

States evolve w/ H_I $\quad |\psi(t)\rangle_I = e^{iH_0 t}|\psi(t)\rangle_S$

$$i\frac{d}{dt}|\psi\rangle_S = H_S|\psi\rangle_S$$

$$i\frac{d}{dt}\left(e^{-iH_0 t}|\psi\rangle_I\right) = (H_0 + H_{int})_S\, e^{-iH_0 t}|\psi\rangle_I \quad \rightarrow = (H_{int})_I = H_I$$

$$i\frac{d}{dt}|\psi\rangle_I = e^{iH_0 t}(H_{int})_S e^{-iH_0 t}|\psi\rangle_I$$

$U_I(t,t_2)\,U_I(t_2,$

$U_I(t,t) = 1$

$i\frac{d}{dt}U_I(t,t_0) = H_I(t)U$

$U_I(t,t_0) \stackrel{?}{=} \exp(-i$

Fields in int. pic

$$\varphi(t_0,\vec{x}) = \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}}\left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + a^{\dagger}_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}}\right)$$

$$\varphi_H(t,\vec{x}) = e^{iH(t-t_0)}\,\varphi(t_0,\vec{x})\,e^{-iH(t-t_0)}, \qquad \Delta t = t - t_0$$

States evolve w/ H_I $\quad |\psi(t)\rangle_I = e^{iH_0 t}|\psi(t)\rangle_S$

$$i\frac{d}{dt}|\psi\rangle_S = H_S|\psi\rangle_S$$

$$i\frac{d}{dt}\left(e^{-iH_0 t}|\psi\rangle_I\right) = (H_0 + H_{int})_S\, e^{-iH_0 t}|\psi\rangle_I \quad \rightarrow = (H_{int})_I = H_I$$

$$i\frac{d}{dt}|\psi\rangle_I = e^{iH_0 t}(H_{int})_S e^{-iH_0 t}|\psi\rangle_I$$

$U_I(t_1,t_2)\,U_I(t_2,$

$U_I(t,t) = 1$

$i\frac{d}{dt}U_I(t,t_0) = H_I(t)U$

$U_I(t,t_0) \stackrel{?}{=} \exp(-i$

Fields in int. pic

$$\varphi(t_0,\vec{x}) = \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}}\left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}}\right)$$

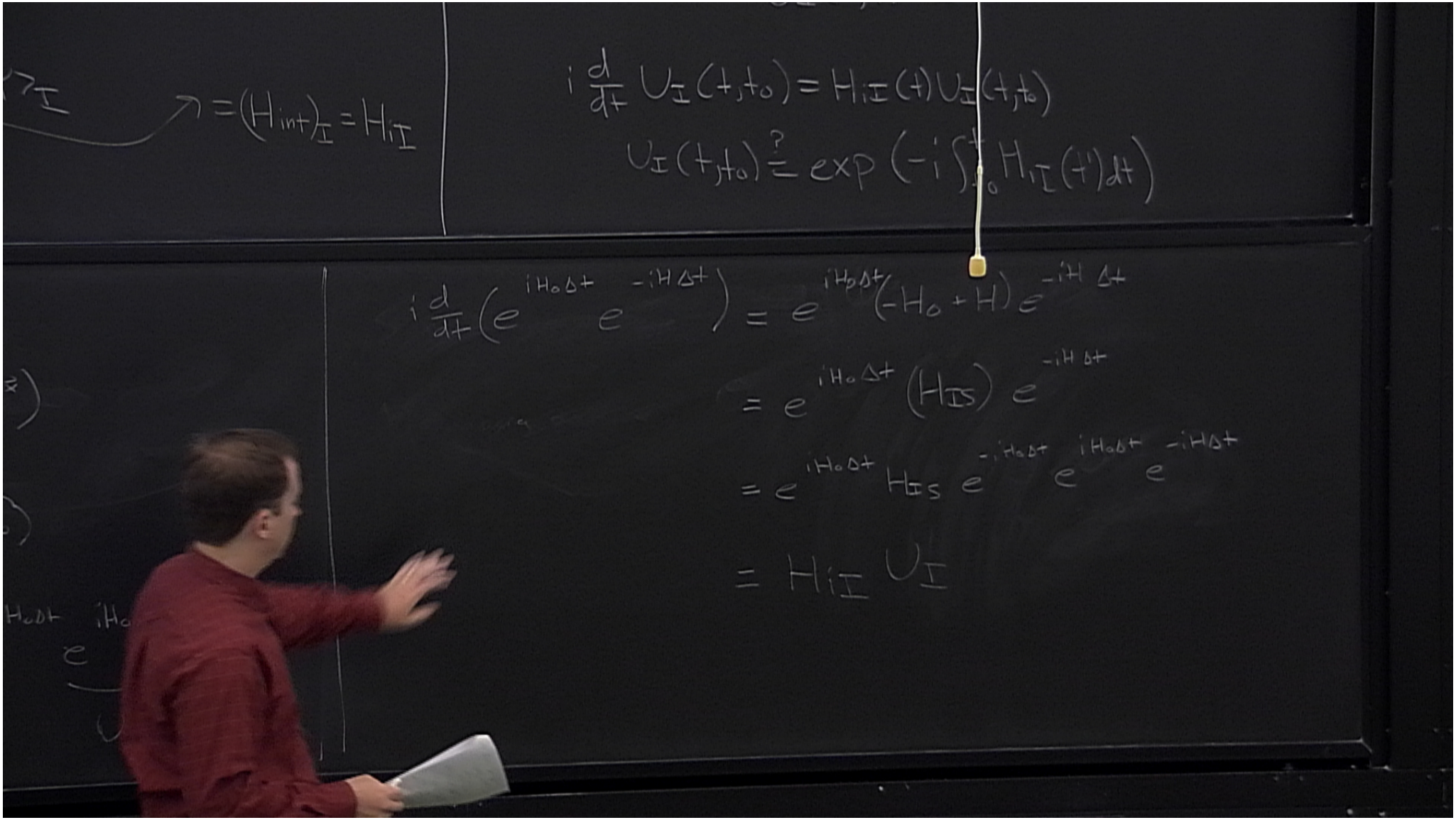
$$\varphi_H(t,\vec{x}) = e^{iH(t-t_0)}\,\varphi(t_0,\vec{x})\,e^{-iH(t-t_0)}$$

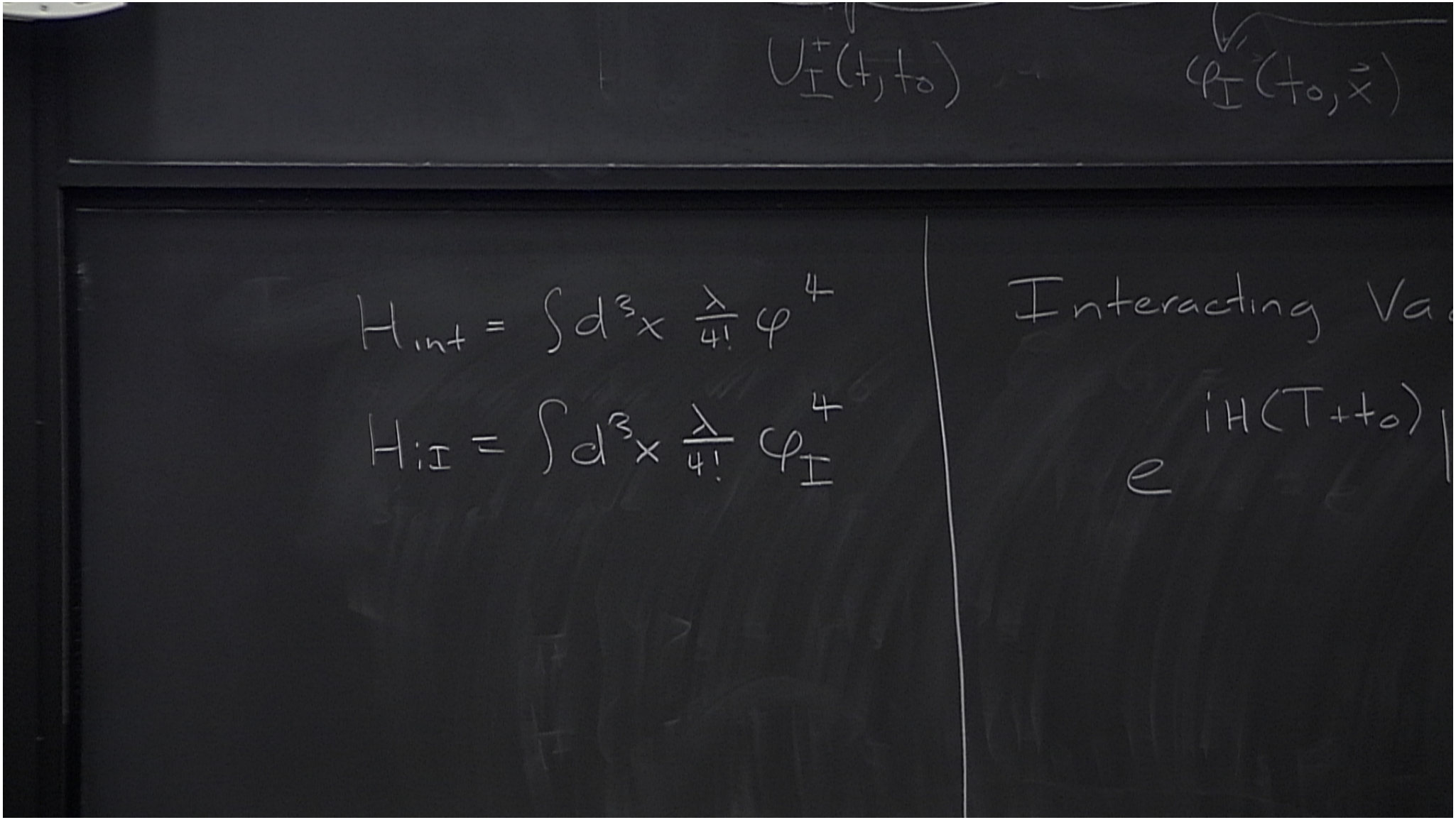
(Δt over $t-t_0$)

$$= e^{iH\Delta t}\, e^{-iH_0\Delta t}\, e^{iH_0\Delta t}\,\varphi(t_0,\vec{x})\, e^{-iH_0\Delta t}\, e^{iH_0\Delta t}\, e^{-iH\Delta t}$$

$e^{iH\Delta t}e^{-iH_0\Delta t} = U_I^\dagger(t,t_0)$ $\quad$ $e^{iH_0\Delta t}\varphi(t_0,\vec{x})e^{-iH_0\Delta t} = \varphi_I(t_0,\vec{x})$ $\quad$ $e^{iH_0\Delta t}e^{-iH\Delta t} = U_I(t,t_0)$

$i\frac{d}{dt}e$


$= (H_{int})_I = H_{iI}$
$i\frac{d}{dt} U_I(t,t_0) = H_{iI}(t) U_I(t,t_0)$
$U_I(t,t_0) \stackrel{?}{=} \exp\left(-i \int_{t_0}^{t} H_{iI}(t')dt\right)$
$i\frac{d}{dt}\left(e^{iH_0\Delta t} e^{-iH\Delta t}\right) = e^{iH_0\Delta t}(-H_0 + H) e^{-iH\Delta t}$
$= e^{iH_0\Delta t}(H_{IS}) e^{-iH\Delta t}$
$= e^{iH_0\Delta t} H_{IS} e^{-iH_0\Delta t} e^{iH_0\Delta t} e^{-iH\Delta t}$
$= H_{iI} U_I$


$U_I^+(t, t_0)$
$\varphi_I(t_0, \vec{x})$
$H_{int} = \int d^3x \frac{\lambda}{4!} \varphi^4$
$H_{I:I} = \int d^3x \frac{\lambda}{4!} \varphi_I^4$
Interacting Va
$e^{iH(T+t_0)}$

$e^{-iH_0\Delta t}\ e^{iH_0\Delta t}\ \varphi(t_0,\vec{x})\ e^{-iH_0\Delta t}\ e^{iH_0\Delta t}\ e^{-iH\Delta t}$

$\varphi_I(t_0,\vec{x})$ $\qquad$ $U_I(t,t_0)$

$= H_{iI}\ U_I$

Interacting Vacuum $|\Omega\rangle$

$$e^{iH(T+t_0)}|0\rangle = e^{iH(T+t_0)}\,\mathbb{1}\,|0\rangle$$

$\mathbb{1}$ ↑ complete set of states

$$\mathbb{1} = |\Omega\rangle\langle\Omega| + \sum_{n\neq 0} |n\rangle\langle n|$$

$$= e^{-iE_0(T+t_0)}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_n(T+t_0)}|n\rangle\langle n|0\rangle$$

↑ all other states

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \, H_{iI}(t_1) H_{iI}(t_2) = \frac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \, T\, H_{iI}(t_1) H_{iI}(t_2)$$

t_2

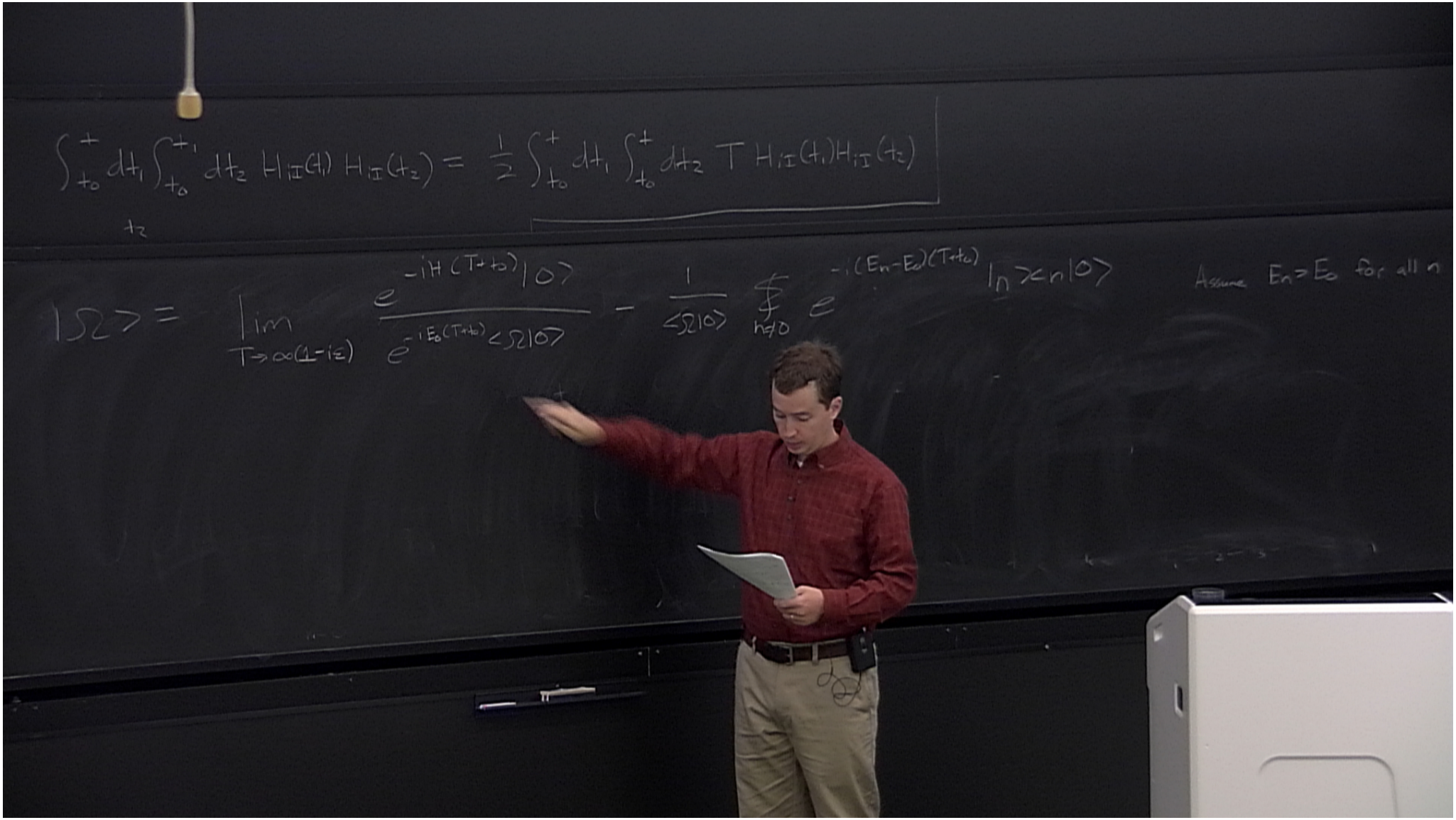
$$|S2\rangle = \frac{e^{-iH(T+t_0)}|0\rangle}{e^{-iE_0(T+t_0)}\langle S2|0\rangle} - \frac{1}{\langle S2|0\rangle} \oint_{h\neq 0} e^{-i}$$

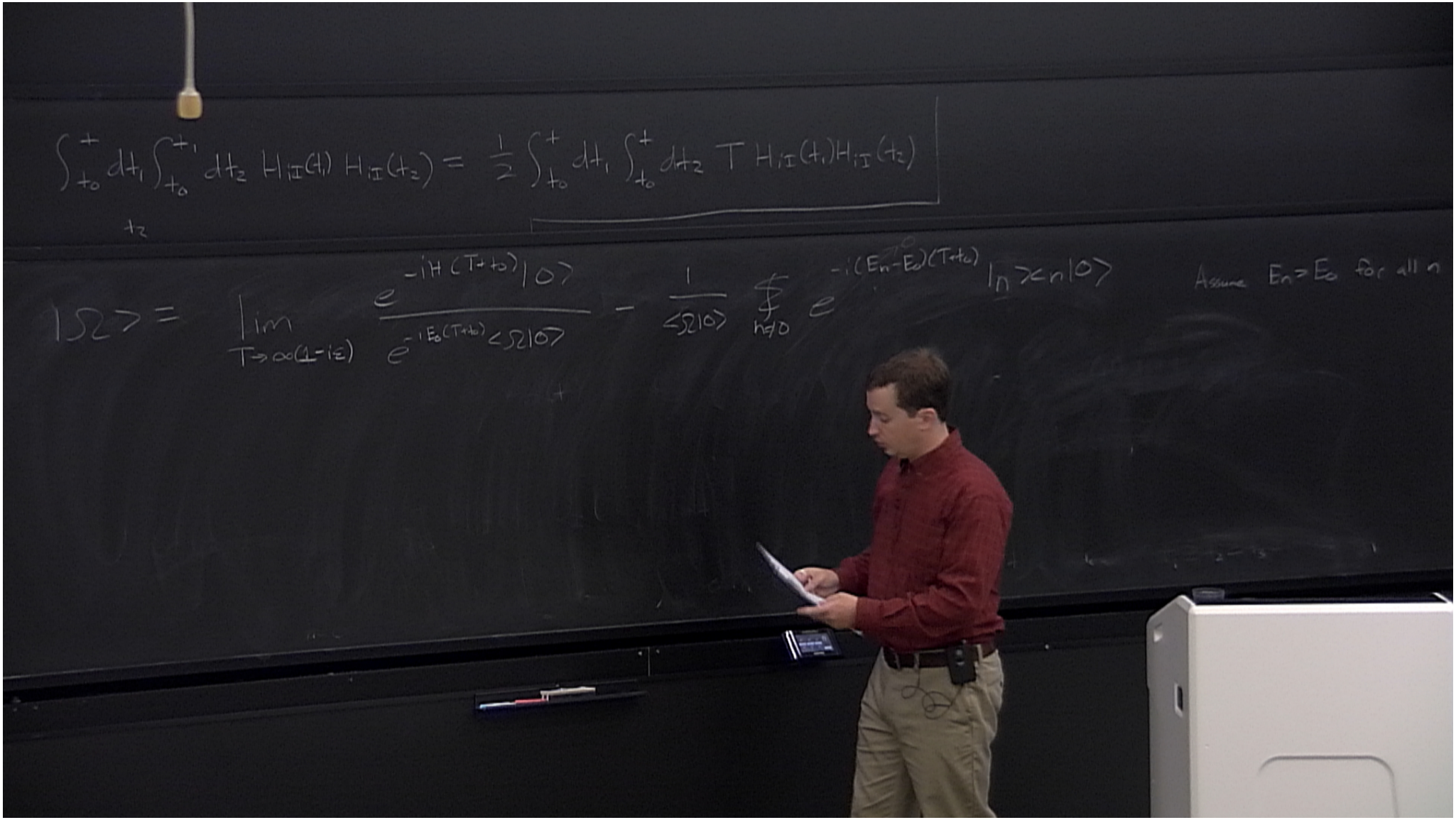
$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_{iI}(t_1) H_{iI}(t_2) = \frac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, T\, H_{iI}(t_1) H_{iI}(t_2)$$

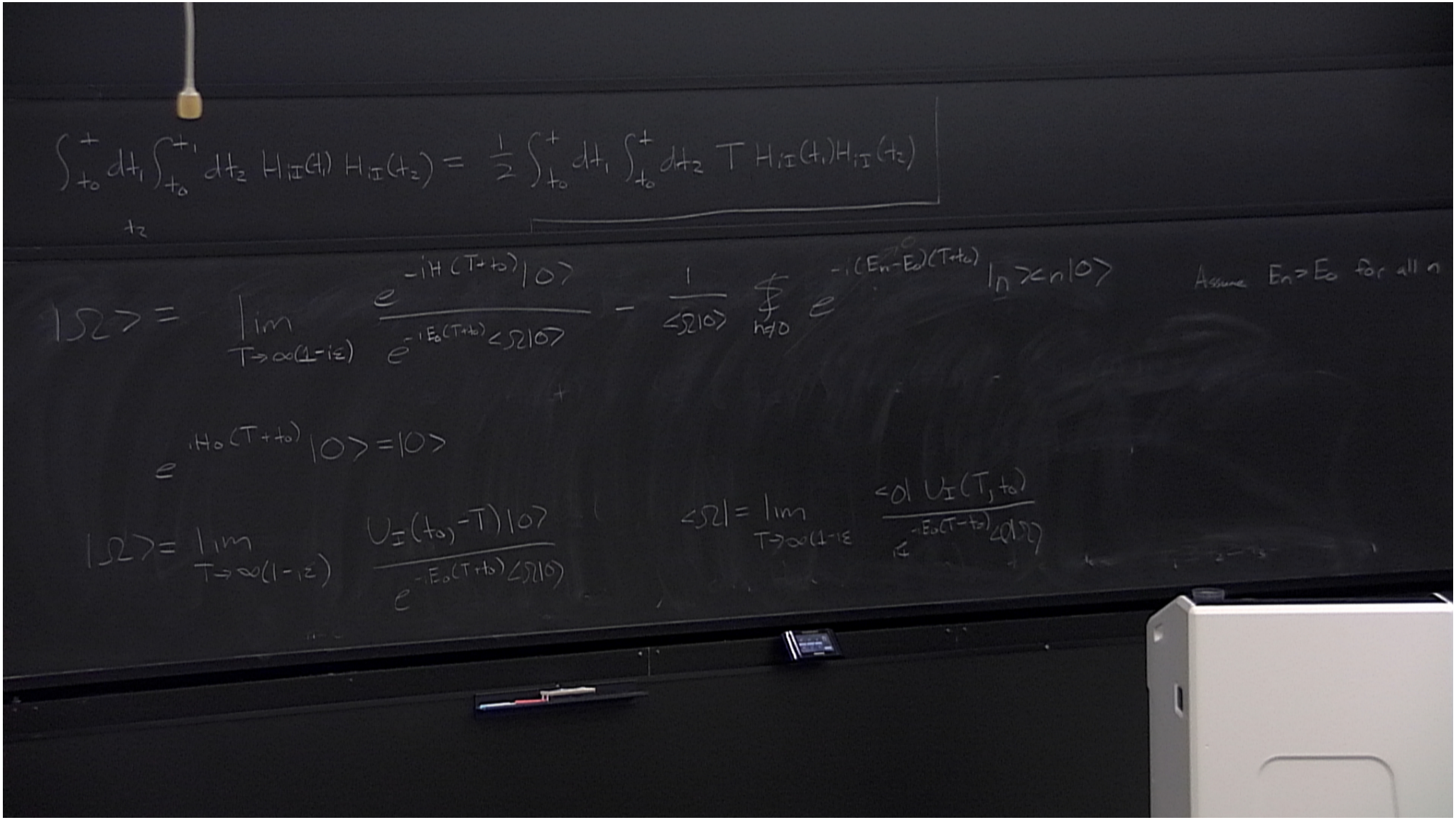
t_2

$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{e^{-iH(T+t_0)}|0\rangle}{e^{-iE_0(T+t_0)}\langle\Omega|0\rangle} - \frac{1}{\langle\Omega|0\rangle} \sum_{n\neq 0} e^{-i(E_n-E_0)(T+t_0)} |n\rangle\langle n|0\rangle$$

Assume $E_n > E_0$ for all n




$\int_{t_0}^{+} dt_1 \int_{t_0}^{t_1} dt_2\, H_{iI}(t_1) H_{iI}(t_2) = \frac{1}{2} \int_{t_0}^{+} dt_1 \int_{t_0}^{+} dt_2\, T\, H_{iI}(t_1) H_{iI}(t_2)$
t_2
$|S\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{e^{-iH(T+t_0)}|0\rangle}{e^{-iE_0(T+t_0)}\langle S|0\rangle} - \frac{1}{\langle S|0\rangle} \sum_{n\neq 0} e^{-i(E_n-E_0)(T+t_0)} |n\rangle\langle n|0\rangle$
Assume $E_n > E_0$ for all n


$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_{iI}(t_1) H_{iI}(t_2) = \frac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, T\, H_{iI}(t_1) H_{iI}(t_2)$
t_2
$|S2\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{e^{-iH(T+t_0)}|0\rangle}{e^{-iE_0(T+t_0)}\langle S2|0\rangle} - \frac{1}{\langle S2|0\rangle} \sum_{n\neq 0} e^{-i(E_n-E_0)(T+t_0)} |n\rangle\langle n|0\rangle$
Assume $E_n > E_0$ for all n
$e^{iH_0(T+t_0)}|0\rangle = |0\rangle$
$|S2\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{U_I(t_0,-T)|0\rangle}{e^{-iE_0(T+t_0)}\langle S2|0\rangle}$
$\langle S2| = \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0| U_I(T,t_0)}{e^{-iE_0(T-t_0)}\langle 0|S2\rangle}$

$$\varphi(t,\vec{x}) = e^{iH(t-t_0)}\,\varphi(t_0,\vec{x})\,e^{-iH(t-t_0)}$$

$$= e^{iH\Delta t}\,e^{-iH_0\Delta t}\,e^{iH_0\Delta t}\,\varphi(t_0,\vec{x})\,e^{-iH_0\Delta t}\,e^{iH_0\Delta t}\,e^{-iH\Delta t}$$

$$U_I^{\dagger}(t,t_0) \qquad \varphi_I(t_0,\vec{x}) \qquad U_I(t,t_0)$$

$$= e^{iH_0\Delta t}\,H_{IS}\,e$$

$$= H_I\,U_I$$

$$H_I = \int d^3x\,\frac{\lambda}{4!}\,\varphi^4$$

$$H_I = \int d^3x\,\frac{\lambda}{4!}\,\varphi_I^4$$

Interacting Vacuum $|\Omega\rangle$

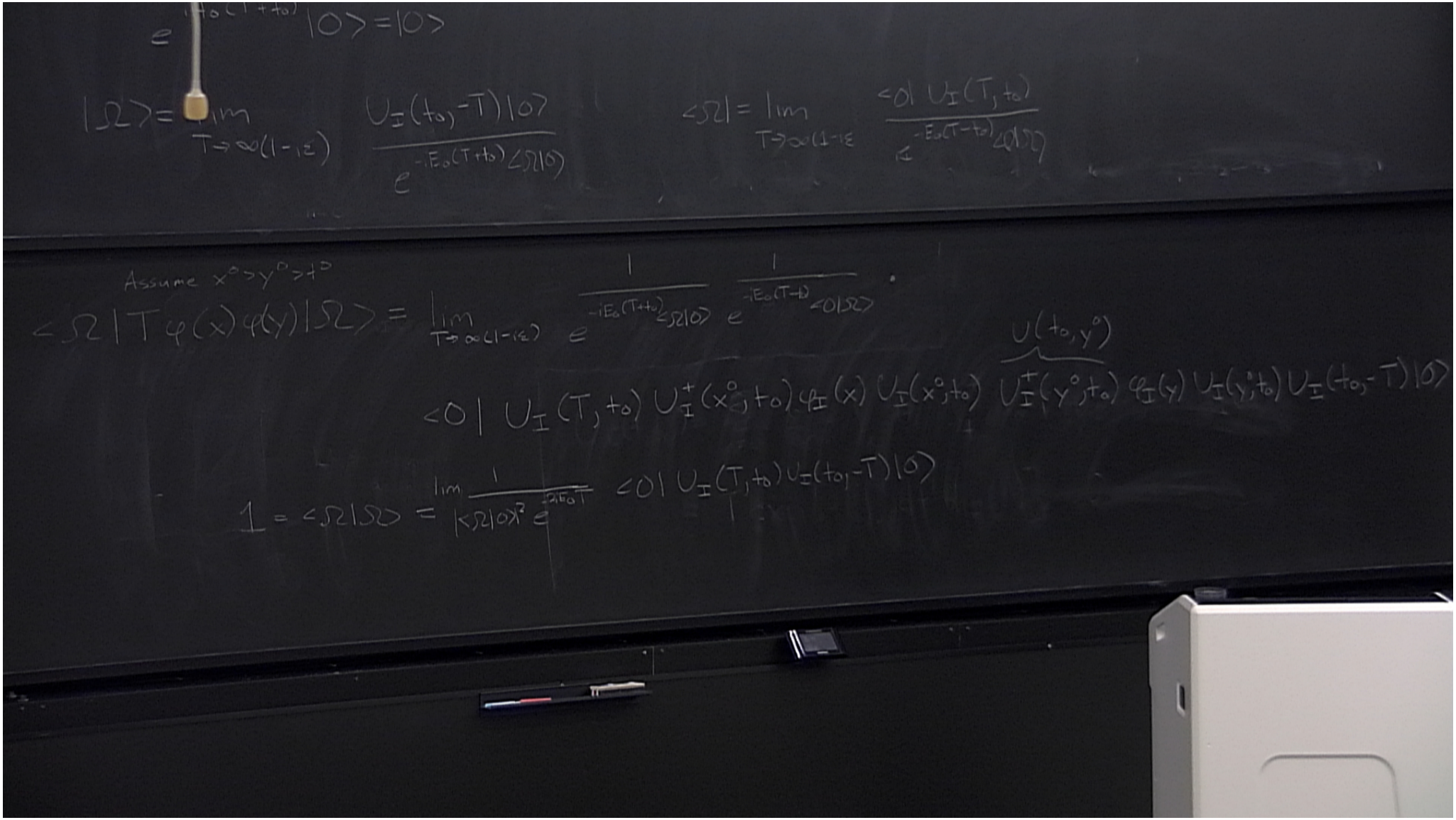
$$e^{-iH(T+t_0)}|0\rangle = e^{-iH(T+t_0)}\,\mathbb{1}\,|0\rangle$$

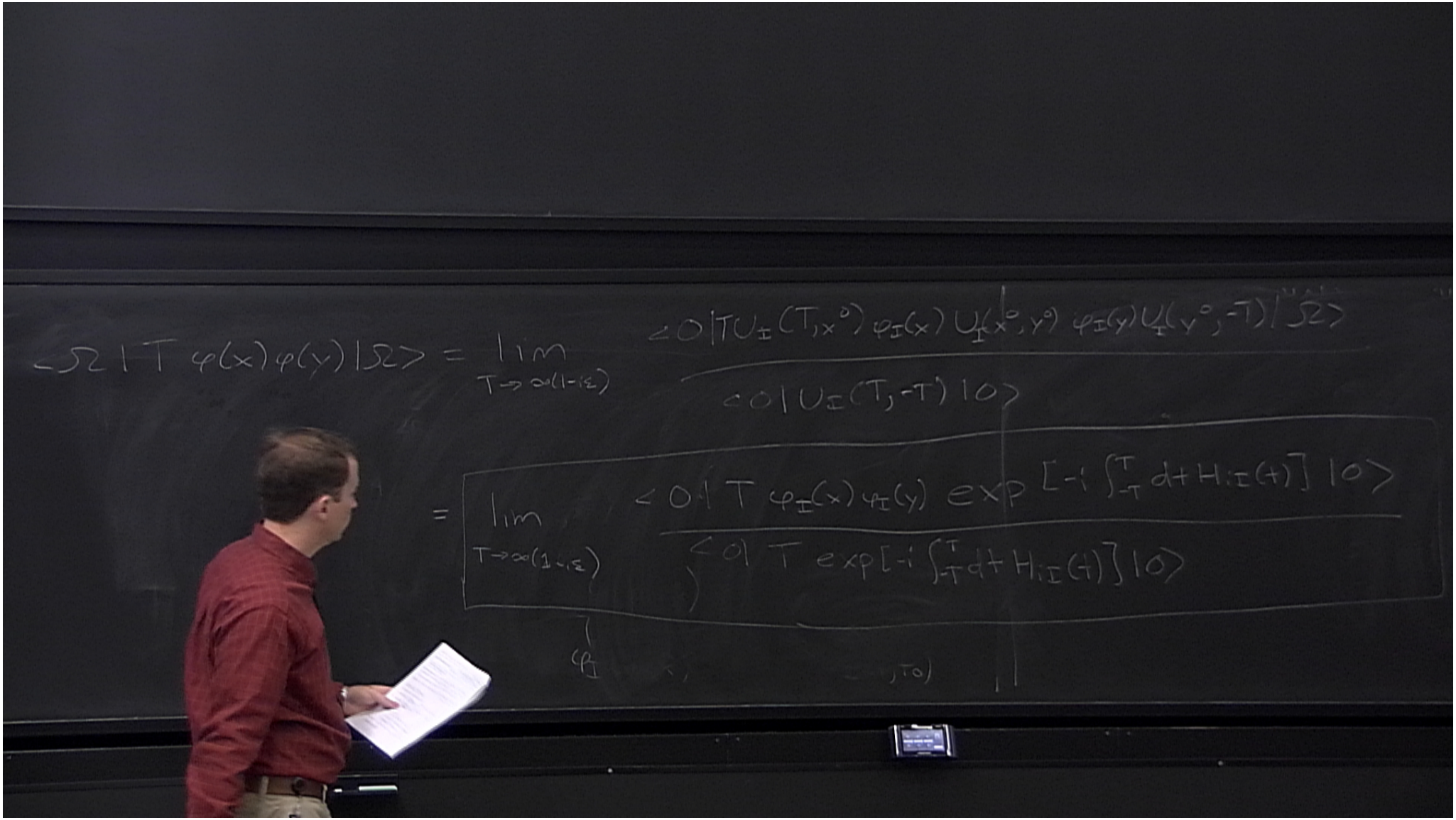
↑ complete set of states

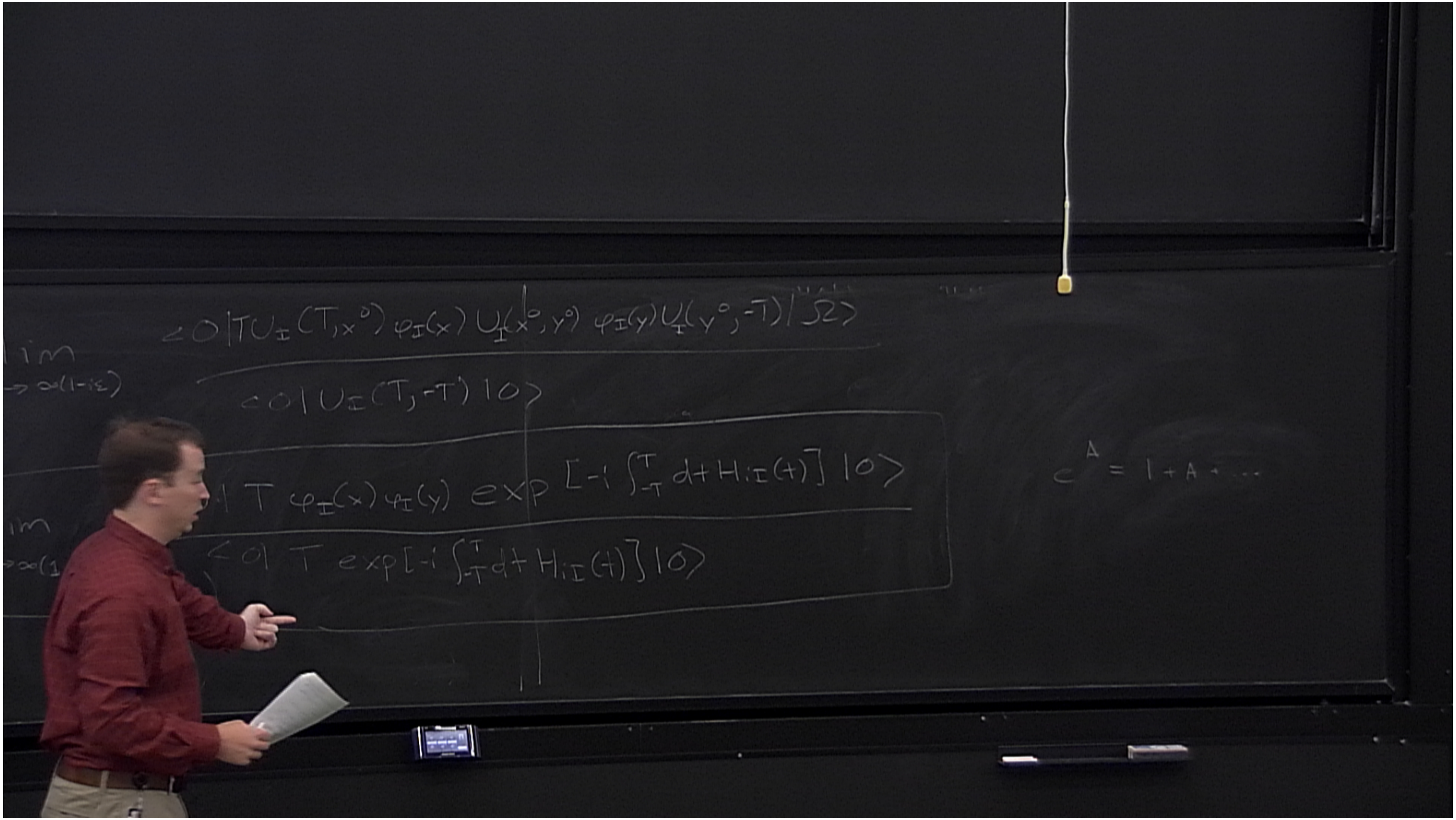
$$\mathbb{1} = |\Omega\rangle\langle\Omega| + \sum_{n\neq 0}|n\rangle\langle n|$$

$$= e^{-iE_0(T+t_0)}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_n(T+t_0)}|n\rangle\langle n|0\rangle$$

↑ all other states


$e^{-iH_0(T+t_0)}|0\rangle = |0\rangle$
$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{U_I(t_0,-T)|0\rangle}{e^{-iE_0(T+t_0)}\langle\Omega|0\rangle}$
$\langle\Omega| = \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|U_I(T,t_0)}{e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle}$
Assume $x^0 > y^0 > t^0$
$\langle\Omega|T\varphi(x)\varphi(y)|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{1}{e^{-iE_0(T+t_0)}\langle\Omega|0\rangle} \frac{1}{e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle}$
$\langle 0|U_I(T,t_0)\, U_I^+(x^0,t_0)\,\varphi_I(x)\,U_I(x^0,t_0)\, \underbrace{U_I^+(y^0,t_0)}_{U(t_0,y^0)}\,\varphi_I(y)\,U_I(y^0,t_0)\,U_I(t_0,-T)|0\rangle$
$1 = \langle\Omega|\Omega\rangle = \lim_{|\langle\Omega|0\rangle|^2 e^{-2E_0T}} \langle 0|U_I(T,t_0)\,U_I(t_0,-T)|0\rangle$


$\langle \Omega | T\,\varphi(x)\varphi(y) | \Omega \rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0 | T U_I(T,x^0)\,\varphi_I(x)\,U_I(x^0,y^0)\,\varphi_I(y)\,U_I(y^0,-T) | \Omega \rangle}{\langle 0 | U_I(T,-T) | 0 \rangle}$
$= \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0 | T\,\varphi_I(x)\varphi_I(y) \exp\left[-i\int_{-T}^{T} dt\, H_I(t)\right] | 0 \rangle}{\langle 0 | T \exp\left[-i\int_{-T}^{T} dt\, H_I(t)\right] | 0 \rangle}$


$\langle 0|T U_I(T,x^0)\,\varphi_I(x)\,U_I(x^0,y^0)\,\varphi_I(y)\,U_I(y^0,-T)|\Omega\rangle$
$\langle 0|U_I(T,-T)|0\rangle$
$\langle 0|T\,\varphi_I(x)\,\varphi_I(y)\exp\left[-i\int_{-T}^{T} dt\, H_I(t)\right]|0\rangle$
$\langle 0|T\exp\left[-i\int_{-T}^{T} dt\, H_I(t)\right]|0\rangle$
$e^A = 1 + A + \cdots$

$$= \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|T\,\varphi_I(x)\varphi_I(y)\exp\left[-i\int_{-T}^{T}dt\,H_I(t)\right]|0\rangle}{\langle 0|T\exp\left[-i\int_{-T}^{T}dt\,H_I(t)\right]|0\rangle}$$

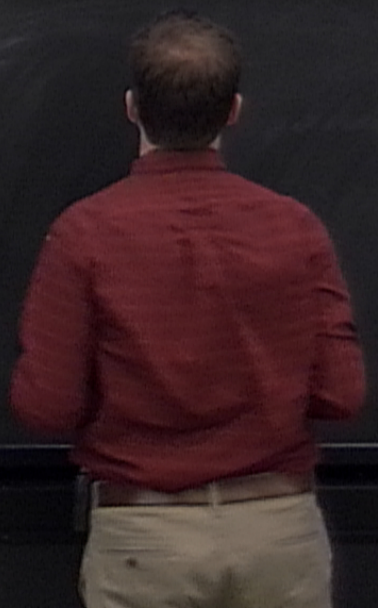
Wick's Theorem

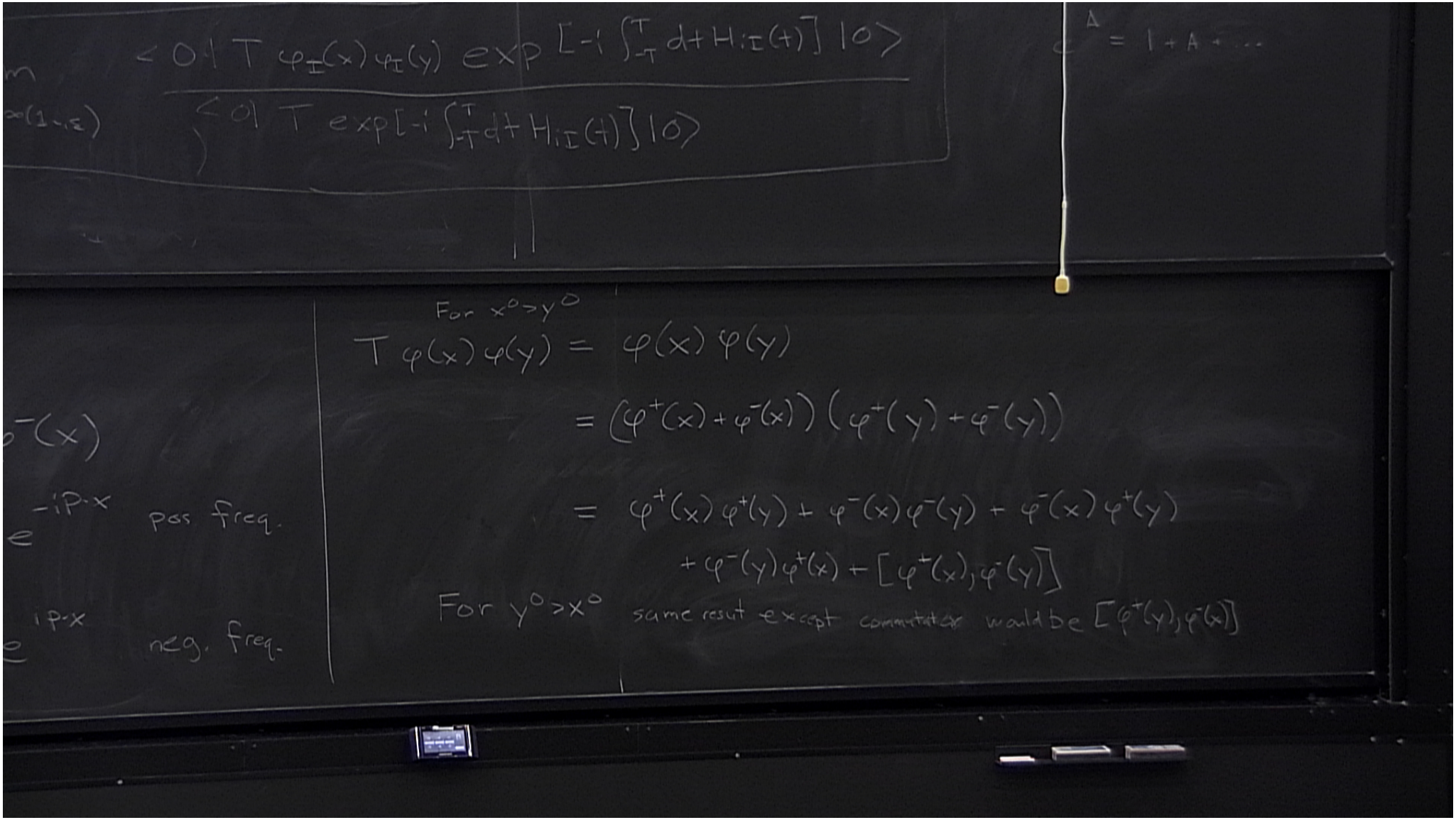
$$\varphi(x) = \varphi^+(x) + \varphi^-(x)$$

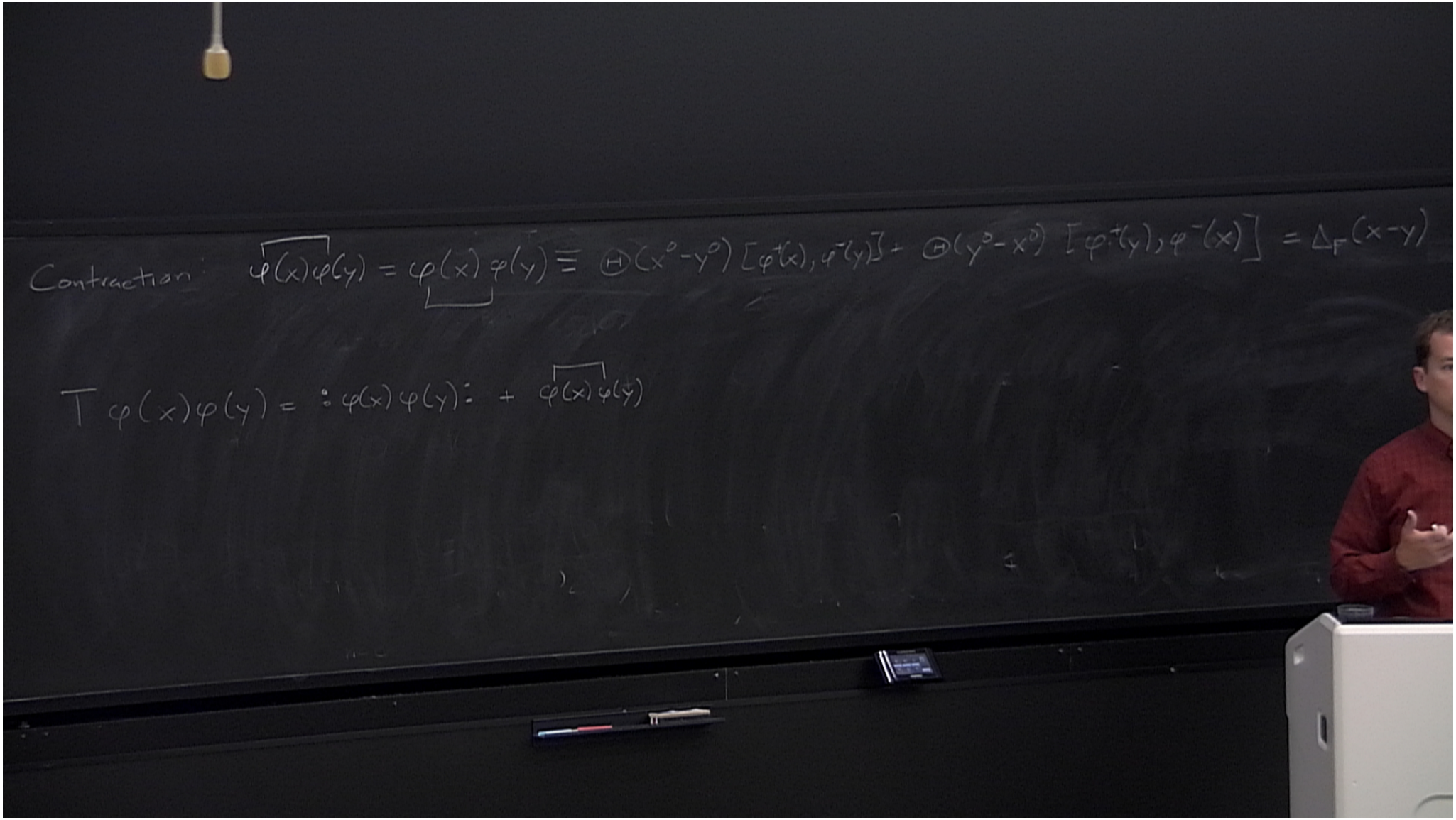
$$\varphi^+(x) = \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}}\, a_{\vec{p}}\, e^{-ip\cdot x} \quad \text{pos. freq.}$$

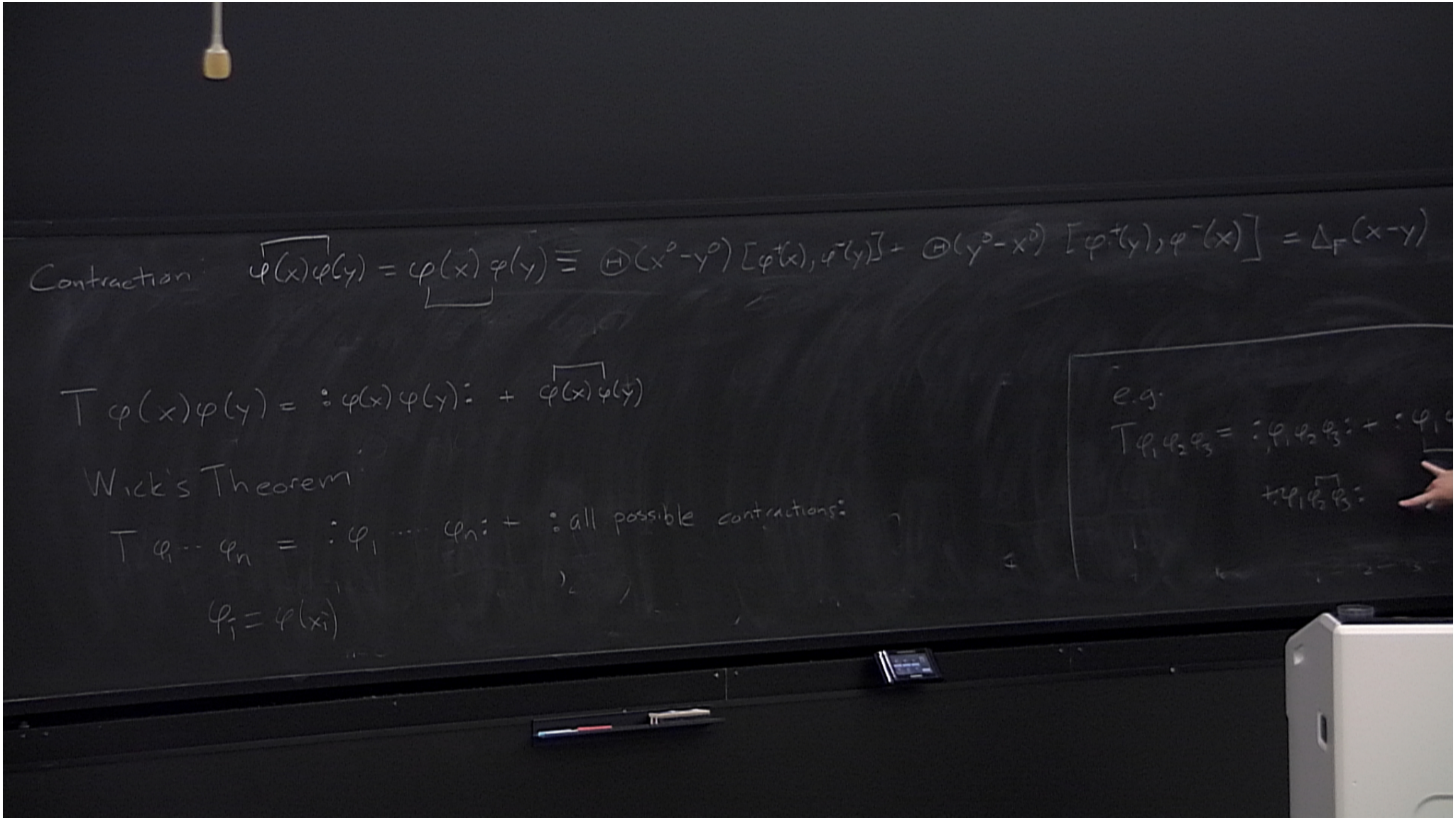
$$\varphi^-(x) = \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}}\, a^\dagger_{\vec{p}}\, e^{ip\cdot x} \quad \text{neg. freq.}$$

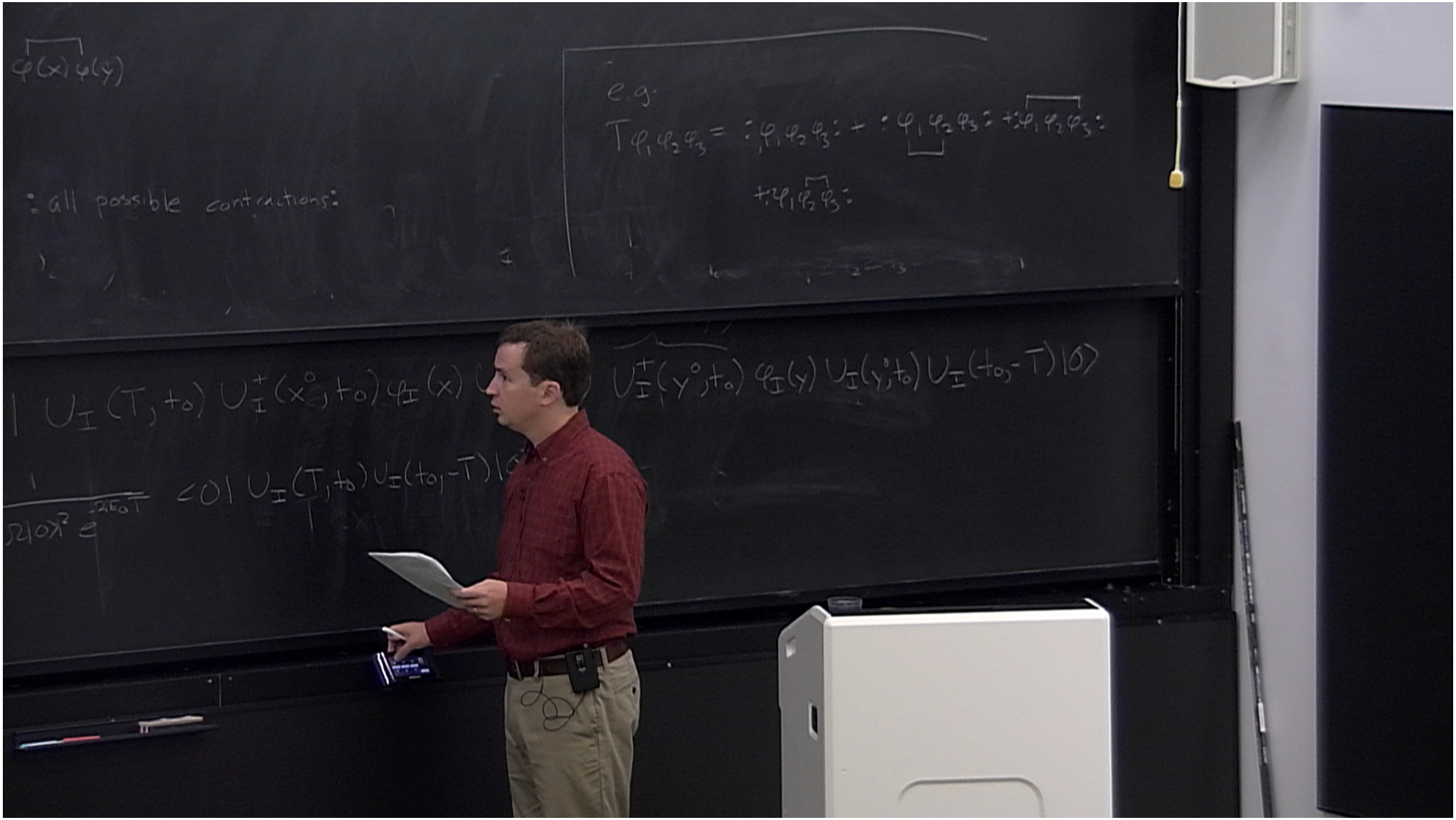
$T\,\varphi(x)\varphi(y) =$




$\langle 0|T\,\varphi_I(x)\varphi_I(y)\exp\left[-i\int_{-T}^{T}dt\,H_I(t)\right]|0\rangle$
$\langle 0|T\exp\left[-i\int_{-T}^{T}dt\,H_I(t)\right]|0\rangle$
$e^A = 1 + A + \ldots$
$\varphi^-(x)$
$e^{-ip\cdot x}$ pos freq.
$e^{ip\cdot x}$ neg. freq.
For $x^0 > y^0$
$T\,\varphi(x)\varphi(y) = \varphi(x)\varphi(y)$
$= (\varphi^+(x) + \varphi^-(x))(\varphi^+(y) + \varphi^-(y))$
$= \varphi^+(x)\varphi^+(y) + \varphi^-(x)\varphi^-(y) + \varphi^-(x)\varphi^+(y)$
$+ \varphi^-(y)\varphi^+(x) + [\varphi^+(x), \varphi^-(y)]$
For $y^0 > x^0$ same result except commutator would be $[\varphi^+(y), \varphi^-(x)]$


Contraction $\overbrace{\varphi(x)\varphi(y)} = \varphi(x)\varphi(y) \equiv \Theta(x^0-y^0)[\varphi^+(x),\varphi^-(y)] + \Theta(y^0-x^0)[\varphi^+(y),\varphi^-(x)] = \Delta_F(x-y)$
$T\varphi(x)\varphi(y) = :\varphi(x)\varphi(y): + \overbrace{\varphi(x)\varphi(y)}$


Contraction $\overbrace{\varphi(x)\varphi(y)} = \varphi(x)\varphi(y) \equiv \Theta(x^0-y^0)[\varphi^+(x),\varphi^-(y)] + \Theta(y^0-x^0)[\varphi^+(y),\varphi^-(x)] = \Delta_F(x-y)$
$T\varphi(x)\varphi(y) = :\varphi(x)\varphi(y): + \overbrace{\varphi(x)\varphi(y)}$
Wick's Theorem
$T\varphi_1\cdots\varphi_n = :\varphi_1\cdots\varphi_n: + :\text{all possible contractions}:$
$\varphi_i = \varphi(x_i)$
e.g.
$T\varphi_1\varphi_2\varphi_3 = :\varphi_1\varphi_2\varphi_3: + :\varphi_1$
$+ :\varphi_1\overbrace{\varphi_2\varphi_3}:$


φ(x) φ(y)
: all possible contractions :
e.g.
T φ1 φ2 φ3 = : φ1 φ2 φ3 : + : φ1 φ2 φ3 : + : φ1 φ2 φ3 :
+ : φ1 φ2 φ3 :

$T\varphi_1 \cdots \varphi_n = :\varphi_1 \cdots \varphi_n: + :\text{all possible contractions}:$

$\varphi_i = \varphi(x_i)$

Proof: Assume for $n = m-1$

Also assume $x_1^0 \geq x_2^0 \geq \cdots \geq x_m^0$

$$T\varphi_1 \cdots \varphi_m = \varphi_1 T\varphi_2 \cdots \varphi_m$$

$$= (\varphi_1^+ + \varphi_1^-)(:\varphi_2 \cdots \varphi_m: + :\text{all contractions that don't involve } \varphi_1:)$$

↑ can bring inside $:\,:$

$$= \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|T\,\varphi_I(x)\varphi_I(y)\exp\left[-i\int_{-T}^{T}dt\,H_I(t)\right]|0\rangle}{\langle 0|T\exp\left[-i\int_{-T}^{T}dt\,H_I(t)\right]|0\rangle}$$

$$\varphi^+ :\varphi_2\cdots\varphi_m: = :\varphi_2\cdots\varphi_m:\varphi_1^+ + [\varphi_1^+, :\varphi_2\cdots\varphi_m:]$$

$$= :\varphi_2\cdots\varphi_m:\varphi_1^+ + :[\varphi_1^+,\varphi_2^-\}\varphi_3\cdots\varphi_m: + \cdots + :\varphi_2\cdots[\varphi_1^+,\varphi_m^-]:$$

$$= :\varphi_1^+\varphi_2\cdots\varphi_m: + :\overbrace{\varphi_1\varphi_2}\cdots\varphi_m: + \cdots + :\overbrace{\varphi_1\cdots\varphi_m}:$$

$$= \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|T\,\varphi_I(x)\varphi_I(y)\exp\left[-i\int_{-T}^{T}dt\,H_I(t)\right]|0\rangle}{\langle 0|T\exp\left[-i\int_{-T}^{T}dt\,H_I(t)\right]|0\rangle}$$

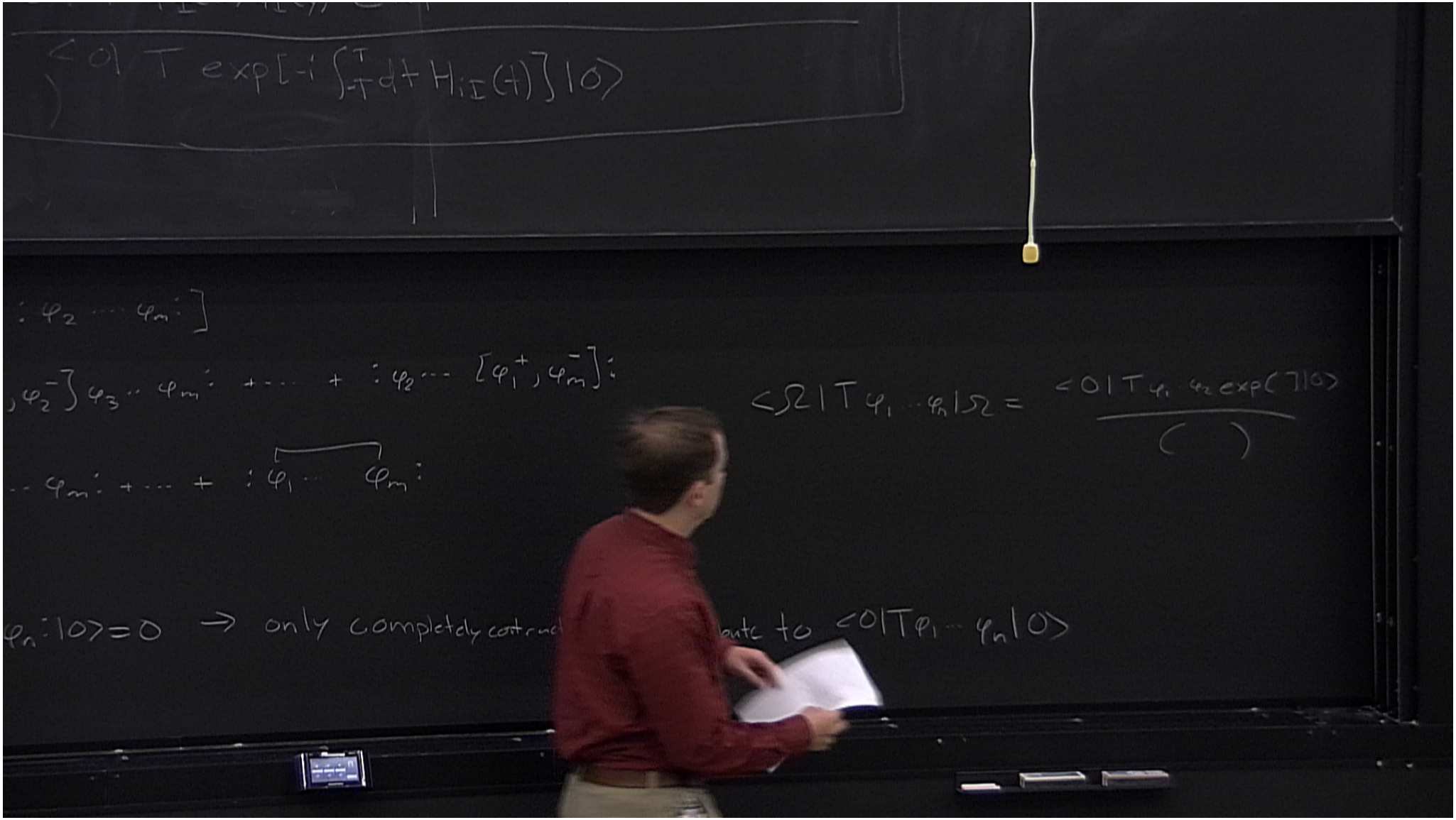
$$\varphi^+ :\varphi_2\cdots\varphi_m: = :\varphi_2\cdots\varphi_m:\varphi_1^+ + [\varphi_1^+, :\varphi_2\cdots\varphi_m:]$$

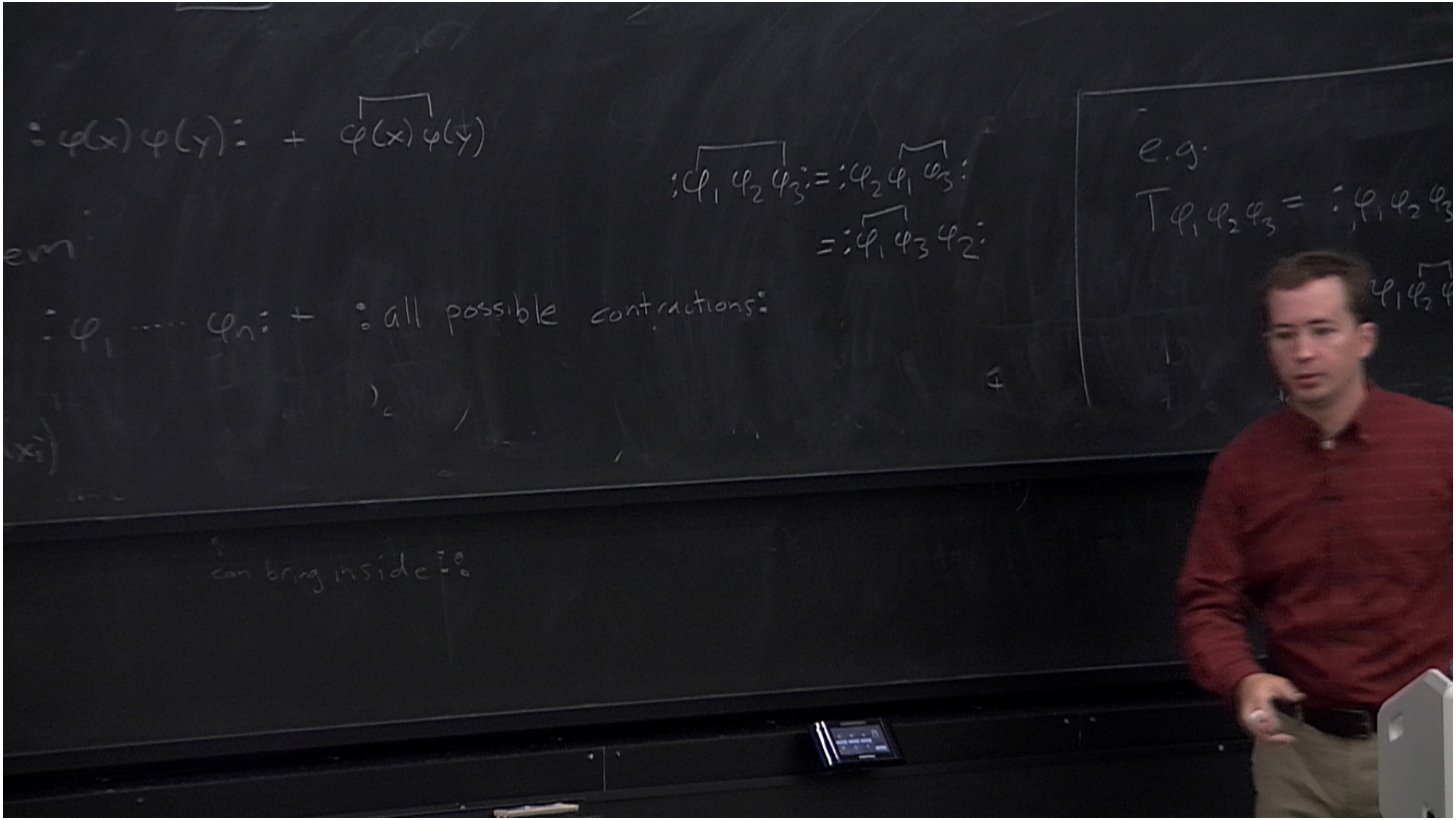
$$= :\varphi_2\cdots\varphi_m:\varphi_1^+ + :[\varphi_1^+,\varphi_2^-]\varphi_3\cdots\varphi_m: + \cdots + :\varphi_2\cdots[\varphi_1^+,\varphi_m^-]:$$

$$= :\varphi_1^+\varphi_2\cdots\varphi_m: + :\overbrace{\varphi_1\varphi_2}\cdots\varphi_m: + \cdots + :\overbrace{\varphi_1\cdots\varphi_m}:$$

so holds for m fields as well

$\langle 0|:\varphi_1\cdots\varphi_n:|0\rangle = 0$ → only completely contracted terms contribute to $\langle 0|T\varphi_1\cdots\varphi_n|0\rangle$


$\langle 0| T \exp[-i \int_{-T}^{T} dt\, H_{I\Sigma}(t)] |0\rangle$
$: \varphi_2 \cdots \varphi_m :]$
$\varphi_2^-\} \varphi_3 \cdots \varphi_m : + \cdots + : \varphi_2 \cdots [\varphi_1^+, \varphi_m^-]:$
$\cdots \varphi_m : + \cdots + : \varphi_1 \cdots \varphi_m :$
$\langle \Omega | T \varphi_1 \cdots \varphi_n | \Omega \rangle = \frac{\langle 0 | T \varphi_1 \cdots \varphi_n \exp(\,) | 0 \rangle}{(\,)}$
$\varphi_n : |0\rangle = 0 \rightarrow$ only completely contr
$\ldots$ to $\langle 0 | T \varphi_1 \cdots \varphi_n | 0 \rangle$


$:\varphi(x)\varphi(y): + \overbrace{\varphi(x)\varphi(y)}$
$:\varphi_1 \cdots \varphi_n: + :\text{all possible contractions}:$
$:\overbrace{\varphi_1\varphi_2}\varphi_3: = :\varphi_2\overbrace{\varphi_1\varphi_3}: = :\overbrace{\varphi_1\varphi_3}\varphi_2:$
e.g.
can bring inside